On the identification of finite non-group semigroups of a given order

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Abstract

Identifying finite non-group semigroups for every positive integer is significant because of many applications of such semigroups are functional in various branches of sciences such as computer science, mathematics and finite machines. The finite non-commutative monoids as a type of such semigroups were identified in 2014, for every positive integer. We here attempt to identify the finite *commutative monoids* and finite *commutative non-monoids* of a given integer $n = p^\alpha q^\beta$, for every integers $\alpha, \beta \geq 2$ and different primes p and q . In order to recognize the commutative monoids, we present a class of 2-generated monoids of a given order, and for the commutative non-monoids of order $n = p^\alpha q^\beta$, we give the minimal generating set. Moreover, we prove that there are exactly $(p^\alpha - 2)(q^\beta - 2)$ non-isomorphic commutative non-monoids of order $p^\alpha q^\beta$. The identification of non-group semigroups for the integers $p^{2\alpha}$ and $2p^\alpha$ is achieved. The automorphism groups of these groups are specified as well. As a result of this study, an interesting difference between the abelian groups and the commutative semigroups of order p^2 is presented.

Keywords Non-monoid monogenic semigroups · Direct product · Presentation of semigroups

Mathematics Subject Classification 20M05 · 20M14

Introduction

Identifying a finite non-group semigroup (commutative or non-commutative) of order of a given positive integer n could be significant because of the interesting applications of finite semigroups in computer science, mathematics and finite machines. As a subclass of non-group semigroups, the non-commutative monoids of a given order were identified in 2014 by Ahmadi et al. [2] to show that for every positive integer $n \geq 4$, there exists a non-commutative monoid of order n . In this paper, we intend to identify two types of finite semigroups, commutative non-monoids of order $p^\alpha q^\beta$ and commutative monoids of order of a given positive integer $n \geq 4$. By giving the unique minimal generating set for the commutative non-monoids, we specify the number of all non-isomorphic semigroups of this type. Subsequently, we

present a class of finitely presented non-isomorphic commutative monoids of a given positive integer n .

Our notation is merely standard, and we follow [3–7]. The detailed information on semigroup (or monoid) presentations may be found in [3, 4, 7]. These articles investigate the efficiency of finitely presented semigroups and the related monoids. We prefer to give a brief history on the finitely presented semigroups and monoids. A semigroup (or monoid) S is said to be presented by a semigroup (or monoid) presentation $\langle A \mid R \rangle$ if $S \cong A^+/\rho$ (or $S \cong A^*/\rho$) where A is an alphabet, A^+ is the free semigroup on A , $A^* = A \cup \{1\}$, ρ is a congruence on A (or A^*) generated by R , and $R \subseteq A^+ \times A^+$ (or $R \subseteq A^* \times A^*$). As usual, we will use the notation $Sg(\pi)$ for the semigroup presented by the presentation $\pi = \langle A \mid R \rangle$. Also, the deficiency of a finite semigroup presentation $\pi = \langle A \mid R \rangle$ is defined to be $|R| - |A|$ and is denoted by $def(\pi)$. The semigroup deficiency $def_s(S)$ of a finitely presented semigroup S is given by

$$def_s(S) = \min\{def(\pi) \mid \pi \text{ is a finite presentation for } S\}$$

as well.

For every positive integers m , r and s , we define the presentations:

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$$\pi_{m,r} = \langle a \mid a^{m+r} = a^r \rangle, \quad (m \geq 2, r \geq 1),$$

$$\pi'_{r,s} = \langle a, b \mid a^{r+1} = a^r, b^{s+1} = b^s, ab = ba = a \rangle, \quad (r, s \geq 2),$$

and let $S_{m,r} = Sg(\pi_{m,r})$. Obviously, the semigroup $S_{m,1}$ is the cyclic group of order $m - 1$; however, $S_{m,r}$ is a non-monoid monogenic semigroup of order $m + r - 1$, for every positive integer $r \geq 2$. These last monogenic semigroups will be used in construction of the commutative non-monoid semigroups. By using the well-known definition of direct product of semigroups [8], we may easily check that the direct product of two non-monoid monogenic semigroups is also a non-monoid semigroup. Note that, the direct product of semigroups $S_{m,1}$ and $S_{m',1}$, as studied in [1], is a 2-generated semigroup. However, the situation is completely different in the direct product of non-monoid monogenic semigroups as we will see in the following sections.

Main results

As a preliminary result, we have the following lemma on the monogenic semigroups.

