ORIGINAL RESEARCH

Solving smoking epidemic model of fractional order using a modified homotopy analysis transform method

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Abstract

The pivotal aim of the present work is to obtain an approximated analytical solution for the fractional smoking epidemic model with the aid of a novel technique called q-homotopy analysis transform method (q-HATM). The considered nonlinear mathematical model has been effectively employed to elucidate the evolution of smoking in a population and its impact on public health in a community. We find some new approximate solutions in a series form, which converges rapidly, and the proposed algorithm provides auxiliary parameters, which are very reliable and feasible in controlling the convergence of obtained approximate solutions. Further, we present novel simulations for all cases of results to validate the applicability and effectiveness of proposed scheme. The outcomes of the study reveal that the q-HATM is computationally very effective to analyse nonlinear fractional differential equations arises in daily life problems.

Introduction

In 1766, Swiss mathematician and physicist Bernoulli [1] established and nurtured the idea of mathematical modelling for spread of disease, which gave birth to the start of modern epidemiology. Further, Ross [2] also presented the modelling of infectious disease in the beginning of twentieth century and explains the nature of epidemic models by using the law of mass action. Epidemic models have been extensively employed to study epidemiological processes which include transmission of contagious diseases. This kind of model has also been applied to study the dissemination of social habits, such as the alcohol consumption [3], obesity epidemics [4], cocaine consumption [5], smoking habit [6]

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and many more. Among these models, smoking model is one of the most attractive models for the last few decades for researchers. The World Health Organization proclaimed that smoking causes 250 million children and adolescent deaths and predicted that more than 10 million people will perish due to smoking-related diseases every year by 2030 [7]. The main effects of short-term smoking are the high blood pressure, stained teeth, bad breath and coughing. In recent years, lung cancer, throat cancer, mouth cancer, stomach ulcers, heart disease and gum disease are the main threatening due to long-term smoking. Therefore, use of smoking treated as a leading global public health problem. Smoking can also spread in similar with the spread of many infectious diseases, through social contact. Thus, mathematical modelling has been widely used to investigate the behaviour of smoking. In 2000, Castillo-Garsow et al. [8] first time suggested a simple mathematical model for giving up smoking. They address a scheme by dividing the population into three classes: smokers (S), potential smokers (P) and quit smokers (Q). The local stability was proved by using the Routh–Hurwitz criterion while the global stability was only conjectured by using numerous simulations. In the recent years, many researches started to work on mathematical modelling of smoking phenomena [9–15] and these models

give us an analytic framework in which to address specific reasons regarding smoking dynamics.

The concept of fractional calculus (FC) debated on 1695 and was firstly put forward by Guillaume de l'Hopital. FC recently showed a tremendous consideration and attention, which is the general expansion of the integer-order calculus to arbitrary order. Modelling by using the perception of FC penetrates the essential fundamentals for many dynamical systems. The biological process that modelled through arbitrary-order derivatives carries information about its present as well as past states [16, 17]. Derivatives and integrals of fractional order consider the system memory, hereditary properties and non-local distributed effects. These effects are essential for portraying the real-world problems [18, 19].

Mathematical models involving fractional differential equations have been proven valuable in understanding the dynamics of smoking. Many researchers developed the fractional-order model to study diffusion equation to predict the effect of smoking. Several powerful techniques for finding solutions for these models have been found in the literature [20–22]. In 1992, Liao [23, 24] proposed homotopy analysis method (HAM), and it has been effectively employing to find the solution for problems arises in science and technology. The HAM is based on construction of a homotopy which continuously deforms an initial guess approximation to the exact solution of the given problem. The q-HATM was proposed by Singh et al. [25], and which is an elegant amalgamation of q-HAM and Laplace transform.

The enhancement of proposed technique is its proficiency of amalgamating two strong algorithms to solve linear and nonlinear fractional differential equations both numerically as well analytically. Since, the techniques with perturbation, linearization or discretization we obtained only approximate solutions for nonlinear complex problems. These problems were appraised by exerting different schemes having their own limitations and weakness including; more time for evaluation, massive computational work and obtaining the divergent results. But, the proposed technique is free from any assumption, discretization and perturbation. The proposed algorithm has many sturdy properties including straightforward solution procedure and promising large convergence region. It is worth revealing that the Laplace transform with semi-analytical techniques requires less C.P.U time to evaluate solution for nonlinear complex models arised in science and technology. The q-HATM solution involves with two auxiliary parameters $\hbar$ and n, which helps us to adjust and control the convergence of the solution. We can say that the proposed technique can decrease the computation of the time and work as compared with other traditional techniques while maintaining the great efficiency of the obtained results. Due to these factors, recently, many authors are employed q-HATM to analyse the complex problems occur in science and engineering [26–29].

In present framework, we find the new approximate analytic solution for fractional smoking epidemic model. The proposed model permits incorporation of the memory effect involved in the fractional order, and this property is important in describing the biological problems. Moreover, q-HATM is applied to find the numerical solutions for the system of equations describing smoking model.

Preliminaries

In this section, we recall some definitions and properties of fractional calculus and Laplace transform

Definition 1 The fractional integral of a function $f(t) \in C_\xi (\xi \geq -1)$ and of order $\mu > 0$, initially defined by Riemann–Liouville which is presented [30, 31] as

$$J^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - \vartheta)^{\mu-1} f(\vartheta) \mathrm{d}\vartheta,$$

$$J^0 f(t) = f(t).$$

(1)

Definition 2 The fractional derivative of $f \in C_{-1}^n$ in the Caputo [32] sense is defined as

$$D_t^\mu f(t) = \begin{cases} \frac{d^n f(t)}{dt^n}, & \mu = n \in N, \\ \frac{1}{\Gamma(n-\mu)} \int_0^t (t - \vartheta)^{n-\mu-1} f^{(n)}(\vartheta) \mathrm{d}\vartheta, & n-1 < \mu < n, n \in \mathbb{N}. \end{cases}$$

(2)

Definition 3 The Laplace transform (LT) of a Caputo fractional derivative $D_t^\mu f(t)$ is represented [33] as

$$\mathcal{L}\left[D_t^\mu f(t)\right] = s^\mu F(s) - \sum_{r=0}^{n-1} s^{\mu-r-1} f^{(r)}\left(0^+\right), (n-1 < \mu \leq n),$$

(3)

where $F(s)$ symbolizes the Laplace transform of the function $f(t)$.

Fractional mathematical model of smoking epidemic model in Caputo fractional derivatives

It is an important and effective way to understand the biological problems by establishing mathematical models and analyzing their dynamical behaviours. In the present investigation, we consider the system of five nonlinear differential

equations describing the smoking epidemic model. Let the total population size at time t is represented by $N(t)$. We separate the population $N(t)$ into five subgroups, namely potential smoker $P(t)$, smoker $S(t)$, occasional smoker $O(t)$, permanently quit smoker $L(t)$ and temporarily quit smoker $Q(t)$. The proposed smoking model is presented as [34]:

$$\frac{\partial P}{\partial t} = \lambda - \beta PS - \eta P, \tag{4}$$

$$\frac{\partial O}{\partial t} = \beta PS - \alpha_1 O - \eta O, \tag{5}$$

$$\frac{\partial S}{\partial t} = \alpha_1 O + \alpha_2 SQ - (\eta + \gamma)S, \tag{6}$$

$$\frac{\partial Q}{\partial t} = -\alpha_2 SQ - \eta Q + \gamma(1 - \delta)S, \tag{7}$$

$$\frac{\partial L}{\partial t} = \delta\gamma S - \eta L. \tag{8}$$

In the above system, λ denotes the recruitment rate in the potential smoker, β indicates the effective contact rate between potential smoker and smoker, η symbolizes the natural death rate, γ stands for rate of quitting smoking, δ represents the remaining fraction of smoking who permanently quit smoking, α_1 is the rate at which occasional smokers becomes regular smoker, and α_2 indicates the contact rate between smoker and temporary quitters who revert back to smoking. The parameters used in the system of Eqs. 4–8 are specified in Table 1.

