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ORIGINAL RESEARCH

Fuzzy reproducing kernel space method for solving fuzzy boundary value problems

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Abstract

In the beginning, we describe the fuzzy inner product space and the fuzzy Hilbert space. Our goal is to use the fuzzy reproducing kernel method to solve the second-order fuzzy boundary value problem. The fuzzy convergence analysis of introduced method is discussed in detail. We present some examples in the end.

Keywords Fuzzy Hilbert space · Fuzzy convergence · Fuzzy reproducing kernel method

Introduction

Reproducing kernel method is one of the most basic methods for approximation. In doing so, our main goal is to solve fuzzy boundary value problems using fuzzy reproducing kernel methods. We discuss the convergence of our procedure using the concept of fuzzy distance. At the end, there are some examples. We describe the fuzzy reproducing kernel method (FRKM for short) for solving the following fuzzy boundary value problem:

$$\begin{cases} y''(x) + m(x)y'(x) + n(x)y(x) = f(x) \quad & 0 < x < 1, \\ y(0) = 0, \quad y(1) = 0, \end{cases} \tag{1.1}$$

here, $f(x)$ is a fuzzy function. The functions of $m(x)$ and $n(x)$ are ordinary and continuous. The process of doing this paper is as follows: In the part of preliminaries, the fundamental concepts that have been used in later sections are presented. In "Fuzzy reproducing kernel space" section, we have defined the fuzzy reproducing kernel (FRK for short). Fuzzy inner product spaces and fuzzy reproducing kernel spaces have been introduced. Also, the solution to Eq. (1.1) is given with the initial boundary value conditions. In "Fuzzy convergence analysis" section, we bring the fuzzy convergence theorem. In the next section, we give some examples for a better understanding. At the end of this work, we bring the results. For more information on the reproducing kernel method and fuzzy convergence, see [1–3, 11].

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Preliminaries

We introduce the primary definitions of the generalized Hukuhara difference, generalized Hukuhara derivative, Hausdorff distance and fuzzy continuous function. For definition of the fuzzy number and α-level set see [10].

Remark 2.1 We assume that in the whole of this paper the generalized Hukuhara difference exists. We denote generalized Hukuhara difference by $\ominus_{gH}$.

Definition 2.2 [5] Let x and y be two fuzzy numbers in $\mathbb{R}_F$, where $\mathbb{R}_F$ is the set of all fuzzy numbers. If the exists a fuzzy number as z that satisfies in follows condition, then the generalized Hukuhara difference (gH-difference for short) is defined as

$$x \ominus_{gH} y = z \Leftrightarrow \begin{cases} (i)\ x = y + z; \\ or\ (ii)\ y = x + (-1)z. \end{cases} \tag{2.2}$$

where (i) and (ii) are both valid if and only if z is a crisp number.

Definition 2.3 [4] Let x, y, z, w are fuzzy numbers and $\beta \in \mathbb{R}$. Define the Hausdorff distance as in [9]:
$D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}^+ \cup \{0\}$
$D(x, y) = \sup_{\alpha\in[0,1]} \max\{|x^-(\alpha) - y^-(\alpha)|, |x^+(\alpha) - y^+(\alpha)|\}$.
By [13], we have:

1. $D(x \oplus z, y \oplus z) = D(x, y)$,
2. $D(\lambda x, \lambda y) = |\lambda| D(x, y)$,
3. $D(x \oplus y, z \oplus w) \le D(x, z) + D(y, w)$,
4. $D(x \ominus y, z \ominus w) \le D(x, z) + D(y, w)$, until $x \ominus y$ and $z \ominus w$ are available.

In the above features, $\ominus$ is the Hukuhara difference. It is equal to $z \ominus y = x$ iff $x \oplus y = z$.

In the following, we denote the Hausdorff distance by D.

Definition 2.4 [14] Let $f : (\alpha, \beta) \to \mathbb{R}_F$ be a fuzzy-valued function. In this case, the generalized Hukuhara derivative of f at $x_o \in (\alpha, \beta)$ is defined as

$$f'_{gH}(x_0) = \lim_{\varepsilon\to 0} \frac{f(x_0 + \varepsilon) \ominus_{gH} f(x_0)}{\varepsilon}. \tag{2.3}$$

If $f'_{gH}(x_0) \in \mathbb{R}_F$ satisfying (2.3) exists, f is called generalized Hukuhara differentiable (gH-differentiable for short) at x_0.

