On controllability for a nondensely defined fractional differential equation with a deviated argument

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Abstract

This article deals with a fractional differential equation with a deviated argument defined on a nondense set. A fixed-point theorem and the concept of measure of noncompactness are used to prove the existence of a mild solution. Furthermore, by using the compactness of associated cosine family, we proved that system is approximately controllable and obtains an optimal control which minimizes the performance index. To illustrate the abstract result, we included an example.

Keywords Approximate controllability · Optimal control · Deviated argument · Fractional differential equation · Cosine family

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Introduction

Exact controllability and approximate controllability are some fundamental concepts in control problems. These concepts help to design and analyze the various kinds of control dynamics process in finite- and infinite-dimensional spaces. In the last few decades, control problems have been a research area of great interest. An extensive study on controllability for various kinds of control problems in abstract spaces has been done by many authors [1–15]. In [1], authors proved the approximate controllability for semi-linear noninteger differential equations by using fixed-point technique and semigroup theory of bounded linear operators in a Hilbert space. Das et al. [2] used the concept of measure of noncompactness and the compactness of associated families of linear operators to investigate the sufficient conditions for approximate controllability of a noninteger-order control system with delay depending on state variable. More works on approximate controllability for state-dependent noninteger control have been done in [12, 13]. Most of the papers available in the literature deal with densely defined

systems in abstract spaces. There are only few papers in which authors considered problems involving nondensely defined linear operators. Fu [5] has proved some results on controllability for a nondensely defined abstract differential equations. Other results on controllability for nondensely defined problems have been done in [5, 15, 16]. In [17], authors investigated some sufficient conditions for approximate controllability of delayed semilinear system by using sequence method and C_0 -semigroup of bounded linear operators.

In the last decade, many authors extended the study of approximate controllability to higher order differential equations in abstract spaces [2, 4, 8, 11]. Shukla et al. [4, 8] have studied the second-order semilinear control system and proved that under some assumptions, system is approximately controllable. In [11], stochastic differential equation of order 2 has been studied. The study of controllability to noninteger-order differential equations has been done in [9, 10, 18]. In [10], authors discussed the approximate controllability for a noninteger-order stochastic control system of order $1 < \alpha \leq 2$. Few authors [18, 19] investigated the optimal controllability for noninteger-order systems. For more work on controllability of fractional differential equations, we refer [20], and papers cited in this paper.

The present work is motivated by the work of Fu [5], in which authors considered a semilinear control system with linear part is defined on a nondense domain and satisfies the Hille–Yosida condition and established some results

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on controllability. We extended the study of approximate and optimal controllability to a class of nondensely defined fractional differential equations with deviated argument in a Hilbert space by using the measure of noncompactness and Darbo–Sadovskii fixed-point theorem. Consider the following fractional differential equations with deviated argument in a Hilbert space $(Z, \|\cdot\|)$:

$$\begin{cases} {}_C D^\alpha z(t) = Az(t) + Ew(t) + g(t, z(t), z(b(z(t), t))), & 0 < t \leq a_0, \\ z(0) = z_0, & z'(0) = z_1, \end{cases} \quad (1)$$

where $1 < \alpha \leq 2$, ${}_C D^\alpha$ denotes the (Caputo) fractional derivative, $A : D(A) \subset Z \rightarrow Z$ is a nondensely defined linear operator which satisfies the Hille–Yosida condition, E denotes a bounded linear operator defined on W and $w \in L^2([0, a_0], W)$ denotes the control function, which takes values in a Banach space W . The maps g, b are suitably defined functions satisfying some suitable conditions and $z_0, z_1 \in \overline{D(A)}$. Let $I_0 = [0, a_0]$.

The organization of the rest of the manuscript is as follows. In Section 2, we state some basic concepts and assumptions. In Section 3, first we prove the existence of mild solution, then the approximate and optimal controllability for problem (1) by using a fixed-point theorem and the concept of measure of noncompactness. In the last section to illustrate the abstract result, we included an example.

Preliminaries and assumptions

In this section, we state some basic definitions, assumptions and notations.

It is well known that if A satisfies Hille–Yosida condition, then A generates an integrated semigroup which is nondegenerate and locally Lipschitz continuous [21, 22].

