The harmonic index of graphs based on some operations related to the lexicographic product

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Abstract

The harmonic index of a graph G is defined as the sum of the weights $\frac{2}{\deg_G(u) + \deg_G(v)}$ of all edges uv of G , where $\deg_G(u)$ denotes the degree of a vertex u in G . In this paper, we investigate the harmonic index of graphs based on operations related to the lexicographic product, subdivision graph, t -subdivision graph, vertex-semitotal graph, edge-semitotal graph and total graph.

Keywords Harmonic index · Subdivision · Lexicographic product · F -product

Mathematics Subject Classification 05C07 · 05C76

Introduction

Throughout this paper, all graphs are finite, simple, undirected and connected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We will use P_n to denote the path of order n .

The inverse degree and first Zagreb index of a graph G are defined as $r(G) = \sum_{u \in V(G)} \frac{1}{\deg_G(u)}$ and $M_1(G) = \sum_{u \in V(G)} \deg_G^2(u) = \sum_{uv \in E(G)} (\deg_G(u) + \deg_G(v))$, respectively [1, 7].

The lexicographic product $G_1[G_2]$ of two graphs G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$ in which two vertices (u_1, v_1) and (u_2, v_2) of $G_1[G_2]$ are adjacent if and only if $[u_1 = u_2 \text{ and } v_1 v_2 \in E(G_2)]$ or $[u_1 u_2 \in E(G_1) \text{ and } v_1, v_2 \in V(G_2)]$.

For a graph G , there are several related graphs such as line graph $L(G)$, subdivision graph $S(G)$, t -subdivision graph $S_t(G)$, vertex-semitotal graph $R(G)$, edge-semitotal graph $Q(G)$ and total graph $T(G)$, defined as follows [2, 6, 15]:

$L(G)$ has as its vertices the edges of G ; adjacency in $L(G)$ is defined as adjacency for the corresponding edges of G .

$S(G)$ is obtained from G by replacing each edge of G by a path of length 2.

$S_t(G)$ is obtained from G by replacing each edge of G by a path of length $t + 1$. Obviously, $S_1(G) = S(G)$.

$R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G and then joining each new vertex to the end vertices of the corresponding edge.

$Q(G)$ is obtained from G by inserting a new vertex into each edge of G and then joining with edges those pairs of new vertices on adjacent edges of G .

$T(G)$ has as its vertices, the edges and vertices of G ; adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G .

Let $F \in \{S, R, Q, T\}$. The F -product $G_1[G_2]_F$ of two graphs G_1 and G_2 is a graph with vertex set $(V(G_1) \cup E(G_1)) \times V(G_2)$ in which two vertices (u_1, v_1) and (u_2, v_2) of $G_1[G_2]_F$ are adjacent if and only if $[u_1 = u_2 \in V(G_1) \text{ and } v_1 v_2 \in E(G_2)]$ or $[u_1 u_2 \in E(F(G_1)) \text{ and } v_1, v_2 \in V(G_2)]$ [12].

We refer to the vertices in $V(G_1)$ and $V(F(G_1)) - V(G_1)$ as black vertices and white vertices, respectively, where $F \in \{S, R, Q, T\}$.

$P_4[P_2]_S$, $P_4[P_2]_R$, $P_4[P_2]_Q$ and $P_4[P_2]_T$ are shown in Fig. 1.

For a graph G , the Randić index $R(G)$ is defined as $R(G) = \sum_{uv \in E(G)} (\deg_G(u) \deg_G(v))^{-\frac{1}{2}}$ [11]. The sum-connectivity index $\chi(G)$ and general sum-connectivity index $\chi_\alpha(G)$ are closely related variants of Randić index, defined as $\chi(G) = \sum_{uv \in E(G)} (\deg_G(u) + \deg_G(v))^{-\frac{1}{2}}$ and $\chi_\alpha(G) = \sum_{uv \in E(G)} (\deg_G(u) + \deg_G(v))^\alpha$, where α is a real number [19, 20]. It has been found that the (general) sum-connectivity index and Randić index correlate well among

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Fig. 1 $P_4[P_2]_F$, where $F \in \{S, R, Q, T\}$

themselves and with the π -electronic energies of benzenoid hydrocarbons [4, 5].