Definition 2.5 [7] Let $f : [\alpha, \beta] \to \mathbb{R}_F$ be a fuzzy-valued function. In this case, f at $x_0 \in [\alpha, \beta]$ is continuous if $\forall\varepsilon > 0, \exists\delta > 0 : D(f(x), f(x_0)) < \varepsilon$, when $x \in [\alpha, \beta]$ and $|x - x_0| < \delta$. If f is continuous for all $x_0 \in [\alpha, \beta]$, then f is a fuzzy continuous on interval of $[\alpha, \beta]$.

Fuzzy reproducing kernel space

We will introduce the FRKM to solve the fuzzy boundary value problems. Initially, we create the fuzzy reproducing kernel space (FRKS for short) $W^m[0, 1]$, while each fuzzy function satisfies in the $y(0) = 0$ and $y(1) = 0$.

Definition 3.1 (*Fuzzy absolutely continuous function*) If $f(x)$ is a fuzzy function on $[\alpha, \beta]$, and $\{(\alpha_i, \beta_i)\}_{i=1}^n$ is a set of disjoint open intervals $(\alpha_i, \beta_i) \subset [\alpha, \beta]$.

If for all ε, there is a δ, which has no relation with n, such that $\sum_{k=1}^n D(f(\beta_k), f(\alpha_k)) < \varepsilon$ for $\sum_{k=1}^n (\beta_k - \alpha_k) < \delta$, then $f(x)$ is a fuzzy absolutely continuous function (FACF for short) on $[a, b]$.

Definition 3.2 Define the fuzzy space $W^m[0, 1]$ as $W^m[0, 1] = \{y(x)|y$ is a fuzzy function, $y^{(m-1)}_{gH}(x)$ is a FACF, $y^{(m)}_{gH}(x) \in L^2[0, 1], L^2[0, 1]$ is a fuzzy space, $y(0) = 0, y(1) = 0\}$.

We define the fuzzy inner product (FIP for short) in this space as

$$\langle y, z\rangle_m = \sum_{k=0}^{m-1} y^{(k)}_{gH}(0) \odot z^{(k)}_{gH}(0) \oplus \int_0^1 y^{(m)}_{gH}(x) \odot z^{(m)}_{gH}(x)\mathrm{d}x; \quad y, z \in W^m[0, 1], \tag{3.4}$$

where $\langle y, z\rangle_m$ is the fuzzy inner product in the fuzzy space $W^m[0, 1]$. Also, fuzzy norm in this space is defined as $\|y\|_m = \sqrt{\langle y, y\rangle_m}$.

Definition 3.3 Define the fuzzy space $W^1[0, 1]$ as $W^1[0, 1] = \{y(x)|y$ is a fuzzy function, $y(x)$ is a FACF, $y'_{gH}(x) \in L^2[0, 1], L^2[0, 1]$ is a fuzzy space $\}$. We define the FIP in this space as

$$\langle y, z\rangle_1 = y(0) \odot z(0) \oplus \int_0^1 y'_{gH}(x) \odot z'_{gH}(x)\mathrm{d}x; \quad y, z \in W^1[0, 1], \tag{3.5}$$

where $\langle y, z\rangle_1$ is the fuzzy inner product in the fuzzy space $W^1[0, 1]$.

Definition 3.4 Define the fuzzy space $L^2[0, 1]$ as

$$L^2[0, 1] = \left\{ y(x)|y \text{ is a fuzzy function}, \quad \int_0^1 y^2(x)\mathrm{d}x < \infty \right\}. \tag{3.6}$$

We define the FIP in this space as

$$\langle y, z\rangle_{L^2} = \int_0^1 y(x) \odot z(x)\mathrm{d}x; \quad y, z \in L^2[0, 1], \tag{3.7}$$

where $\langle y, z\rangle_{L^2}$ is the fuzzy inner product in the fuzzy space $L^2[0, 1]$.

Definition 3.5 If $f \in W^m[0, 1]$, we define $\langle f, f\rangle_m = \|f\|^2_m = D^2(f, 0)$.

Using definition 3.1 in [8], we can deduce Lemmas 3.6 and 3.8 .