Lemma 2.1 *Let $\alpha \geq 2$ be an integer, and p be a prime. There are exactly $p^\alpha - 2$ number of non-isomorphic non-monoid monogenic semigroups of order p^α .*

Proof For every values of $r \geq 2$, $S_{m,r} = Sg(\pi_{m,r})$ is a non-monoid monogenic semigroup. Then, the equation $m + r - 1 = p^\alpha$ gives us all of the possible values for m and r satisfying this equation. The number of partitions of $p^\alpha + 1$ as a summation of two positive integers m and r such that $m, r \geq 2$, is equal to $t = p^\alpha - 2$. For two different partitions $m + r = m' + r' = p^\alpha + 1$, consider the semigroups $S_{m,r}$ and $S_{m',r'}$. Evidently, $m \neq m'$ and $r \neq r'$. Hence $S_{m,r}$ and $S_{m',r'}$ are non-isomorphic semigroups because of their non-isomorphic maximal subgroups of orders m and m' , respectively. Consequently, t is the number of non-isomorphic non-monoid monogenic semigroup of order p^α . $\square$

Theorem 2.2 *There are exactly $(p^\alpha - 2)(q^\beta - 2)$ number of non-isomorphic semigroups of order $n = p^\alpha q^\beta$ which are commutative and non-monoid. Moreover, each of these semigroups possesses the unique minimal generating set of the cardinality $p^\alpha + q^\beta - 1$.*

Proof Let $T_1 = S_{m,r}$ and $T_2 = S_{m',r'}$ be two non-monoids of orders p^α and q^β , respectively. So, $m + r = p^\alpha + 1$ and $m' + r' = q^\beta + 1$. Let $S = T_1 \times T_2$ be the direct product of T_1 and T_2 . Every element of S is an ordered pair (a^i, b^j) , where $1 \leq i \leq p^\alpha$ and $1 \leq j \leq q^\beta$. Since T_1 and T_2 are commutative non-monoid semigroups, it may be easily checked that S is also a commutative non-monoid semigroup. Then, by the

results of Lemma 2.1 and the definition of S as above, we conclude that there are exactly $(p^\alpha - 2)(q^\beta - 2)$ number of commutative non-monoid semigroups of order $p^\alpha q^\beta$. To complete the proof, we have to give the minimal generating set for S . Without loss of generality, suppose that $p^\alpha < q^\beta$. First of all by considering the elements

$$x = (a, b),$$

$$A_i = (a^i, b), \quad (i = 2, 3, \dots, p^\alpha),$$

$$B_j = (a, b^j), \quad (j = 2, 3, \dots, q^\beta),$$

of S , we may easily check that every element $c_{ij} = (a^i, b^j)$ of S may be presented as:

$$c_{ij} = \begin{cases} x^i, & \text{if } i = j, \\ x^{i-1} B_{j-i+1}, & \text{if } i < j, \\ x^{j-1} A_{i-j+1}, & \text{if } i > j, \end{cases}$$

where $2 \leq i \leq p^\alpha$ and $2 \leq j \leq q^\beta$. Consequently, the subset

$$X = \{x\} \cup \{A_i \mid i = 2, \dots, p^\alpha\} \cup \{B_j \mid j = 2, \dots, q^\beta\}$$

of cardinality $p^\alpha + q^\beta - 1$ is a generating set for S . To prove that X is the unique minimal generating set for S , it is sufficient to show that $S \setminus S^2 = X$, where $S^2 = \{yz \mid y, z \in S\}$. Remind the relators $a^{p^\alpha+1} = a^r$ and $b^{q^\beta+1} = b^s$ where $r, s \geq 2$, then, any product yz of the elements of X belongs to $S \setminus X$. Also, any product yz of the elements of $S \setminus X$ belongs to $S \setminus X$. Consequently, $S \setminus S^2 = X$. $\square$

To identify the commutative monoids of a given integer $n \geq 4$, we show that the disjoint union of two finite cyclic groups $H = \langle a \rangle$ and $K = \langle b \rangle$ becomes a commutative non-group monoid if the multiplication is defined as $ab = a = ba$.

