Common set of weights and efficiency improvement on the basis of separation vector in two-stage network data envelopment analysis

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Abstract

Common set of weights (CSWs) method is one of the popular ranking methods in DEA which can rank efficient and inefficient units. Based on an identical criterion, the method selects the most favorable weight set for all units. An important issue is that in most common DEA models, the internal structure of the production units is ignored and the units are often considered as black boxes. In this paper, in order to evaluate the units and subunits in the two-stage NDEA based on an identical criterion, it is suggested to use CSWs method on the basis of separation vector. Our research contribution in this paper includes: (1) CSWs method is formulated in two-stage NDEA as a multiple objective fractional programming (MOFP) problem. (2) A method is suggested based on separation vector to change MOFP problem into single objective linear programming (SOLP) problem in two-stage NDEA. In the theorem, it is shown that the obtained solutions from MOFP and SOLP in two-stage NDEA are identical. (3) In the framework of the new models of two-stage NDEA, a process is introduced to improve efficiency evaluation by CSWs on the basis of separation vector which is based on the radial improvement of inputs and final outputs. Finally, an enlightening application is presented.

Keywords Network DEA · Two stage · Common set weights (CSWs) · Separation vector · Ranking

Introduction

Data envelopment analysis (DEA) is a linear programming method that was proposed by Charnes et al. [3] to assess the performance of decision making units that use multiple inputs and produce multiple outputs. To evaluate the performance of the units, DEA calculates the proportion of the weighted outputs to the weighted inputs for each unit as a measure of relative efficiency. To this end, DEA is comparably flexible in selecting the appropriate weights to calculate the highest efficiency score. This flexibility in the selection of weights often causes more than one DMU being evaluated as efficient; therefore, it is not able to distinguish efficient DMUs from one another. To

resolve this problem, different ranking methods have been proposed. Evaluation of DMUs by common set of weights (CSWs) is one of the ranking methods that has gained popularity among the researchers. The use of CSWs makes it possible to compare and rank the performances of the DMUs on the same basis. A number of methods have been suggested in the DEA literature for obtaining CSWs for DMUs. For example, Ganley and Cubbin [12] obtained CSWs by maximizing the total efficiencies of all units. Roll and Golany [37] and Roll et al. [36] proposed some methods such as averaging of different weights obtained from infinite DEA model, maximizing the average efficiency of all units and maximizing some efficient units to determine CSWs. Sinuany-Stern et al. [38] expanded the two-stage linear discriminant analysis to produce common weights. Liu and Peng [28] introduced common weights analysis (CWA) method to obtain the common weights for the units. Hashimoto and Wu [14] proposed compromise programming (CP) model in DEA which aimed at allocating common weights based on the combination of DEA and CP. Kao and Hung [22] also presented the estimation of similar compromise solution for determining the common weights. Zohrehbandian et al. [45] improved Kao and

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Hung's [22] estimation by introducing a MCDM model being resulted from DEA linear model. Wang et al. [41], through imposing the least weight restrictions, obtained CSWs for units ranking. They proposed MCDM model which is resulted from nonlinear DEA model. Wang et al. [40] introduced a method on the basis of regression analysis which is looking for CSWs based on the proximity to the ideal performance of each unit. Jahanshahloo et al. [18] suggested two methods to determine CSWs based on comparing ideal line and special line. Jahanshahloo et al. [21] implemented Zionts–Wallenius technique to find CSWs based on the DM's preference information. Sun et al. [39] introduced two models from MADA viewpoint for searching CSWs which is obtained based on applying ideal and anti-ideal units. Ramezani-Tarkhorani et al. [32] by expanding Liu and Peng's [28] estimation proposed a method through which unique CSWs and as a result unique ranking is represented for the units. Chiang et al. [6] by introducing separation vector changed MOFP problem into SOLP separation vector problem and used it for extracting CSWs. Hosseinzadeh Lotfi et al. [17] by applying goal programming in MOFP problem obtained CSWs and considered it for setting targets and allocating resources. Ramon et al. [33] represented a method to determine CSWs which is obtained from DEA weights profiles without being zero of efficient units based on CSWs minimum deviations. Khalili-Damghani and Fadaei [26] proposed a model which is resulted from the combination of Sun et al. [39] models and used it to extract CSWs in the units with positive, zero or negative data for unit ranking.

Wu et al. [43] by introducing the concept of satisfaction degree for the units represented a model which selects CSWs based on the least satisfaction degree. Pourhabib Yekta et al. [29] suggested weight restrictions method which looks for selecting CSWs to evaluate and rank the units. Jahanshahloo et al. [20] proposed a new effective method for fixed cost allocation based on the efficiency invariance and common set of weights. Puri et al. [31] proposed a new multi-component DEA (MC-DEA) approach using CSWs methodology based on interval arithmetic and unified production frontier. Razavi Hajiagha et al. [34] proposed a method for determining the CSWs in a multi-period DEA.

In most conventional DEA models, the internal structures of the production units are disregarded and the units are often considered as black boxes. Therefore, DEA models are likely to treat a unit as efficient while it includes a number of inefficient sub-processes. In recent DEA literature, there are remarkable studies regarding the modeling of production systems with network structures. For the purpose of simplicity, most such researchers assume the production systems to consist of two stages. The network data envelopment analysis (NDEA) method results in a better knowledge

of the system as it takes account of the internal processes of the decision making unit.

Over the past two decades, there has been a surge of papers associated with the theory, method and the application of NDEA and two-stage NDEA. The first paper to capture this idea was published by Charnes et al. [2]. Färe and Whittaker [10] used a two-stage DEA model to investigate the relative efficiency of dairy products. In a different study, Färe and Grosskopf [9] proposed a NDEA model for the economic health organization in Sweden. Chen and Zhu [5] developed a model that identifies the efficiency frontiers of the two-stage production process related to the intermediate measures. Kao and Hwang [23] suggested to consider a serial relation between the stages of the network systems in traditional DEA and employed it to measure the efficiency of non-life insurance companies in Taiwan. Chen et al. [4] proposed a method that specifies the frontier points for inefficient units within two-stage DEA structure. Kazemi Matin and Azizi [25] proposed a new method of performance evaluation for general network systems regardless of the specific assumptions operating on the internal relations. Wanke et al. [42] introduced the directional distance function for measuring the efficiency in two-stage process. Kao and Liu [24] used the idea of cross-efficiency evaluation for measuring the efficiency of series and parallel basic two-stage network systems. Hassanzadeh and Mostafaei [15] examine the way intermediate products affect overall and stage efficiency scores of units in different scenarios. Feizabadi et al. [11] introduced specifications and weaknesses of Färe & Grasskopf and Kao's network models are investigated from a comparative point of view and a network system is presented for which these two models are equivalent. Recently, using CSWs idea in NDEA is considered by some authors under special conditions, for example, Yang and Liu [44] by applying the idea of combination of CSWs and two-stage NDEA model and fuzzy multiple objective model tried to evaluate the performance of banks' branches of Taiwan. Hatami-Marbini and Saati [16] introduced CSWs method for two-stage structures in fuzzy space to evaluate the performance of units and subunits. Pourmahmoud and Zeynali [30] represented nonlinear model to extract CSWs in NDEA, and avoiding the selection of zero weights evaluated the performance of the units. Gharakhani et al. [13] proposed a new approach for seeking a common set of weights in dynamic network DEA models based on the goal programming (GP) technique. The proposed approach makes it possible to monitor dynamic change of the period efficiency. Kiani Mavi et al. [27] proposed a approach to find the common set of weights in a two-stage network data envelopment analysis based on goal programming to analyze the joint effects of eco-efficiency and eco-innovation, considering the undesirable inputs, intermediate products and the outputs in the context of big data.