In this paper, we consider the harmonic index, which is another variant of Randić index. For a graph G , the harmonic index $H(G)$ is defined as $H(G) = \sum_{uv \in E(G)} \frac{2}{\deg_G(u) + \deg_G(v)}$ [1]. In recent years, this vertex-degree-based topological index has been extensively studied. For example, Zhong [16] gave the minimum and maximum values of the harmonic index for simple connected graphs and trees and characterized the corresponding extremal graphs. Recently, Onagh investigated the harmonic index of product graphs, subdivision graphs, t -subdivision graphs, vertex-semitotal graphs, edge-semitotal graphs, total graphs and F -sum of graphs, where $F \in \{S, S_t, R, Q, T\}$ [8–10]. More results on the harmonic index can be found in [3, 13, 14, 17, 18].

In this paper, we extend the definition of F -product to S_t -product of graphs and give an upper bound for the harmonic index of F -product of graphs, where $F \in \{S, S_t, R, Q, T\}$.

The harmonic index for F -products of graphs

First, we give an upper bound for the harmonic index of $G_1[G_2]_F$ in terms of $H(S(G_1))$, $H(G_2)$, $r(G_1)$ and $r(G_2)$, where $F \in \{S, S_t\}$.

Theorem 2.1 *Let G_1 and G_2 be two graphs. Then*

$$H(G_1[G_2]_S) < \frac{1}{4}n_2H(S(G_1)) + \frac{1}{4}n_1H(G_2) + \frac{1}{4}\frac{m_2}{n_2}r(G_1) + n_2m_1r(G_2),$$

where $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$, $i = 1, 2$.

Proof Let $\deg(u, v) = \deg_{G_1[G_2]_S}(u, v)$ be the degree of a vertex (u, v) in $G_1[G_2]_S$. By the definition of the harmonic index, we have

$$\begin{aligned} H(G_1[G_2]_S) &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{\deg(u, v_1) + \deg(u, v_2)} \\ &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(S(G_1))} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ &=: \sum 1 + \sum 2. \end{aligned}$$

Then,

$$\begin{aligned} \sum 1 &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{(n_2 \deg_{G_1}(u) + \deg_{G_2}(v_1)) + (n_2 \deg_{G_1}(u) + \deg_{G_2}(v_2))} \\ &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{2n_2 \deg_{G_1}(u) + (\deg_{G_2}(v_1) + \deg_{G_2}(v_2))}. \end{aligned}$$

By Jensen's inequality, for every $u \in V(G_1)$ and $v_1 v_2 \in E(G_2)$, we have

$$\begin{aligned} &\frac{2}{2n_2 \deg_{G_1}(u) + (\deg_{G_2}(v_1) + \deg_{G_2}(v_2))} \\ &\leq \frac{1}{4n_2} \frac{1}{\deg_{G_1}(u)} + \frac{1}{4} \frac{2}{\deg_{G_2}(v_1) + \deg_{G_2}(v_2)}, \end{aligned} \quad (2.1)$$

with equality if and only if $2n_2 \deg_{G_1}(u) = \deg_{G_2}(v_1) + \deg_{G_2}(v_2)$.

So,

$$\begin{aligned} \sum 1 &\leq \frac{1}{4n_2} \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \frac{1}{\deg_{G_1}(u)} \\ &\quad + \frac{1}{4} \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \frac{2}{\deg_{G_2}(v_1) + \deg_{G_2}(v_2)} \\ &= \frac{1}{4n_2} \sum_{u \in V(G_1)} \left(m_2 \times \frac{1}{\deg_{G_1}(u)} \right) + \frac{1}{4} \sum_{u \in V(G_1)} H(G_2) \\ &= \frac{1}{4} \frac{m_2}{n_2} r(G_1) + \frac{1}{4} n_1 H(G_2). \end{aligned}$$

Also,

$$\begin{aligned} \sum 2 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(S(G_1))} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(S(G_1)) \\ u_1 \in V(G_1), u_2 \in V(S(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(S(G_1)) \\ u_1 \in V(G_1), u_2 \in V(S(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 \in V(G_1)} \frac{2 \deg_{G_1}(u_1)}{(n_2 \deg_{G_1}(u_1) + \deg_{G_2}(v_1)) + 2n_2} \\ &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 \in V(G_1)} \frac{2 \deg_{G_1}(u_1)}{n_2 (\deg_{G_1}(u_1) + 2) + \deg_{G_2}(v_1)}. \end{aligned}$$