Lemma 3.6 *Suppose that $F(\mathbb{R}, F(\mathbb{R}))$ is a vector space over $\mathbb{R}$. Also, the FIP in the fuzzy space $L^2[0, 1]$ on $F(\mathbb{R}, F(\mathbb{R}))$ is a mapping as $\langle ., .\rangle_{L^2} : F(\mathbb{R}, F(\mathbb{R})) \times F(\mathbb{R}, F(\mathbb{R})) \to F(\mathbb{R})$, with this property for each $k \in F(\mathbb{R})$ and all functions $f_1, f_2, f_3 \in F(\mathbb{R}, F(\mathbb{R}))$, satisfies the following conditions:*

1. $\langle f_1 + f_2, f_3\rangle_{L^2} = \langle f_1, f_3\rangle_{L^2} \oplus \langle f_2, f_3\rangle_{L^2}$,
2. $\langle kf_1, f_2\rangle_{L^2} = k\langle f_1, f_2\rangle_{L^2}$,

3. $\langle f_1, f_2 \rangle_{L^2} = \langle f_2, f_1 \rangle_{L^2}$,
4. $\langle f_1, f_1 \rangle_{L^2} > 0$,
5. if $f_1 \neq 0$, then $inf_{\alpha \in (0,1]} \langle f_1, f_1 \rangle_\alpha^- > 0$,
6. $\langle f_1, f_1 \rangle_{L^2} = 0$ iff $f_1 = 0$.

Proof To prove, we need to apply the fuzzy integral properties, the concept of the Hausdorff distance and the fuzzy norm.

1. $\langle f_1 \oplus f_2, f_3 \rangle_{L^2} = \int_0^1 ((f_1 \oplus f_2) \odot f_3) = \int_0^1 (f_1 \odot f_3) \oplus \int_0^1 (f_2 \odot f_3) = \langle f_1, f_3 \rangle_{L^2} \oplus \langle f_2, f_3 \rangle_{L^2}$ $\langle f_1 \oplus f_2, f_3 \rangle_{L^2} = \int_0^1 ((f_1 \oplus f_2) \odot f_3) = \int_0^1 (f_1 \odot f_3) \oplus \int_0^1 (f_2 \odot f_3) = \langle f_1, f_3 \rangle_{L^2} \oplus \langle f_2, f_3 \rangle_{L^2}$.
2. $\langle k \odot f_1, f_2 \rangle_{L^2} = \int_0^1 (k \odot f_1 \odot f_2) = k \odot \int_0^1 (f_1 \odot f_2) = k \odot \langle f_1, f_2 \rangle_{L^2}$.
3. $\langle f_1, f_2 \rangle_{L^2} = \int_0^1 (f_1 \odot f_2) = \int_0^1 (f_2 \odot f_1) = \langle f_2, f_1 \rangle_{L^2}$.
4. Since $\langle f_1, f_1 \rangle_{L^2} = \|f_1\|^2 = D^2(f_1, 0) > 0$, we have $\langle f_1, f_1 \rangle_{L^2} > 0$.
5. If $\langle f_1, f_1 \rangle_{L^2} > 0$, then $[\langle f_1, f_1 \rangle_{L^2}]_\alpha > 0$. Therefore, we have $[\langle f_1, f_1 \rangle_\alpha^-, \langle f_1, f_1 \rangle_\alpha^+] > 0$ and $inf \langle f_1, f_1 \rangle_\alpha^- > 0$.
6. Since $\langle f_1, f_1 \rangle_{L^2} = 0$, we have $\|f_1\|^2 = 0$ and $D^2(f_1, 0) = 0$. Hence, $f_1 \ominus_{gH} 0 = 0$. It follows that $f_1 = 0$.

Conversely, if $f_1 = 0$, then $f_1 \ominus_{gH} 0 = 0$. Thus, $\|f_1 \ominus_{gH} 0\| = 0$. Since $D^2(f_1, 0) = 0$, we have $\langle f_1, f_1 \rangle_{L^2} = 0$.

Corollary 3.7 *The vector space $F(\mathbb{R}, F(\mathbb{R}))$ with a FIP in the form of $\langle f_1, f_2 \rangle_{L^2} = \int_0^1 f_1(x) \odot f_2(x) dx$ is called a fuzzy inner product space (FIPS for short).*

Lemma 3.8 *Suppose that $F(\mathbb{R}, F(\mathbb{R}))$ is a vector space over $\mathbb{R}$. Also, the FIP in the fuzzy space $W^1[0,1]$ on $F(\mathbb{R}, F(\mathbb{R}))$ is a mapping as $\langle ., . \rangle_1 : F(\mathbb{R}, F(\mathbb{R})) \times F(\mathbb{R}, F(\mathbb{R})) \rightarrow F(\mathbb{R})$, with this property for each $k \in F(\mathbb{R})$ and all functions $f_1, f_2, f_3 \in F(\mathbb{R}, F(\mathbb{R}))$, satisfies the following conditions:*