No doubt, considering the internal structure of data envelopment analysis helps to achieve a higher level of accuracy in efficiency evaluation of the units. Also, evaluated efficiency by CSWs results is a better discrimination of the units and it disregards the impact of unreal weights in efficiency evaluation in DEA. The advantage of developing common set of weights model is that it removes biases from measuring efficiency by different favorite set of weights for each DMU. Rödder and Reucher [35] made an attempt to improve cross-efficiency in DEA framework. They did this using two models based on radial input reduction and changing the inputs in input-oriented models. On the basis of the performed studies, no direct efficiency evaluation by CSWs techniques has been developed in NDEA models to investigate the improvement scenarios of efficiency scores. Therefore, the main purpose of the present paper is to develop CSWs model with separation vector and its improvement scenarios in two-stage NDEA.

The rest of the paper is organized as follows: in “[Common set weights in black-box case](#)” section, CSWs evaluation is introduced in black box. A new approach in CSWs on the basis of separation vector in basic two-stage NDEA is explained in “[A new two-stage CSWs approach on the basis of separation vector](#)” section. In “[Improving the overall and divisional efficiency scores obtained by CSWs on the basis of separation vector in basic two-stage NDEA: a novel approach](#)” section, a novel method is presented to improve the overall and divisional efficiency scores obtained by CSW on the basis of separation vector in basic two-stage NDEA. An illustrative application is presented in “[Illustrative application](#)” section. The final section discusses the conclusions.

Common set weights in black-box case

Suppose that there are n DMUs with m inputs and s outputs. Further, suppose that $\mathbf{x}_j$ and $\mathbf{y}_j$ are input and output vectors, respectively, where $\mathbf{x}_j \geq 0$, $\mathbf{y}_j \geq 0$ and at least one of their components is positive. To evaluate each $\text{DMU}_k (k = 1, \dots, n)$, the efficiency score θ_k can be calculated by the input-oriented CCR multiplier model with constant returns to scale as follows:

$$\begin{aligned} \theta_k^* = \max & \frac{\mathbf{u}^T \mathbf{y}_k}{\mathbf{v}^T \mathbf{x}_k} \\ \text{s.t.} & \frac{\mathbf{u}^T \mathbf{y}_j}{\mathbf{v}^T \mathbf{x}_j} \leq 1, \quad j = 1, \dots, n, \\ & \mathbf{u}, \mathbf{v} \geq 0. \end{aligned} \quad (1)$$

Using the Charnes and Cooper [1] conversion, we changed it to a linear model as follows:

$$\begin{aligned} \theta_k^* = \max & \mathbf{u}^T \mathbf{y}_k \\ \text{s.t.} & \mathbf{v}^T \mathbf{x}_k = 1, \\ & \mathbf{u}^T \mathbf{y}_j - \mathbf{v}^T \mathbf{x}_j \leq 0, \quad j = 1, \dots, n, \\ & \mathbf{u}, \mathbf{v} \geq 0. \end{aligned} \quad (2)$$

where $\mathbf{v} \in \mathbb{R}_+^m$, $\mathbf{u} \in \mathbb{R}_+^s$ are inputs' and outputs' weight vectors for DMU_k , respectively. If $\theta_k^* = 1$, then DMU_k is efficient, otherwise DMU_k is considered as inefficient. Model (2) is solved n -times, and there will be n different sets of the weights of $\mathbf{v}$ and $\mathbf{u}$. On one hand, some units may be efficient and it may not be considered an appropriate discrimination between the units; and on the other hand, units ranking would not be taken place based on an identical criterion. To solve the problem, CSWs method in units ranking was suggested. CSWs method looks for selecting a weight set for units' evaluation, so that the weight set maximizes the efficiency of all units. As a result, the method evaluates and ranks all units based on an identical criterion. The method leads to discriminate appropriately the units; so, in order to determine CSWs in black box, consider the following MOFP model:

$$\begin{aligned} \text{Max}_{\mathbf{v}^T, \mathbf{u}^T} & \left\{ \frac{\mathbf{u}^T \mathbf{y}_1}{\mathbf{v}^T \mathbf{x}_1}, \frac{\mathbf{u}^T \mathbf{y}_2}{\mathbf{v}^T \mathbf{x}_2}, \dots, \frac{\mathbf{u}^T \mathbf{y}_n}{\mathbf{v}^T \mathbf{x}_n} \right\} \\ \text{s.t.} & \frac{\mathbf{u}^T \mathbf{y}_j}{\mathbf{v}^T \mathbf{x}_j} \leq 1, \quad j = 1, \dots, n, \\ & \mathbf{u}, \mathbf{v} \geq \varepsilon. \end{aligned} \quad (3)$$

The obtained solution from model (3) is called CSWs in DEA and is determined by $\mathbf{v}_{\text{CSW}}^*$ and $\mathbf{u}_{\text{CSW}}^*$. In this way, the efficiency of the resulted units from CSWs method is obtained as $\theta_{\text{CSW}-j}^* = \mathbf{u}_{\text{CSW}}^{*T} \mathbf{y}_j / \mathbf{v}_{\text{CSW}}^{*T} \mathbf{x}_j$, ($j = 1, \dots, n$).

To solve the model (3), different methods were proposed by some authors. For instance, Chiang and Tzeng [7] represented a max–min multiple objective model, and then introducing nonnegative variable—which represents the level of achievement—changed it into a nonlinear problem. In fact, they solved a nonlinear programming problem instead of solving n -times the linear programming model (2) and obtained the efficiency of all units. Since that the efficiency score of all units is evaluated only through one weight set, this efficiency index for the units can be accepted in ranking. Jahanshahloo et al. [19] also introduced a similar model to the model of Chiang and Tzeng [7] for determining CSWs. Kao and Hung [22] suggested some models based on the concept of compromise solution. They first calculated the efficiency score for each unit through the model (2) and considered it as the ideal solution for each unit, then obtained CSWs by approximating relative efficiency of each unit to its ideal efficiency through nonlinear model. Cook and Zhu [8] described goal programming model to extract CSWs. Similarly, Liu and Peng [28] for ranking the units on the efficient line through CSWs method proposed a goal programming model. The proposed methods by the above-mentioned

authors are to solve MOFP problem (3) as being on non-linear formula or are viewed in the form of various stages in the process of solving the solution. In order to solve the problem, Chiang et al. [6] represented a nonlinear model which solves MOFP problem (3) through a much simpler process. They changed MOFP problem as two simpler forms by defining the subsidiary variable which appeared as separation vector and showed that the obtained solutions from that model are the same as the solutions of the model (3).

The proposed method by Chiang et al. [6] suggests a simpler process to solve MOFP model (3), but the method cannot discriminate strongly the units, because some units may get the efficiency score of one. It is obvious that by considering the units with network structure instead of black box, more resources of inefficiency may be identified. Now, if the inner structure of the units is also studied instead of considering the units as black box, how would be the efficiency of the units by applying the obtained common weights from the separation vector? In other words, by considering the basic two-stage network structure instead of black box, how would the evaluation of the units' performance through the obtained CSWs from the separation vector affect the better discrimination of the units? The question will be answered in next sections.

A new two-stage CSWs approach on the basis of separation vector

In this section, the focus is on the basic two-stage system. All the inputs, that are supplied externally, are used by the first stage to yield the intermediate products which are consumed in the second stage to obtain the final products. It is assumed that the first stage does not provide any final products. Figure 1 represents the structure of basic two-stage model in which the first stage consumes all the homogeneous inputs $\mathbf{x}_i$, $i = 1, \dots, m$, to derive the intermediate products $\mathbf{z}_g$, $g = 1, \dots, h$, and all of these products are used by the second stage to produce the final products $\mathbf{y}_r$, $r = 1, \dots, s$.