Similarly, for every $u_1 \in V(G_1)$ and every $v_1 \in V(G_2)$, we have

$$\begin{aligned} &\frac{2}{n_2 (\deg_{G_1}(u_1) + 2) + \deg_{G_2}(v_1)} \\ &\leq \frac{1}{4n_2} \frac{2}{\deg_{G_1}(u_1) + 2} + \frac{1}{2} \frac{1}{\deg_{G_2}(v_1)}, \end{aligned} \quad (2.2)$$

with equality if and only if $n_2 (\deg_{G_1}(u_1) + 2) = \deg_{G_2}(v_1)$. Thus,

$$\begin{aligned} \sum 2 &\leq \frac{1}{4n_2} \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 \in V(G_1)} \frac{2 \deg_{G_1}(u_1)}{\deg_{G_1}(u_1) + 2} \\ &\quad + \frac{1}{2} \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 \in V(G_1)} \frac{\deg_{G_1}(u_1)}{\deg_{G_2}(v_1)} \\ &= \frac{1}{4n_2} \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} H(S(G_1)) \\ &\quad + \frac{1}{2} \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \left(2m_1 \times \frac{1}{\deg_{G_2}(v_1)} \right) \\ &= \frac{1}{4} n_2 H(S(G_1)) + n_2 m_1 r(G_2). \end{aligned}$$

Therefore,

$$\begin{aligned} H(G_1[G_2]_S) &\leq \frac{1}{4} n_2 H(S(G_1)) + \frac{1}{4} n_1 H(G_2) \\ &\quad + \frac{1}{4} \frac{m_2}{n_2} r(G_1) + n_2 m_1 r(G_2). \end{aligned}$$

Moreover, equality holds in above inequality if and only if the inequalities (2.1) and (2.2) be equalities, i.e., G_1 and G_2 are k_1 -regular and k_2 -regular graphs, respectively, such that $2n_2 k_1 = k_2 + k_2$ and $n_2(k_1 + 2) = k_2$, a contradiction. This completes the proof. $\square$

Example 2.2 For any $n \geq 2$,

$$\begin{aligned} H(P_n[P_2]_S) &= \left(\frac{2}{3} + \frac{1}{5}(n-2) \right) + \left(\frac{16}{7} + \frac{16}{9}(n-2) \right) \\ &= \frac{62}{21} + \frac{89}{45}(n-2). \end{aligned}$$

In the following, we extend the definition of F -product to S_t -product of graphs.

Definition 2.3 Let G_1 and G_2 be two graphs. The S_t -product $G_1[G_2]_{S_t}$ is a graph with vertex set $V(S_t(G_1)) \times V(G_2)$ in which two vertices (u_1, v_1) and (u_2, v_2) of $G_1[G_2]_{S_t}$ are adjacent if and only if $[u_1 = u_2 \in V(G_1) \text{ and } v_1 v_2 \in E(G_2)]$ or $[u_1 u_2 \in E(S_t(G_1)) \text{ and } v_1, v_2 \in V(G_2)]$.

Similarly, we refer to the vertices in $V(G_1)$ and $V(S_i(G_1)) - V(G_1)$ as black vertices and white vertices, respectively.

$P_4[P_2]_{S_3}$ is shown in Fig. 2.

Theorem 2.4 Let G_1 and G_2 be two graphs. Then

$$H(G_1[G_2]_{S_i}) < \frac{1}{4}n_2H(S(G_1)) + \frac{1}{4}n_1H(G_2) + \frac{1}{4}\frac{m_2}{n_2}r(G_1) + n_2m_1r(G_2) + \frac{1}{2}(t-1)n_2m_1,$$

where $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$, $i = 1, 2$.

Proof Let $\deg(u, v) = \deg_{G_1[G_2]_{S_i}}(u, v)$ be the degree of a vertex (u, v) in $G_1[G_2]_{S_i}$. Then,

$$\begin{aligned} H(G_1[G_2]_{S_i}) &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \frac{2}{\deg(u, v_1) + \deg(u, v_2)} \\ &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(S_i(G_1))} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ &=: \sum 1 + \sum 2. \end{aligned}$$

By the proof of Theorem 2.1, one can get

$$\sum 1 \leq \frac{1}{4}\frac{m_2}{n_2}r(G_1) + \frac{1}{4}n_1H(G_2),$$

with equality $2n_2 \deg_{G_1}(u) = \deg_{G_2}(v_1) + \deg_{G_2}(v_2)$ for all $u \in V(G_1)$ and $v_1, v_2 \in E(G_2)$.

