Application of Chebyshev tau method for bending analysis of elastically restrained edge functionally graded nano/micro-scaled sandwich beams, under non-uniform normal and shear loads

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Abstract

In this study, for the first time, an approximate solution procedure based on the Chebyshev tau method (CTM) is developed for bending analysis of functionally graded nano/micro-scaled sandwich beams. The proposed approach has the advantage of decreasing the problem to the solution of a system of algebraic equations, which may then be solved by any numerical method. In the CTM, the solution is approximated via a truncated Chebyshev series expansion and the Chebyshev polynomials are used as the test function. Based on the proposed technique, sandwich beams with elastically restrained edges under arbitrary non-uniform distributed normal and shear loads can be analyzed. The effectiveness of the CTM is illustrated by comparison of the obtained results for various end supports with those extracted from the ABAQUS software. In each considered cases, the numerical results indicate that the proposed scheme is of high accuracy and is efficient for solving the ordinary differential equations and systems of them.

Keywords Chebyshev tau method · Chebyshev polynomials · Spectral method · Micro/nano-sandwich beam · Non-uniform loads

Introduction

Nowadays, spectral methods are extensively utilized for various applications [1]. Their main appeal relies on their premiere rate of convergence for sufficiently smooth functions [2]. Chebyshev tau method is a particularly efficient spectral scheme in which Chebyshev polynomials are used in the tau method of Lanczos [3]. Numerical programs using this technique are often considerably faster with greater accuracy than other standard methods such as finite differencing [4]. Recently, there have been several published papers on the

applications of the tau method. Siyyam and Syam [5] presented the CTM for the two-dimensional Poisson equation. Ahmadi and Adibi [6] applied the Chebyshev tau method for the Laplace equation. Saadatmandi and Dehghan [7] utilized the CTM to approximate the solution of hyperbolic telegraph problem. Wang [8] applied a time-splitting CT spectral method to the Ginzburg–Landau–Schrödinger equation with zero/nonzero far-field boundary conditions. Lee [9] applied a Chebyshev tau method based on Euler–Bernoulli and Timoshenko beam theories to the free vibration analyses of stepped beams. Functionally graded materials (FGMs) have great practical applications in engineering and industrial fields [10] and represent a novel generation of materials, composed of a mixture of two different materials (metal and ceramic) [11]. They are constructed by a continuous change in composition and do not possess a particular interface [12]. FGMs concepts are applied to metals, ceramics and organic composites to generate improved components with superior physical properties [27]. Investigations in this emerging area [13–26] are only at the beginning stage and are very confined to most of the previous studies devoted to the compression behavior and energy dissipation performance only [28]. In this study, we employ the Chebyshev

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tau spectral scheme for solution of governing equations of a three-layer sandwich beam regarded to non-uniform normal and shear loads. The paper is organized as follows. In “[The tau method](#)” section, the Chebyshev tau spectral method is described briefly. The third section is devoted to solving our intended problem by the CTM and obtaining the approximate results. In “[Results and comparisons](#)” section, a detailed discussion is carried out to show the accuracy, validity and applicability of the technique. Finally, we give our conclusions in “[Conclusion](#)” section.

The tau method

Some fundamental results for Chebyshev approximation [29] are needed. The Chebyshev polynomial of degree n on $[-1, 1]$ is defined by $T_n(\cos \theta) = \cos(n\theta)$. For $x \in [-1, 1]$, $\{T_n(x)\}$ satisfy the following recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad (1)$$

$$n > 0, T_0(x) = 1, T_1(x) = x.$$

They form an orthonormal basis with respect to

$$(u, v)_\omega = \left(\int_{-1}^1 u(x)v(x)\omega(x)dx \right), \quad (2)$$

where the weight function ω is defined by $\omega(x) = \frac{1}{\sqrt{1-x^2}}$.

Proposition 1 The polynomials $T_n(x)$ are orthogonal, i.e.,

$$(T_n, T_m)_{0,\omega} = \frac{\pi}{2} c_n \delta_{n,m}, \quad m, n \in \mathbb{N}, \quad (3)$$

where $\delta_{n,m}$ is

$$\delta_{n,m} = \begin{cases} 0, & n \neq m, \\ 1, & n = m, \end{cases} \quad (4)$$

and the coefficients c_n are

$$c_n = \begin{cases} 2, & n = 0, \\ 1, & n > 0. \end{cases} \quad (5)$$

Some useful properties of Chebyshev polynomials are [30]

$$|T_n(x)| \leq 1, \quad |x| \leq 1, \quad T_n(\pm 1) = (\pm 1)^n, \quad (6)$$

$$T'_n(\pm 1) = (\pm 1)^n n^2,$$

$$T_n(x)T_m(x) = \frac{T_{n+m}(x) + T_{|n-m|}(x)}{2}. \quad (7)$$

The Chebyshev expansion of a function $u(x)$ is $u(x) = \sum_{n=0}^{\infty} b_n T_n(x)$ where $b_n = \frac{(u, T_n)_{0,\omega}}{\|T_n\|_{0,\omega}^2} = \frac{2}{\pi c_n} (u, T_n)_{0,\omega}$.

Consider the expansion of a function $u(x)$ or its derivatives in terms of Chebyshev polynomials on the interval $[-1, 1]$. Suppose that u and its derivatives can be expanded as

$$u(x) = \sum_{n=0}^{\infty} b_n^{(0)} T_n(x), \quad (8)$$

$$\frac{d^m u}{dx^m} = \sum_{n=0}^{\infty} b_n^{(m)} T_n(x), \quad m = 0, 1, \dots$$

Then, recursion relation $c_{n-1} b_{n-1}^{(m)} = 2nb_n^{(m-1)} + b_{n+1}^{(m)}$, $n \geq 1$ gives

$$c_n b_n^{(1)} = 2 \sum_{\substack{p=n+1 \\ p+n \text{ odd}}}^{\infty} p b_p, \quad n \geq 0, \quad (9)$$

$$c_n b_n^{(2)} = 2 \sum_{\substack{p=n+2 \\ p+n \text{ even}}}^{\infty} p[p^2 - n^2] b_p, \quad n \geq 0,$$

where c_n is defined by (5). The formulae given above are utilized to expand products of Chebyshev polynomials and derivatives of Chebyshev polynomials as expansions in Chebyshev polynomials. For instance, if a function $u(x)$ and its first and second derivatives $u'(x)$ and $u''(x)$ have series expansions in terms of Chebyshev polynomials

$$u(x) = \sum_{n=0}^N b_n T_n(x), \quad u'(x) = \sum_{n=0}^{N-1} b_n^{(1)} T_n(x), \quad (10)$$

$$u''(x) = \sum_{n=0}^{N-2} b_n^{(2)} T_n(x),$$

then the coefficients $b_n^{(1)}$ and $b_n^{(2)}$ are related to the coefficients b_n , by

$$c_n b_n^{(1)} = 2 \sum_{\substack{p=N \\ p=n+1 \\ p+n \text{ odd}}}^{\infty} p b_p, \quad (11)$$

$$c_n b_n^{(2)} = 2 \sum_{\substack{p=N \\ p=n+2 \\ p+n \text{ even}}}^{\infty} p[p^2 - n^2] b_p, \quad n \geq 0,$$

where c_n is defined by (5).