Define A_0 on $D(A_0) \subset \overline{D(A)}$ into $\overline{D(A)}$ by

$$A_0 z = Az,$$

where

$$D(A_0) = \{z \in D(A) | Az \in \overline{D(A)}\}.$$

Then, A_0 generates (see [21]), a compact family $\{C_\alpha(t)\}_{t \geq 0}$ of cosines on $\overline{D(A)}$. For $t \geq 0$, we define

$$\begin{aligned} S_\alpha(t) &= \int_0^t C_\alpha(s) ds, \\ P_\alpha(t) &= \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-2} C_\alpha(s) ds. \end{aligned}$$

Let

$$C(I_0, \overline{D(A)}) = \{z : I_0 \rightarrow \overline{D(A)} | z \text{ is continuous on } I_0\},$$

$$C_L(I_0, \overline{D(A)}) = \{z \in C(I_0, \overline{D(A)}) | \|z(t) - z(s)\| \leq L|t - s|\}.$$

Obviously, $(C_L(I_0, \overline{D(A)}), \|z\|_{a_0})$ is a Banach space, where

$$\|z\|_{a_0} = \sup_{t \in I_0} \|z(t)\|.$$

Definition 1 A mild solution of problem (1) is a function $z \in C_L(I_0, \overline{D(A)})$ satisfying the integral equation:

$$\begin{aligned} z(t) &= C_\alpha(t)z_0 + S_\alpha(t)z_1 \\ &\quad + \int_0^t P_\alpha(t-s)[g(s, z(s), z(b(z(s), s))) + Ew(s)]ds, \\ &\quad 0 \leq t \leq a_0. \end{aligned} \quad (2)$$

Next, we define the approximate controllability on a non-densely defined domain.

Definition 2 System (1) is approximately controllable on I_0 , if for every $\epsilon_0 > 0$ and $z_0, z_1, z_{a_0} \in \overline{D(A)}$, there is an admissible control $w \in L^2(I_0, W)$ such that the solution $z(t)$ of (1) satisfies

$$\|z(a_0) - z_{a_0}\| < \epsilon_0.$$

Definition 3 The map $Q_0 : Z_1 \rightarrow Z_1$ is said to be a ϕ -contraction if there exists a positive constant $k_0 < 1$ such that $\phi(Q_0 C_0) \leq k_0 \phi(C_0)$ for any bounded closed subset $C_0 \subseteq Z_1$, where Z_1 is a Banach space and ϕ denotes the Hausdorff measure of noncompactness defined on bounded subsets of Z_1 .

It is convenient to define the operator

$$\Omega_0^{a_0} = \int_0^{a_0} P_\alpha(t-s) E E^* P_\alpha^*(t-s) ds,$$

where E^* and P_α^* are the adjoint operators of E and P_α , respectively.

For $\delta > 0$, let

$$R(\delta, \Omega_0^{a_0}) = (\delta I + \Omega_0^{a_0})^{-1}$$

and

$$\begin{aligned} q(z(t)) &= z_{a_0} - C_\alpha(a_0)z_0 - S_\alpha(a_0)z_1 \\ &\quad - \int_0^{a_0} P_\alpha(a_0-s)g(s, z(s), b(z(s), s))ds, \end{aligned}$$

then we have

$$w_z(t) = E^* P_\alpha^*(a_0 - t) R(\delta, \Omega_0^{a_0}) q(z(t)).$$

Lemma 4 If the map $Q_0 : D(Q_0) \subseteq Z_1 \rightarrow Z_1$ is a Lipschitz continuous with constant k_0 , then $\phi(QC_0) \leq k_0\phi(C_0)$, for any bounded set C_0 .

Lemma 5 [23] (Darbo–Sadovskii fixed-point theorem) A ϕ -contraction continuous map $Q_0 : W_0 \rightarrow W_0$, defined on a convex, closed and bounded set $W_0 \subseteq Z_1$, has at least one fixed point in W_0 .