1. $\langle f_1 + f_2, f_3 \rangle_1 = \langle f_1, f_3 \rangle_1 \oplus \langle f_2, f_3 \rangle_1$,
2. $\langle k f_1, f_2 \rangle_1 = k \langle f_1, f_2 \rangle_1$,
3. $\langle f_1, f_2 \rangle_1 = \langle f_2, f_1 \rangle_1$,
4. $\langle f_1, f_1 \rangle_1 > 0$,
5. if $f_1 \neq 0$, then $inf_{\alpha \in (0,1]} \langle f_1, f_1 \rangle_\alpha^- > 0$,
6. $\langle f_1, f_1 \rangle_1 = 0$ iff $f_1 = 0$.

Proof To prove, we need to apply the fuzzy integral concepts, the Hausdorff distance and the fuzzy norm.

1. $\langle f_1 \oplus f_2, f_3 \rangle_1 = (f_1(0) \oplus f_2(0)) \odot f_3(0) \oplus \int_0^1 (f_1' \oplus f_2') \odot f_3' = f_1(0) \odot f_3(0) \oplus f_2(0) \odot f_3(0) \oplus \int_0^1 (f_1' \odot f_3') \oplus \int_0^1 (f_2' \odot f_3') = \langle f_1, f_3 \rangle_1 \oplus \langle f_2, f_3 \rangle_1$.
2. $\langle k \odot f_1, f_2 \rangle_1 = k \odot f_1(0) \odot f_2(0) \oplus \int_0^1 (k \odot f_1' \odot f_2') = k \odot f_1(0) \odot f_2(0) \oplus k \odot \int_0^1 (f_1' \odot f_2') = k \odot (f_1(0) \odot f_2(0) \oplus \int_0^1 (f_1' \odot f_2')) = k \odot \langle f_1, f_2 \rangle_1$.
3. $\langle f_1, f_2 \rangle_1 = f_1(0) \odot f_2(0) \oplus \int_0^1 (f_1' \odot f_2') = f_2(0) \odot f_1(0) \oplus \int_0^1 (f_2' \odot f_1') = \langle f_2, f_1 \rangle_1$.

The remaining proofs follow from the previous lemma.

Corollary 3.9 *The vector space $F(\mathbb{R}, F(\mathbb{R}))$ with a FIP in the form $\langle f_1, f_2 \rangle_1 = f_1(0) \odot f_2(0) \oplus \int_0^1 f_1'(x) \odot f_2'(x) dx$ is called a FIPS.*

In the following, assume that $\langle ., . \rangle_m$ is the fuzzy inner product in the fuzzy space $W^m[0,1]$. Also, $\|.\|$ is the fuzzy norm.

Definition 3.10 A fuzzy sequence $\{(h_n, \alpha_n)\}$ in a FIPS $(W^m[0,1], \langle ., . \rangle_m)$ is a fuzzy convergent if there is a fuzzy function $h \in W^m[0,1]$ such that $\lim_{n \to \infty} \|h_n \ominus_{gH} h\| = 0$ or $\lim_{n \to \infty} D(h_n, h) = 0$, in which $\|h\| = \sqrt{\langle h, h \rangle}$.

Definition 3.11 A fuzzy sequence $\{(h_n, \alpha_n)\}$ in a FIPS $(W^m[0,1], \langle ., . \rangle_m)$ is a fuzzy Cauchy if $\forall \varepsilon > 0, \exists M > 0 : \forall m, n > M$, we have $\|h_m \ominus_{gH} h_n\| < \varepsilon$, in which $\|h\| = \sqrt{\langle h, h \rangle}$.

Definition 3.12 The FIPS $(W^m[0,1], \langle ., . \rangle_m)$ is a fuzzy complete if each fuzzy Cauchy sequence in $W^m[0,1]$ is a fuzzy convergent.

Definition 3.13 The FIPS is a fuzzy Hilbert space (FHS for short), while it is a fuzzy complete in the fuzzy norm $\|x\| = \sqrt{\langle x, x \rangle_m}$.