The tau approach was first suggested by Lanczos [3, 31], and its use with Chebyshev polynomials was later developed widely by Fox [32] and was applied by Orszag for an extensive variety of problems [33–35]. Consider the following linear two-point boundary value problem

$$(LP) \begin{cases} Lu = f, x \in (-1, 1), \\ Bu = 0, \end{cases} \quad (12)$$

where L is a linear differential operator acting in a Hilbert space X and B stands for a set of linear differential operators defined on -1 and 1 . The Chebyshev tau method is characterized by the following choice:

$$X = L^2_\omega(-1, 1), \quad X_N = \{v \in \mathcal{P}_N | Bv = 0\}, \quad Y_N = \mathcal{P}_{N-\beta}, \quad (13)$$

where X_N denotes the space of trial functions and Y_N is that of test functions. Moreover, $\mathcal{P}_N$ is the space of algebraic polynomials of degree at most $N \in \mathbb{N}$, $N > 0$ and β stands for the number of boundary conditions. Accepting the family $\{T_n, n = 0, 1, 2, \dots, N\}$ as a basis for the finite dimensional space X_N and the family $\{T_m, m = 0, 1, 2, \dots, N - \beta\}$ as a set of test functions in Y_N , the variational formulation corresponding to (LP) is:

$$(CT) \begin{cases} \text{find the coefficients } b_n \text{ of } u_N(x) \in \mathcal{P}_N, \\ u_N(x) = \sum_{n=0}^N b_n T_n(x), \text{ such that} \\ \int_{-1}^1 (Lu_N(x) - f(x)) \psi_m dx = 0, \quad m = 0, 1, 2, \dots, N - \beta, \\ \text{and } \sum_{n=0}^N b_n B(T_n) = 0, \end{cases} \quad (14)$$

Fig. 1 Schematic diagram of an elastically restrained sandwich beam under arbitrary non-uniform normal and shear loads

where

$$\psi_m(x) = \frac{2}{\pi c_m} T_m(x) \omega(x) = \frac{2}{\pi c_m} T_m(x) \frac{1}{\sqrt{1-x^2}}. \quad (15)$$

Governing equations of the FG micro/nano-sandwich beams

A three-layer sandwich beam with regard to non-uniform normal and shear loads is investigated, as shown in Fig. 1. The thickness of the top and bottom face sheets and core are denoted, respectively, by h_t , h_b and h_c . The general boundary conditions are simulated by employing the in-plane and transverse translational springs at the mid-plane of the core and the rotational springs at the mid-plane of each layer. By setting the spring coefficients equal to infinite or zero, the classical boundary conditions can be modeled as some particular cases of the elastic supports. Based on the theory of layerwise [36–38], the displacement field of the layers can be expressed as:

$$\begin{cases} u_t = u_0 + \left(z - \frac{h_c}{2}\right) \varphi_x^{(t)} + \frac{h_c}{2} \varphi_x^{(c)}, & \frac{h_c}{2} \leq z \leq \frac{h_c}{2} + h_t, \\ u_c = u_0 + z \varphi_x^{(c)}, & -\frac{h_c}{2} \leq z \leq \frac{h_c}{2}, \\ u_b = u_0 + \left(z + \frac{h_c}{2}\right) \varphi_x^{(b)} - \frac{h_c}{2} \varphi_x^{(c)}, & -h_b - \frac{h_c}{2} \leq z \leq -\frac{h_c}{2}, \\ w = w_0, \end{cases} \quad (16)$$

where u_t , u_c and u_b are the in-plane displacement of the top, core and bottom layers, $\varphi_x^{(t)}$, $\varphi_x^{(c)}$ and $\varphi_x^{(b)}$ are the rotation of the section of the top, core and bottom layers, and w is the

transverse displacement of sandwich beam. For small deflections of beam, strains of each layer are given as:

$$\varepsilon_x^{(i)} = \frac{\partial u_i}{\partial x}, \quad \gamma_{xz}^{(i)} = \frac{\partial u_i}{\partial z} + \frac{\partial w_i}{\partial x}, \quad i = t, c, b. \quad (17)$$

Constitutive model that expresses the nonlocal stress tensor is as follow:

$$\begin{aligned} (1 - \mu \nabla^2) \sigma_x^{(i)} &= \frac{E_i}{1 - \nu_i^2} \varepsilon_x^{(i)}, \\ (1 - \mu \nabla^2) \tau_{xz}^{(i)} &= \frac{E_i}{2(1 + \nu_i)} \gamma_{xz}^{(i)} \quad i = t, c, b. \end{aligned} \quad (18)$$

Based on the minimum total potential energy principle, the governing equations of the elastically restrained sandwich beams may be derived as:

$$\delta \Pi = \delta U - \delta W = 0, \quad (19)$$

where W is the work of the externally applied loads and U is the potential energy.

$$\begin{aligned} \delta U &= \int_A \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_t} (\sigma_x^{(t)} \delta \varepsilon_x^{(t)} + \tau_{xz}^{(t)} \delta \gamma_{xz}^{(t)}) dz dA \\ &+ \int_A \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} (\sigma_x^{(c)} \delta \varepsilon_x^{(c)} + \tau_{xz}^{(c)} \delta \gamma_{xz}^{(c)}) dz dA \\ &+ \int_A \int_{-\frac{h_c}{2} - h_b}^{-\frac{h_c}{2}} (\sigma_x^{(b)} \delta \varepsilon_x^{(b)} + \tau_{xz}^{(b)} \delta \gamma_{xz}^{(b)}) dz dA, \end{aligned} \quad (20)$$

$$\begin{aligned} \delta W &= \int_A \left(T \delta u_t \Big|_{z=\frac{h_c}{2} + h_t} + q \delta w \right) dA + V_t \delta w \Big|_{x=r} + V_c \delta w \Big|_{x=s} \\ &+ K_u^{(r)} \delta u_0 \Big|_{x=r} + K_u^{(s)} \delta u_0 \Big|_{x=s} + K_w^{(r)} \delta w \Big|_{x=r} + K_w^{(s)} \delta w \Big|_{x=s} \\ &+ K_\varphi^{(r)} (\delta \varphi^{(t)} + \delta \varphi^{(c)} + \delta \varphi^{(b)}) \Big|_{x=r} \\ &+ K_\varphi^{(s)} (\delta \varphi^{(t)} + \delta \varphi^{(c)} + \delta \varphi^{(b)}) \Big|_{x=s}, \end{aligned} \quad (21)$$

where $K_u^{(i)}$, $K_\varphi^{(i)}$ and $K_w^{(i)}$, $i = r, s$, are the stiffness parameters of the in-plane translational, rotational and transverse translational springs. $q(x)$ and $T(x)$ are the arbitrary distributed transverse and shear loads of the top surface of the sandwich beam, respectively. Substituting Eqs. (20) and (21) into Eq. (19) and using Eqs. (17) and (18), by some manipulation the governing differential equations of the FG sandwich micro/nano-beam may be expressed as:

$$\begin{aligned} &(A^{(t)} + A^{(c)} + A^{(b)}) u_{0,xx} + \left(B^{(t)} - \frac{h_2}{2} A^{(t)} \right) \varphi_{x,xx}^{(t)} \\ &+ \left(\frac{h_c}{2} A^{(t)} + B^{(c)} - \frac{h_c}{2} A^{(b)} \right) \varphi_{x,xx}^{(c)} \\ &+ \left(B^{(b)} + \frac{h_c}{2} A^{(b)} \right) \varphi_{x,xx}^{(b)} = -(1 - \mu \nabla^2) T, \\ &\left(B^{(t)} - \frac{h_c}{2} A^{(t)} \right) u_{0,xx} + \left(D^{(t)} - h_c B^{(t)} + \frac{h_c^2}{4} A^{(t)} \right) \varphi_{x,xx}^{(t)} \\ &+ \left(\frac{h_c}{2} B^{(t)} - \frac{h_c^2}{4} A^{(t)} \right) \varphi_{x,xx}^{(c)} - \bar{A}^{(t)} (\varphi_x^{(t)} + w_{,x}) \\ &= -h_t (1 - \mu \nabla^2) T, \\ &\left(\frac{h_c}{2} A^{(t)} + B^{(c)} - \frac{h_c}{2} A^{(b)} \right) u_{0,xx} + \frac{h_c}{2} \left(B^{(t)} - \frac{h_c}{2} A^{(t)} \right) \varphi_{x,xx}^{(t)} \\ &+ \left(\frac{h_c^2}{4} A^{(t)} + D^{(c)} + \frac{h_c^2}{4} A^{(b)} \right) \varphi_{x,xx}^{(c)} - \frac{h_c}{2} \left(B^{(b)} + \frac{h_c}{2} A^{(b)} \right) \varphi_{x,xx}^{(b)} \\ &- \bar{A}^{(c)} (\varphi_x^{(c)} + w_{,x}) = -\frac{h_c}{2} (1 - \mu \nabla^2) T, \\ &\left(B^{(b)} + \frac{h_c}{2} A^{(b)} \right) u_{0,xx} - \frac{h_c}{2} \left(B^{(b)} + \frac{h_c}{2} A^{(b)} \right) \varphi_{x,xx}^{(c)} \\ &+ \left(\frac{h_c^2}{4} A^{(b)} + h_c B^{(b)} + D^{(b)} \right) \varphi_{x,xx}^{(b)} - \bar{A}^{(b)} (\varphi_x^{(b)} + w_{,x}) = 0, \\ &\bar{A}^{(t)} \varphi_{x,x}^{(t)} + \bar{A}^{(c)} \varphi_{x,x}^{(c)} + \bar{A}^{(b)} \varphi_{x,x}^{(b)} + (\bar{A}^{(t)} + \bar{A}^{(c)} + \bar{A}^{(b)}) \\ &w_{,xx} = (1 - \mu \nabla^2) q, \end{aligned} \quad (22)$$

where $x \in [r, s] = [-1, 1]$ and $\mu = (e_0 a)^2$ are the nonlocal parameter which incorporates the small-scale effect, a is the internal characteristic length and e_0 is a material constant. $A^{(i)}$, $B^{(i)}$, $D^{(i)}$ and $\bar{A}^{(i)}$ for $i = t, c, b$ are defined as:

$$\begin{Bmatrix} A^{(t)} \\ B^{(t)} \\ D^{(t)} \end{Bmatrix} = \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_t} \frac{E_t}{1 - \nu_t^2} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} dz, \quad (23)$$

$$\begin{Bmatrix} A^{(c)} \\ B^{(c)} \\ D^{(c)} \end{Bmatrix} = \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \frac{E_c}{1 - \nu_c^2} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} dz, \quad (24)$$

$$\begin{Bmatrix} A^{(b)} \\ B^{(b)} \\ D^{(b)} \end{Bmatrix} = \int_{-\frac{h_c}{2} - h_b}^{-\frac{h_c}{2}} \frac{E_b}{1 - \nu_b^2} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} dz, \quad (25)$$

$$\bar{A}^{(t)} = \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_t} \frac{E_t}{2(1 + \nu_t)} dz, \quad (26)$$

$$\bar{A}^{(c)} = \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \frac{E_c}{2(1+\nu_c)} dz, \quad (27)$$

$$\bar{A}^{(b)} = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} \frac{E_b}{2(1+\nu_b)} dz, \quad (28)$$

where E_i and ν_i , $i = t, c, b$ denote Young's modulus and Poissons ratio of each layer, respectively.

The general boundary conditions can be written as:

$$\begin{cases} N_x^{(t)} + N_x^{(c)} + N_x^{(b)} + K_u^{(i)} u_0 = 0, \\ M_x^{(t)} - \frac{h_c}{2} N_x^{(t)} + K_\varphi^{(i)} \varphi_x^{(t)} = 0, \\ \frac{h_c}{2} N_x^{(t)} + M_x^{(c)} - \frac{h_c}{2} N_x^{(b)} + K_\varphi^{(i)} \varphi_x^{(c)} = 0, \\ M_x^{(b)} + \frac{h_c}{2} N_x^{(b)} + K_\varphi^{(i)} \varphi_x^{(b)} = 0, \\ Q_x^{(t)} + Q_x^{(c)} + Q_x^{(b)} + K_w^{(i)} w = V, \end{cases} \quad i = r, s, \quad (29)$$

where V is shear load at the free edge of beam and

$$\begin{cases} N_x^{(t)} = A^{(t)} u_{0,x} + \left(B^{(t)} - \frac{h_c}{2} A^{(t)} \right) \varphi_{x,x}^{(t)} + \frac{h_c}{2} A^{(t)} \varphi_{x,x}^{(c)}, \\ M_x^{(t)} = B^{(t)} u_{0,x} + \left(D^{(t)} - \frac{h_c}{2} B^{(t)} \right) \varphi_{x,x}^{(t)} + \frac{h_c}{2} B^{(t)} \varphi_{x,x}^{(c)}, \end{cases} \quad (30)$$

$$\begin{cases} N_x^{(c)} = A^{(c)} u_{0,x} + B^{(c)} \varphi_{x,x}^{(c)}, \\ M_x^{(c)} = B^{(c)} u_{0,x} + D^{(c)} \varphi_{x,x}^{(c)}, \end{cases} \quad (31)$$

$$\begin{cases} N_x^{(b)} = A^{(b)} u_{0,x} + \left(B^{(b)} + \frac{h_c}{2} A^{(b)} \right) \varphi_{x,x}^{(b)} - \frac{h_c}{2} A^{(b)} \varphi_{x,x}^{(c)}, \\ M_x^{(b)} = B^{(b)} u_{0,x} + \left(D^{(b)} + \frac{h_c}{2} B^{(b)} \right) \varphi_{x,x}^{(b)} - \frac{h_c}{2} B^{(b)} \varphi_{x,x}^{(c)}, \end{cases} \quad (32)$$

$$\begin{cases} Q_x^{(t)} = \bar{A}^{(t)} (\varphi_x^{(t)} + w_x), \\ Q_x^{(c)} = \bar{A}^{(c)} (\varphi_x^{(c)} + w_x), \\ Q_x^{(b)} = \bar{A}^{(b)} (\varphi_x^{(b)} + w_x). \end{cases} \quad (33)$$