Thus, in order to introduce the effect of non-locality, we moderate the above system by substituting the time derivative by the Caputo fractional derivative of the form:

$$\begin{aligned} D_t^\mu P &= \lambda - \beta PS - \eta P, \\ D_t^\mu O &= \beta PS - \alpha_1 O - \eta O, \\ D_t^\mu S &= \alpha_1 O + \alpha_2 SQ - (\eta + \gamma)S, \quad 0 < \mu \leq 1 \\ D_t^\mu Q &= -\alpha_2 SQ - \eta Q + \gamma(1 - \delta)S, \\ D_t^\mu L &= \delta\gamma S - \eta L, \end{aligned} \tag{9}$$

with initial conditions

$$P(0) = \epsilon_1, O(0) = \epsilon_2, S(0) = \epsilon_3, Q(0) = \epsilon_4, L(0) = \epsilon_5. \tag{10}$$

Equilibrium point and stability

For equilibrium point [35], in system (9) we consider

$$D_t^\mu P(t) = D_t^\mu O(t) = D_t^\mu S(t) = D_t^\mu Q(t) = D_t^\mu L(t) = 0; \tag{11}$$

we obtained disease-free equilibria as

$$E_0 = (P, 0, S, 0, 0),$$

and endemic equilibria of the system is

$$E^* = (P^*, O^* S^*, Q^*, L^*),$$

where

$$P^* = \frac{\lambda}{\beta + \eta}, O^* = \frac{\beta\lambda S}{(\beta S + \eta)(\alpha_1 + \eta)}, Q^* = \frac{\gamma(1 - \delta)S}{\alpha_2 S + \eta}, L^* = \frac{\delta\gamma S}{\eta}. \tag{12}$$

Theorem 1 [35] *The disease-free equilibrium E^* is locally asymptotically stable if $R_0 < 1$, otherwise unstable.*

Fundamental idea of *q*-homotopy analysis transform method

To present the fundamental idea of proposed method [36, 37], consider the nonlinear non-homogeneous partial differential equation of fractional order:

$$D_t^\mu v(x,t) + \mathcal{R}v(x,t) + \mathcal{N}v(x,t) = f(x,t), \quad n - 1 < \mu \leq n, \tag{13}$$

where $D_t^\mu v(x,t)$ represents the Caputo fractional derivative of the function $v(x,t)$, $\mathcal{R}$ and $\mathcal{N}$, respectively, specifies the linear and nonlinear differential operator, and $f(x,t)$ represents the source term. Now, by employing the *LT* on Eq. (13), we get

$$\begin{aligned} s^\mu \mathcal{L}[v(x,t)] &- \sum_{k=0}^{n-1} s^{\mu-k-1} v^{(k)}(x,0) \\ &+ \mathcal{L}[\mathcal{R}v(x,t)] + \mathcal{L}[\mathcal{N}v(x,t)] = \mathcal{L}[f(x,t)]. \end{aligned} \tag{14}$$

Table 1 Description of parameters used in system (9) and their specific values

Parameters	Descriptions	Values (unit: 1/time)
λ	Recruitment rate in P	1
β	Effective contact rate between S and P	0.14
η	Natural death rate	0.05
α_1	Rate at which occasional smokers become regular smokers	0.002
α_2	Contact rate between smokers and temporary quitters who revert back to smoking	0.0025
γ	Rate of quitting smoking	0.8
δ	Remaining fraction of smoking who permanently quit smoking	0.1

On simplifying Eq. (14), we have

$$\mathcal{L}[v(x,t)] - \frac{1}{s^\mu} \sum_{k=0}^{n-1} s^{\mu-k-1} v^k(x,0)$$

$$+ \frac{1}{s^\mu} \left\{ \mathcal{L}[\mathcal{R}v(x,t)] + \mathcal{L}[\mathcal{N}v(x,t)] - \mathcal{L}[f(x,t)] \right\} = 0. \quad (15)$$

According to homotopy analysis method [24], the nonlinear operator defined as

$$\mathcal{N}[\varphi(x,t;q)] = \mathcal{L}[\varphi(x,t;q)] - \frac{1}{s^\mu} \sum_{k=0}^{n-1} s^{\mu-k-1} \varphi^{(k)}(x,t;q)(0^+)$$

$$+ \frac{1}{s^\mu} \left\{ \mathcal{L}[\mathcal{R}\varphi(x,t;q)] + L[\mathcal{N}\varphi(x,t;q)] - L[f(x,t)] \right\}, \quad (16)$$

where $q \in \left[0, \frac{1}{n}\right]$, and $\varphi(x,t;q)$ is a real function of x, t and q.

We construct a homotopy for nonzero auxiliary function as follows:

$$(1 - nq)\mathcal{L}[\varphi(x,t;q) - v_0(x,t)] = \hbar q \mathcal{N}[\varphi(x,t;q)], \quad (17)$$

where $\mathcal{L}$ be a symbol of the LT, $\hbar \neq 0$ is an auxiliary parameter, $q \in \left[0, \frac{1}{n}\right]$ ($n \geq 1$) is the embedding parameter, $v_0(x,t)$ is an initial guess of $v(x,t)$ and $\varphi(x,t;q)$ is an unknown function. The following results hold for $q = 0$ and $q = \frac{1}{n}$:

$$\varphi(x,t;0) = v_0(x,t), \quad \varphi\left(x,t;\frac{1}{n}\right) = v(x,t), \quad (18)$$

respectively. Thus, by amplifying q from 0 to $\frac{1}{n}$, the solution $\varphi(x,t;q)$ converges from $v_0(x,t)$ to the solution $v(x,t)$. Expanding the function $\varphi(x,t;q)$ in series form by employing Taylor theorem near to q, one can get

$$\varphi(x,t;q) = v_0(x,t) + \sum_{m=1}^\infty v_m(x,t) q^m, \quad (19)$$

where

$$v_m(x,t) = \frac{1}{m!} \frac{\partial^m \varphi(x,t;q)}{\partial q^m} \Big|_{q=0}. \quad (20)$$

On choosing the auxiliary linear operator, the initial guess is $v_0(x,t), n$, and $\hbar$, and the series (19) converges at $q = \frac{1}{n}$; then, it gives one of the solutions of the original nonlinear equation of the form

$$v(x,t) = v_0(x,t) + \sum_{m=1}^\infty v_m(x,t) \left(\frac{1}{n}\right)^m. \quad (21)$$

Now, differentiating Eq. (17) m times with respect to q and then dividing by $m!$ and lastly taking $q = 0$, which yields

$$\mathcal{L}[v_m(x,t) - \mathrm{k}_m v_{m-1}(x,t)] = \hbar \mathfrak{R}_m(\vec{v}_{m-1}), \quad (22)$$

where the vectors are defined as

$$\vec{v}_m = \{v_0(x,t), v_1(x,t), \ldots, v_m(x,t)\}. \quad (23)$$

Applying the inverse LT on Eq. (22), it provides the following recursive equation

$$v_m(x,t) = \mathrm{k}_m v_{m-1}(x,t) + \hbar \mathcal{L}^{-1}[\mathfrak{R}_m(\vec{v}_{m-1})], \quad (24)$$

where

$$\mathfrak{R}_m(\vec{v}_{m-1}) = L[v_{m-1}(x,t)] - \left(1 - \frac{\mathrm{k}_m}{n}\right)$$

$$\times \left(\sum_{k=0}^{n-1} s^{\mu-k-1} v^{(k)}(x,0) + \frac{1}{s^\mu} L[f(x,t)]\right) \quad (25)$$

$$+ \frac{1}{s^\mu} L[Rv_{m-1} + \mathcal{H}_{m-1}],$$

and

$$\mathrm{k}_m = \begin{cases} 0, & m \leq 1, \\ n, & m > 1. \end{cases} \quad (26)$$

In Eq. (25), $\mathcal{H}_m$ denotes homotopy polynomial and defined as

$$\mathcal{H}_m = \frac{1}{m!} \left[\frac{\partial^m \varphi(x,t;q)}{\partial q^m}\right]_{q=0} \quad \text{and} \quad \varphi(x,t;q) = \varphi_0 + q\varphi_1 + q^2\varphi_2 + \cdots. \quad (27)$$