We consider the following assumptions:

(H1) Map g satisfies:

- (1) $\|g(t_1, z_1, \tilde{z}_1) - g(t_2, z_2, \tilde{z}_2)\|_{a_0} \leq L_g[|t_1 - t_2| + \|z_1 - z_2\|_{a_0} + \|\tilde{z}_1 - \tilde{z}_2\|_{a_0}]$,
- (2) $\|g(t, z, \tilde{z})\| \leq G_0$,

for some $L_g, G_0 > 0$, and for all $t, t_1, t_2 \in I_0$, $z, z_1, z_2 \in C(I_0, \overline{D(A)})$ and $\tilde{z}, \tilde{z}_1, \tilde{z}_2 \in C_L(I_0, \overline{D(A)})$.

(H2) There exists a constant $L_b > 0$ such that

$$\|b(z(t), t) - b(\tilde{z}(t), t)\| \leq L_b \|z(t) - \tilde{z}(t)\|.$$

(H3) There exist constants $M_1 > 0, M_2 > 0$ and $r_1 > 0$ such that

$$\|C_\alpha(t)\| \leq M_1, \quad \|S_\alpha(t)\| \leq M_2, \quad \|P_\alpha(t)\| \leq r_1, \quad t \in I_0.$$

(H4) There exists a constant $M_3 > 0$ such that $\|E\| \leq M_3$.

Lemma 6 Under assumptions (H1)–(H4), there exists $K_w > 0$ s.t. $\|w_z(t)\| \leq K_w$.

Proof Using the definition of control function $w_z(t)$, and (H4), we have

$$\begin{aligned} \|w_z(t)\| &= \|E^* \|P_\alpha^*(a_0 - t)\| \|R(\delta, \Omega_0^{a_0})\| \|q(z(t))\| \\ &\leq \frac{M_3 r_1}{\delta} \|q(z(t))\|. \end{aligned}$$

In view of assumptions (H1)–(H3), we can find a constant $K_w > 0$, such that

$$\|w_z(t)\| \leq K_w.$$

□

Main results

Theorem 1 If (H1)–(H4) hold, then for every $z_0, z_1, z_{a_0} \in \overline{D(A)}$, system (1) has a mild solution on I_0 , if

$$r_1 a_0 L_g (1 + LL_b) \left[1 + \frac{r_1^2 a_0 M_3^2}{\delta} \right] < \frac{1}{2}.$$

Proof To show that system (1) has a mild solution, we need to show that, there exists $z \in C_L(I_0, \overline{D(A)})$, such that

$$\begin{aligned} z(t) &= C_\alpha(t)z_0 + S_\alpha(t)z_1 \\ &\quad + \int_0^t P_\alpha(t-s)[g(s, z(s), z(b(z(s), s))) + Ew_z(s)]ds \\ &\quad 0 \leq t \leq a_0. \end{aligned}$$

We define the operator $\tilde{G}_\delta$ on $C_L(I_0, \overline{D(A)})$, by

$$\begin{aligned} (\tilde{G}_\delta z)(t) &= C_\alpha(t)z_0 + S_\alpha(t)z_1 \\ &\quad + \int_0^t P_\alpha(t-s)[g(s, z(s), z(b(z(s), s))) + Ew_z(s)]ds, \\ &\quad 0 \leq t \leq a_0. \end{aligned}$$

Let

$$B_{r_0} = \left\{ z \in C_L(I_0, \overline{D(A)}) \mid z(0) = z_0, z'(0) = z_1, \|z\|_{a_0} \leq r_0 \right\}.$$

If $z(t)$ satisfies (2), then $z(t)$ can be written as $z(t) = x_0(t) + x(t)$, where

$$\begin{aligned} x_0(t) &= C_\alpha(t)z_0 + S_\alpha(t)z_1 \\ x(t) &= \int_0^t P_\alpha(t-s)[g(s, x_0(s) + x(s), x_0(b(x_0(s) + x(s), s))) \\ &\quad + x(b(x_0(s) + x(s), s))) + Ew_{x_0+x}(s)]ds. \end{aligned}$$

Define $\tilde{F}_\delta : C_L(I_0, \overline{D(A)}) \rightarrow C_L(I_0, \overline{D(A)})$, by

$$\begin{aligned} (\tilde{F}_\delta x)(t) &= \int_0^t P_\alpha(t-s)[g(s, x_0(s) + x(s), x_0(b(x_0(s) + x(s), s))) \\ &\quad + x(b(x_0(s) + x(s), s))) + Ew_{x_0+x}(s)]ds. \end{aligned}$$

Obviously, map $\tilde{G}_\delta$ has a fixed point if and only if $\tilde{F}_\delta$ has a fixed point.