Substituting

$$\{K_u^{(r)}, K_\varphi^{(r)}, K_w^{(r)}, K_u^{(s)}, K_\varphi^{(s)}, K_w^{(s)}\} = \{\infty, \infty, \infty, \infty, \infty, \infty\}, \quad (34)$$

$$\{K_u^{(r)}, K_\varphi^{(r)}, K_w^{(r)}, K_u^{(s)}, K_\varphi^{(s)}, K_w^{(s)}\} = \{\infty, \infty, \infty, 0, 0, 0\}, \quad (35)$$

$$\{K_u^{(r)}, K_\varphi^{(r)}, K_w^{(r)}, K_u^{(s)}, K_\varphi^{(s)}, K_w^{(s)}\} = \{\infty, \infty, \infty, 0, 0, \infty\}, \quad (36)$$

in (29), we obtain various edge conditions, respectively, as clamped–clamped edge (C–C):

$$\{u_0(\pm 1) = \varphi_x^{(t)}(\pm 1) = \varphi_x^{(c)}(\pm 1) = \varphi_x^{(b)}(\pm 1) = w(\pm 1) = 0. \quad (37)$$

clamped–free edge (C–F):

$$\begin{cases} u_0(-1) = \varphi_x^{(t)}(-1) = \varphi_x^{(c)}(-1) = \varphi_x^{(b)}(-1) = w(-1) = 0, \\ N_x^{(t)}(1) + N_x^{(c)}(1) + N_x^{(b)}(1) = 0, \\ M_x^{(t)}(1) - \frac{h_c}{2} N_x^{(t)}(1) = 0, \\ \frac{h_c}{2} N_x^{(t)}(1) + M_x^{(c)}(1) - \frac{h_c}{2} N_x^{(b)}(1) = 0, \\ M_x^{(b)}(1) + \frac{h_c}{2} N_x^{(b)}(1) = 0, \\ Q_x^{(t)}(1) + Q_x^{(c)}(1) + Q_x^{(b)}(1) = V. \end{cases} \quad (38)$$

clamped–simply supported edge (C–S):

$$\begin{cases} u_0(-1) = \varphi_x^{(t)}(-1) = \varphi_x^{(c)}(-1) = \varphi_x^{(b)}(-1) = w(-1) = 0, \\ N_x^{(t)}(1) + N_x^{(c)}(1) + N_x^{(b)}(1) = 0, \\ M_x^{(t)}(1) - \frac{h_c}{2} N_x^{(t)}(1) = 0, \\ \frac{h_c}{2} N_x^{(t)}(1) + M_x^{(c)}(1) - \frac{h_c}{2} N_x^{(b)}(1) = 0, \\ M_x^{(b)}(1) + \frac{h_c}{2} N_x^{(b)}(1) = 0, \\ w(1) = 0. \end{cases} \quad (39)$$

In order to solve the differential equation system (22) by the CTM, we expand the solution functions u_0 , w and $\varphi_x^{(i)}$, ($i = t, c, b$) as a finite series of basis functions $\{T_n(x)\}_{n=0}^N$ as given below

$$(u_0)_N(x) = \sum_{n=0}^N b_n T_n(x), \quad w_N(x) = \sum_{n=0}^N a_n T_n(x), \quad (40)$$

$$(\varphi_x^{(i)})_N(x) = \sum_{n=0}^N c_{jn}^{(i)} T_n(x), \quad i = t, c, b, \quad j = 1, 2, 3. \quad (41)$$

We will need the first- and second-order derivatives of $(u_0)_N$, $(\varphi_x^{(t)})_N$, $(\varphi_x^{(c)})_N$, $(\varphi_x^{(b)})_N$ and w_N as they relate to $T_n(x)$. For this, Eq. (11) will be used.

$$\frac{d(u_0)_N}{dx} = \sum_{n=0}^{N-1} b_n^{(1)} T_n(x), \quad \frac{d^2(u_0)_N}{dx^2} = \sum_{n=0}^{N-2} b_n^{(2)} T_n(x), \quad (42)$$

$$\frac{d(\varphi_x^{(t)})_N}{dx} = \sum_{n=0}^{N-1} c_{1n}^{(1)} T_n(x), \quad \frac{d^2(\varphi_x^{(t)})_N}{dx^2} = \sum_{n=0}^{N-2} c_{1n}^{(2)} T_n(x), \quad (43)$$

$$\frac{d(\varphi_x^{(c)})_N}{dx} = \sum_{n=0}^{N-1} c_{2n}^{(1)} T_n(x), \quad \frac{d^2(\varphi_x^{(c)})_N}{dx^2} = \sum_{n=0}^{N-2} c_{2n}^{(2)} T_n(x), \quad (44)$$

$$\frac{d(\varphi_x^{(b)})_N}{dx} = \sum_{n=0}^{N-1} c_{3n}^{(1)} T_n(x), \quad \frac{d^2(\varphi_x^{(b)})_N}{dx^2} = \sum_{n=0}^{N-2} c_{3n}^{(2)} T_n(x), \quad (45)$$

$$\frac{dw_N}{dx} = \sum_{n=0}^{N-1} a_n^{(1)} T_n(x), \quad \frac{d^2 w_N}{dx^2} = \sum_{n=0}^{N-2} a_n^{(2)} T_n(x). \quad (46)$$

Let

$$\begin{aligned} T(x) &= t_0 + t_1x + t_2x^2 + t_3x^3, t_i \in \mathbb{R}, i = 0, 1, 2, 3, \\ q(x) &= q_0 + q_1x + q_2x^2 + q_3x^3, q_j \in \mathbb{R}, j = 0, 1, 2, 3. \end{aligned} \quad (47)$$

Then, we have

$$\begin{aligned} T(x) &= t_0 + \frac{1}{2}t_2T_0 + \left(\frac{3}{4}t_3 + t_1\right)T_1 + \frac{1}{2}t_2T_2 + \frac{1}{4}t_3T_3, \\ q(x) &= q_0 + \frac{1}{2}q_2T_0 + \left(\frac{3}{4}q_3 + q_1\right)T_1 + \frac{1}{2}q_2T_2 + \frac{1}{4}q_3T_3. \end{aligned} \quad (48)$$

inner product of the obtained system of equations with $\psi_m(x)$ where

$$\begin{aligned} (T_n(x), \psi_m(x)) &= \frac{2}{\pi c_m} \int_{-1}^1 T_n(x) T_m(x) \frac{1}{\sqrt{1-x^2}} dx = \delta_{nm}, \\ m &= 0, 1, \dots, N-2. \end{aligned} \quad (49)$$