By Eqs. (24) and (25), we have

$$v_m(x,t) = (\mathrm{k}_m + \hbar)v_{m-1}(x,t) - \left(1 - \frac{\mathrm{k}_m}{n}\right)\mathcal{L}^{-1}\left(\sum_{k=0}^{n-1} s^{\mu-k-1} v^{(k)}(x,0)\right)$$

$$+ \frac{1}{s^\mu}\mathcal{L}[f(x,t)]\right) + \hbar\mathcal{L}^{-1}\left\{\frac{1}{s^\mu} L[Rv_{m-1} + \mathcal{H}_{m-1}]\right\}. \quad (28)$$

Finally, on solving Eq. (28), we get the iterative terms of $v_m(x,t)$. The q-HATM series solution is presented by

$$v(x,t) = \sum_{m=0}^\infty v_m(x,t). \quad (29)$$

Convergence analysis of *q*-HATM solution

Theorem 2 (Uniqueness theorem) [38] *The solution for the nonlinear fractional differential Eq. (13) obtained by q-HATM is unique for every $\beta \in (0,1)$, where $\beta = (n + \hbar) + \hbar(\epsilon + \mu)T$.*

Theorem 3 (Convergence theorem) [38] *Let X be a Banach space and F : X → X be a nonlinear mapping. Assume that*

$$||F(v) - F(w)|| \leq \beta||v - w||, \quad \forall a, b \in X,$$

then F has a fixed point in view of Banach fixed point theory [39]. Moreover, for the arbitrary selection of $a_0, b_0 \in X$, the sequence generated by the q-HATM converges to fixed point of F and

$$\|v_m - v_n\| \le \frac{\beta^n}{1 - \beta} \|v_1 - v_0\|, \quad \forall a, b \in X. \tag{30}$$

q-HATM solution for system of time-fractional smoking epidemic effect equations

In this section, we illustrate some numerical solutions of the model (9) for different values of the parameters, and we show that these solutions are in agreement with the qualitative behaviour of the solutions. Consider time-fractional smoking epidemic effect equations

$$\frac{d^\mu P}{dt^\mu} = \lambda - \beta P S - \eta P,$$

$$\frac{d^\mu O}{dt^\mu} = \beta P S - \alpha_1 O - \eta O,$$

$$\frac{d^\mu S}{dt^\mu} = \alpha_1 O + \alpha_2 S Q - (\eta + \gamma) S, \quad 0 < \mu \le 1, \tag{31}$$

$$\frac{d^\mu Q}{dt^\mu} = -\alpha_2 S Q - \eta Q + \gamma (1 - \delta) S,$$

$$\frac{d^\mu L}{dt^\mu} = \delta \gamma S - \eta L,$$

with initial conditions

$$P(0) = \epsilon_1, O(0) = \epsilon_2, S(0) = \epsilon_3, Q(0) = \epsilon_4, L(0) = \epsilon_5. \tag{32}$$

Now, by performing *LT* on system of Eqs. (31) and using conditions cited in Eq. (32), we have

$$\mathcal{L}[P(t)] - \frac{1}{s}(\epsilon_1) - \frac{1}{s^\mu}\mathcal{L}\{\lambda - \beta P S - \eta P\} = 0,$$

$$\mathcal{L}[O(t)] - \frac{1}{s}(\epsilon_2) - \frac{1}{s^\mu}\mathcal{L}\{\beta P S - \alpha_1 O - \eta O\},$$

$$\mathcal{L}[S(t)] - \frac{1}{s}(\epsilon_3) - \frac{1}{s^\mu}\mathcal{L}\{\alpha_1 O + \alpha_2 S Q - (\eta + \gamma)S\}, \tag{33}$$

$$\mathcal{L}[Q(t)] - \frac{1}{s}(\epsilon_4) - \frac{1}{s^\mu}\mathcal{L}\{-\alpha_2 S Q - \eta Q + \gamma(1 - \delta)S\},$$

$$\mathcal{L}[L(t)] - \frac{1}{s}(\epsilon_5) - \frac{1}{s^\mu}\mathcal{L}\{\delta \gamma S - \eta L\}.$$

Define the nonlinear operator as

$$N^1\left[\varphi_1(t;q), \varphi_2(t;q), \varphi_3(t;q), \varphi_4(t;q), \varphi_5(t;q)\right]$$
$$= \mathcal{L}\left[\varphi_1(t;q)\right] - \frac{1}{s}(\epsilon_1) - \frac{1}{s^\mu}\mathcal{L}\{\lambda - \beta\varphi_1(t;q)\varphi_3(t;q) - \eta\varphi_1(t;q)\},$$

$$N^2\left[\varphi_1(t;q), \varphi_2(t;q), \varphi_3(t;q), \varphi_4(t;q), \varphi_5(t;q)\right]$$
$$= \mathcal{L}\left[\varphi_2(t;q)\right] - \frac{1}{s}(\epsilon_2) - \frac{1}{s^\mu}\mathcal{L}\{\beta\varphi_1(t;q)\varphi_3(t;q) - \alpha_1\varphi_2(t;q) - \eta\varphi_2(t;q)\},$$

$$N^3\left[\varphi_1(t;q), \varphi_2(t;q), \varphi_3(t;q), \varphi_4(t;q), \varphi_5(t;q)\right]$$
$$= \mathcal{L}\left[\varphi_3(t;q)\right] - \frac{1}{s}(\epsilon_3) - \frac{1}{s^\mu}\mathcal{L}\{\alpha_1\varphi_2(t;q) + \alpha_2\varphi_3(t;q)\varphi_4(t;q) - (\eta + \gamma)\varphi_3(t;q)\}, \tag{34}$$

$$N^4\left[\varphi_1(t;q), \varphi_2(t;q), \varphi_3(t;q), \varphi_4(t;q), \varphi_5(t;q)\right]$$
$$= \mathcal{L}\left[\varphi_4(t;q)\right] - \frac{1}{s}(\epsilon_4) - \frac{1}{s^\mu}\mathcal{L}\{-\alpha_2\varphi_3(t;q)\varphi_4(t;q) - \eta\varphi_4(t;q) + \gamma(1 - \delta)\varphi_3(t;q)\},$$

$$N^5\left[\varphi_1(t;q), \varphi_2(t;q), \varphi_3(t;q), \varphi_4(t;q), \varphi_5(t;q)\right]$$
$$= \mathcal{L}\left[\varphi_5(t;q)\right] - \frac{1}{s}(\epsilon_5) - \frac{1}{s^\mu}\mathcal{L}\{\delta\gamma\varphi_3(t;q) - \eta\varphi_5(t;q)\}.$$

By applying suggested algorithm, the deformation equation of mth order is given as

$$
\begin{aligned}
\mathcal{L}\left[P_m(x,t) - \mathrm{k}_m P_{m-1}(x,t)\right] &= \hbar \mathcal{L}^{-1}\left\{\Re_{1,m}\left[\vec{P}_{m-1}, \vec{O}_{m-1}, \vec{S}_{m-1}, \vec{Q}_{m-1}, \vec{L}_{m-1}\right]\right\}, \\
\mathcal{L}\left[O_m(x,t) - \mathrm{k}_m O_{m-1}(x,t)\right] &= \hbar \mathcal{L}^{-1}\left\{\Re_{2,m}\left[\vec{P}_{m-1}, \vec{O}_{m-1}, \vec{S}_{m-1}, \vec{Q}_{m-1}, \vec{L}_{m-1}\right]\right\}, \\
\mathcal{L}\left[S_m(x,t) - \mathrm{k}_m S_{m-1}(x,t)\right] &= \hbar \mathcal{L}^{-1}\left\{\Re_{3,m}\left[\vec{P}_{m-1}, \vec{O}_{m-1}, \vec{S}_{m-1}, \vec{Q}_{m-1}, \vec{L}_{m-1}\right]\right\}, \\
\mathcal{L}\left[Q_m(x,t) - \mathrm{k}_m Q_{m-1}(x,t)\right] &= \hbar \mathcal{L}^{-1}\left\{\Re_{4,m}\left[\vec{P}_{m-1}, \vec{O}_{m-1}, \vec{S}_{m-1}, \vec{Q}_{m-1}, \vec{L}_{m-1}\right]\right\}, \\
\mathcal{L}\left[L_m(x,t) - \mathrm{k}_m L_{m-1}(x,t)\right] &= \hbar \mathcal{L}^{-1}\left\{\Re_{5,m}\left[\vec{P}_{m-1}, \vec{O}_{m-1}, \vec{S}_{m-1}, \vec{Q}_{m-1}, \vec{L}_{m-1}\right]\right\},
\end{aligned}
\tag{35}
$$