To reflect the appropriate performance of the basic two-stage system, the operations of the two stages should be included in measuring the efficiency of a DMU. The operations of the two stages require the aggregated output to be smaller than or equal to the aggregated input and the same factor to

be used. Put differently, system efficiency can be measured in terms of system technologies in two stages. Let's assume that there are n observed units, in which DMU_j ($j = 1, \dots, n$) is represented as $(\mathbf{x}_j, \mathbf{z}_j, \mathbf{y}_j)$. We assume the unit under evaluation to be $(\mathbf{x}_k, \mathbf{z}_k, \mathbf{y}_k)$.

We also assume that at least one input, one intermediate input and output and one output observed unit are positive. Kao and Hwang [23] suggested the following model under constant returns to scale in the input-oriented multiplier form as follows:

$$\begin{aligned} \theta_k^* = \max \quad & \mathbf{u}^T \mathbf{y}_k \\ \text{s.t.} \quad & \mathbf{v}^T \mathbf{x}_k = 1, \\ & \mathbf{u}^T \mathbf{y}_j - \mathbf{v}^T \mathbf{x}_j \leq 0, \quad j = 1, \dots, n, \\ & \mathbf{w}^T \mathbf{z}_j - \mathbf{v}^T \mathbf{x}_j \leq 0, \quad j = 1, \dots, n, \\ & \mathbf{u}^T \mathbf{y}_j - \mathbf{w}^T \mathbf{z}_j \leq 0, \quad j = 1, \dots, n, \\ & \mathbf{u}, \mathbf{v}, \mathbf{w} \geq \mathbf{0}. \end{aligned} \quad (4)$$

The striking feature of this model is that the coefficient $\mathbf{w} \in \mathbb{R}_+^h$ corresponding to the intermediate production $\mathbf{z}_g$ as the output of the first stage requires the same coefficient in the input of the second stage. Kao and Hwang [23] called this model a relational model.

In optimality, system efficiency and divisional efficiencies are expressed in terms of the restrictions of the model above as follows:

$$\begin{aligned} \theta_k^* &= \frac{\mathbf{u}_k^{*T} \mathbf{y}_k}{\mathbf{v}_k^{*T} \mathbf{x}_k}, \quad k = 1, \dots, n, \\ \theta_k^{*(1)} &= \frac{\mathbf{w}_k^{*T} \mathbf{z}_k}{\mathbf{v}_k^{*T} \mathbf{x}_k}, \quad k = 1, \dots, n, \\ \theta_k^{*(2)} &= \frac{\mathbf{u}_k^{*T} \mathbf{y}_k}{\mathbf{w}_k^{*T} \mathbf{z}_k}, \quad k = 1, \dots, n. \end{aligned} \quad (5)$$

Note that system efficiency is expressed as the product of the two stages:

$$\theta_k^{*(1)} \times \theta_k^{*(2)} = \frac{\mathbf{w}_k^{*T} \mathbf{z}_k}{\mathbf{v}_k^{*T} \mathbf{x}_k} \times \frac{\mathbf{u}_k^{*T} \mathbf{y}_k}{\mathbf{w}_k^{*T} \mathbf{z}_k} = \frac{\mathbf{u}_k^{*T} \mathbf{y}_k}{\mathbf{v}_k^{*T} \mathbf{x}_k} = \theta_k^* \quad (6)$$

where $\mathbf{w}_k^{*T}$, $\mathbf{u}_k^{*T}$, $\mathbf{v}_k^{*T}$ is the optimal solution derived from model (4) in the evaluation of k th unit. Since some units and subunits may get the same efficiency score from the relation (5), and the units and subunits are not evaluated on the basis of an identical index, CSWs method is represented in multiplier model of Kao and Hwang [23] as MOFP problem in two-stage NDEA.

Fig. 1 Structure of basic two-stage model

$$\begin{aligned}
& \text{Max}_{\nu^T, w^T} \left\{ \frac{w^T z_1}{\nu^T x_1}, \frac{w^T z_2}{\nu^T x_2}, \dots, \frac{w^T z_n}{\nu^T x_n} \right\} \\
& \text{Max}_{w^T, u^T} \left\{ \frac{u^T y_1}{w^T z_1}, \frac{u^T y_2}{w^T z_2}, \dots, \frac{u^T y_n}{w^T z_n} \right\} \\
& \text{s.t.} \quad \frac{w^T z_j}{\nu^T x_j} \leq 1, \quad j = 1, \dots, n, \\
& \quad \frac{u^T y_j}{w^T z_j} \leq 1, \quad j = 1, \dots, n, \\
& \quad u, w, \nu \geq \varepsilon.
\end{aligned} \quad (7)$$

Suppose ν^* , w^* and u^* are the obtained solutions from the model (7), so it is called common set of weights in two-stage basic NDEA and is showed by ν_{CSWN}^* , w_{CSWN}^* and u_{CSWN}^* . Therefore, overall efficiency and the resulted stages from basic two-stage network CSWs are obtained as follows:

$$\begin{aligned}
\theta_{\text{CSWN}-j}^* &= \frac{u_{\text{CSWN}}^{*T} y_j}{\nu_{\text{CSWN}}^{*T} x_j}, \quad j = 1, \dots, n, \\
\theta_{\text{CSWN}-j}^{(1)} &= \frac{w_{\text{CSWN}}^{*T} z_j}{\nu_{\text{CSWN}}^{*T} x_j}, \quad j = 1, \dots, n, \\
\theta_{\text{CSWN}-j}^{(2)} &= \frac{u_{\text{CSWN}}^{*T} y_j}{w_{\text{CSWN}}^{*T} z_j}, \quad j = 1, \dots, n.
\end{aligned} \quad (8)$$

In model (7), there are two stages of variables in each stage. In the first stage, the first stage of the variables, w , is just appeared in the numerator, while in the other stage, ν , is represented in the denominator. Similarly, in the second stage, the first stage of the variable u is appeared in the numerator, but in the second stage, the variable of w is in the denominator. It should be considered that the variable of w is the same in both stages.

By generalizing the method of Chiang et al. [6], a simpler process to solve the model (7) using the separation vector is represented. To do this, the following model is considered.

$$\begin{aligned}
& \text{Min}_{\nu^T, w^T, \lambda} \left\{ \nu^T x_1 - w^T z_1, \nu^T x_2 - w^T z_2, \dots, \nu^T x_n - w^T z_n \right\} \\
& \text{Min}_{w^T, u^T, \lambda'} \left\{ w^T z_1 - u^T y_1, w^T z_2 - u^T y_2, \dots, w^T z_n - u^T y_n \right\} \\
& \text{s.t.} \quad \nu^T x_j \geq \lambda_j, \quad j = 1, \dots, n, \\
& \quad w^T z_j \leq \lambda_j, \quad j = 1, \dots, n, \\
& \quad w^T z_j \geq \lambda'_j, \quad j = 1, \dots, n, \\
& \quad u^T y_j \leq \lambda'_j, \quad j = 1, \dots, n, \\
& \quad u, w, \nu \geq \varepsilon.
\end{aligned} \quad (9)$$

λ_j and λ'_j are, respectively, the separation variables for the first and second stages in basic two-stage NDEA.

Theorem 1 The model (7) can change into multiple objective linear programming problem by the separation variables of λ_j and λ'_j (9).