Let $N = 3$. Using the recurrence relationships for the first and second derivative expansion coefficients from Eq. (11) and orthogonality properties of the Chebyshev polynomials, the resulted algebraic equation system leads to

$$\begin{aligned} &4(A^{(t)} + A^{(c)} + A^{(b)})b_2 + 4\left(B^{(t)} - \frac{h_c}{2}A^{(t)}\right)c_{12} + 4\left(\frac{h_c}{2}A^{(t)} + B^{(c)} - \frac{h_c}{2}A^{(b)}\right)c_{22} \\ &+ 4\left(B^{(b)} + \frac{h_c}{2}A^{(b)}\right)c_{32} + t_0 + \frac{t_2}{2} - 2t_2\mu = 0, (A^{(t)} + A^{(c)} + A^{(b)})b_3 \\ &+ 24\left(B^{(t)} - \frac{h_c}{2}A^{(t)}\right)c_{13} + 24\left(\frac{h_c}{2}A^{(t)} + B^{(c)} - \frac{h_c}{2}A^{(b)}\right)c_{23} \\ &+ 24\left(B^{(b)} + \frac{h_c}{2}A^{(b)}\right)c_{33} + t_1 + \frac{3}{4}t_3 - 6t_3\mu = 0, 4\left(B^{(t)} - \frac{h_c}{2}A^{(t)}\right)b_2 \\ &+ 4\left(D^{(t)} - h_cB^{(t)} + \frac{h_c^2}{4}A^{(t)}\right)c_{12} + 4\left(\frac{h_c}{2}B^{(t)} - \frac{h_c^2}{4}A^{(t)}\right)c_{22} \\ &- \bar{A}^{(t)}(c_{10} + a_1 + 3a_3) + t_0h_t + \frac{h_t}{2}t_2 - 2t_2h_t\mu = 0, \\ &24\left(B^{(t)} - \frac{h_c}{2}A^{(t)}\right)b_3 + 24\left(D^{(t)} - h_cB^{(t)} + \frac{h_c^2}{4}A^{(t)}\right)c_{13} + 24\left(\frac{h_c}{2}B^{(t)} - \frac{h_c^2}{4}A^{(t)}\right)c_{23} \\ &- \bar{A}^{(t)}(c_{11} + 4a_2) + t_1h_t + \frac{3}{4}t_3h_t - 6t_3h_t\mu = 0, 4\left(\frac{h_c}{2}A^{(t)} + B^{(c)} - \frac{h_c}{2}A^{(b)}\right)b_2 \\ &+ 2h_c\left(B^{(t)} - \frac{h_c}{2}A^{(t)}\right)c_{12} + 4\left(\frac{h_c^2}{4}A^{(t)} + D^{(c)} + \frac{h_c^2}{4}A^{(b)}\right)c_{22} \\ &- 2h_c\left(B^{(b)} + \frac{h_c}{2}A^{(b)}\right)c_{32} - \bar{A}^{(c)}(c_{20} + a_1 + 3a_3) + \frac{h_c}{2}t_0 + \frac{h_c}{4}t_2 - t_2h_c\mu = 0, \\ &24\left(\frac{1}{2}h_2A^{(1)} + B^{(2)} - \frac{1}{2}h_2A^{(3)}\right)b_3 + 12h_2\left(B^{(1)} - \frac{1}{2}h_2A^{(1)}\right)c_{13} \\ &+ 24\left(\frac{1}{4}h_2^2A^{(1)} + D^{(2)} + \frac{1}{4}h_2^2A^{(3)}\right)c_{23} - 12h_2\left(B^{(3)} + \frac{1}{2}h_2A^{(3)}\right)c_{33} \\ &- \bar{A}^{(2)}(c_{21} + 4a_2) + \frac{h_c}{2}t_1 + \frac{3}{8}t_3h_c - 3t_3h_c\mu = 0, 4\left(B^{(b)} + \frac{h_c}{2}A^{(b)}\right)b_2 \\ &- 2h_c\left(B^{(b)} + \frac{h_c}{2}A^{(b)}\right)c_{22} + 4\left(\frac{h_c^2}{4}A^{(b)} + D^{(b)} + h_cB^{(b)}\right)c_{32} \\ &- \bar{A}^{(b)}(c_{30} + a_1 + 3a_3) = 0, 24\left(B^{(b)} + \frac{h_c}{2}A^{(b)}\right)b_3 - 12h_c\left(B^{(b)} + \frac{h_c}{2}A^{(b)}\right)c_{23} \\ &+ 24\left(\frac{h_c^2}{4}A^{(b)} + D^{(b)} + h_cB^{(b)}\right)c_{33} - \bar{A}^{(b)}(c_{31} + 4a_2) = 0, \bar{A}^{(t)}(c_{11} + 3c_{13}) + \bar{A}^{(c)}(c_{21} + 3c_{23}) \\ &+ \bar{A}^{(b)}(c_{31} + 3c_{33}) + 4(\bar{A}^{(t)} + \bar{A}^{(c)} + \bar{A}^{(b)})a_2 - q_0 - \frac{q_2}{2} + 2q_2\mu = 0, 4\bar{A}^{(t)}c_{12} + 4\bar{A}^{(c)}c_{22} + 4\bar{A}^{(b)}c_{32} + 24(\bar{A}^{(t)} + \bar{A}^{(c)} + \bar{A}^{(b)})a_3 \\ &- q_1 - \frac{3}{4}q_3 + 6q_3\mu = 0. \end{aligned} \quad (50)$$

The substitution of the expansions of Eqs. (40), (41) and (48) into the system of differential equations (22) yields an algebraic equation system. The first step will be to take the

In the similar way, using the recurrence relationships for the first and second derivative expansion coefficients from Eq. (11) and the Chebyshev expansions of Eqs. (40) and (41),

the boundary conditions (37), (38) or (39) lead to the ten linear algebraic equations. For instance, (38) can be written as

Boundary conditions $C-F$:

$+380 \times 10^9$, $g_c = 1$ and $E_b = 380 \times 10^9$. Then, the solutions to (50) plus the ten boundary equations $C-C$, $C-F$ or $C-S$ are, respectively, as