where

$$
\begin{aligned}
&\Re_{1,m}\left[\vec{P}_{m-1}, \vec{O}_{m-1}, \vec{S}_{m-1}, \vec{Q}_{m-1}, \vec{L}_{m-1}\right] \\
&\quad = \mathcal{L}\left[P_{m-1}(x,t)\right] - \left(1 - \frac{k_m}{n}\right)\frac{1}{s}(\epsilon_1) - \frac{1}{s^\mu}\mathcal{L}\left\{\lambda - \beta \sum_{i=0}^{m-1} P_i S_{m-1-i} - \eta P_{m-1}\right\}, \\
&\Re_{2,m}\left[\vec{P}_{m-1}, \vec{O}_{m-1}, \vec{S}_{m-1}, \vec{Q}_{m-1}, \vec{L}_{m-1}\right] \\
&\quad = \mathcal{L}\left[O_{m-1}(x,t)\right] - \left(1 - \frac{k_m}{n}\right)\frac{1}{s}(\epsilon_2) - \frac{1}{s^\mu}\mathcal{L}\left\{\beta \sum_{i=0}^{m-1} P_i S_{m-1-i} - \alpha_1 O_{m-1} - \eta O_{m-1}\right\}, \\
&\Re_{3,m}\left[\vec{P}_{m-1}, \vec{O}_{m-1}, \vec{S}_{m-1}, \vec{Q}_{m-1}, \vec{L}_{m-1}\right] \\
&\quad = \mathcal{L}\left[S_{m-1}(x,t)\right] - \left(1 - \frac{k_m}{n}\right)\frac{1}{s}(\epsilon_3) - \frac{1}{s^\mu}\mathcal{L}\left\{\alpha_1 O_{m-1} + \alpha_2 \sum_{i=0}^{m-1} S_i Q_{m-1-i} - (\eta + \gamma)S_{m-1}\right\}, \\
&\Re_{4,m}\left[\vec{P}_{m-1}, \vec{O}_{m-1}, \vec{S}_{m-1}, \vec{Q}_{m-1}, \vec{L}_{m-1}\right] \\
&\quad = \mathcal{L}\left[Q_{m-1}(x,t)\right] - \left(1 - \frac{k_m}{n}\right)\frac{1}{s}(\epsilon_4) - \frac{1}{s^\mu}\mathcal{L}\left\{-\alpha_2 \sum_{i=0}^{m-1} S_i Q_{m-1-i} - \eta Q_{m-1} + \gamma(1-\delta)S_{m-1}\right\}, \\
&\Re_{5,m}\left[\vec{P}_{m-1}, \vec{O}_{m-1}, \vec{S}_{m-1}, \vec{Q}_{m-1}, \vec{L}_{m-1}\right] \\
&\quad = \mathcal{L}\left[L_{m-1}(x,t)\right] - \left(1 - \frac{k_m}{n}\right)\frac{1}{s}(\epsilon_5) - \frac{1}{s^\mu}\mathcal{L}\left\{\delta \gamma S_{m-1} - \eta L_{m-1}\right\}.
\end{aligned}
\tag{36}
$$

By applying inverse Laplace transform on system of Eq. (35), we get

$$
\begin{aligned}
P_m(x,t) &= \mathrm{k}_m P_{m-1}(x,t) + \hbar \mathcal{L}^{-1}\left\{\Re_{1,m}\left[\vec{P}_{m-1}, \vec{O}_{m-1}, \vec{S}_{m-1}, \vec{Q}_{m-1}, \vec{L}_{m-1}\right]\right\}, \\
O_m(x,t) &= \mathrm{k}_m O_{m-1}(x,t) + \hbar \mathcal{L}^{-1}\left\{\Re_{2,m}\left[\vec{P}_{m-1}, \vec{O}_{m-1}, \vec{S}_{m-1}, \vec{Q}_{m-1}, \vec{L}_{m-1}\right]\right\}, \\
S_m(x,t) &= \mathrm{k}_m S_{m-1}(x,t) + \hbar \mathcal{L}^{-1}\left\{\Re_{3,m}\left[\vec{P}_{m-1}, \vec{O}_{m-1}, \vec{S}_{m-1}, \vec{Q}_{m-1}, \vec{L}_{m-1}\right]\right\}, \\
Q_m(x,t) &= \mathrm{k}_m Q_{m-1}(x,t) + \hbar \mathcal{L}^{-1}\left\{\Re_{4,m}\left[\vec{P}_{m-1}, \vec{O}_{m-1}, \vec{S}_{m-1}, \vec{Q}_{m-1}, \vec{L}_{m-1}\right]\right\}, \\
L_m(x,t) &= \mathrm{k}_m L_{m-1}(x,t) + \hbar \mathcal{L}^{-1}\left\{\Re_{5,m}\left[\vec{P}_{m-1}, \vec{O}_{m-1}, \vec{S}_{m-1}, \vec{Q}_{m-1}, \vec{L}_{m-1}\right]\right\}.
\end{aligned}
\tag{37}
$$

On solving above system of equations by using initial conditions $P(0) = \epsilon_1 = 40, O(0) = \epsilon_2 = 10, S(0) = \epsilon_3 = 20, Q(0) = \epsilon_4 = 10, L(0) = \epsilon_5 = 5$, we have $P_0(t) = 40, O_0(t) = 10, S_0(t) = 20, Q_0(t) = 10, L_0(t) = 5,$

$$P_1(t) = \frac{\hbar(800\beta + 40\eta - \lambda)t^\mu}{\Gamma[\mu+1]},$$

$$O_1(t) = \frac{\hbar\big(10(-80\beta + \eta + \alpha_1)\big)t^\mu}{\Gamma[\mu+1]},$$

$$S_1(t) = \frac{\hbar\big(-10\alpha_1 + 20(\gamma + \eta - 10\alpha_2)\big)t^\mu}{\Gamma[\mu+1]},$$

$$Q_1(t) = \frac{\hbar\big(10(2\gamma(-1+\delta) + \eta + 20\alpha_2)\big)t^\mu}{\Gamma[\mu+1]},$$

$$L_1(t) = \frac{\hbar(-20\gamma\delta + 5\eta)t^\mu}{\Gamma[\mu+1]},$$

$$P_2(t) = \frac{(n+\hbar)\hbar(800\beta + 40\eta - \lambda)t^\mu}{\Gamma[\mu+1]} + \frac{\hbar^2\big(-\lambda + \eta(800\beta + 40\eta - \lambda) + 20\beta(800\beta + 40\gamma + 80\eta - \lambda - 20\alpha_1 - 400\alpha_2)\big)t^{2\mu}}{\Gamma[2\mu+1]},$$

$$O_2(t) = \frac{(n+\hbar)\hbar\big(10(-80\beta + \eta + \alpha_1)\big)t^\mu}{\Gamma[\mu+1]} + \frac{10\hbar^2\big(-1600\beta^2 - 80\beta\gamma - 240\beta\eta + \eta^2 + 2\beta\lambda + (-40\beta + 2\eta)\alpha_1 + \alpha_1^2 + 800\beta\alpha_2\big)t^{2\mu}}{\Gamma[2\mu+1]},$$

$$S_2(t) = \frac{(n+\hbar)\hbar\big(-10\alpha_1 + 20(\gamma + \eta - 10\alpha_2)\big)t^\mu}{\Gamma[\mu+1]} - \frac{10\hbar^2\big(\alpha_1^2 + \alpha_1(-80\beta + \gamma + 2\eta - 10\alpha_2) - 2((\gamma+\eta)^2 - 10(2\gamma\delta + 3\eta)\alpha_2 - 100\alpha_2^2)\big)t^{2\mu}}{\Gamma[2\mu+1]},$$