For $r > 0$, we define

$$C_r = \left\{ z \in C_L(I_0, \overline{D(A)}) : z(0) = 0, z'(0) = 0, \|z\|_{a_0} \leq r \right\}.$$

Clearly, $\tilde{F}_\delta$ is well defined on C_r which is a closed convex set in $C_L(I_0; \overline{D(A)})$.

To show that $\tilde{F}_\delta$ has a fixed point, we use Darbo–Sadovskii fixed-point theorem. For this, we need to follow the two steps:

Step 1: There exists $r > 0$, s.t. $\tilde{F}_\delta(C_r) \subseteq C_r$. Let $x \in C_r$. Then, using (H2), we have

$$\begin{aligned} \|(\tilde{F}_\delta x)(t)\| &\leq r_1 \int_0^t \|g(s, x_0(s) + x(s), x_0(b(x_0(s) + x(s), s)) \\ &\quad + x(b(x_0(s) + x(s), s))) + Ew_{x_0+x}(s)\| ds. \end{aligned}$$

Using (H1), (H4) and Lemma 6, we get

$$\|(\tilde{F}_\delta x)(t)\| \leq r_1(G_0 a_0 + M_3 K_w).$$

We can choose $r > 0$, sufficiently large such that

$$\|(\tilde{F}_\delta x)(t)\| \leq r.$$

Thus, there exists $r > 0$ such that

$$\tilde{F}(C_r) \subseteq C_r.$$

Step 2: $\tilde{F}_\delta$ is a ϕ -contraction map. For this, first, we need to show that $\tilde{F}_\delta$ is Lipschitz continuous. Using (H3) and (H4), we have

$$\begin{aligned} \|(\tilde{F}_\delta x_1)(t) - (\tilde{F}_\delta x_2)(t)\| &\leq r_1 \int_0^t \|g(s, x_0(s) + x_1(s), x_0(b(x_0(s) + x_1(s), s)) \\ &\quad + x_1(b(x_0(s) + x_1(s), s))) \\ &\quad - g(s, x_0(s) + x_2(s), x_0(b(x_0(s) + x_2(s), s)) \\ &\quad + x_2(b(x_0(s) + x_2(s), s)))\| ds \\ &\quad + r_1 M_3 \int_0^t \|w_{x_0+x_1}(s) - w_{x_0+x_2}(s)\| ds. \end{aligned} \quad (3)$$

Using (H1), (H2) and (H3), we get

$$\begin{aligned} &\|g(s, x_0(s) + x_1(s), x_0(b(x_0(s) + x_1(s), s)) \\ &\quad + x_1(b(x_0(s) + x_1(s), s))) \\ &\quad - g(s, x_0(s) + x_2(s), x_0(b(x_0(s) + x_2(s), s)) \\ &\quad + x_2(b(x_0(s) + x_2(s), s)))\| \\ &\leq L_g [\|x_1(s) - x_2(s)\| + L \|b(x_0(s) + x_1(s), s) - b(x_0(s) + x_2(s), s)\| \\ &\quad + \|x_1(b(x_0(s) + x_1(s), s)) - x_2(b(x_0(s) + x_2(s), s))\|] \\ &\leq L_g [\|x_1(s) - x_2(s)\| + LL_b \|x_1(s) - x_2(s)\| \\ &\quad + \|x_1(b(x_0(s) + x_1(s), s)) - x_2(b(x_0(s) + x_1(s), s))\| \\ &\quad + \|x_2(b(x_0(s) + x_1(s), s)) - x_2(b(x_0(s) + x_2(s), s))\|] \\ &\leq 2L_g [1 + LL_b] \|x_1 - x_2\|_{a_0}. \end{aligned}$$

and

$$\begin{aligned} &\|w_{x_0+x_1}(s) - w_{x_0+x_2}(s)\| \\ &\leq \frac{M_3 r_1}{\delta} \|q((x_0 + x_1)(s)) - q((x_0 + x_2)(s))\| \\ &\leq \frac{2M_3 r_1^2}{\delta} a_0 L_g (1 + LL_b) \|x_1 - x_2\|_{a_0}. \end{aligned}$$