Proof The restrictions of the model (7), the restrictions of $\frac{w^T z_j}{\nu^T x_j} \leq 1$, $\frac{u^T y_j}{w^T z_j} \leq 1$, $j = 1, \dots, n$, can be, respectively, ordered as $w^T z_j \leq \nu^T x_j$, $u^T y_j \leq w^T z_j$, $j = 1, \dots, n$. There are the positive separation vectors of $\Lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]$ and $\Lambda' = [\lambda'_1, \lambda'_2, \dots, \lambda'_n]$, respectively, between the points of $[w^T z_1, w^T z_2, \dots, w^T z_n]$ and the points of $[\nu^T x_1, \nu^T x_2, \dots, \nu^T x_n]$, also the points of $[u^T y_1, u^T y_2, \dots, u^T y_n]$ and the points of $[w^T z_1, w^T z_2, \dots, w^T z_n]$, so that $w^T z_j \leq \lambda_j \leq \nu^T x_j$, $u^T y_j \leq \lambda'_j \leq w^T z_j$, $j = 1, \dots, n$. In addition, the value of each target function in the model (7) is between 0 and 1, i.e., $0 \leq \frac{w^T z_j}{\nu^T x_j} \leq 1$, $0 \leq \frac{u^T y_j}{w^T z_j} \leq 1$, $j = 1, \dots, n$. Inequality stages are multiplied, respectively, in $-\nu^T x_j$ and $-w^T z_j$, then the expressions of $\nu^T x_j$ and $w^T z_j$ are, respectively, added to them. So, the restrictions of $\nu^T x_j \geq \nu^T x_j - w^T z_j \geq 0$, $w^T z_j \geq w^T z_j - u^T y_j \geq 0$ are obtained. As a result, the target function of the model (7) is changed into the target function of the model (9). In other words, maximizing the ratio of $w^T z_j$ to $\nu^T x_j$ and the ratio of $u^T y_j$ to $w^T z_j$ with the restrictions of $\frac{w^T z_j}{\nu^T x_j} \leq 1$, $\frac{u^T y_j}{w^T z_j} \leq 1$, $j = 1, \dots, n$, in the model (7) equals minimizing the difference between $\nu^T x_j$ and $w^T z_j$, also between $w^T z_j$ and $u^T y_j$ in the model (9). Therefore, using the separation vectors, the model (7) and the model (9) are equivalence.

Consider the following model:

$$\begin{aligned}
& \text{Min}_{\nu^T, w^T, \lambda} \left\{ \nu^T x_1 - w^T z_1, \nu^T x_2 - w^T z_2, \dots, \nu^T x_n - w^T z_n \right\} \\
& \text{Min}_{w^T, u^T, \lambda'} \left\{ w^T z_1 - u^T y_1, w^T z_2 - u^T y_2, \dots, w^T z_n - u^T y_n \right\} \\
& \text{s.t.} \quad \nu^T x_j \geq \lambda_j, \quad j = 1, \dots, n, \\
& \quad w^T z_j \leq \lambda_j, \quad j = 1, \dots, n, \\
& \quad w^T z_j \geq \lambda'_j, \quad j = 1, \dots, n, \\
& \quad u^T y_j \leq \lambda'_j, \quad j = 1, \dots, n, \\
& \quad u, w, \nu \geq \varepsilon.
\end{aligned} \quad (10)$$

In the following theorem, it is shown that the resulted u , w , ν from the model (9) equals the model (10).

Theorem 2 The obtained CSWs from the model (10) are equal to the obtained CSWs from (9).

Proof Suppose $W = (u^*, w^*, \nu^*, \lambda^*)$ that is an optimal solution of the model (10). So there is $u^{*T} y_j \leq \lambda_j^* \leq w^{*T} z_j \leq \lambda_j^* \leq \nu^{*T} x_j$, $j = 1, \dots, n$. Therefore, $u^{*T} y_j \leq w^{*T} z_j$, $w^{*T} z_j \leq \nu^{*T} x_j$, $j = 1, \dots, n$, and as a

result, $W_{CSW} = (\mathbf{u}^*, \mathbf{w}^*, \mathbf{v}^*)$ is a feasible solution for the model (9). Now suppose that $W = (\mathbf{u}^*, \mathbf{w}^*, \mathbf{v}^*, \lambda^*, \lambda'^*)$ is an optimal solution of the model (9). Therefore, $\mathbf{w}^{*T} \mathbf{z}_j \leq \lambda_j^* \leq \mathbf{v}^{*T} \mathbf{x}_j$, $\mathbf{u}^{*T} \mathbf{y}_j \leq \lambda_j'^* \leq \mathbf{w}^{*T} \mathbf{z}_j$, $j = 1, \dots, n$; i.e., $\mathbf{u}^{*T} \mathbf{y}_j \leq \mathbf{w}^{*T} \mathbf{z}_j$, $\mathbf{w}^{*T} \mathbf{z}_j \leq \mathbf{v}^{*T} \mathbf{x}_j$, $j = 1, \dots, n$, so $W_{CSW} = (\mathbf{u}^*, \mathbf{w}^*, \mathbf{v}^*)$ is a feasible solution for the model (10). Consequently, the resulted CSWs from the two models of (9) and (10) are the same.

The multiple objective model of (10) can be changed into the single objective model. In the following theorem, it is shown that the solutions of the model (11) are the same as the solutions of the model (7).

Theorem 3 *If there are the resulted optimal solutions from the following single objective model, then these solutions will be efficient solutions for the model (7).*

$$\begin{aligned} & \text{Min}_{\mathbf{v}^T, \mathbf{w}^T, \mathbf{u}^T, \lambda} \sum_{j=1}^n \mathbf{v}^T \mathbf{x}_j - \mathbf{w}^T \mathbf{z}_j + \sum_{j=1}^n \mathbf{w}^T \mathbf{z}_j - \mathbf{u}^T \mathbf{y}_j \\ \text{s.t.} \quad & \mathbf{v}^T \mathbf{x}_j \geq \lambda_j, \quad j = 1, \dots, n, \\ & \mathbf{w}^T \mathbf{z}_j \leq \lambda_j, \quad j = 1, \dots, n, \\ & \mathbf{w}^T \mathbf{z}_j \geq \lambda_j, \quad j = 1, \dots, n, \\ & \mathbf{u}^T \mathbf{y}_j \leq \lambda_j, \quad j = 1, \dots, n, \\ & \mathbf{u}, \mathbf{w}, \mathbf{v} \geq \varepsilon. \end{aligned} \quad (11)$$

Proof Suppose $W = (\mathbf{u}^*, \mathbf{w}^*, \mathbf{v}^*)$ is the optimal solution resulted from the model (11). Suppose that this solution is not an efficient solution in the model (10). Therefore, there is the vector $W' = (\mathbf{u}', \mathbf{w}', \mathbf{v}')$, so that $\mathbf{v}'^T \mathbf{x}_j - \mathbf{w}'^T \mathbf{z}_j > \mathbf{v}^{*T} \mathbf{x}_j - \mathbf{w}^{*T} \mathbf{z}_j$, $\mathbf{w}'^T \mathbf{z}_j - \mathbf{u}'^T \mathbf{y}_j > \mathbf{w}^{*T} \mathbf{z}_j - \mathbf{u}^{*T} \mathbf{y}_j$ for each $j = 1, \dots, n$, $j \neq l$ and $\mathbf{v}'^T \mathbf{x}_l - \mathbf{w}'^T \mathbf{z}_l \geq \mathbf{v}^{*T} \mathbf{x}_l - \mathbf{w}^{*T} \mathbf{z}_l$, $\mathbf{w}'^T \mathbf{z}_l - \mathbf{u}'^T \mathbf{y}_l \geq \mathbf{w}^{*T} \mathbf{z}_l - \mathbf{u}^{*T} \mathbf{y}_l$. So, $\sum_{j=1}^n \mathbf{v}'^T \mathbf{x}_j - \mathbf{w}'^T \mathbf{z}_j + \sum_{j=1}^n \mathbf{w}'^T \mathbf{z}_j - \mathbf{u}'^T \mathbf{y}_j > \sum_{j=1}^n \mathbf{v}^{*T} \mathbf{x}_j - \mathbf{w}^{*T} \mathbf{z}_j + \sum_{j=1}^n \mathbf{w}^{*T} \mathbf{z}_j - \mathbf{u}^{*T} \mathbf{y}_j$. It is contradiction to the optimality solution of $W = (\mathbf{u}^*, \mathbf{w}^*, \mathbf{v}^*)$ in the model (11).

According to the Theorem 3, the resulted solutions from the model (11) are the same as CSWs in two-stage basic NDEA. Therefore, overall and divisional efficiency scores resulted from CSWs method are obtained based on the separation vector by replacing the obtained solutions from the model (11) in the relation (8).