$$Q_2(t) = \frac{(n+\hbar)\hbar\big(10(2\gamma(-1+\delta) + \eta + 20\alpha_2)\big)t^\mu}{\Gamma[\mu+1]} + \frac{10\hbar^2\big(-2\gamma^2 + 2\gamma^2\delta - 4\gamma\eta + 4\gamma\delta\eta + \eta^2 + \alpha_1(\gamma - \gamma\delta - 10\alpha_2) + 20(\gamma\delta + 3\eta)\alpha_2 + 200\alpha_2^2\big)t^{2\mu}}{\Gamma[2\mu+1]},$$

$$L_2(t) = \frac{(n+\hbar)\hbar(-20\gamma\delta + 5\eta)t^\mu}{\Gamma[\mu+1]} + \frac{5\hbar^2\big(-4\gamma^2\delta - 8\gamma\delta\eta + \eta^2 + 2\gamma\delta\alpha_1 + 40\gamma\delta\alpha_2\big)t^{2\mu}}{\Gamma[2\mu+1]}$$

$$
\begin{aligned}
P_3(t) = {} & \frac{(n+\hbar)^2\hbar(800\beta + 40\eta - \lambda)t^\mu}{\Gamma[\mu+1]} \\
& + \frac{(n+\hbar)\hbar^2\big(-\lambda + \eta(800\beta + 40\eta - \lambda) + 20\beta(800\beta + 40\gamma + 80\eta - \lambda - 20\alpha_1 - 400\alpha_2)\big)t^{2\mu}}{\Gamma[2\mu+1]} \\
& + \frac{\hbar^3 t^{3\mu}}{\Gamma[\mu+1]^2\Gamma[3\mu+1]}\big(\big(10t^{3\mu}\hbar^3\beta(800\beta + 40\eta - \lambda)(-\alpha_1 + 2(\gamma + \eta - 10\alpha_2))\big)\Gamma[2\mu+1] \\
& + \Gamma[\mu+1]^2\big(-\lambda + 40(8000\beta^3 + \eta^3 + 400\beta^2(\gamma + 4\eta) + 20\beta(\gamma^2 + 3\gamma\eta + 5\eta^2)) \\
& - (20\beta + \eta)(1 + 20\beta + \eta)\lambda + 400\beta(-\alpha_1(-60\beta + \gamma + 3\eta + \alpha_1) \\
& + 10(-4(10\beta + \gamma\delta + 2\eta) + \alpha_1)\alpha_2 - 200\alpha_2^2)\big)\big),
\end{aligned}
$$

Table 2 Numerical simulation using q-HATM for $P(t)$ at $n = 1, \hbar = -1$ and using Table 1 for different μ with diverse values t

$P(t)$	$\mu = 0.8$	$\mu = 0.9$	$\mu = 1$
$t = 0$	40	40	40
$t = 0.1$	28.0338	29.1162	30.7667
$t = 0.2$	28.5365	26.0052	25.6668
$t = 0.3$	35.8119	28.4733	24.7002
$t = 0.4$	48.4484	35.8752	27.8670
$t = 0.5$	65.6923	47.8402	35.1672
$t = 0.6$	87.0555	64.1152	46.6008
$t = 0.7$	112.189	84.5111	62.1678
$t = 0.8$	140.825	108.878	81.8682
$t = 0.9$	172.753	137.095	105.702
$t = 1.0$	207.799	169.057	133.669

Table 5 Numerical study by using q-HATM for $Q(t)$ at $n = 1, \hbar = -1$ and using Table 1 for different μ with different t

$Q(t)$	$\mu = 0.8$	$\mu = 0.9$	$\mu = 1$
$t = 0$	10	10	10
$t = 0.1$	12.0554	11.6331	11.2760
$t = 0.2$	13.2887	12.8520	12.4241
$t = 0.3$	14.1876	13.8408	13.4443
$t = 0.4$	14.8466	14.6413	14.3365
$t = 0.5$	15.3112	15.2749	15.1008
$t = 0.6$	15.6089	15.7550	15.7372
$t = 0.7$	15.7583	16.0913	16.2456
$t = 0.8$	15.7730	16.2908	16.6260
$t = 0.9$	15.6636	16.3594	16.8786
$t = 1.0$	15.4383	16.3015	17.0032

Table 3 Numerical study by employing q-HATM for $O(t)$ at $n = 1, \hbar = -1$ and using Table 1 for different μ with distinct t

$O(t)$	$\mu = 0.8$	$\mu = 0.9$	$\mu = 1$
$t = 0$	10	10	10
$t = 0.05$	18.4931	16.6946	15.0559
$t = 0.10$	21.6874	20.6741	19.0756
$t = 0.15$	22.3060	22.8892	22.0590
$t = 0.20$	20.9520	23.5858	24.0063
$t = 0.25$	17.9345	22.8977	24.9173
$t = 0.30$	13.4483	20.9135	24.7922
$t = 0.35$	7.63003	17.6978	23.6308
$t = 0.40$	3.58228	13.3004	21.4332
$t = 0.45$	1.61433	7.76208	18.1994
$t = 0.50$	0.58228	1.11636	13.9294

Table 6 Numerical simulation using q-HATM for $L(t)$ at $n = 1, \hbar = -1$ and using Table 1 for different μ with diverse t

$L(t)$	$\mu = 0.8$	$\mu = 0.9$	$\mu = 1$
$t = 0$	5	5	5
$t = 0.1$	5.20537	5.16361	5.12807
$t = 0.2$	5.32615	5.28413	5.24228
$t = 0.3$	5.41201	5.38032	5.34263
$t = 0.4$	5.47262	5.45648	5.42913
$t = 0.5$	5.51271	5.51481	5.50176
$t = 0.6$	5.53511	5.55673	5.56054
$t = 0.7$	5.54178	5.58323	5.60545
$t = 0.8$	5.53414	5.59506	5.63651
$t = 0.9$	5.51326	5.59282	5.65371
$t = 1.0$	5.48004	5.57701	5.65705

Table 4 Numerical simulation using q-HATM for $S(t)$ at $n = 1, \hbar = -1$ and using Table 1 for different μ with diverse t

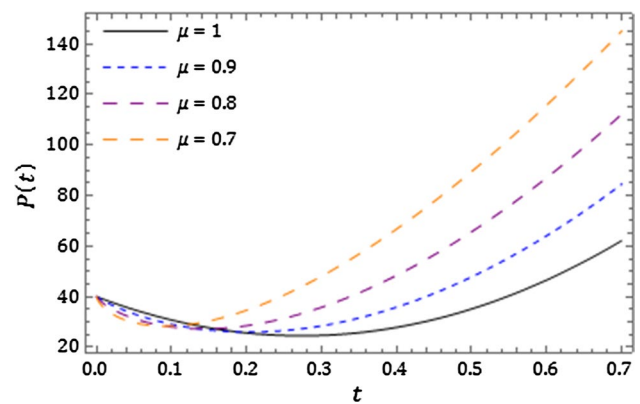| $S(t)$	$\mu = 0.8$	$\mu = 0.9$	$\mu = 1$
$t = 0$	20	20	20
$t = 0.1$	17.4502	17.9798	18.4244
$t = 0.2$	15.8891	16.4515	16.9938
$t = 0.3$	14.7230	15.1913	15.7080
$t = 0.4$	13.8383	14.1491	14.5671
$t = 0.5$	13.1806	13.2994	13.5711
$t = 0.6$	12.7172	12.6260	12.7201
$t = 0.7$	12.4259	12.1178	12.0138
$t = 0.8$	12.2905	11.7662	11.4525
$t = 0.9$	12.2987	11.5645	11.0360
$t = 1.0$	12.4407	11.5073	10.7645