$$\left\{ \begin{array}{l} b_0 - b_1 + b_2 - b_3 = 0, \\ A^{(t)}(b_1 + 9b_3 + 4b_2) \\ + \left(B^{(t)} - \frac{h_c}{2}A^{(t)}\right)(c1_1 + 9c1_3 + 4c1_2) + \frac{h_c}{2}A^{(t)}(c2_1 + 9c2_3 + 4c2_2) \\ + A^{(c)}(b_1 + 9b_3 + 4b_2) + B^{(c)}(c2_1 + 9c2_3 + 4c2_2) + A^{(b)}(b_1 + 9b_3 + 4b_2) \\ + \left(B^{(b)} + \frac{h_c}{2}A^{(b)}\right)(c3_1 + 9c3_3 + 4c3_2) - \frac{h_c}{2}A^{(b)}(c2_1 + 9c2_3 + 4c2_2) = 0, \\ c1_0 - c1_1 + c1_2 - c1_3 = 0, \\ B^{(t)}(b_1 + 9b_3 + 4b_2) \\ + \left(D^{(t)} - \frac{h_c}{2}B^{(t)}\right)(c1_1 + 9c1_3 + 4c1_2) + \frac{h_c}{2}B^{(t)}(c2_1 + 9c2_3 + 4c2_2) \\ - \frac{h_c}{2}A^{(t)}(b_1 + 9b_3 + 4b_2) - \frac{1}{2}h_c\left(B^{(1)} - \frac{1}{2}h_cA^{(1)}\right)(c1_1 + 4c1_2 + 9c1_3) \\ - \frac{h_c^2}{4}A^{(t)}(c2_1 + 4c2_2 + 9c2_3) = 0, \\ c2_0 - c2_1 + c2_2 - c2_3 = 0, \\ \frac{h_c}{2}A^{(t)}(b_1 + 9b_3 + 4b_2) \\ + \frac{h_c}{2}\left(B^{(t)} - \frac{h_c}{2}A^{(t)}\right)(c1_1 + 9c1_3 + 4c1_2) + \frac{h_c^2}{4}A^{(t)}(c2_1 + 9c2_3 + 4c2_2) \\ + B^{(c)}(b_1 + 9b_3 + 4b_2) + D^{(c)}(c2_1 + 4c2_2 + 9c2_3) - \frac{h_c}{2}A^{(b)}(b_1 + 4b_2 + 9b_3) \\ - \frac{h_c}{2}\left(B^{(b)} + \frac{h_c}{2}A^{(b)}\right)(c3_1 + 9c3_3 + 4c3_2) + \frac{h_c^2}{4}A^{(b)}(c2_1 + 9c2_3 + 4c2_2) = 0, \\ c3_0 - c3_1 + c3_2 - c3_3 = 0, \\ B^{(b)}(b_1 + 9b_3 + 4b_2) \\ + \left(D^{(b)} + \frac{h_c}{2}B^{(b)}\right)(c3_1 + 9c3_3 + 4c3_2) - \frac{h_c}{2}B^{(b)}(c2_1 + 9c2_3 + 4c2_2) \\ + \frac{h_c}{2}A^{(b)}(b_1 + 9b_3 + 4b_2) + \frac{h_c}{2}\left(B^{(b)} + \frac{h_c}{2}A^{(b)}\right)(c3_1 + 9c3_3 + 4c3_2) \\ - \frac{h_c^2}{4}A^{(b)}(c2_1 + 9c2_3 + 4c2_2) = 0, \\ a_0 - a_1 + a_2 - a_3 = 0, \\ \bar{A}^{(t)}(c1_0 + c1_1 + c1_2 + c1_3 + a_1 + 9a_3 + 4a_2) \\ + \bar{A}^{(c)}(c2_0 + c2_1 + c2_2 + c2_3 + a_1 + 9a_3 + 4a_2) \\ + \bar{A}^{(b)}(c3_0 + c3_1 + c3_2 + c3_3 + a_1 + 9a_3 + 4a_2) = V. \end{array} \right. \quad (51)$$

$$\text{Let } q = 10^7, T = 0, \mu = 0, V = 0, h_t = h_b = 0.1, h_c = 0.2, \quad \left\{ \begin{array}{l} a = (-3.20021 \times 10^{-5}, 0, 3.20021 \times 10^{-5}, 0), \\ b = (0, -4.83486 \times 10^{-6}, 0, 4.83486 \times 10^{-6}), \\ c1 = (0, -7.23547 \times 10^{-5}, 0, 7.23547 \times 10^{-5}), \\ c2 = (0, -6.57372 \times 10^{-5}, 0, 6.57372 \times 10^{-5}), \\ c3 = (0, -9.93407 \times 10^{-5}, 0, 9.93407 \times 10^{-5}), \end{array} \right. \quad (52)$$

$$\left\{ \begin{array}{l} a = (-4.91479 \times 10^{-3}, -6.13221 \times 10^{-3}, -1.01589 \times 10^{-3}, 2.01537 \times 10^{-4}), \\ b = (3.40334 \times 10^{-4}, 2.59792 \times 10^{-4}, -7.57436 \times 10^{-5}, 4.79798 \times 10^{-6}), \\ c1 = (5.37224 \times 10^{-3}, 4.11718 \times 10^{-3}, -1.18559 \times 10^{-3}, 6.94700 \times 10^{-5}), \\ c2 = (5.35739 \times 10^{-3}, 4.12423 \times 10^{-3}, -1.17098 \times 10^{-3}, 6.21869 \times 10^{-5}), \\ c3 = (5.45602 \times 10^{-3}, 4.09237 \times 10^{-3}, -1.25886 \times 10^{-3}, 1.04786 \times 10^{-4}), \end{array} \right. \quad (53)$$

Fig. 2 The values of w_{25} for the C–C sandwich beams regarded to the uniform load for various values of N

Fig. 3 The values of w_{25} for sandwich beams regarded to the uniform load for different values of $K_u^{(s)}$, $K_\varphi^{(s)}$ and $K_w^{(s)}$

Fig. 4 The values of w_{25} for sandwich beams regarded to the uniform load for $K_w^{(s)} = \infty (10^{100})$ and different values of $K_u^{(s)}$ and $K_\phi^{(s)}$

$$\begin{cases} a = (-7.44699 \times 10^{-5}, -1.95995 \times 10^{-5}, 7.44699 \times 10^{-5}, 1.95995 \times 10^{-5}), \\ b = (-3.29550 \times 10^{-6}, -1.54451 \times 10^{-5}, -7.22321 \times 10^{-6}, 4.92639 \times 10^{-6}), \\ c1 = (-5.35185 \times 10^{-5}, -2.38879 \times 10^{-4}, -1.09947 \times 10^{-4}, 7.54129 \times 10^{-5}), \\ c2 = (-5.50621 \times 10^{-5}, -2.32667 \times 10^{-4}, -1.05061 \times 10^{-4}, 7.25444 \times 10^{-5}), \\ c3 = (-4.61740 \times 10^{-5}, -2.69457 \times 10^{-4}, -1.33852 \times 10^{-4}, 8.94309 \times 10^{-5}). \end{cases} \quad (54)$$

Fig. 5 The values of w_{25} for sandwich beams regarded to the uniform shear load for $K_u^{(s)} = K_\phi^{(s)} = 0$ and different values of $K_w^{(s)}$

Table 1 w_N for clamped–clamped (C–C) end supports

xlN	5	6	10	25	26
–1	0	0	0	0	0
–0.8	–0.000118317	–0.000117006	–0.000115237	–0.000115240	–0.000115240
–0.6	–0.000303853	–0.000291497	–0.000287530	–0.000287644	–0.000287644
–0.4	–0.000486476	–0.000456357	–0.000454298	–0.000454177	–0.000454177
–0.2	–0.000616088	–0.000570078	–0.000571434	–0.000571412	–0.000571412
0	–0.000662632	–0.000610323	–0.000613187	–0.000613376	–0.000613376
0.2	–0.000616088	–0.000570078	–0.000571434	–0.000571412	–0.000571412
0.4	–0.000486476	–0.000456357	–0.000454298	–0.000454177	–0.000454177
0.6	–0.000303853	–0.000291497	–0.000287530	–0.000287644	–0.000287644
0.8	–0.000118317	–0.000117006	–0.000115237	–0.000115240	–0.000115240
1	0	0	0	0	0

Table 2 Comparison the CTM and FEM results for w_{25}

x	C–C CTM	C–C FEM	C–F CTM	C–F FEM	C–S CTM	C–S FEM
–1	0	0	0	0	0	0
–0.8	–0.000115240	–0.000117742	–0.00052494	–0.00051581	–0.000160906	–0.00016286
–0.6	–0.000287644	–0.000304835	–0.00174432	–0.00183505	–0.000425826	–0.000448896
–0.4	–0.000454177	–0.000486541	–0.00350617	–0.00375483	–0.000709959	–0.000757281
–0.2	–0.000571412	–0.000614643	–0.00568032	–0.00613143	–0.000948686	–0.00101680
0	–0.000613376	–0.000660515	–0.00815425	–0.00884097	–0.00109487	–0.00117595
0.2	–0.000571412	–0.000614643	–0.0108328	–0.0117789	–0.00111867	–0.00120239
0.4	–0.000454177	–0.000486541	–0.0136379	–0.0148595	–0.00100763	–0.00108277
0.6	–0.000287644	–0.000304835	–0.0165092	–0.0180165	–0.000766492	–0.000822802
0.8	–0.000115240	–0.000117742	–0.0194034	–0.0212022	–0.000417422	–0.000447286
1	0	0	–0.0222944	–0.0243880	0	0