Improving the overall and divisional efficiency scores obtained by CSWs on the basis of separation vector in basic two-stage NDEA: a novel approach

Rödter and Reucher [35] proposed two strategies to improve cross-efficiency in DEA. One is radial input reduction and the other changes the inputs (not necessarily reduce them in input-oriented CCR model). The first strategy is to decrease the inputs in production possibility set (PPS) to look for the cross-efficiency of i th unit from the perspective of k th unit. In the second strategy, they proposed a different method to improve cross-efficiency that instead of radial input reduction it sought to change the inputs.

To improve the overall and divisional efficiency scores obtained by CSWs on the basis of separation vector in basic two-stage NDEA, we suggest increasing the outputs of the second as well as reducing the inputs of the first stage (the intermediate measures stay unchanged).

Suppose $\mathbf{v}_{CSWn}^{*T}, \mathbf{w}_{CSWn}^{*T}$ and $\mathbf{u}_{CSWn}^{*T}$ are the resulted solutions from the model (11).

Assume the divisional efficiency scores obtained by CSWs on the basis of separation vector of the first and the second subunits of the l th unit are less than 1. Then, we have $\mathbf{w}_{CSWn}^{*T} \mathbf{z}_l - \mathbf{v}_{CSWn}^{*T} \mathbf{x}_l < 0$ and $\mathbf{u}_{CSWn}^{*T} \mathbf{y}_l - \mathbf{w}_{CSWn}^{*T} \mathbf{z}_l < 0$. To improve the divisional efficiency scores obtained by CSWs on the basis of separation vector of the subunits of the l th unit, we reduce the inputs and increase the outputs, respectively, as: $\mathbf{x}'_l = \mathbf{x}_l \cdot \frac{\mathbf{w}_{CSWn}^{*T} \mathbf{z}_l}{\mathbf{v}_{CSWn}^{*T} \mathbf{x}_l}$ and $\mathbf{y}'_l = \mathbf{y}_l \cdot \frac{\mathbf{w}_{CSWn}^{*T} \mathbf{z}_l}{\mathbf{u}_{CSWn}^{*T} \mathbf{y}_l}$. Therefore, the inputs of the first stage are reduced and the outputs of the second stage are increased, respectively. As the result, the new divisional efficiency scores of the inefficient units become: $\frac{\mathbf{w}_{CSWn}^{*T} \mathbf{z}_l}{\mathbf{v}_{CSWn}^{*T} \mathbf{x}'_l} = 1$ and $\frac{\mathbf{u}_{CSWn}^{*T} \mathbf{y}'_l}{\mathbf{w}_{CSWn}^{*T} \mathbf{z}_l} = 1$. Note that these improvements do not necessarily affect all the divisional efficiency scores obtained by CSWs on the basis of separation vector of the subunits.

To improve the divisional efficiency scores obtained by CSWs on the basis of separation vector of the subunits in the context of PPS, we introduce the following model:

$$\begin{aligned}
\min \quad & p_l - q_l \\
\text{s.t.} \quad & \mathbf{w}_{\text{cswn}}^{*T} \mathbf{z}_l - \mathbf{v}_{\text{cswn}}^{*T} p_l \mathbf{x}_l \leq 0, \\
& p_l \mathbf{x}_l - \sum_{j=1}^n \lambda_j \mathbf{x}_j \geq 0, \\
& \mathbf{z}_l - \sum_{j=1}^n \lambda_j \mathbf{z}_j \leq 0, \\
& \mathbf{u}_{\text{cswn}}^{*T} q_l \mathbf{y}_l - \mathbf{w}_{\text{cswn}}^{*T} \mathbf{z}_l \leq 0, \\
& q_l \mathbf{y}_l - \sum_{j=1}^n \mu_j \mathbf{y}_j \leq 0, \\
& \mathbf{z}_l - \sum_{j=1}^n \mu_j \mathbf{z}_j \geq 0, \\
& \lambda_j, \mu_j \geq 0, \quad \forall j.
\end{aligned} \tag{12}$$

Here, p_l is the decreasing factor of the inputs of the first stage of DMU_l and q_l is the increasing factor of the outputs of the second stage of DMU_l . In the model (12), with respect to the objective function, the first restriction by decreasing the inputs of the first stage of DMU_l subunit seeks to increase the divisional efficiency scores obtained by CSWs on the basis of separation vector of the first stage of DMU_l . The second and third restrictions ensure that the inputs of the first stage of subunit should be decreased. The fourth restriction by increasing the outputs of the second stage of DMU_l seeks to increase the divisional efficiency scores obtained by CSWs on the basis of separation vector of the second stage of DMU_l . The fifth and sixth restrictions also seek to increase the outputs of the second stage of DMU_l in the production set.

Let p_j^*, q_j^* be the optimal solutions obtained from the model above; in this case, while outputs are increased and inputs are decreased, the overall and the divisional efficiency scores obtained by CSWs on the basis of separation vector of DMU_j are improved as follows:

$$\begin{aligned}
\theta_{\text{cswn}-j}^{*(1)} &= \frac{\mathbf{w}_{\text{cswn}}^{*T} \mathbf{z}_j}{\mathbf{v}_{\text{cswn}}^{*T} p_j^* \mathbf{x}_j}, \quad j = 1, \dots, n, \\
\theta_{\text{cswn}-j}^{*(2)} &= \frac{\mathbf{u}_{\text{cswn}}^{*T} q_j^* \mathbf{y}_j}{\mathbf{w}_{\text{cswn}}^{*T} \mathbf{z}_j}, \quad j = 1, \dots, n, \\
\theta_{\text{cswn}-j}^* &= \frac{\mathbf{u}_{\text{cswn}}^{*T} q_j^* \mathbf{y}_j}{\mathbf{v}_{\text{cswn}}^{*T} p_j^* \mathbf{x}_j}, \quad j = 1, \dots, n.
\end{aligned} \tag{13}$$

In a similar way, it is also possible to present a different approach by considering input and output vectors as decision variables in the production set, instead of simultaneous decrease and increase in the inputs and outputs under evaluation unit, in such a way that the overall and the divisional efficiency scores obtained by CSWs on the

basis of separation vector of the subunits are improved. To do so, consider the following model:

$$\begin{aligned}
\min \quad & \mathbf{v}_{\text{cswn}}^{*T} \mathbf{x}'_l - \mathbf{u}_{\text{cswn}}^{*T} \mathbf{y}'_l \\
\text{s.t.} \quad & \mathbf{w}_{\text{cswn}}^{*T} \mathbf{z}_l - \mathbf{v}_{\text{cswn}}^{*T} \mathbf{x}'_l \leq 0, \\
& \mathbf{x}'_l - \sum_{j=1}^n \lambda_j \mathbf{x}_j \geq 0, \\
& \mathbf{z}_l - \sum_{j=1}^n \lambda_j \mathbf{z}_j \leq 0, \\
& \mathbf{u}_{\text{cswn}}^{*T} \mathbf{y}'_l - \mathbf{w}_{\text{cswn}}^{*T} \mathbf{z}_l \leq 0, \\
& \mathbf{y}'_l - \sum_{j=1}^n \mu_j \mathbf{y}_j \leq 0, \\
& \mathbf{z}_l - \sum_{j=1}^n \mu_j \mathbf{z}_j \geq 0, \\
& \lambda_j, \mu_j \geq 0, \quad \forall j.
\end{aligned} \tag{14}$$

Here, $\mathbf{x}'_l, \mathbf{y}'_l$ are decision variables for inputs and outputs of DMU_l , respectively, that show the change in inputs of the first stage and the outputs of the second stage.

Note that the intensity weights λ_j and μ_j in both models (12) and (14) are different for the first and second stages, respectively.