Theorem 2.3 *For a given integer $n \geq 4$, the semigroup $Sg(\pi'_{r,s})$ is a commutative monoid of order n , for all values of the integers $r, s \geq 2$, where $n = r + s$. Moreover, there are exactly $n - 3$ non-isomorphic commutative monoids of order n .*

Proof Let $n = r + s$. Then, the set of elements of the semigroup $Sg(\pi'_{r,s})$ may be decomposed into the union of two groups as

$$\{a^i \mid i = 1, 2, \dots, r\} \cup \{b^j \mid j = 1, 2, \dots, s\},$$

because of the relators $ab = ba = a$. So, $Sg(\pi'_{r,s})$ is of order n . The element $e = b^s$ is the identity element of $Sg(\pi'_{r,s})$ because of the relators

$$e.a^i = b^{s-1}.(ba).a^{i-1} = b^{s-1}.$$

$$a.a^{i-1} = b^{s-1}.a^i = \dots = ba^i = (ba).a^{i-1} = a^i,$$

for every $i = 1, 2, \dots, r$. So, $Sg(\pi'_{r,s})$ is a commutative monoid. Although $Sg(\pi'_{r,s})$ is the disjoint union of two cyclic groups generated by a and b , it is not a group, because the element a has not the group inverse in $Sg(\pi'_{r,s})$.

The number of different pairs (r, s) of the integers $r, s \geq 2$ satisfying the equation $n = r + s$, is equal to $n - 3$. Also, it is not very hard to see that the semigroups $Sg(\pi'_{r,s})$ and $Sg(\pi'_{r',s'})$ are non-isomorphic for any two different pairs (r, s) and (r', s') . Hence there are exactly $n - 3$ non-isomorphic commutative monoids of order n . $\square$

Non-isomorphic semigroups

Two special cases of non-group commutative semigroups of orders $2p^\alpha$ and $p^{2\alpha}$ are of interest to consider, for every odd prime p and every integer $\alpha \geq 2$. Our results on the identification of semigroups are collected in the following theorems.

Theorem 3.1 *There are exactly $2p^\alpha$ non-isomorphic commutative non-monoid monogenic semigroups of order $2p^\alpha$, for every odd prime p and every integer $\alpha \geq 2$.*

Proof A similar method as in Lemma 2.1 may be used here to get the number of non-isomorphic semigroups. $\square$

Theorem 3.2 *There are exactly $p^\alpha - 2$ non-isomorphic commutative non-monoid semigroups of order $p^{2\alpha}$, for every integer $\alpha \geq 2$ and every odd prime p . Moreover, each semigroup possesses the unique minimal generating set of cardinality $2p^\alpha - 1$.*

Proof A commutative non-monoid semigroup of order $p^{2\alpha}$ is indeed the direct product $S_{m,r} \times S_{m,r}$, where $m + r = p^\alpha + 1$ and $r \geq 2$. The first part may be verified in a similar method as in Theorem 2.3. Also, the subset

$$X = \{x, A_2, \dots, A_{p^\alpha}, B_2, \dots, B_{p^\alpha}\}$$

is the unique minimal generating set for each semigroup $S_{m,r} \times S_{m,r}$ if $m + r = p^\alpha + 1$ and $r = 2, 3, \dots, p^\alpha - 1$. Obviously, $|X| = 2p^\alpha - 1$. $\square$

Remark 3.3 The proof of Lemma 2.1 may be used to identify the commutative non-monoid semigroups of order p^2 . Indeed, we deduce that there exists exactly $p - 2$

commutative non-monoid semigroups of order p^2 , for every odd prime p . This is a substantial difference between the groups and semigroups of order p^2 .

Conclusion

As an example we will give an efficient presentation for the commutative non-monoid semigroup of order 3^2 . Before presenting this example, we have a look at the automorphism groups of the commutative non-monoid semigroups in the following theorem.

Theorem 4.1 *For every integer $\alpha \geq 2$ and every odd prime p , the following statements hold:*

- (i) *The automorphism group of every non-monoid monogenic semigroup is the trivial group.*
- (ii) *The automorphism group of every commutative non-monoid semigroup of order $p^{2\alpha}$ is a non-trivial group.*
- (iii) *For a given integer $n \geq 4$ and the positive integers r and s as in Theorem 2.3, the automorphism group of the semigroup $Sg(\pi'_{r,s})$ is a non-trivial group of order at least $\phi(r)\phi(s)$, where ϕ is the well-known Eulerian function.*

Proof To prove (i), observe that any automorphism $\theta \in \text{Aut}(S_{m,r})$, where $r \geq 2$, has to map a generator to a generator. Since $S_{m,r}$ possesses a unique generating element, θ is the identity map.