For $V = -0.4 \times 10^7$ and $q = 0$, the solution to (50) plus the ten boundary equations C–F (51) is:

$$\begin{cases} a = (-4.77632 \times 10^{-3}, -6.03179 \times 10^{-3}, -1.07594 \times 10^{-3}, 1.79532 \times 10^{-4}), \\ b = (3.39086 \times 10^{-4}, 2.71598 \times 10^{-4}, -6.76144 \times 10^{-5}, -1.26712 \times 10^{-7}), \\ c1 = (5.35402 \times 10^{-3}, 4.29846 \times 10^{-3}, -1.06142 \times 10^{-3}, -5.86433 \times 10^{-6}), \\ c2 = (.534089 \times 10^{-3}, 4.29929 \times 10^{-3}, -1.05182 \times 10^{-3}, -1.02206 \times 10^{-5}), \\ c3 = (5.42944 \times 10^{-3}, 4.30415 \times 10^{-3}, -1.11013 \times 10^{-3}, 1.51518 \times 10^{-5}). \end{cases} \quad (55)$$

The substitution of (52)–(55) into the (40) and (41), respectively, leads to

$$\begin{cases} (u_0)_3(x) = (-1.93394 \times 10^{-5})x + (1.93394 \times 10^{-5})x^3, \\ (\varphi_x^{(i)})_3(x) = (-2.89418 \times 10^{-4})x + (2.89418 \times 10^{-4})x^3, \\ (\varphi_x^{(c)})_3(x) = (-2.62948 \times 10^{-4})x + (2.62948 \times 10^{-4})x^3, \\ (\varphi_x^{(b)})_3(x) = (-3.97362 \times 10^{-4})x + (3.97362 \times 10^{-4})x^3, \\ w_3(x) = (-6.40042 \times 10^{-5}) + (6.40042 \times 10^{-5})x^2, \end{cases} \quad (56)$$

$$\begin{cases} (u_0)_3(x) = (4.16078 \times 10^{-4}) + (2.45398 \times 10^{-4})x \\ \quad - (1.51487 \times 10^{-4})x^2 + (1.91919 \times 10^{-5})x^3, \\ (\varphi_x^{(i)})_3(x) = (6.55783 \times 10^{-3}) + (3.90877 \times 10^{-3})x \\ \quad - (2.37118 \times 10^{-3})x^2 + (2.77880 \times 10^{-4})x^3, \\ (\varphi_x^{(c)})_3(x) = (6.52837 \times 10^{-3}) + (3.93767 \times 10^{-3})x \\ \quad - (2.34196 \times 10^{-3})x^2 + (2.48748 \times 10^{-4})x^3, \\ (\varphi_x^{(b)})_3(x) = (6.71488 \times 10^{-3}) + (3.77801 \times 10^{-3})x \\ \quad - (2.51772 \times 10^{-3})x^2 + (4.19144 \times 10^{-4})x^3, \\ w_3(x) = (-3.89890 \times 10^{-3}) - (6.73682 \times 10^{-3})x \\ \quad - (2.03178 \times 10^{-3})x^2 + (8.06148 \times 10^{-4})x^3, \end{cases} \quad (57)$$

Fig. 6 The values of w_{25} for sandwich beams regarded to the uniform load for $K_w^{(s)} = 10^9$ and different values of $K_u^{(s)}$ and $K_\phi^{(s)}$

Fig. 7 The values of w_{25} for sandwich beams regarded to the uniform load for different values of $K_u^{(s)}$, $K_\phi^{(s)}$ and $K_w^{(s)}$

$$\left\{ \begin{aligned} (u_0)_3(x) &= (3.92771 \times 10^{-6}) - (3.02243 \times 10^{-5})x \\ &\quad - (1.44464 \times 10^{-5})x^2 + (1.97056 \times 10^{-5})x^3 \\ (\varphi_x^{(i)})_3(x) &= (5.64285 \times 10^{-5}) - (4.65118 \times 10^{-4})x \\ &\quad - (2.19894 \times 10^{-4})x^2 + (3.01652 \times 10^{-4})x^3, \\ (\varphi_x^{(c)})_3(x) &= (4.99989 \times 10^{-5}) - (4.50300 \times 10^{-4})x \\ &\quad - (2.10122 \times 10^{-4})x^2 + (2.90178 \times 10^{-4})x^3, \\ (\varphi_x^{(b)})_3(x) &= (8.76780 \times 10^{-5}) - (5.37750 \times 10^{-4})x \\ &\quad - (2.67704 \times 10^{-4})x^2 + (3.57724 \times 10^{-4})x^3, \\ w_3(x) &= (-1.48940 \times 10^{-4}) - (7.83980 \times 10^{-5})x \\ &\quad + (1.48940 \times 10^{-4})x^2 + (7.783980 \times 10^{-5})x^3. \end{aligned} \right. \quad (58)$$

$$\left\{ \begin{aligned} (u_0)_3(x) &= (4.06700 \times 10^{-4}) + (2.71978 \times 10^{-4})x \\ &\quad - (1.35229 \times 10^{-4})x^2 - (5.06848 \times 10^{-7})x^3 \\ (\varphi_x^{(i)})_3(x) &= (6.41544 \times 10^{-3}) + (4.31605 \times 10^{-3})x \\ &\quad - (2.12284 \times 10^{-3})x^2 - (2.34574 \times 10^{-5})x^3, \\ (\varphi_x^{(c)})_3(x) &= (6.39271 \times 10^{-3}) + (4.32995 \times 10^{-3})x \\ &\quad - (2.10364 \times 10^{-3})x^2 - (4.08824 \times 10^{-5})x^3, \\ (\varphi_x^{(b)})_3(x) &= (6.53957 \times 10^{-3}) + (4.25870 \times 10^{-3})x \\ &\quad - (2.22026 \times 10^{-3})x^2 + (6.06072 \times 10^{-5})x^3, \\ w_3(x) &= (-3.70038 \times 10^{-3}) - (6.57038 \times 10^{-3})x \\ &\quad - (2.15188 \times 10^{-3})x^2 + (7.18128 \times 10^{-4})x^3. \end{aligned} \right. \quad (59)$$

Fig. 8 Variations of the in-plane displacement regarded to the non-uniformly load for different values of $K_u^{(s)}$, $K_\varphi^{(s)}$ and $K_w^{(s)}$, $x = 0.2$

Results and comparisons

In the current section, different examples of FG micro/nano-sandwich beams are investigated. To demonstrate the accuracy and reliability of the CTM, the obtained results of FG sandwich beam in especial cases (with classical edge conditions and $\mu = 0$) are compared with the extracted results of ABAQUS software by using the finite element method. It can be mentioned that the extracted results from the ABAQUS software are obtained based on the 3D elasticity theory and the obtained results in this study are presented based on the theory of layerwise, so the difference between the results is expected. In all examples, it is assumed that $h_t = h_b = 0.1$, $h_c = 0.2$, $\nu_t = \nu_c = \nu_b = 0.3$, $E_t = 70 \times 10^9$, $E_c = -310 \times 10^9 \left(\frac{z}{h_c} + \frac{1}{2} \right)_{t_{g_c}}^{g_c} + 380 \times 10^9$, $g_c = 1$, $E_b = 380 \times 10^9$.