From (3), we have

$$\begin{aligned} &\|(\tilde{F}_\delta x_1)(t) - (\tilde{F}_\delta x_2)(t)\| \\ &\leq 2r_1 a_0 L_g (1 + LL_b) \left[1 + \frac{r_1^2 a_0 M_3^2}{\delta} \right] \|x_1 - x_2\|_{a_0}. \end{aligned}$$

Thus, $\tilde{F}_\delta$ is Lipschitz continuous. By using Lemma 4, we have

$$\phi(\tilde{F}_\delta C_r) \leq 2r_1 a_0 L_g (1 + LL_b) \left[1 + \frac{r_1^2 a_0 M_3^2}{\delta} \right] \phi(C_r).$$

Since $2r_1 a_0 L_g (1 + LL_b) \left[1 + \frac{r_1^2 a_0 M_3^2}{\delta} \right] < 1$, $\tilde{F}_\delta$ is a ϕ -contraction, by using Lemma 5, $\tilde{F}_\delta$ has a fixed point and consequently $\tilde{G}_\delta$ has a fixed point, say z_δ . $\square$

Theorem 2 Under all the assumptions of Theorem 1, system (1) is approximately controllable on I_0 .

Proof The fixed point z_δ obtained in Theorem 1 is given by:

$$\begin{aligned} z_\delta(t) &= C_\alpha(t) z_0 + S_\alpha(t) z_1 \\ &\quad + \int_0^t P_\alpha(t-s) [g(s, z_\delta(s), z_\delta(b(z_\delta(s), s))) \\ &\quad + Ew_{z_\delta}(s)] ds, \end{aligned}$$

where

$$w_{z_\delta}(t) = D^* P_\alpha^*(a_0 - t) R(\delta, \Omega_0^{a_0}) q(z_\delta(t)),$$

and

$$\begin{aligned} q(z_\delta(t)) &= z_{a_0} - C_\alpha(a_0) z_0 - S_\alpha(a_0) z_1 \\ &\quad - \int_0^{a_0} P_\alpha(a_0 - s) g(s, z_\delta(s), b(z_\delta(s), s)) ds. \end{aligned}$$

Thus, we have

$$\begin{aligned} z_\delta(a_0) - z_{a_0} &= -q(z_\delta(t)) + \Omega_0^{a_0} R(\delta, \Omega_0^{a_0}) q(z_\delta(t)) \\ &= -(\delta I + \Omega_0^{a_0}) R(\delta, \Omega_0^{a_0}) q(z_\delta(t)) + \Omega_0^{a_0} R(\delta, \Omega_0^{a_0}) q(z_\delta(t)) \\ &= -\delta R(\delta, \Omega_0^{a_0}) q(z_\delta(t)). \end{aligned} \quad (4)$$

Now

$$\int_0^{a_0} \|g(s, z_\delta(s), b(z_\delta(s), s))\|^2 ds \leq G_0^2 a_0.$$

Thus, the sequence $\{g(t, z_\delta(t), b(z_\delta(t), t))\}$ is bounded in the Hilbert space $L^2(I_0, Z)$. Therefore, there exists a weakly convergent subsequence $\{g(t, z_\delta(t), b(z_\delta(t), t))\}$ (again denoted by same notation) converging to $v(\cdot) \in L^2(I_0, Z)$. Let

$$\mu = z_{a_0} - C_\alpha(a_0)z_0 - S_\alpha(a_0)z_1 - \int_0^{a_0} \tilde{P}_\alpha(a_0 - s)v(s)ds.$$

Thus,

$$\|q(z_\delta(t)) - \mu\| \leq \left\| \int_0^{a_0} \tilde{P}_\alpha(a_0 - s)[g(s, z_\delta(s), b(z_\delta(s), s)) - v(s)]ds \right\|.$$

By using the compactness of $P_\alpha(t)$, the Ascoli–Arzela theorem, and following the idea used in [1], one can prove that the mapping

$$z(t) \rightarrow \int_0^{a_0} P_\alpha(t - s)z(s)ds$$

is compact. Therefore, we have

$$\|q(z_\delta(t)) - \mu\| \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