Note that

Fig. 2 $P_4[P_2]_{S_3}$

$$\begin{aligned} \sum 2 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(S_i(G_1)) \\ u_1 \in V(G_1), u_2 \in V(S_i(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(S_i(G_1)) \\ u_1, u_2 \in V(S_i(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(S_i(G_1)) \\ u_1 \in V(G_1), u_2 \in V(S_i(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(S_i(G_1)) \\ u_1, u_2 \in V(S_i(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(S_i(G_1)) \\ u_1 \in V(G_1), u_2 \in V(S_i(G_1)) - V(G_1)}} \frac{2}{2n_2 + 2n_2} \\ &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(S_i(G_1)) \\ u_1 \in V(G_1), u_2 \in V(S_i(G_1)) - V(G_1)}} \frac{2}{2n_2 + 2n_2} \\ &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(S_i(G_1)) \\ u_1 \in V(G_1), u_2 \in V(S_i(G_1)) - V(G_1)}} \frac{2}{2n_2 + 2n_2} \\ &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(S_i(G_1)) \\ u_1 \in V(G_1), u_2 \in V(S_i(G_1)) - V(G_1)}} \frac{1}{2n_2}m_1(t-1) \\ &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(S_i(G_1)) \\ u_1 \in V(G_1), u_2 \in V(S_i(G_1)) - V(G_1)}} \frac{2}{2n_2 + 2n_2} \\ &\quad + \frac{1}{2}(t-1)n_2m_1. \end{aligned}$$

According to the proof of Theorem 2.1, we have

$$\sum 2 \leq \frac{1}{4}n_2H(S(G_1)) + n_2m_1r(G_2) + \frac{1}{2}(t-1)n_2m_1,$$

with equality if and only if $n_2(\deg_{G_1}(u_1) + 2) = \deg_{G_2}(v_1)$ for all $u_1 \in V(G_1)$ and $v_1 \in V(G_2)$.

Therefore,

$$H(G_1[G_2]_{S_i}) \leq \frac{1}{4}n_2H(S(G_1)) + \frac{1}{4}n_1H(G_2) + \frac{1}{4}\frac{m_2}{n_2}r(G_1) \\ + n_2m_1r(G_2) + \frac{1}{2}(t-1)n_2m_1.$$

One can see that equality in above inequality cannot occur. This completes the proof. $\square$

Example 2.5 For any $n \geq 2$,

$$H(P_n[P_2]_{S_i}) = \left(\frac{2}{3} + \frac{1}{5}(n-2)\right) \\ + \left(\frac{16}{7} + \frac{16}{9}(n-2) + (t-1)(n-1)\right) \\ = \frac{62}{21} + \frac{89}{45}(n-2) + (t-1)(n-1).$$

In the following, we give an upper bound for the harmonic index of $G_1[G_2]_R$ in terms of $H(R(G_1))$, $H(G_1)$, $H(G_2)$, $r(G_1)$ and $r(G_2)$.

Theorem 2.6 Let G_1 and G_2 be two graphs. Then

$$H(G_1[G_2]_R) < \frac{1}{4}n_2H(R(G_1)) + \frac{1}{4}n_1H(G_2) - \frac{5}{72}n_2H(G_1) \\ + \frac{1}{8}\frac{m_2}{n_2}r(G_1) + \frac{13}{9}n_2m_1r(G_2),$$

where $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$, $i = 1, 2$.

Proof Let $\deg(u, v) = \deg_{G_1[G_2]_R}(u, v)$ be the degree of a vertex (u, v) in $G_1[G_2]_R$. Then,

$$H(G_1[G_2]_R) = \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{\deg(u, v_1) + \deg(u, v_2)} \\ + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(R(G_1))} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ =: \sum 1 + \sum 2.$$

Note that

$$\sum 1 = \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{(2n_2 \deg_{G_1}(u) + \deg_{G_2}(v_1)) + (2n_2 \deg_{G_1}(u) + \deg_{G_2}(v_2))} \\ = \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{4n_2 \deg_{G_1}(u) + (\deg_{G_2}(v_1) + \deg_{G_2}(v_2))}.$$

Similar to the proof of Theorem 2.1, one can get

$$\sum 1 \leq \frac{1}{8}\frac{m_2}{n_2}r(G_1) + \frac{1}{4}n_1H(G_2), \quad (2.3)$$

with equality $4n_2 \deg_{G_1}(u) = \deg_{G_2}(v_1) + \deg_{G_2}(v_2)$ for all $u \in V(G_1)$ and $v_1 v_2 \in E(G_2)$.