Suppose that $\mathbf{x}_l^*, \mathbf{y}_l^*$ are the optimal solutions computed from the above model. Then, upon the simultaneous change of the inputs and outputs, the overall and the divisional efficiency scores obtained by CSWs on the basis of separation vector of the subunits DMU_l are expressed as follows:

$$\begin{aligned}
\theta_{\text{cswn}-l}^{*(1)} &= \frac{\mathbf{w}_{\text{cswn}}^{*T} \mathbf{z}_l}{\mathbf{v}_{\text{cswn}}^{*T} \mathbf{x}_l^*}, \quad l = 1, \dots, n, \\
\theta_{\text{cswn}-l}^{*(2)} &= \frac{\mathbf{u}_{\text{cswn}}^{*T} \mathbf{y}_l^*}{\mathbf{w}_{\text{cswn}}^{*T} \mathbf{z}_l}, \quad l = 1, \dots, n, \\
\theta_{\text{cswn}-l}^* &= \frac{\mathbf{u}_{\text{cswn}}^{*T} \mathbf{y}_l^*}{\mathbf{v}_{\text{cswn}}^{*T} \mathbf{x}_l^*}, \quad l = 1, \dots, n.
\end{aligned} \tag{15}$$

This model, upon the simultaneous change of the inputs and outputs, produces the overall and the divisional efficiency scores obtained by CSWs on the basis of separation vector of the subunits. Note that the intermediate products do not change during the calculation of the overall and the divisional efficiency scores obtained by CSWs on the basis of separation vector.

In the provided illustrative application, we show that model (14) performs better than model (12) to simultaneously improve the overall and the divisional efficiency scores obtained by CSWs on the basis of separation vector of units.

Illustrative application

Kao and Hwang [23] evaluated the performance of the Taiwanese non-life insurance companies (DMUs) based on a two-stage NDEA approach. In this study, we use the same data set to apply the above-discussed CSWs on the basis of separation vector approach. The following inputs, intermediate and outputs variables are used in this evaluation:

- Inputs: operation expenses (x_1) and insurance expenses (x_2)
- Intermediate products: direct written premiums (z_1) and reinsurance premiums (z_2)
- Outputs: underwriting profit (y_1) and investment profit (y_2)

Table 1 shows the data for 24 Taiwanese non-life insurance companies.

Table 2 shows the overall efficiency scores and the resulted stages from the models of (4) and (11). As it can be observed, by applying CSWs method based on the separation vector, a better discrimination can be made between units and subunits. In this method, units and subunits are ranked based on an identical criterion. For example, the

second, ninth, twelfth, fifteenth and nineteenth subunits in the first stage receive the same rank based on the model (4), while in the model (11), only the fifteenth and nineteenth subunits have received the same rank. Also, the third and twenty third subunits in the second stage have been equally ranked based on the model (4), but the model (11) has given a distinct rank to these subunits. Table 2 shows that the best and the worst units from CSWs viewpoint based on the separation vector are DMU₂₂ and DMU₂₃, respectively, while these units have received the ranks of (10) and (10), respectively, based on the conventional evaluation.

Table 3 represents the comparison of efficiency scores and units' ranking in two-stage NDEA and DEA based on the CSWs separation vector. It can be observed that if the units are evaluated based on the conventional model of DEA, DMU₂, DMU₃, DMU₅, DMU₁₂, DMU₁₅ and DMU₂₂ will get the efficiency score of one. As a result, a complete discrimination cannot be considered between the units. But if these units are evaluated based on the model (4), a better discrimination will be taken place. For example, only DMU₅, DMU₁₂ and DMU₂₂ receive the efficiency score of one. In one word, CSWs in DEA based on the separation vector lead to a better discrimination, but it cannot discriminate completely. If CSWs method is applied in two-stage NDEA

Table 1 Data set for 24 Taiwanese non-life insurance companies

DMU	Bank	x_1	x_2	z_1	z_2	y_1	y_2
1	Taiwan Fire	1,178,744	673,512	7,451,757	856,735	984,143	681,687
2	Chung Kuo	1,381,822	1,352,755	10,020,274	1,812,894	1,228,502	834,754
3	TaiPing	1,177,494	592,790	4,776,548	560,244	293,613	658,428
4	China Mariners	601,320	594,259	3,174,851	371,863	248,709	177,331
5	Fubon	6,699,063	3,531,614	37,392,862	1,753,794	7,851,229	3,925,272
6	Zurich	2,627,707	668,363	9,747,908	952,326	1,713,598	415,058
7	Taian	1,942,833	1,443,100	10,685,457	643,412	2,239,593	439,039
8	Ming Tai	3,789,001	1,873,530	17,267,266	1,134,600	3,899,530	622,868
9	Central	1,567,746	950,432	11,473,162	546,337	1,043,778	264,098
10	The First	1,303,249	1,298,470	8,210,389	504,528	1,697,941	554,806
11	Kuo Hua	1,962,448	672,414	7,222,378	643,178	1,486,014	18,259
12	Union	2,592,790	650,952	9,434,406	1,118,489	1,574,191	909,295
13	Shingkong	2,609,941	1,368,802	13,921,464	811,343	3,609,236	223,047
14	South Ch	1,396,002	988,888	7,396,396	465,509	1,401,200	332,283
15	Cathay Century	2,184,944	651,063	10,422,297	749,893	3,355,197	555,482
16	Allianz President	1,211,716	415,071	5,606,013	402,881	854,054	197,947
17	Newa	1,453,797	1,085,019	7,695,461	342,489	3,144,484	371,984
18	AIU	757,515	547,997	3,631,484	995,620	692,731	163,927
19	North America	159,422	182,338	1,141,951	483,291	519,121	46,857
20	Federal	145,442	53,518	316,829	131,920	355,624	26,537
21	Royal & Sunalliance	84,171	26,224	225,888	40,542	51,950	6491
22	Aisa	15,993	10,502	52,063	14,574	82,141	4181
23	AXA	54,693	28,408	245,910	49,864	0.1	18,980
24	Mitsui Sumitomo	163,297	235,094	476,419	644,816	142,370	16,976

Table 2 The overall and divisional efficiency scores and the ranks obtained from models (4) and (11)