Corollary 3.14 *The FIPS $(W^m[0,1], \langle ., . \rangle_m)$ is a FHS, when it is a fuzzy complete in the fuzzy norm $\|.\| = \sqrt{\langle ., . \rangle_m}$.*

Definition 3.15 We consider H as a FHS, with FIP $\langle f_1(x), f_2(x) \rangle_H$, in which $f_1(x)$ and $f_2(x)$ belong to H. If there is a fuzzy function such as $R_y(x) \in H$, where every $f_1(x)$ satisfies the condition $\langle f_1(x), R_y(x) \rangle_H = f_1(y)$, then $R_y(x)$ is called the FRK of H. Also, FHS of H is called the FRKS.

The exact and approximate solutions

Definition 3.16 (*Fuzzy Hilbert adjoint operator*) Assume that H_1 and H_2 are fuzzy Hilbert spaces, and $L : H_1 \rightarrow H_2$ is a fuzzy bounded linear operator, then $L^* : H_2 \rightarrow H_1$ is the fuzzy Hilbert adjoint operator, while for each $x_1 \in H_1$ and $x_2 \in H_2$, we have $\langle L x_1, x_2 \rangle_{H_2} = \langle x_1, L^* x_2 \rangle_{H_1}$, where $\langle ., . \rangle_{H_i}$ is the FIP in the FHS of H_i, in which $i = 1, 2$.

Definition 3.17 (*Fuzzy Gram–Schmidt process*) Given an arbitrary basis $\varphi_1, \varphi_2, \ldots, \varphi_n$ for a FIPS, if all the gH-differences are present, the fuzzy Gram–Schmidt process constructs by the fuzzy orthogonal basis $A_1, A_2, \ldots, A_n$:

Step 1 Let $A_1 = \varphi_1$,

Step 2 Let $A_2 = \varphi_2 \ominus_{gH} \dfrac{\langle \varphi_1, \varphi_2 \rangle}{\|\varphi_1\|^2} \varphi_1$, Step 3 Let $A_3 = \varphi_3 \ominus_{gH} \frac{\langle \varphi_1, \varphi_3 \rangle}{\|\varphi_1\|^2}\varphi_1 \ominus_{gH} \frac{\langle \varphi_2 \ominus_{gH} \frac{\langle \varphi_1, \varphi_2 \rangle}{\|\varphi_1\|^2}\varphi_1, \varphi_3 \rangle}{\|\varphi_2 \ominus_{gH} \frac{\langle \varphi_1, \varphi_2 \rangle}{\|\varphi_1\|^2}\varphi_1\|^2}\varphi_2 \ominus_{gH} \frac{\langle \varphi_1, \varphi_2 \rangle}{\|\varphi_1\|^2}\varphi_1,$

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Step n Let $A_n = \varphi_n \ominus_{gH} \frac{\langle A_1, \varphi_n \rangle}{\|A_1\|^2}A_1 \ominus_{gH} \frac{\langle A_2, \varphi_n \rangle}{\|A_2\|^2}A_2 \ldots \ominus_{gH} \frac{\langle A_{n-1}, \varphi_n \rangle}{\|A_{n-1}\|^2}A_{n-1}.$

By normalizing $A_1, \ldots, A_n$ vectors, we can obtain the fuzzy normal orthogonal vectors of the form as follows:

$$\overline{\varphi_1} = \frac{1}{\|A_1\|}A_1, \overline{\varphi_2} = \frac{1}{\|A_2}\|A_2, \ldots, \overline{\varphi_n} = \frac{1}{\|A_n\|}A_n.$$

The fuzzy orthonormal function system $\{\overline{\varphi_i}(x)\}_{i=1}^{\infty}$ of the fuzzy space $W^m[0,1]$ will be obtained by fuzzy Gram–Schmidt orthogonalization process of $\{\varphi_i(x)\}_{i=1}^{\infty}$ as $\overline{\varphi_i}(x) = \sum_{k=1}^{i} \beta_{ik}\varphi_k(x)$, where β_{ik} are orthogonalization coefficients given as $\beta_{11} = \frac{1}{\|\varphi_1\|}$, $\beta_{ii} = \frac{1}{d_{ik}}$, and $\beta_{ij} = -(\frac{1}{d_{ik}})\sum_{k=j}^{i-1} c_{ik}\beta_{kj}$ for $j < i$, while $d_{ik} = \sqrt{\|\varphi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}$, $c_{ik} = \langle \varphi_i, \overline{\varphi_k} \rangle$, and $\{\varphi_i(x)\}_{i=1}^{\infty}$ is the orthonormal system in the space $W^m[0,1]$.