For (ii), let $S = S_{m,r} \times S_{m,r}$ where $m + r - 1 = p^\alpha$. We may define the involution automorphism $\theta \in \text{Aut}(S)$ by $\theta(a^i, b^j) = (a^j, b^i)$, where $1 \leq i, j \leq p^\alpha + 1$. By considering the generating set

$$X = \{x\} \cup \{A_2, A_3, \dots, A_{p^\alpha}\} \cup \{B_2, B_3, \dots, B_{p^\alpha}\},$$

we get $\theta(x) = x$, $\theta(A_i) = B_i$ and $\theta(B_i) = A_i$, for every $i = 2, 3, \dots, p^\alpha$. Hence, we have to verify the relators:

$$\theta(A_i A_j) = \theta(A_i) \theta(A_j),$$

$$\theta(B_i B_j) = \theta(B_i) \theta(B_j),$$

$$\theta(A_i B_j) = \theta(A_i) \theta(B_j),$$

$$\theta(x A_i) = x \theta(A_i),$$

$$\theta(x B_i) = x \theta(B_i).$$

The proofs are straightforward because of the commutativity of S and the key relators $A_i A_j = x A_{i+j-1}$, $B_i B_j = x B_{i+j-1}$ and

$$A_i B_j = \begin{cases} x^j A_{i-j+1}, & \text{if } i > j, \\ x^i B_{j-i+1}, & \text{if } i < j, \\ x^{i+1}, & \text{if } i = j. \end{cases}$$

Finally, consider the subgroups $H = \{a, a^2, \dots, a^r\}$ and $K = \{b, b^2, \dots, b^s\}$ of the semigroup $Sg(\pi'_{r,s})$. For any $\theta \in \text{Aut}(Sg(\pi'_{r,s}))$, the restrictions $\theta|_H$ and $\theta|_K$ are automorphisms of H and K , respectively. As a well-known result of the cyclic groups, there are exactly $\phi(r)$ number of generating elements for H , and then, $|\text{Aut}(H)| = \phi(r)$. Similarly, $|\text{Aut}(K)| = \phi(s)$. Since the cross-product of two automorphisms is an automorphism, (iii) follows at once. $\square$

Obtaining a finite presentation for the non-group commutative semigroups of order $n = p^\alpha q^\beta$ requires too long and tedious hand calculation. As an example, we give here a finite presentation for the non-group commutative non-monoid semigroup of order p^2 where $p = 3$. Let $T = Sg(S_{2,2} \times S_{2,2})$. Then, T is of order 9 and possesses an efficient presentation as in the following example.

Example 4.2 The semigroup T may be presented as $\langle X \mid R \rangle$ where $X = \{x, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and R is the set of 29 relators:

$$\begin{aligned} x^4 &= x^2, \quad \alpha_i^4 = \alpha_i^2 = x\alpha_i & (1 \leq i \leq 4), \\ x\alpha_i &= \alpha_i x, \quad \alpha_i \alpha_j = \alpha_j \alpha_i & (1 \leq i < j \leq 4), \\ x\alpha_1 &= \alpha_1 \alpha_2, \quad x\alpha_3 = \alpha_3 \alpha_4, \quad x\alpha_2 = x^2 = x\alpha_4, \\ \alpha_1 \alpha_3 &= x^3, \quad \alpha_2 \alpha_4 = x^2, \\ x\alpha_1 \alpha_4 &= \alpha_2 \alpha_3 = x\alpha_1 \alpha_2, \\ x\alpha_2 \alpha_3 &= \alpha_1 \alpha_4 = x\alpha_3 \alpha_4. \end{aligned}$$

To verify these relators, one may use the original definition of the minimal generating set X . Indeed, by letting $\alpha_1 = A_2 = (a^2, b)$, $\alpha_2 = A_3 = (a^3, b)$, $\alpha_3 = B_2 = (a, b^2)$, $\alpha_4 = B_3 = (a, b^3)$, all of the products of two elements of X could be calculated by using Lemma 2.1. Then, the redundant relators have to be deleted. By using Gap code [9], we now believe that this presentation is an efficient semigroup presentation for T .

This example along with our computations results the following conjecture on the non-group commutative non-monoid semigroups of order p^2 .

Conjecture 4.3 *For every odd prime p , there are $p - 2$ number of non-group commutative non-monoid semigroups of order p^2 . Each semigroup of this type has an efficient presentation of semigroup deficiency $4p(p - 1)$.*

The first part of this conjecture may be verified in a similar method as in Theorem 3.2. For $p = 3$, the above example verifies the conjecture and shows that $\text{def}(T) = 29 - 5 = 24 = 4 \times 3 \times 2$. Also, for $p = 5$, we examined the semigroup $T_1 = Sg(S_{4,2} \times S_{4,2})$ of order p^2 with the generating set $X = \{x, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$. As well as in the above example, we managed to get an efficient presentation with 89 relators. This verifies the conjecture showing that $\text{def}(T_1) = 89 - 9 = 80 = 4 \times 5 \times 4$.

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