DMU	Efficiency stage 1	Efficiency stage 2	Overall efficiency	Efficiency stage 1 by CSWs	Efficiency stage 2 by CSWs	Overall efficiency by CSWs
1	0.9926 (6)	0.7045 (5)	0.6992 (3)	0.9568 (3)	0.1760 (15)	0.1684 (9)
2	1.0000 (1)	0.6275 (6)	0.6275 (5)	0.8208 (9)	0.1445 (19)	0.1186 (18)
3	0.6900 (16)	1.0000 (1)	0.6900 (4)	0.6727 (17)	0.1562 (18)	0.1051 (20)
4	0.7286 (15)	0.4197 (13)	0.3058 (15)	0.5307 (24)	0.1052 (21)	0.0558 (23)
5	0.8307 (12)	0.9233 (3)	0.7670 (1)	0.7807 (11)	0.2837 (5)	0.2215 (6)
6	0.9606 (7)	0.4057 (17)	0.3897 (12)	0.8896 (7)	0.1776 (14)	0.1580 (10)
7	0.6706 (18)	0.4124 (15)	0.2766 (17)	0.6180 (20)	0.2196 (10)	0.1357 (13)
8	0.6630 (20)	0.4150 (14)	0.2752 (18)	0.6922 (15)	0.2269 (9)	0.1570 (11)
9	1.0000 (1)	0.2233 (24)	0.2233 (20)	0.9297 (4)	0.1025 (22)	0.0953 (21)
10	0.9430 (8)	0.5115 (10)	0.4823 (9)	0.5646 (23)	0.2397 (8)	0.1354 (14)
11	0.5989 (22)	0.2866 (22)	0.1716 (23)	0.7372 (13)	0.1722 (16)	0.1270 (15)
12	1.0000 (1)	0.7596 (4)	0.7596 (2)	0.9135 (5)	0.2059 (11)	0.1881 (7)
13	0.6720 (17)	0.3093 (21)	0.2078 (21)	0.7667 (12)	0.2421 (7)	0.1856 (8)
14	0.6699 (19)	0.4309 (12)	0.2886 (16)	0.6203 (19)	0.2042 (12)	0.1267 (16)
15	1.0000 (1)	0.6138 (7)	0.6138 (6)	1.0000 (1)	0.3209 (4)	0.3209 (3)
16	0.8856 (10)	0.3615 (18)	0.3202 (14)	0.8960 (6)	0.1605 (17)	0.1438 (12)
17	0.6276 (21)	0.5736 (9)	0.3600 (13)	0.5735 (22)	0.4137 (3)	0.2372 (5)
18	0.7935 (13)	0.3262 (19)	0.2588 (19)	0.7916 (10)	0.1434 (20)	0.1136 (19)
19	1.0000 (1)	0.4112 (16)	0.4112 (11)	1.0000 (1)	0.2485 (6)	0.2485 (4)
20	0.9332 (9)	0.5857 (8)	0.5465 (8)	0.6844 (16)	0.6096 (2)	0.4172 (2)
21	0.7321 (14)	0.2743 (23)	0.2008 (22)	0.6675 (18)	0.1820 (13)	0.1215 (17)
22	0.5895 (23)	1.0000 (1)	0.5895 (7)	0.5821 (21)	1.0000 (1)	0.5821 (1)
23	0.8426 (11)	0.4989 (11)	0.4203 (10)	0.8449 (8)	0.0523 (24)	0.0442 (24)
24	0.4287 (24)	0.3145 (20)	0.1348 (24)	0.7037 (14)	0.0800 (23)	0.0563 (22)

based on the separation vector, the units will be discriminated completely. In this method, DMU₂₂ is ranked the best, while it has received the seventh rank based on the model (4). Although the model (4) without CSWs can do ranking completely, its evaluation is based on different and occasionally unreal set of weights and does not benefit from assessment with the same criterion.

Table 4 compares the resulted CSWs from the models of (3) and (11). As it can be seen, these weights are positive and zero weights are avoided in evaluating the units. As well, the shares of the first and second inputs in evaluating the units' performance in two-stage NDEA are more than black box.

The improved overall and divisional efficiency scores by CSWs on the basis of separation vector and the amount of changes in inputs and outputs, derived from the models (12) and (14), are presented in Tables 5 and 6. As the result shows, the overall and divisional efficiency scores of the models (12) and (14) are higher than those of model (11). Also, the model (14), compared to model (12), caused a great increase in the overall and divisional efficiency scores. In other words, the change in inputs and outputs (fixed intermediate measure) in the two-stage NDEA by CSWs results

in a better improvement in the overall and divisional efficiency scores in comparison to reducing inputs and increasing the outputs. Upon the simultaneous decrease in the inputs and increase in the outputs or their changes in basic two-stage NDEA by CSWs, in addition to improvement of the divisional efficiency scores, some ranks of the units and subunits have also changed.

For example, to improve the overall and divisional efficiency scores of DMU₄ based on model (12), it is enough to reduce the first and second inputs from 601,320 and 594,259 to 435,566 and 430,451, respectively, and increase the first and the second outputs from 248,709 and 177,331 to 575,291 and 410,186, respectively. As a result of these changes, the overall and divisional efficiency scores (rank) are increased from 0.0558(23), 0.5307(24) and 0.1052(21) to 0.1783(21), 0.7327(22) and 0.2434(20), respectively. Now, if we want to make the maximum improvement in the overall and divisional efficiency scores of DMU₄, it is enough to change the first and second inputs to 636,982 and 238,018 according to model (14) and convert the first and the second outputs to 2,442,269 and 184,775, respectively. Due to these changes, the overall and divisional efficiency scores (rank)

Table 3 The efficiency scores and efficiency scores by CSWs on the basis of separation vector derived from the black box and the two-stage NDEA

DMU	Efficiency for DEA model (1)	Efficiency for DEA by CSWs model (3)	Overall efficiency for NDEA model (4)	Overall efficiency for NDEA by CSWs model (11)
1	0.9984 (7)	0.8763 (5)	0.6992 (3)	0.1684 (9)
2	1.0000 (1)	0.5821 (8)	0.6275 (5)	0.1186 (18)
3	1.0000 (1)	0.8434 (7)	0.6900 (4)	0.1051 (20)
4	0.4882 (14)	0.2799 (22)	0.3058 (15)	0.0558 (23)
5	1.0000 (1)	1.0000 (1)	0.7670 (1)	0.2215 (6)
6	0.5938 (13)	0.5512 (9)	0.3897 (12)	0.1580 (10)
7	0.4711 (16)	0.3768 (17)	0.2766 (17)	0.1357 (13)
8	0.4148 (19)	0.4148 (15)	0.2752 (18)	0.1570 (11)
9	0.3272 (22)	0.3046 (21)	0.2233 (20)	0.0953 (21)
10	0.7807 (10)	0.4683 (14)	0.4823 (9)	0.1354 (14)
11	0.2826 (23)	0.1883 (23)	0.1716 (23)	0.1270 (15)
12	1.0000 (1)	1.0000 (1)	0.7596 (2)	0.1881 (7)
13	0.3578 (20)	0.3441 (19)	0.2078 (21)	0.1856 (8)
14	0.4707 (17)	0.3863 (16)	0.2886 (16)	0.1267 (16)
15	1.0000 (1)	0.9185 (4)	0.6138 (6)	0.3209 (3)
16	0.4717 (15)	0.4717 (13)	0.3202 (14)	0.1438 (12)
17	0.6364 (11)	0.5296 (10)	0.3600 (13)	0.2372 (5)
18	0.4283 (18)	0.3453 (18)	0.2588 (19)	0.1136 (19)
19	0.8220 (9)	0.4843 (12)	0.4112 (11)	0.2485 (4)
20	0.9351 (8)	0.8547 (6)	0.5465 (8)	0.4172 (2)
21	0.3328 (21)	0.3066 (20)	0.2008 (22)	0.1215 (17)
22	1.0000 (1)	1.0000 (1)	0.5895 (7)	0.5821 (1)
23	0.5990 (12)	0.4852 (11)	0.4203 (10)	0.0442 (24)
24	0.2571 (24)	0.1200 (24)	0.1348 (24)	0.0563 (22)

Table 4 Common weights in derived by models (3) and (11) on the basis of separation vector

Model	v_1	v_2	w_1	w_2	u_1	u_2
Model (3)	0.00126	0.00929	–	–	0.00100	0.00851
Model (11)	0.00251	0.01030	0.00100	0.00100	0.00100	0.00235

are increased to 0.6488(10), 1.0000(1), and 0.6488(10), respectively. In summary, using the improvement model (12), overall and divisional efficiencies of DMU_4 equal 3.20, 1.38 and 2.31, respectively, while if the improvement model (14) is used, overall and divisional efficiencies of DMU_4 equal 11.63, 1.88 and 6.17, respectively. As a result, the model (14) gains more improvement than the model (12).

Conclusions

Data envelopment analysis (DEA) selects the most desirable weights for evaluating the performance of decision maker units (DMU). This flexibility in selecting weights makes some units efficient, and as a result, the discrimination of

ranks between the units is not happened, and on the other hand, because of selecting different weight sets for units, they are ranked discriminately in their performance evaluation. In order to solve the problem, common sets of weights (CSWs) method was introduced on the basis of its different solving methods. CSWs method leads to evaluate and rank units only from the view of one weight which is the most desirable one for all units. As well, the method leads to discriminate the units appropriately.