Also,

$$\sum 2 = \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R(G_1)) \\ u_1 \in V(G_1), u_2 \in V(R(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ =: \sum 2' + \sum 2''.$$

Note that

$$\sum 2' = \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ = \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} \frac{2}{(2n_2 \deg_{G_1}(u_1) + \deg_{G_2}(v_1)) + (2n_2 \deg_{G_1}(u_2) + \deg_{G_2}(v_2))} \\ = \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} \frac{2}{2n_2 (\deg_{G_1}(u_1) + \deg_{G_1}(u_2)) + \deg_{G_2}(v_1) + \deg_{G_2}(v_2)}.$$

By Jensen's inequality, for every $u_1 u_2 \in E(G_1)$ and $v_1, v_2 \in V(G_2)$, we have

$$\frac{2}{2n_2 (\deg_{G_1}(u_1) + \deg_{G_1}(u_2)) + \deg_{G_2}(v_1) + \deg_{G_2}(v_2)} \\ \leq \frac{1}{18n_2} \frac{2}{\deg_{G_1}(u_1) + \deg_{G_1}(u_2)} + \frac{2}{9} \frac{1}{\deg_{G_2}(v_1)} + \frac{2}{9} \frac{1}{\deg_{G_2}(v_2)}, \quad (2.4)$$

with equality if and only if $2n_2 (\deg_{G_1}(u_1) + \deg_{G_1}(u_2)) = \deg_{G_2}(v_1) = \deg_{G_2}(v_2)$.

So, $\sum 2' \leq \frac{1}{18}n_2H(G_1) + \frac{4}{9}m_1n_2r(G_2)$.

Finally,

$$\begin{aligned} \sum 2'' &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R(G_1)) \\ u_1 \in V(G_1), u_2 \in V(R(G_1)) - V(G_1)}} \frac{2}{(2n_2 \deg_{G_1}(u_1) + \deg_{G_2}(v_1)) + 2n_2} \\ &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R(G_1)) \\ u_1 \in V(G_1), u_2 \in V(R(G_1)) - V(G_1)}} \frac{2}{2n_2 (\deg_{G_1}(u_1) + 1) + \deg_{G_2}(v_1)}. \end{aligned}$$

By the similar argument as above, for every $u_1 \in V(G_1)$ and $v_1 \in V(G_2)$, we have

$$\begin{aligned} &\frac{2}{2n_2 (\deg_{G_1}(u_1) + 1) + \deg_{G_2}(v_1)} \\ &\leq \frac{1}{4n_2} \frac{1}{\deg_{G_1}(u_1) + 1} + \frac{1}{2} \frac{1}{\deg_{G_2}(v_1)}, \end{aligned} \quad (2.5)$$

with equality if and only if $2n_2 (\deg_{G_1}(u_1) + 1) = \deg_{G_2}(v_1)$.

Hence,

$$\begin{aligned} \sum 2'' &\leq \frac{1}{4n_2} \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 \in V(G_1)} \frac{\deg_{G_1}(u_1)}{\deg_{G_1}(u_1) + 1} \\ &\quad + \frac{1}{2} \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \left(2m_1 \times \frac{1}{\deg_{G_2}(v_1)} \right) \\ &= \frac{1}{4} n_2 \left(H(R(G_1)) - \frac{1}{2} H(G_1) \right) + n_2 m_1 r(G_2). \end{aligned}$$

Therefore,

$$\begin{aligned} H(G_1[G_2]_R) &\leq \frac{1}{4} n_2 H(R(G_1)) + \frac{1}{4} n_1 H(G_2) - \frac{5}{72} n_2 H(G_1) \\ &\quad + \frac{1}{8} \frac{m_2}{n_2} r(G_1) + \frac{13}{9} n_2 m_1 r(G_2). \end{aligned}$$

Furthermore, equality holds in above inequality if and only if the inequalities (2.3), (2.4) and (2.5) be equalities, i.e., G_1 and G_2 are k_1 -regular and k_2 -regular graphs, respectively, such that $4n_2 k_1 = k_2 + k_2$, $2n_2(k_1 + k_1) = k_2$ and $2n_2(k_1 + 1) = k_2$, a contradiction. This completes the proof. $\square$

Example 2.7 For any $n \geq 3$,

$$\begin{aligned} H(P_n[P_2]_R) &= \left(\frac{2}{5} + \frac{1}{9}(n-2) \right) + \left(\frac{8}{7} + \frac{4}{9}(n-3) \right) \\ &\quad + \left(\frac{16}{9} + \frac{16}{13}(n-2) \right) \\ &= \frac{1046}{315} + \frac{157}{117}(n-2) + \frac{4}{9}(n-3). \end{aligned}$$

Now, we give an upper bound for the harmonic index of $G_1[G_2]_Q$ in terms of $H(Q(G_1))$, $H(L(G_1))$, $H(G_2)$, $M_1(G_1)$, $r(G_1)$ and $r(G_2)$.