Fig. 1 Plots of q-HATM solution for potential smoker $P(t)$ with respect to t at $\hbar = -1, n = 1$ and using Table 1 for diverse μ
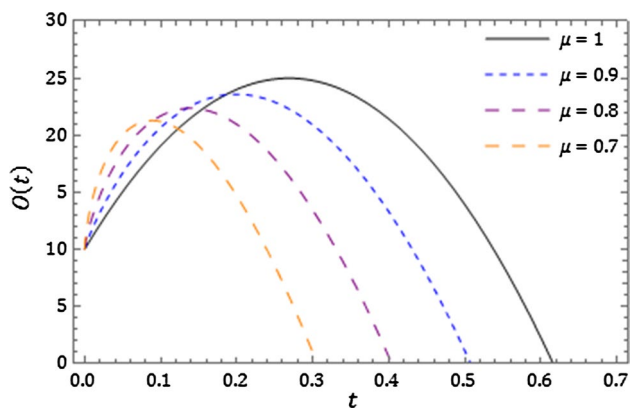
Fig. 2 Response of q-HATM solution for occasional smoker $O(t)$ with respect to t at $\hbar = -1, n = 1$ and using Table 1 for different μ
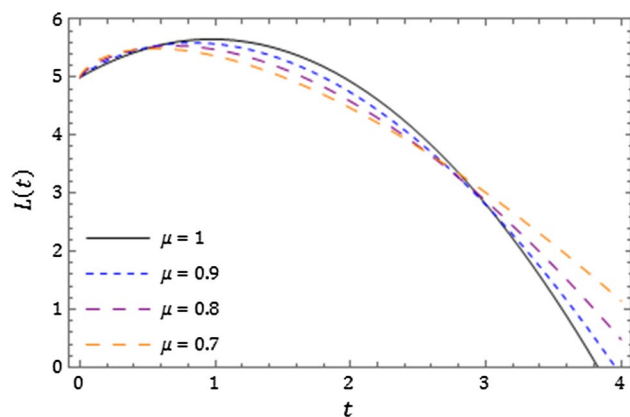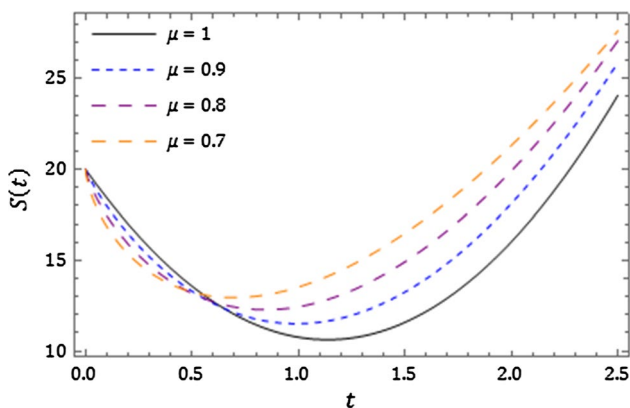

Fig. 3 Nature of q-HATM solution for smoker $S(t)$ with respect to t at $\hbar = -1, n = 1$ and using Table 1 for diverse μ
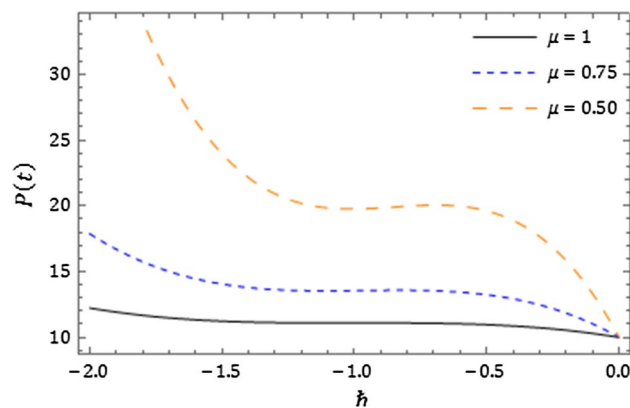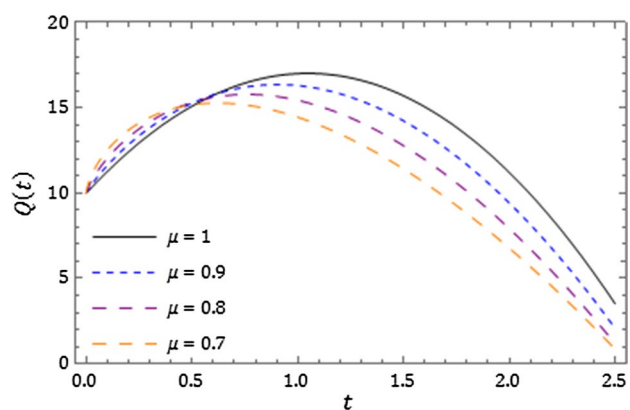

Fig. 4 Plot of q-HATM solution for temporarily quit smoker $Q(t)$ with respect to t at $\hbar = -1, n = 1$ and using Table 1 for different μ
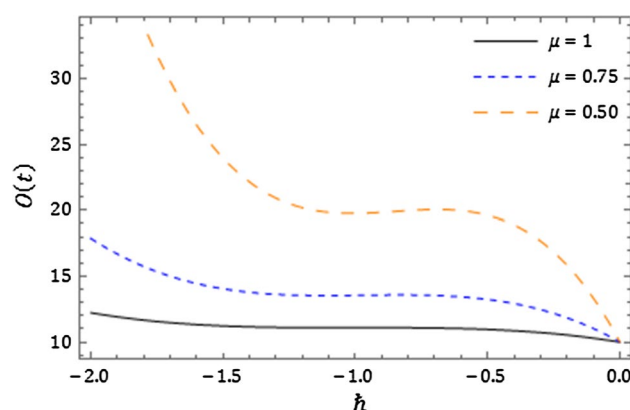

Fig. 5 Nature of q-HATM solution for permanently quit smoker $Q(t)$ with respect to t at $\hbar = -1, n = 1$ and using Table 1 for distinct μ

Fig. 6 $\hbar$-curves drown for $P(t)$ with diverse μ at $t = 0.01, n = 1$ and using Table 1

Fig. 7 $\hbar$-curves drown for $O(t)$ with distinct μ at $t = 0.01, n = 1$ and using Table 1
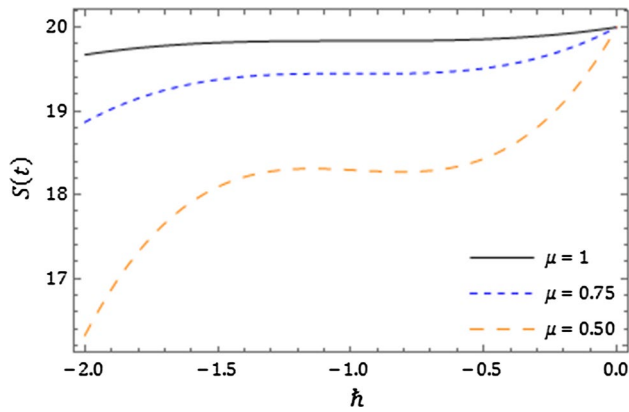
Fig. 8 $\hbar$-curves drown for $S(t)$ with different μ at $t = 0.01, n = 1$ and using Table 1
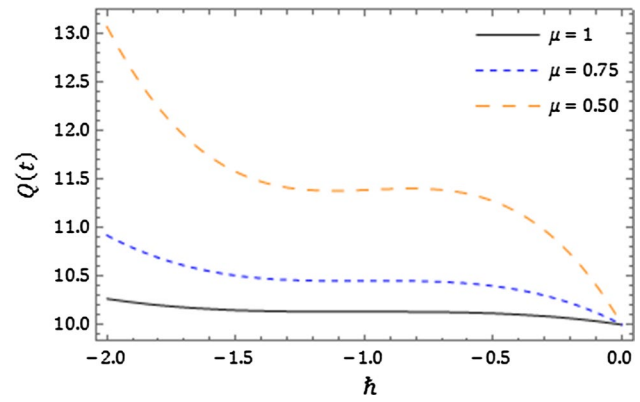