In most conventional DEA models, the internal structure of the production units is disregarded and the units are often considered as black box. Therefore, DEA models are likely to recognize an efficient unit as black box though it consists of several inefficient sub-processes. To resolve this problem, numerous ideas have been developed from the standard

Table 5 Decrease in the inputs and increase in the outputs, the overall and divisional efficiency scores derived by CSWs and the ranks obtained from model (12)

DMU	xc_1	xc_2	yc_1	yc_2	Improve efficiency stage 1	Improve efficiency stage 2	Improve overall efficiency
1	1,169,991	668,511	1,379,560	955,581	0.9639 (9)	0.2467 (19)	0.2378 (18)
2	1,379,744	1,350,721	1,957,840	1,330,332	0.8220 (18)	0.2302 (22)	0.1893 (20)
3	812,495	409,037	293,613	658,428	0.9748 (8)	0.1562 (23)	0.1523 (23)
4	435,566	430,451	575,291	410,186	0.7327 (22)	0.2434 (20)	0.1783 (21)
5	5,610,573	2,957,783	7,851,229	3,925,272	0.9322 (12)	0.2837 (17)	0.2644 (16)
6	2,532,297	644,095	4,224,258	1,023,176	0.9232 (14)	0.4378 (9)	0.4041 (10)
7	1,461,171	1,085,331	4,164,069	816,304	0.8218 (19)	0.4083 (12)	0.3355 (13)
8	2,749,309	1,359,438	7,625,997	1,218,093	0.9539 (10)	0.4437 (8)	0.4232 (7)
9	1,567,746	950,432	3,575,048	904,563	0.9297 (13)	0.3511 (16)	0.3265 (15)
10	1,122,801	1,118,683	2,520,683	823,639	0.6553 (24)	0.3559 (15)	0.2332 (19)
11	1,453,284	497,954	4,548,938	55,894	0.9955 (4)	0.5272 (5)	0.5249 (4)
12	2,592,790	650,952	2,072,448	1,197,102	0.9135 (15)	0.2710 (18)	0.2476 (17)
13	2,115,838	1,109,666	6,640,814	410,395	0.9458 (11)	0.4455 (7)	0.4213 (8)
14	1,011,571	716,568	2,705,967	641,698	0.8561 (17)	0.3943 (13)	0.3375 (12)
15	2,184,944	651,063	4,760,948	788,216	1.0000 (1)	0.4554 (6)	0.4554 (6)
16	1,099,272	376,553	2,219,789	514,488	0.9876 (6)	0.4173 (10)	0.4121 (9)
17	1,051,543	784,803	3,144,484	371,984	0.7929 (20)	0.4137 (11)	0.3280 (14)
18	601,119	434,858	1,853,898	438,704	0.9975 (3)	0.3839 (14)	0.3830 (11)
19	159,422	182,338	1,248,548	112,697	1.0000 (1)	0.5976 (4)	0.5976 (2)
20	135,729	49,944	394,536	29,441	0.7334 (21)	0.6763 (2)	0.4960 (5)
21	63,172	19,682	185,864	23,223	0.8894 (16)	0.6510 (3)	0.5790 (3)
22	9428	6191	82,141	4181	0.9875 (7)	1.0000 (1)	0.9875 (1)
23	46,492	24,148	0.18	33,898	0.9940 (5)	0.0934 (24)	0.0928 (24)
24	163,297	235,094	424,868	50,661	0.7037 (23)	0.2387 (21)	0.1680 (22)

DEA to design the models which can measure the efficiency of production systems with different network structures. A distinctive feature of a network system is intermediate production.

In this method, it was studied CSWs evaluation method and the improvement of the resulted efficiency from CSWs on the basis of the separation vector in two-stage NDEA. CSWs were obtained by making single objective linear model in two-stage NDEA through the separation vector, and the units and subunits were evaluated based on it. It was observed that CSWs have made a better discrimination between units and subunits on the basis of the separation vector. On one hand, units and subunits have been evaluated based on an identical criterion, and it has been avoided to rank on the basis of different and occasionally unreal weights (zero weight and large weights). The advantage of the extended CSWs in two-stage NDEA based on the separation vector is that it has avoided to evaluate discriminately

in measuring the efficiency of different weight sets for units and subunits.

To improve efficiency score in basic two-stage NDEA by CSWs on the basis of separation vector, we presented two approaches including simultaneous decrease and increase in the inputs and outputs, respectively, as well as simultaneous change of the inputs and outputs (without any changes in intermediate measures). It was observed that the overall and divisional efficiency scores are improved better upon the simultaneous change of the inputs and outputs than the simultaneous decrease and increase in the input and outputs, respectively. In this study, we used efficiency evaluation by CSWs on the basis of separation vector in two-stage network with constant returns to scale. The efficiency evaluation by CSW on the basis of separation vector in the basic two-stage NDEA with variable returns to scale and in general NDEA would be considered as two interesting topics that can be addressed in future studies.

Table 6 Change of the inputs and outputs, the overall and divisional efficiency scores derived by CSWs and the ranks obtained from model (14)

DMU	xc_1	xc_2	yc_1	yc_2	Improve efficiency stage 1	Improve efficiency stage 2	Improve overall efficiency
1	1,498,279	554,210	5,652,484	431,506	1.0000 (1)	0.6427 (11)	0.6427 (11)
2	1,882,951	928,114	10,882,587	669,978	1.0000 (1)	0.8089 (6)	0.8089 (6)
3	958,182	358,313	3,678,244	278,098	1.0000 (1)	0.6493 (9)	0.6493 (9)
4	636,982	238,018	2,442,269	184,775	1.0000 (1)	0.6488 (10)	0.6488 (10)
5	7,839,089	2,335,868	15,724,311	1,819,669	0.9496 (22)	0.4226 (22)	0.4013 (22)
6	1,993,528	678,381	6,557,952	541,602	1.0000 (1)	0.5923 (12)	0.5923 (12)
7	2,240,113	667,502	5,199,788	539,307	0.9764 (19)	0.4705 (19)	0.4594 (19)
8	3,619,932	1,078,656	8,873,779	884,382	0.9874 (16)	0.4895 (16)	0.4834 (16)
9	2,405,249	716,709	4,865,496	559,441	0.9511 (21)	0.4252 (21)	0.4044 (21)
10	1,721,237	512,889	4,045,764	415,766	0.9789 (18)	0.4748 (18)	0.4648 (18)
11	1,489,481	485,351	4,548,938	392,806	1.0000 (1)	0.5658 (13)	0.5658 (13)
12	1,890,177	711,021	7,324,295	550,906	1.0000 (1)	0.6528 (8)	0.6528 (8)
13	2,918,514	869,650	6,640,814	698,977	0.9725 (20)	0.4637 (20)	0.4510 (20)
14	1,550,590	462,040	3,699,289	376,040	0.9819 (17)	0.4800 (17)	0.4713 (17)
15	2,184,944	651,063	5,679,198	542,636	1.0000 (1)	0.5106 (14)	0.5106 (14)
16	1,175,252	350,198	3,052,399	291,812	0.9998 (15)	0.5103 (15)	0.5102 (15)
17	1,613,287	480,722	3,144,484	371,984	0.9448 (23)	0.4137 (23)	0.3908 (23)
18	614,895	430,061	5,625,481	288,788	1.0000 (1)	0.9904 (2)	0.9904 (2)
19	159,422	182,338	1,801,683	91,706	1.0000 (1)	0.8312 (5)	0.8312 (5)
20	44,663	49,989	499,868	25,443	1.0000 (1)	0.8379 (4)	0.8379 (4)
21	42,513	20,832	243,707	15,059	1.0000 (1)	0.8057 (7)	0.8057 (7)
22	8756	6249	82,141	4181	1.0000 (1)	1.0000 (1)	1.0000 (1)
23	45,139	24,264	293,756	17,172	1.0000 (1)	0.8563 (3)	0.8563 (3)
24	163,297	235,094	751,657	38,260	0.7037 (24)	0.3965 (24)	0.2790 (24)

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