Theorem 2.8 Let G_1 and G_2 be two graphs. Then

$$\begin{aligned} H(G_1[G_2]_Q) &< \frac{1}{4} n_2 H(Q(G_1)) + \frac{3}{16} n_2 H(L(G_1)) \\ &\quad + \frac{1}{4} n_1 H(G_2) + \frac{3}{64} n_2 M_1(G_1) \\ &\quad + \frac{1}{4} \frac{m_2}{n_2} r(G_1) + n_2 m_1 r(G_2) - \frac{3}{32} n_2 m_1, \end{aligned}$$

where $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$, $i = 1, 2$.

Proof Let $\deg(u, v) = \deg_{G_1[G_2]_Q}(u, v)$ be the degree of a vertex (u, v) in $G_1[G_2]_Q$. Then,

$$\begin{aligned} H(G_1[G_2]_Q) &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{\deg(u, v_1) + \deg(u, v_2)} \\ &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(Q(G_1))} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ &=: \sum 1 + \sum 2. \end{aligned}$$

Note that

$$\begin{aligned} \sum 1 &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{(n_2 \deg_{G_1}(u) + \deg_{G_2}(v_1)) + (n_2 \deg_{G_1}(u) + \deg_{G_2}(v_2))} \\ &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{2n_2 \deg_{G_1}(u) + (\deg_{G_2}(v_1) + \deg_{G_2}(v_2))}. \end{aligned}$$

By the proof of Theorem 2.1, we have

$$\sum 1 \leq \frac{1}{4} \frac{m_2}{n_2} r(G_1) + \frac{1}{4} n_1 H(G_2), \quad (2.6)$$

with equality $2n_2 \deg_{G_1}(u) = \deg_{G_2}(v_1) + \deg_{G_2}(v_2)$ for all $u \in V(G_1)$ and $v_1, v_2 \in E(G_2)$.

Also,

$$\begin{aligned} \sum 2 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} \\ &\quad \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1, u_2 \in V(Q(G_1)) - V(G_1)}} \\ &\quad \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\ &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} \\ &\quad \frac{2}{n_2 \deg_{Q(G_1)}(u_1) + \deg_{G_2}(v_1) + n_2 \deg_{Q(G_1)}(u_2)} \\ &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1, u_2 \in V(Q(G_1)) - V(G_1)}} \\ &\quad \frac{2}{n_2 \deg_{Q(G_1)}(u_1) + n_2 \deg_{Q(G_1)}(u_2)} \\ &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} \\ &\quad \frac{2}{n_2 (\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)) + \deg_{G_2}(v_1)} \\ &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1, u_2 \in V(Q(G_1)) - V(G_1)}} \\ &\quad \frac{2}{n_2 (\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2))}. \end{aligned}$$

For every $v \in V(G_2)$ and $u_1 u_2 \in E(Q(G_1))$ with $u_1 \in V(G_1)$ and $u_2 \in V(Q(G_1)) - V(G_1)$, we have

$$\begin{aligned} &\frac{2}{n_2 (\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)) + \deg_{G_2}(v_1)} \\ &\leq \frac{1}{4n_2} \frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)} + \frac{1}{2} \frac{1}{\deg_{G_2}(v)}, \quad (2.7) \end{aligned}$$

with equality if and only if $n_2 (\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)) = \deg_{G_2}(v_1)$.

Thus,

$$\begin{aligned} &\sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} \\ &\quad \frac{2}{n_2 (\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)) + \deg_{G_2}(v_1)} \\ &\leq \frac{1}{4n_2} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} \\ &\quad \frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)} \\ &\quad + \frac{1}{2} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} \frac{1}{\deg_{G_2}(v_1)} \\ &= \frac{1}{4n_2} \sum_{u_1 u'_1 \in E(G_1)} \left(\frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)} \right. \\ &\quad \left. + \frac{2}{\deg_{Q(G_1)}(u_2) + \deg_{Q(G_1)}(u'_1)} \right) \\ &\quad \text{(where } u_2 \text{ is the vertex inserted into the edge } u_1 u'_1 \text{ of } G_1) \\ &\quad + \frac{m_1}{\deg_{G_2}(v_1)} \\ &= \frac{1}{4n_2} \left(H(Q(G_1)) - \sum_{ww' \in E(L(G_1))} \frac{2}{\deg_{Q(G_1)}(w) + \deg_{Q(G_1)}(w')} \right) \\ &\quad + \frac{m_1}{\deg_{G_2}(v_1)}. \end{aligned}$$