Definition 3.18 We consider $\{\overline{\varphi_i}(x)\}_{i=1}^{\infty}$ as a fuzzy orthonormal system, we have:

$$\langle \overline{\varphi_i}(x), \overline{\varphi_j}(x) \rangle = \begin{cases} 0 & i \neq j; \\ 1 & i = j. \end{cases} \tag{3.8}$$

where 0 and 1 are fuzzy numbers.

Assume that $Ly(x) = y''(x) + m(x)y'(x) + n(x)y(x)$ in the equation of (1.1). In this case, $L : W^m[0,1] \to W^1[0,1]$ is a fuzzy bounded linear operator. We take $\delta_i(x) = R_{x_i}(x)$ and $\varphi_i(x) = L^*\delta_i(x)$ where $R_{x_i}(x)$ is the FRK. Also, L^* is the fuzzy adjoint operator of L. Using the concepts of the FRK, for each $y(x)$, the following equality is true:

$$\langle y(x), \delta_i(x) \rangle = \langle y(x), R_{x_i}(x) \rangle = y(x_i). \tag{3.9}$$

Regarding the above and using the properties of $R_x(t)$, we get

$$\langle y(x), \varphi_i(x) \rangle = \langle y(x), L^*\delta_i(x) \rangle = \langle Ly(x), \delta_i(x) \rangle = Ly(x_i), \quad i \in \mathbb{N}. \tag{3.10}$$

Moreover,

$$\varphi_i(x) = \langle \varphi_i(t), R_x(t) \rangle = \langle L^*\delta_i(t), R_x(t) \rangle = \langle \delta_i(t), L_t R_x(t) \rangle = L_t R_x(t)|_{t=x_i}. \tag{3.11}$$

In this case, $\varphi_i(x) = L_t R_x(t)|_{t=x_i}$, where L_t is the fuzzy operator L that applies to the function of t.

By the fuzzy Gram–Schmidt process, we orthonormalize the sequence $\{\varphi_i(x)\}_{i=1}^{\infty}$ and we get the fuzzy orthonormal system $\{\overline{\varphi_i}(x)\}_{i=1}^{\infty}$, that is, $\overline{\varphi_i}(x) = \sum_{k=1}^{i} \beta_{ik}\varphi_k(x)$, $(\beta_{ii} > 0, i = 1, 2, \ldots)$, where β_{ik} are coefficients of Gram–Schmidt orthonormalization and $\{\overline{\varphi_i}(x)\}_{i=1}^{\infty}$ is a fuzzy orthonormal basis of the fuzzy space $W^m[0,1]$.

Theorem 3.19 *Assume that the solution of equation of (1.1) is unique. The exact solution of this equation is as* $y(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\varphi_i}(x)$.

Proof Suppose that $y(x)$ is the solution of equation of (1.1) given that $\{\overline{\varphi_i}(x)\}_{i=1}^{\infty}$ is a fuzzy orthonormal system. In this case, the following equalities are true:

$$\begin{aligned} y(x) &= \sum_{i=1}^{\infty} \langle y(x), \overline{\varphi_i}(x) \rangle \overline{\varphi_i}(x) \\ &= \sum_{i=1}^{\infty} \langle y(x), \sum_{k=1}^{i} \beta_{ik}\varphi_k(x) \rangle \overline{\varphi_i}(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle y(x), \varphi_k(x) \rangle \overline{\varphi_i}(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle y(x), L^*\delta_k(x) \rangle \overline{\varphi_i}(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle Ly(x), \delta_k(x) \rangle \overline{\varphi_i}(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle Ly(x), R_{x_k}(x) \rangle \overline{\varphi_i}(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} Ly(x_k) \overline{\varphi_i}(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k) \overline{\varphi_i}(x), \end{aligned} \tag{3.12}$$

where $\langle ., . \rangle$ is the fuzzy inner product. Also, the approximate solution of (1.1) is as

$$y_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(x_k) \overline{\varphi_i}(x). \tag{3.13}$$

□

Fuzzy convergence analysis

Here, we examine the fuzzy convergence.

Lemma 4.1 *Let* $(W^m[0,1], \langle .,. \rangle_m)$ *be a FIPS. Also,* $y(x) \in W^m[0,1]$. *In this case, there is a constant n such that*

1. $|y(x)| \leq n\|y(x)\|_m$,
2. $|y_{gH}^{(k)}(x)| \leq n\|y(x)\|_m, \quad (k = 1, 2, \dots, m-1)$.