Fig. 9 $\hbar$-curves plotted for $Q(t)$ with distinct μ at $t = 0.01, n = 1$ and using Table 1

$$
\begin{aligned}
O_3(t) = {}& \frac{(n+\hbar)^2\hbar\big(10(-80\beta+\eta+\alpha_1)\big)t^\mu}{\Gamma[\mu+1]} \\
& + \frac{10(n+\hbar)\hbar^2\big(-1600\beta^2-80\beta\gamma-240\beta\eta+\eta^2+2\beta\lambda+(-40\beta+2\eta)\alpha_1+\alpha_1^2+800\beta\alpha_2\big)t^{2\mu}}{\Gamma[2\mu+1]} \\
& + \frac{10\hbar^3 t^{3\mu}}{\Gamma[\mu+1]^2\Gamma[3\mu+1]}\Big(\big((\beta(800\beta+40\eta-\lambda)(\alpha_1-2(\gamma+\eta-10\alpha_2))\big)\Gamma[2\mu+1] \\
& + \Gamma[\mu+1]^2\big(-32000\beta^3+\eta^3-80\beta(\gamma^2+3\gamma\eta+5\eta^2)-40\beta^2(40(\gamma+4\eta)-\lambda) \\
& + 2\beta(1+2\eta)\lambda+\alpha_1(-4000\beta^2+3\eta^2+2\beta(-20(\gamma+5\eta)+\lambda)+3\eta\alpha_1+\alpha_1^2) \\
& + 400\beta(40\beta+4\gamma\delta+8\eta+\alpha_1)\alpha_2+8000\beta\alpha_2^2\big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
S_3(t) = {}& \frac{(n+\hbar)^2\hbar\big(-10\alpha_1+20(\gamma+\eta-10\alpha_2)\big)t^\mu}{\Gamma[\mu+1]} \\
& - \frac{10(n+\hbar)\hbar^2 t^{2\mu}}{\Gamma[2\mu+1]}\big(\alpha_1^2+\alpha_1(-80\beta+\gamma+2\eta-10\alpha_2) \\
& - 2((\gamma+\eta)^2-10(2\gamma\delta+3\eta)\alpha_2-100\alpha_2^2)\big) - \frac{10\hbar^3 t^{3\mu}}{\Gamma[\mu+1]^2\Gamma[3\mu+1]}\big((-2(\gamma+\eta)^3 \\
& + 20(\gamma^2(-1+4\delta)+5\eta^2+\gamma(\eta+6\delta\eta))+\alpha_1^3+\alpha_1^2(-40\beta+\gamma+3\eta-10\alpha_2) \\
& + 200(\gamma+4\eta)\alpha_2^2+2000\alpha_2^3-\alpha_1(1600\beta^2-\gamma^2-3\gamma\eta-3\eta^2+2\beta(80\gamma+160\eta-\lambda) \\
& + (-1600\beta+20\gamma\delta+30\eta)\alpha_2+100\alpha_2^2))\Gamma[\mu+1]^2 \\
& + 10\Gamma[2\mu+1](-\alpha_1+2(\gamma+\eta-10\alpha_2))\alpha_2(2\gamma(-1+\delta)+\eta+20\alpha_2)\big),
\end{aligned}
$$

$$Q_3(t) = \frac{(n+\hbar)^2\hbar\left(10\left(2\gamma(-1+\delta)+\eta+20\alpha_2\right)\right)t^\mu}{\Gamma[\mu+1]}$$
$$+ \frac{10(n+\hbar)\hbar^2 t^{2\mu}}{\Gamma[2\mu+1]}\left(-2\gamma^2+2\gamma^2\delta-4\gamma\eta+4\gamma\delta\eta+\eta^2\right.$$
$$\left.+\alpha_1(\gamma-\gamma\delta-10\alpha_2)+20(\gamma\delta+3\eta)\alpha_2+200\alpha_2^2\right) - \frac{10\hbar^3 t^{3\mu}}{\Gamma[\mu+1]^2\Gamma[3\mu+1]}\left(\left(-2\gamma^3(-1+\delta)\right.\right.$$
$$-2\gamma^2(-1+\delta)\eta+2\gamma(-1+\delta)\eta^2+\eta^3+\alpha_1^2\left(\gamma(-1+\delta)-10\alpha_2\right)+20\left(\gamma^2\left(-1+2\delta^2\right)\right.$$
$$+\gamma(-5+8\delta)\eta+5\eta^2\left.\right)\alpha_2+200(\gamma(-1+\delta)+4\eta)\alpha_2^2+2000\alpha_2^3$$
$$-\alpha_1\left(-\gamma(-1+\delta)(-80\beta+\gamma+\eta)-10(80\beta+2\gamma-3\gamma\delta-3\eta)\alpha_2+100\alpha_2^2\right)\left.\right)\Gamma[\mu+1]^2$$
$$+10\Gamma[2\mu+1](-\alpha_1+2(\gamma+\eta-10\alpha_2))\alpha_2\left(2\gamma(-1+\delta)+\eta+20\alpha_2\right)\left.\right),$$

$$L_3(t) = \frac{(n+\hbar)^2\hbar(-20\gamma\delta+5\eta)t^\mu}{\Gamma[\mu+1]}$$
$$+\frac{5(n+\hbar)\hbar^2\left(-4\gamma^2\delta-8\gamma\delta\eta+\eta^2+2\gamma\delta\alpha_1+40\gamma\delta\alpha_2\right)t^{2\mu}}{\Gamma[2\mu+1]}$$
$$-\frac{5\hbar^3\left(4\gamma^3\delta+12\gamma^2\delta\eta+12\gamma\delta\eta^2-\eta^3-2\gamma\delta\alpha_1^2-2\gamma\delta\alpha_1\left(-80\beta+\gamma+3\eta-10\alpha_2\right)-80\gamma\delta(\gamma\delta+2\eta)\alpha_2-400\gamma\delta\alpha_2^2\right)t^{3\mu}}{\Gamma[3\mu+1]}$$

$\vdots$

In this way, the rest of the iterative terms can be obtained. Then, the *q*-HATM series solution for the system of Eq. (31) is given by

$$P(t) = P_0(t) + \sum_{m=1}^{\infty} P_m(t)\left(\frac{1}{n}\right)^m,$$

$$O(t) = O_0(t) + \sum_{m=1}^{\infty} O_m(t)\left(\frac{1}{n}\right)^m,$$

$$S(t) = S_0(t) + \sum_{m=1}^{\infty} S_m(t)\left(\frac{1}{n}\right)^m, \quad (38)$$

$$Q(t) = Q_0(t) + \sum_{m=1}^{\infty} Q_m(t)\left(\frac{1}{n}\right)^m,$$

$$L(t) = L_0(t) + \sum_{m=1}^{\infty} L_m(t)\left(\frac{1}{n}\right)^m.$$

Numerical results and discussion

In order to validate the proposed technique is reliable and efficient; the approximated analytical solutions have been evaluated. Tables 2, 3, 4, 5 and 6 show that the considered model is noticeably depends on time-fractional-order derivative. This guides us to understand the behaviour of smoking evolution. Figures 1, 2, 3, 4 and 5 explore the nature of obtained solutions for potential smoker $P(t)$, smoker $S(t)$, occasional smoker $O(t)$, permanently quit smoker $L(t)$ and

temporarily quit smoker $Q(t)$ with change in time (t) for distinct Brownian motion and standard motion ($\mu = 1$), and from these we see that the arbitrary-order smoking model has more degree of flexibility. Additionally, from these plots, we absorbed that the consequence of considered model is evidently depends on its history and the parameters mentioned in the system of Eq. (9). The $\hbar$-curves for the system considered in Eq. (31) at distinct values of μ are presented in Figs. 6, 7, 8, 9 and 10. For appropriate value of $\hbar$, series solutions rapidly converge to the exact solution in the cited scheme. Moreover, this helps us to control and adjust the region of convergence for the obtained series solution. From
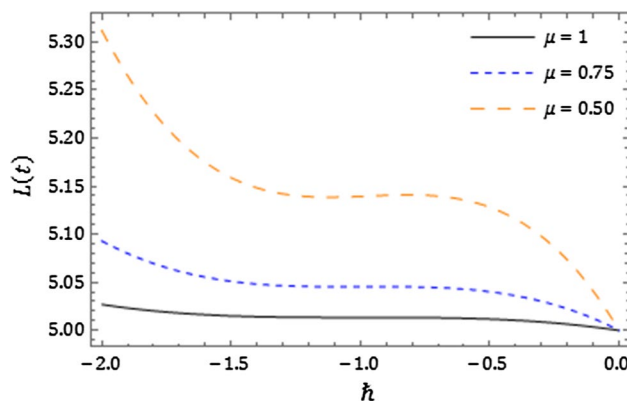

Fig. 10 $\hbar$-curves drown for $L(t)$ with diverse μ at $t = 0.01$, $n = 1$ and using Table 1

the obtained results, we can see that the proposed scheme is helpful to understand the behaviour with the aid of fractional derivative.