So,

$$\sum 2 \leq \frac{1}{4n_2} \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)}$$

$$\left(H(Q(G_1)) - \sum_{ww' \in E(L(G_1))} \frac{2}{\deg_{Q(G_1)}(w) + \deg_{Q(G_1)}(w')} \right)$$

$$+ \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \frac{m_1}{\deg_{G_2}(v_1)}$$

$$+ \frac{1}{n_2} \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1, u_2 \in V(Q(G_1)) - V(G_1)}}$$

$$\frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)}$$

$$= \frac{1}{4} n_2 H(Q(G_1)) + n_2 m_1 r(G_2)$$

$$- \frac{1}{4} n_2 \sum_{ww' \in E(L(G_1))} \frac{2}{\deg_{Q(G_1)}(w) + \deg_{Q(G_1)}(w')}$$

$$+ n_2 \sum_{u_1 u_2 \in E(L(G_1))} \frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)}$$

$$= \frac{1}{4} n_2 H(Q(G_1)) + n_2 m_1 r(G_2) + \frac{3}{4} n_2$$

$$\sum_{ww' \in E(L(G_1))} \frac{2}{\deg_{Q(G_1)}(w) + \deg_{Q(G_1)}(w')}$$

$$= \frac{1}{4} n_2 H(Q(G_1)) + n_2 m_1 r(G_2) + \frac{3}{4} n_2$$

$$\sum_{ww' \in E(L(G_1))} \frac{2}{4 + (\deg_{L(G_1)}(w) + \deg_{L(G_1)}(w'))}.$$

Similarly, one can verify that

$$\begin{aligned} & \sum_{ww' \in E(L(G_1))} \frac{2}{4 + (\deg_{L(G_1)}(w) + \deg_{L(G_1)}(w'))} \\ & \leq \frac{1}{4} H(L(G_1)) + \frac{1}{16} M_1(G_1) - \frac{1}{8} m_1, \end{aligned} \quad (2.8)$$

with equality if and only if $\deg_{L(G_1)}(w) + \deg_{L(G_1)}(w') = 4$ for all $ww' \in E(L(G_1))$.

Therefore,

$$\begin{aligned} H(G_1[G_2]_Q) & \leq \frac{1}{4} n_2 H(Q(G_1)) + \frac{3}{16} n_2 H(L(G_1)) \\ & + \frac{1}{4} n_1 H(G_2) + \frac{3}{64} n_2 M_1(G_1) \\ & + \frac{1}{4} \frac{m_2}{n_2} r(G_1) + n_2 m_1 r(G_2) - \frac{3}{32} n_2 m_1. \end{aligned}$$

Moreover, equality holds in above inequality if and only if the inequalities (2.6), (2.7) and (2.8) be equalities, i.e., G_1 and G_2 are k_1 -regular and k_2 -regular graphs, respectively, such that $2n_2 k_1 = k_2 + k_2$, $n_2(k_1 + 2k_1) = k_2$ and $(2k_1 - 2) + (2k_1 - 2) = 4$, a contradiction. This completes the proof. $\square$

Example 2.9 For any $n \geq 4$,

$$\begin{aligned} H(P_n[P_2]_Q) & = \left(\frac{2}{3} + \frac{1}{5}(n-2) \right) + \left(\frac{320}{99} + \frac{16}{13}(n-3) \right) \\ & + \left(\frac{8}{7} + \frac{1}{2}(n-4) \right) \\ & = \frac{3494}{693} + \frac{1}{5}(n-2) + \frac{16}{13}(n-3) + \frac{1}{2}(n-4). \end{aligned}$$

Finally, we give an upper bound for the harmonic index of $G_1[G_2]_T$ in terms of $H(T(G_1))$, $H(L(G_1))$, $H(G_1)$, $H(G_2)$, $M_1(G_1)$, $r(G_1)$ and $r(G_2)$.