From (4), we have

$$\begin{aligned} \|z_\delta(a_0) - z_0\| &= \|\delta R(\delta, \Omega_0^{a_0})q(z_\delta(t))\| \\ &\leq \|\delta R(\delta, \Omega_0^{a_0})\| \|q(z_\delta(t)) - \mu\| + \|\delta R(\delta, \Omega_0^{a_0})\mu\|. \end{aligned}$$

This implies that $\|z_\delta(a_0) - z_0\| \rightarrow 0$ as $\delta \rightarrow 0$. Thus, system (1) is approximately controllable on I_0 . $\square$

Next, we discuss the optimal controllability of problem (1). In order to proceed further, we define performance index

$$\tilde{J}(w) = \int_0^{a_0} \tilde{F}(t, z(t), z(b(z(t), t)), w(t))dt,$$

where $\tilde{F}$ is a functional defined on $I_0 \times C(I_0, Z) \times C_L(I_0, Z) \times W_{ad}$ and W_{ad} denotes the set of all admissible control and consequently is closed and convex in $L^2(I_0, W)$.

Theorem 3 Under all the assumptions of Theorem 1, there exists an optimal control of problem (1) if $r_1 a_0 L_g(2 + LL_b) < 1$.

Proof To complete the proof of the theorem, it is sufficient to find a control $w^0 \in L^2(I_0, W)$ that minimizes the performance index $\tilde{J}(w)$.

If $\inf\{\tilde{J}(w) | w \in W_{ad}\} = \infty$, then result trivially holds.

If $\inf\{\tilde{J}(w) | w \in W_{ad}\} = \epsilon_0 < \infty$, then there exists a sequence $\{w^n\}$ in W_{ad} such that $\tilde{J}(w^n) \rightarrow \epsilon_0$. As W_{ad} is a closed and convex subset of $L_2(I_0, W)$, the sequence $\{w^n\}$ has a weakly convergent subsequence $\{w^m\}$ converging to $w^0 \in W_{ad}$. Using Theorem 1, for each $w^m \in W_{ad}$, there exists a mild solution z^m of (1) satisfying:

$$\begin{aligned} z^m(t) &= C_\alpha(t)z_0 + S_\alpha(t)z_1 \\ &\quad + \int_0^t P_\alpha(t - s)[g(s, z^m(s), z^m(b(z^m(s), s))) \\ &\quad + Ew^m(s)]ds, \quad 0 \leq t \leq a_0. \end{aligned}$$

Similarly, corresponding to w^0 , there exists a mild solution z^0 of (1) satisfying:

$$\begin{aligned} z^0(t) &= C_\alpha(t)z_0 + S_\alpha(t)z_1 \\ &\quad + \int_0^t P_\alpha(t - s)[g(s, z^0(s), z^0(b(z^0(s), s))) \\ &\quad + Ew^0(s)]ds, \quad 0 \leq t \leq a_0. \end{aligned}$$

Using (H3), for $t \in [0, a_0]$, we have

$$\begin{aligned} \|z^m(t) - z^0(t)\| &\leq r_1 \int_0^t \|g(s, z^m, z^m(b(z^m(s), s))) - g(s, z^0, z^0(b(z^0(s), s)))\| \\ &\quad + \|Ew^m(s) - Ew^0(s)\| ds. \end{aligned} \quad (5)$$

Using (H1), (H2), we get

$$\begin{aligned} \|g(s, z^m(s), z^m(b(z^m(s), s))) - g(s, z^0(s), z^0(b(z^0(s), s)))\| &\leq L_g[\|z^m(s) - z^0(s)\| + \|z^m(b(z^m(s), s)) - z^0(b(z^0(s), s))\|] \\ &\leq L_g[\|z^m(s) - z^0(s)\| + \|z^m(b(z^m(s), s)) - z^m(b(z^0(s), s))\| \\ &\quad + \|z^m(b(z^0(s), s)) - z^0(b(z^0(s), s))\|] \\ &\leq L_g(2 + LL_b)\|z^m - z^0\|_{a_0}. \end{aligned}$$

From (5), we have

$$\begin{aligned} \|z^m(t) - z^0(t)\| &\leq r_1 \left[L_g(2 + LL_b)a_0 \|z^m - z^0\|_{a_0} + \int_0^t \|Ew^m(s) - Ew^0(s)\| ds \right] \\ &\leq r_1 a_0 [L_g(2 + LL_b) \|z^m - z^0\|_{a_0} + \|Ew^m - Ew^0\|_{a_0}]. \end{aligned}$$

Since $r_1 a_0 L_g(2 + LL_b) < 1$ and $\|Ew^m - Ew^0\|_{a_0} \rightarrow 0$, we conclude that $z^m \rightarrow z^0$.