Conclusion

In present framework, the approximated analytical solutions for the smoking epidemic model through q-homotopy analysis transform method were investigated. The fractional equations involved in the considered model are of the Caputo sense. The obtained solutions are given in a series form, which converges rapidly. The graphical representation shows that the model depends notably on the fractional order and the selected system of parameters, and these parameters can influence the stability of the model. The dynamic behaviour of smoking model depends on both the time instant and the time history, which can be efficiently modelled using the

theory of fractional calculus. By considering the Caputo fractional derivatives, we realize that we have stabilized a more competent realistic model. Finally, we conclude that the proposed technique is highly methodical as well as more effective and it can be used to examine nonlinear fractional mathematical models describing biological phenomena, and the use of fractional calculus opens the new paradigms in the area of mathematical modelling.

Appendix

Here, we present proof for the Theorems 2 and 3.

Theorem 2 (Uniqueness theorem) [38] *The solution for the nonlinear fractional differential Eq. (13) obtained by q-HATM is unique for every $\beta \in (0, 1)$, where $\beta = (n + \hbar) + \hbar(\epsilon + \mu)T$.*

Proof For Eq. (13), the solution is defined by

$$v(x, t) = \sum_{m=0}^{\infty} v_m(x, t),$$

where

$$v_m(x, t) = (k_m + \hbar)v_{m-1}(x, t) - \left(1 - \frac{k_m}{n}\right)L^{-1}\left(\sum_{k=0}^{n-1} s^{\mu-k-1}v^{(k)}(x, 0)\right.$$
$$\left. + \frac{1}{s^{\mu}}L[f(x, t)]\right) + \hbar L^{-1}\left[\frac{1}{s^{\mu}}L[Rv_{m-1} + \mathcal{H}_{m-1}]\right].$$
(39)

If possible, let v and $v^{\star}$ be two distinct solutions of Eq. (13); then, with the help of the above equation, we obtain

$$\left|v - v^{\star}\right| = \left|(n + \hbar)(v - v^{\star}) + \hbar L^{-1}\left(\frac{1}{s^{\alpha}}L(N(v - v^{\star}) + R(v - v^{\star}))\right)\right|.$$

Then, by using the convolution theorem for LT, we have

$$\left|v - v^{\star}\right| \le (n + \hbar)\left|v - v^{\star}\right| + \hbar \int_0^t \left(\left|N(v - v^{\star})\right| + \left|R(v - v^{\star})\right|\right)\frac{(t - \xi)^{\mu}}{\Gamma(\mu + 1)}\mathrm{d}\xi$$

$$\le (n + \hbar)\left|v - v^{\star}\right| + \hbar \int_0^t \left(\epsilon\left|(v - v^{\star})\right| + \mu\left|(v - v^{\star})\right|\right)\frac{(t - \xi)^{\mu}}{\Gamma(\mu + 1)}\mathrm{d}\xi.$$

By the aid of integral mean value theorem, the above equation reduces to

$$\left|v - v^{\star}\right| \le (n + \hbar)\left|v - v^{\star}\right| + \hbar\left(\epsilon\left|(v - v^{\star})\right| + \mu\left|(v - v^{\star})\right|\right)T.$$

Here, $\beta = (n + \hbar) + \hbar(\epsilon + \mu)T$; thus,

$$\left|v - v^{\star}\right| \le \beta\left|v - v^{\star}\right| \Rightarrow (1 - \beta)\left|v - v^{\star}\right| \le 0$$

Since $0 < \beta < 1$, then $v - v^{\star} = 0 \Rightarrow v = v^{\star}$. Hence, the solution for Eq. (5) is unique.

Theorem 3 (Convergence theorem) [38] *Let X be a Banach space and $F : X \to X$ be a nonlinear mapping. Assume that*

$$||F(v) - F(w)|| \le \beta||v - w||, \quad \forall a, b \in X,$$

then F has a fixed point in view of Banach fixed point theory. Moreover, for the arbitrary selection of $a_0, b_0 \in X$, the sequence generated by the q-HATM converges to fixed point of F and

$$v_m - v_n \le \frac{\beta^n}{1 - \beta}v_1 - v_0, \quad \forall a, b \in X.$$
(40)

Proof For all continuous functions, let us consider Banach space $(C[I], ||\cdot||)$ on I with norm is given by $||g(\lambda)|| = \max_{\lambda \in I} |g(\lambda)|$. First, we prove that $\{v_n\}$ is Cauchy sequence in X.

Now consider,

$$
\begin{aligned}
||v_m - v_n|| &= \max_{\lambda \in I} |v_m - v_n| \\
&= \max_{\lambda \in I} \left| (n + \hbar)(v_{m-1} - v_{n-1}) + \hbar L^{-1}\left(\frac{1}{s^\alpha} L(N(v_{m-1} - v_{n-1}) + R(v_{m-1} - v_{n-1}))\right)\right| \\
&\leq \max_{\lambda \in I} [(n + \hbar)|(v_{m-1} - v_{n-1})| + \hbar L^{-1}\left(\frac{1}{s^\alpha} L(N|v_{m-1} - v_{n-1}| + R(|v_{m-1} - v_{n-1}|))\right).
\end{aligned}
\tag{41}
$$

By the convolution theorem for Laplace transform, Eq. (40) becomes

$$
\begin{aligned}
||v_m - v_n|| &\leq \max_{\lambda \in I} [(n + \hbar)|(v_{m-1} - v_{n-1})| \\
&\quad + \hbar \int_0^t \left(|N(v_{m-1} - v_{n-1})| + |R(v_{m-1} - v_{n-1})|\right) \frac{(t-\xi)^\alpha}{\Gamma(\alpha+1)} d\xi \\
&\leq \max_{\lambda \in I} [(n + \hbar)|(v_{m-1} - v_{n-1})| \\
&\quad + \hbar \int_0^t \left(|N(v_{m-1} - v_{n-1})| + |R(v_{m-1} - v_{n-1})|\right) \frac{(t-\xi)^\alpha}{\Gamma(\alpha+1)} d\xi.
\end{aligned}
\tag{42}
$$

By the aid of integral mean value theorem, the above inequality reduces to

$$
\begin{aligned}
||v_m - v_n|| &\leq \max_{\lambda \in I} \left[(n + \hbar)|(v_{m-1} - v_{n-1})| \right. \\
&\quad \left. + \hbar(\epsilon|v_{m-1} - v_{n-1}| + \mu|v_{m-1} - v_{n-1}|)T\right] \\
&\leq \beta ||v_{m-1} - v_{n-1}||.
\end{aligned}
$$

For $m = n + 1$, one can get

$$
\begin{aligned}
||v_{n+1} - v_n|| &\leq ||\beta v_n - v_{n-1}|| \leq \beta^2 ||v_{n-1} - v_{n-2}|| \\
&\leq \beta^3 ||v_{n-2} - v_{n-3}|| \leq \cdots \leq \beta^n ||v_1 - v_0||.
\end{aligned}
$$

In view of triangular inequality, we have

$$
\begin{aligned}
||v_m - v_n|| &\leq ||v_{n+1} - v_n|| + ||v_{n+2} - v_{n+1}|| + \cdots + ||v_m - v_{m-1}|| \\
&\leq [\beta^n + \beta^{n+1} + \cdots + \beta^{m-1}] ||v_1 - v_0|| \\
&= \beta^n [1 + \beta + \cdots + \beta^{m-n-1}] ||v_1 - v_0|| \\
&\leq \beta^n \left[\frac{1 - \beta^{m-n-1}}{1 - \beta}\right] ||v_1 - v_0||.
\end{aligned}
$$

Clearly, $1 - \beta^{m-n-1} < 1$ (since $0 < \beta < 1$). Therefore, the above inequality becomes

$$
||v_m - v_n|| \leq \frac{\beta^n}{1 - \beta} ||v_1 - v_0||.
\tag{43}
$$

But $||v_1 - v_0|| < \infty$, consequently as $m \to \infty$, then $||v_m - v_n|| \to 0$.

It provides $\{v_n\}$ is Cauchy sequence in $C[I]$, and every Cauchy sequence is convergent sequence. Hence, $\{v_n\}$ is convergent sequence.

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