Proof Using the property of the FRK in the Definition 3.15, we have $y(x) = \langle y(.), R_x(.)\rangle_m$. Also, by Lemma 3.2 in [8], we can conclude that $|y(x)| = |\langle y(.), R_x(.)\rangle_m| \leq \|y(.)\|_m \|R_x(.)\|_m$. Therefore, there is a constant n such that $|y(x)| \leq n\|y\|_m$.

Also, $y_{gH}^{(k)}(x) = \langle y(.), R_x^{(k)}(.)\rangle_m$. In this case,

$$\begin{aligned}|y_{gH}^{(k)}(x)| &= |\langle y(.), R_x^{(k)}(.)\rangle_m| \\ &\leq \|y(.)\|_m \|R_x^{(k)}(.)\|_m \leq n_k \|y\|_m, \\ &(k = 1, \dots, m-1).\end{aligned} \tag{4.14}$$

Hence, it is enough to suppose $n = \max_{1 \leq k \leq m-1}\{n_k\}$. □

Corollary 4.2 *By Definition 3.10, a fuzzy sequence* $\{(y_n, \alpha_n)\}$ *in a FIPS* $(W^m[0,1], \langle .,. \rangle_m)$ *is called a fuzzy convergent when there is a fuzzy function* y_n *in the fuzzy space* $W^m[0,1]$ *such that*

$$\lim_{n\to\infty} \|y_n \ominus_{gH} y\| = 0. \tag{4.15}$$

where y is an exact solution and y_n *is an approximate solution of* (1.1).

Theorem 4.3 *The approximate solution* $y_n(x)$ *is uniformly fuzzy convergent. Also, for all* $k = 1, 2, \dots, m-1$, *the derivatives* $y_n^{(k)}(x)$ *are all uniformly fuzzy convergent.*

Proof If $y(x)$ be solution of equation of (1.1) in the fuzzy space $W^m[0,1]$, then we have:

$$\begin{aligned}|y_n(x) \ominus_{gH} y(x)| &= |\langle y_n(.) \ominus_{gH} y(.), R_x(.)\rangle| \\ &\leq \|y_n(.) \ominus_{gH} y(.)\| \|R_x(.)\| \\ &\leq M\|y_n(.) \ominus_{gH} y(.)\|\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}|y_n^{(k)}(x) \ominus_{gH} y_{gH}^{(k)}(x)| &= |\langle y_n(.) \ominus_{gH} y(.), R_x^{(k)}(.)\rangle| \\ &\leq \|y_n(.) \ominus_{gH} y(.)\| \|R_x^{(k)}(.)\| \\ &\leq N_k\|y_n(.) \ominus_{gH} y(.)\|, \ (k = 1, 2, \dots, m-1).\end{aligned} \tag{4.17}$$

Hence, it is enough to suppose $N = \max_{1 \leq k \leq m-1}\{N_k\}$ where $M, N > 0$ are constants. If $\lim_{n\to\infty} \|y_n(.) \ominus_{gH} y(.)\| = 0$, then the approximate solution $y_n(x)$(resp. $y_n^{(k)}(x)$) is uniformly fuzzy convergent to the exact solution $y(x)$(resp. $y_{gH}^{(k)}(x)$).

To solve examples of "Examples" section, by [6], we need to define the space $W^3[0,1]$ and the inner product in this space as $W^3[0,1] = \{y(x) | y''$ is an absolutely continuous function, $y'''(x) \in L^2[0,1], y(0) = 0, y(1) = 0\}$.

$$\begin{aligned}\langle y, z\rangle &= y(0)z(0) + y'(0)z'(0) + y''(0)z''(0) \\ &+ \int_0^1 y'''(x)z'''(x)\mathrm{d}x; \ y, z \in W^3[0,1].\end{aligned} \tag{4.18}$$