Theorem 2.10 Let G_1 and G_2 be two graphs. Then

$$\begin{aligned} H(G_1[G_2]_T) & < \frac{1}{4} n_2 H(T(G_1)) + \frac{3}{16} n_2 H(L(G_1)) \\ & + \frac{1}{4} n_1 H(G_2) - \frac{5}{72} n_2 H(G_1) \\ & + \frac{3}{64} n_2 M_1(G_1) + \frac{1}{8} \frac{m_2}{n_2} r(G_1) \\ & + \frac{13}{9} n_2 m_1 r(G_2) - \frac{3}{32} n_2 m_1, \end{aligned}$$

where $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$, $i = 1, 2$.

Proof Let $\deg(u, v) = \deg_{G_1[G_2]_T}(u, v)$ be the degree of a vertex (u, v) in $G_1[G_2]_T$. Then,

$$\begin{aligned}
H(G_1[G_2]_T) &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \frac{2}{\deg(u, v_1) + \deg(u, v_2)} \\
&+ \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(T(G_1))} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\
&= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \frac{2}{\deg(u, v_1) + \deg(u, v_2)} \\
&+ \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(G_1)} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\
&+ \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(T(G_1)) \\ u_1 \in V(G_1), u_2 \in V(T(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\
&+ \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(T(G_1)) \\ u_1, u_2 \in V(T(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v_1) + \deg(u_2, v_2)} \\
&= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \frac{2}{4n_2 \deg_{G_1}(u) + (\deg_{G_2}(v_1) + \deg_{G_2}(v_2))} \\
&+ \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(G_1)} \frac{2}{2n_2 (\deg_{G_1}(u_1) + \deg_{G_1}(u_2)) + (\deg_{G_2}(v_1) + \deg_{G_2}(v_2))} \\
&+ \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(T(G_1)) \\ u_1 \in V(G_1), u_2 \in V(T(G_1)) - V(G_1)}} \frac{2}{n_2 (\deg_{T(G_1)}(u_1) + \deg_{T(G_1)}(u_2)) + \deg_{G_2}(v_1)} \\
&+ \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(T(G_1)) \\ u_1, u_2 \in V(T(G_1)) - V(G_1)}} \frac{2}{\deg_{T(G_1)}(u_1) + \deg_{T(G_1)}(u_2)}.
\end{aligned}$$

By the similar argument as proof of Theorems 2.6 and 2.8, we have

$$\begin{aligned}
H(G_1[G_2]_T) &\leq \frac{1}{8} \frac{m_2}{n_2} r(G_1) + \frac{1}{4} n_1 H(G_2) \\
&+ \frac{1}{18} n_2 H(G_1) + \frac{4}{9} n_2 m_1 r(G_2) \\
&+ \frac{1}{4} n_2 H(T(G_1)) - \frac{1}{8} n_2 H(G_1) + n_2 m_1 r(G_2) \\
&+ \frac{3}{4} n_2 \left(\frac{1}{4} H(L(G_1)) + \frac{1}{16} M_1(G_1) - \frac{1}{8} m_1 \right).
\end{aligned}$$

So,

$$\begin{aligned}
H(G_1[G_2]_T) &\leq \frac{1}{4} n_2 H(T(G_1)) + \frac{3}{16} n_2 H(L(G_1)) \\
&+ \frac{1}{4} n_1 H(G_2) - \frac{5}{72} n_2 H(G_1) \\
&+ \frac{3}{64} n_2 M_1(G_1) + \frac{1}{8} \frac{m_2}{n_2} r(G_1) \\
&+ \frac{13}{9} n_2 m_1 r(G_2) - \frac{3}{32} n_2 m_1.
\end{aligned}$$

Moreover, equality holds in above inequality if and only if G_1 and G_2 be k_1 -regular and k_2 -regular graphs, respectively, such that $4n_2 k_1 = k_2 + k_2$, $2n_2(k_1 + k_1) = k_2$, $n_2(2k_1 + 2k_1) = k_2$ and $(2k_1 - 2) + (2k_1 - 2) = 4$, a contradiction. This completes the proof. $\square$

Example 2.11 For any $n \geq 4$,

$$\begin{aligned}
H(P_n[P_2]_T) &= \left(\frac{2}{5} + \frac{1}{9}(n-2) \right) + \left(\frac{8}{7} + \frac{4}{9}(n-3) \right) \\
&+ \left(\frac{416}{165} + \frac{16}{17}(n-3) \right) + \left(\frac{8}{7} + \frac{1}{2}(n-4) \right) \\
&= \frac{6014}{1155} + \frac{1}{9}(n-2) + \frac{212}{153}(n-3) + \frac{1}{2}(n-4).
\end{aligned}$$

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