Applying Balder's theorem, we get

$$\begin{aligned} \epsilon_0 &= \lim_{n \rightarrow \infty} \int_0^{a_0} \tilde{F}(t, z^n(t), z^n(b(z^n(t), t)), w^n(t)) dt \\ &\leq \int_0^{a_0} \tilde{F}(t, z^0(t), z^0(b(z^0(t), 0)), w^0(t)) dt \\ &= \tilde{J}(w^0) \geq \epsilon_0. \end{aligned}$$

This shows that $\tilde{J}(w^0) = \epsilon_0$, i.e., $\tilde{J}$ attains its minimum value at $w^0 \in L^2(I_0, W)$. $\square$

Application

Consider the following example:

$$\begin{cases} {}_C D^{\frac{3}{2}} z(y, s) - \frac{\partial^2}{\partial y^2} z(y, s) = v_0(y, s) + g_1(y, z(y, s)) + g_2(s, y, z(y, s)), \\ \quad y \in (0, \pi) \quad s \in (0, a_0], \\ z(0) = z(\pi) = 0, \\ z(y, 0) = z_0(y), \quad \frac{\partial z}{\partial s}(y, 0) = z_1(y), \quad y \in (0, \pi), \end{cases} \quad (6)$$

where

$$g_1(y, z(y, s)) = \int_0^y K(y, x) z(y, a_1 h(t)) |z(x, t)| dx,$$

and the function $g_2 : R_+ \times [0, 1] \times R \rightarrow R$ is locally Hölder continuous in t , locally Lipschitz continuous in z , uniformly in y and measurable in y . Let $I_0 = [0, a_0]$.

Choose $Z = C([0, \pi], R)$ and the operator A defined by $Az = z''$ with domain

$$D(A) = \{z(\cdot) \in Z | z'' \in Z, z(0) = z(\pi) = 0\}.$$

Obviously, $\overline{D(A)} \neq Z$. Thus, A is nondensely defined on Z .

Also $\rho(A) \supseteq (0, +\infty)$, $\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$ for $\lambda > 0$. This shows that A satisfies the Hille–Yosida on Z . Therefore, A generates a compact C_0 -semigroup $\{T(t)\}$ on $\overline{D(A)}$ and consequently A generates a compact cosine family $\{C_\alpha(t)\}_{t \geq 0}$ on $D(A)$.

Let $C(I_0, \overline{D(A)})$ denote the set of all continuous functions on I_0 , and

$$C_L(I_0, \overline{D(A)}) = \left\{ z \in C(I_0, \overline{D(A)}) \mid \|z(s_1) - z(s_2)\| \leq L|s_1 - s_2| \right\}.$$

We define $g : I_0 \times C(I_0, \overline{D(A)}) \times C_L(I_0, \overline{D(A)}) \rightarrow X$, by

$$g(s, z, \phi)(y) = g_1(y, \phi) + g_2(s, y, z),$$

and $(Ev)(s)(y) = v_0(s, y)$, $z(s)(y) = z(y, s)$; then, abstract formulation of problem (6) is

$$\begin{aligned} {}_C D^{\frac{3}{2}} z(s) &= Az(s) + Ev(s) + g(s, z(s), b(z(s), s)) \\ z(0) &= z_0, \quad \frac{\partial z}{\partial s}(0) = z_1. \end{aligned}$$

We can easily prove that all assumptions of Theorem 2 are satisfied; therefore by using theorem, we conclude that system (6) is approximately controllability on I_0 .

Define performance index

$$\tilde{J}(w) = \int_0^{a_0} (\|z(t)\|^2 + \|w(t)\|^2) dt.$$

It can be easily checked that all the assumptions of Theorem 3 are satisfied; therefore, by using Theorem 3, we find an admissible control w_0 that minimizes $\tilde{J}(w)$.

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