In this case, reproducing kernel $R_v(u)$ is as

$$R_v(u) = \begin{cases} lR_v(u) = \sum_{i=1}^6 c_i(v)u^{i-1}, & u < v, \\ rR_v(u) = \sum_{i=1}^6 d_i(v)u^{i-1}, & u \geqslant v. \end{cases} \tag{4.19}$$

where $c_i(v)$ and $d_i(v)$ are obtained by the following relationships:

$$\frac{\partial^i lR_v(v)}{\partial u^i} = \frac{\partial^i rR_v(v)}{\partial u^i}, \quad i = 0, 1, \dots, 4; \tag{4.20}$$

$$(-1)^3\left(\frac{\partial^5 R_v(v^+)}{\partial u^5} - \frac{\partial^5 R_v(v^-)}{\partial u^5}\right) = 1, \tag{4.21}$$

and

$$\begin{cases} \dfrac{\partial^i R_v(0)}{\partial u^i} - (-1)^{2-i}\dfrac{\partial^{5-i} R_v(0)}{\partial u^{5-i}} = 0, & i = 1, 2; \\ \dfrac{\partial^{5-i} R_v(1)}{\partial u^{5-i}} = 0, & i = 1, 2; \\ R_v(0) = 0; \\ R_v(1) = 0. \end{cases} \tag{4.22}$$

Thus, the representation of the reproducing kernel in $W^3[0,1]$ is as

$$R_v(u)=\begin{cases}\frac{1}{48}u(-1+v)(6(-2+v)v+3u(-2+v)v-u^2(-4+v)(2+v)), & u<v;\\ -\frac{1}{48}(-1+u)v(-8v^2-2u(-6-3v+v^2)+u^2(-6-3v+v^2)), & u\geqslant v.\end{cases} \quad (4.23)$$

Therefore,

$$R_v(u)=\begin{cases}\frac{1}{48}u(-1+v)(-12v+6v^2-6uv+3uv^2+8u^2+2u^2v-u^2v^2), & u<v;\\ \frac{1}{48}v(-1+u)(8v^2-12u-6uv+2uv^2+6u^2+3u^2v-u^2v^2), & u\geqslant v.\end{cases} \quad (4.24)$$

□

Examples

Here are some examples to better understand. In the following examples, taking into account the reproducing kernel space $W^m[a,b]$ and the reproducing kernel in this space and applying the reproducing kernel method, the exact solution of the equation is obtained. Also, we can obtain the approximate solution of this equation using the exact solution. Using the method presented in this paper (i.e, reproducing kernel method), taking:

$$x_i=a+(i-1)dx, \quad dx=\frac{b-a}{n-1}, \quad i=1,2,\dots,n. \quad (5.25)$$

we can simply discuss about $|y(x)-y_n(x)|$, for every positive n, in the different reproducing kernel spaces. This process is similar to [12].

Example 5.1 We consider the following boundary value problem. Then, we find its exact solution.

$$\begin{cases} y''(x)+200e^x y'(x)+300\sin(x)y(x)=f(x), & 0<x<1,\\ y(0)=0, \quad y(1)=0, \end{cases} \quad (5.26)$$

where $f(x)=\sinh(x)+200e^x(\cosh(x)-\sinh(1))+300\sin(x)(\sinh(x)-x\sinh(1))$. The exact solution is given by $y(x)=\sinh(x)-x\sinh(1)$.

Example 5.2 Suppose the equation of boundary value is as

$$\begin{cases} y''(x)+200e^x y'(x)+300\sin(x)y(x)=f(x), & 0<x<1,\\ y(0)=0, \quad y(1)=0, \end{cases} \quad (5.27)$$

where $f(x)=2\sinh(x)+x\cosh(x)+200e^x(\cosh(x)+x\sinh(x)-\cosh(1))+300\sin(x)(x\cosh(x)-x\cosh(1))$. Then, we simply see that the exact solution is as $y(x)=x\cosh(x)-x\cosh(1)$.

Example 5.3 If the boundary value problem is as

$$\begin{cases} y''(x)+200e^x y'(x)+300\cos(x)y(x)=f(x), & 0<x<1,\\ y(0)=0, \quad y(1)=0, \end{cases} \quad (5.28)$$

where $f(x)=\sinh(x)+200e^x(\cosh(x)-\sinh(1))+300\cos(x)(\sinh(x)-x\sinh(1))$, then the exact solution is as $y(x)=\sinh(x)-x\sinh(1)$.

Conclusion

In this paper, the definitions of fuzzy Cauchy, fuzzy complete and fuzzy inner product were studied. Moreover, we presented the solution of fuzzy second-order two-point boundary value problem by the fuzzy reproducing kernel method. We had a lot of limitations compared to the real state, and all the lemmas and the theorems were not easily verifiable. The basis for what we did was based on the existence of the gH-differences.

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