Example 1 In this example, results of FG sandwich beams subjected to $\mu = 0$, $V = 0$, $q = 10^7$ and $T = 0$ are obtained. Table 1 and Fig. 2 show the values of w_N for clamped-clamped edge (C-C) and different amounts of N . It is observed that increasing the number of polynomial items N improves the accuracy of results and leads to convergent solutions at $N = 25$. Hence, $N = 25$ is utilized in our numerical calculations. In Fig. 3, the values of in-plane translational, rotation and transverse translational stiffness parameters are supposed to be equal ($K_u^{(s)} = K_\varphi^{(s)} = K_w^{(s)}$). Nine different

Fig. 9 Variations of the in-plane displacement regarded to the non-uniformly load for different values of $K_u^{(s)}$, $K_\varphi^{(s)}$ and $K_w^{(s)}$, $x = 0.9$

Fig. 10 Variations of the in-plane displacement regarded to the non-uniformly load for $K_w^{(s)} = \infty (10^{100})$ and different values of $K_u^{(s)}$ and $K_\varphi^{(s)}$, $x = 0.2$

values, $K_u^{(s)} = K_\varphi^{(s)} = K_w^{(s)} = 0$, $K_u^{(s)} = K_\varphi^{(s)} = K_w^{(s)} = \infty (10^{100})$, $K_u^{(s)} = K_\varphi^{(s)} = K_w^{(s)} = 10^7, 5 \times 10^7, 10^8, 5 \times 10^8$,

$K_u^{(s)} = K_\varphi^{(s)} = K_w^{(s)} = 10^7, 5 \times 10^7, 10^8$, are considered and variations of w when the boundary condition tends from the

Table 3 Comparison the CTM and FEM results regarded to $C-F$ edge for w_{25}

x	CTM	FEM
-1	0	0
-0.8	-0.00019272	-0.00018583
-0.6	-0.00069808	-0.00072984
-0.4	-0.00148046	-0.00157826
-0.2	-0.00250520	-0.00269294
0	-0.00373766	-0.00403576
0.2	-0.00514322	-0.00556869
0.4	-0.00668726	-0.00725371
0.6	-0.00833516	-0.00905274
0.8	-0.0100523	-0.0109277
1	-0.0118040	-0.0128451

$C-F$ to the $C-C$ edge are demonstrated. It can be observed from Fig. 3 that increasing $K_u^{(s)}$, $K_\phi^{(s)}$ and $K_w^{(s)}$ from zero to infinity decreases the lateral deflections. In Fig. 4, the values of in-plane translational and rotation stiffness parameters are supposed to be equal ($K_u^{(s)} = K_\phi^{(s)}$) and transverse translational stiffness parameter tends infinity $K_w^{(s)} = \infty (10^{100})$. Seven different values $K_u^{(s)} = K_\phi^{(s)} = 0, K_u^{(s)} = K_\phi^{(s)} = \infty (10^{100}), K_u^{(s)} = K_\phi^{(s)} = 10^8, 5 \times 10^8, 10^9, 5 \times 10^9, 10^{10}$ are considered and variations of w when the boundary condition tends from the $C-S$ to the $C-C$ edge are shown. It is seen from Fig. 4 that

increasing $K_u^{(s)}$ and $K_\phi^{(s)}$ from zero to infinity reduces the lateral deflections. In Fig. 5, the values of in-plane translational and rotation stiffness parameters are assumed to be ($K_u^{(s)} = K_\phi^{(s)} = 0$). Seven different values $K_w^{(s)} = 0, K_w^{(s)} = \infty (10^{100}), K_w^{(s)} = 5 \times 10^7, 10^8, 2 \times 10^8, 5 \times 10^8, 10^9$ are considered and variations of w when the boundary condition tends from the $C-F$ to the $C-S$ edge are depicted. As it can be observed from Fig. 5, increasing $K_w^{(s)}$ from zero to infinity decreases the lateral deflections. In Fig. 6, the values of in-plane and rotation stiffness parameters are supposed to be equal ($K_u^{(s)} = K_\phi^{(s)}$) and transverse translational stiffness parameter is $K_w^{(s)} = 10^9$. Six different values $K_u^{(s)} = K_\phi^{(s)} = 0, K_u^{(s)} = K_\phi^{(s)} = \infty (10^{100}), K_u^{(s)} = K_\phi^{(s)} = 10^8, 5 \times 10^8, 10^9, 2 \times 10^9$ are considered, and variations of w are depicted. It may be observed from Fig. 6 that when $K_u^{(s)}$ and $K_\phi^{(s)}$ increase from zero to infinity, the lateral deflections decrease. In Table 2, the lateral deflections at the $C-C$, $C-F$ and $C-S$ edges of the mentioned sandwich beams based on the CTM are compared with those extracted from the ABAQUS software based on the 3D theory of elasticity. Although the difference between the theories of Chebyshev tau method and ABAQUS software may causes some differences between the results of CTM and FEM, it can be observed from Table 2, there are good agreement between present results and results of the FEM for various edge conditions $C-C$, $C-F$ and $C-S$.

Fig. 11 Variations of the in-plane displacement regarded to the non-uniformly load for $K_w^{(s)} = \infty (10^{100})$ and different values of $K_u^{(s)}$ and $K_\phi^{(s)}$, $x = 0.5$

Fig. 12 Variations of the in-plane displacement regarded to the non-uniformly load for $K_w^{(s)} = \infty (10^{100})$ and different values of $K_u^{(s)}$ and $K_\phi^{(s)}$, $x = 0.9$

Fig. 13 Variations of the in-plane displacement regarded to the non-uniformly load for $K_u^{(s)} = K_\phi^{(s)} = 0$ and different values of $K_w^{(s)}$, $x = 0.2$

Fig. 14 Variations of the in-plane displacement regarded to the non-uniformly load for $K_u^{(s)} = K_\varphi^{(s)} = 0$ and different values of $K_w^{(s)}$, $x = 0.9$

Fig. 15 The values of w_{25} for C-C sandwich beams regarded to the non-uniformly load for different values of μ

Example 2 In this example, results of FG sandwich beams subjected to $\mu = 0$, $V = -0.4 \times 10^7$, $q = 0$ and $T = 0$ are obtained. In Fig. 7, the values of in-plane translational,

rotation and transverse translational stiffness parameters are supposed to be equal ($K_u^{(s)} = K_\varphi^{(s)} = K_w^{(s)}$). Nine different values $K_u^{(s)} = K_\varphi^{(s)} = K_w^{(s)} = 0$, $K_u^{(s)} = K_\varphi^{(s)} = K_w^{(s)} = \infty (10^{100})$,

Fig. 16 The values of w_{25} for $C-F$ sandwich beams regarded to the non-uniformly load for different values of μ

Fig. 17 The values of w_{25} for $C-S$ sandwich beams regarded to the non-uniformly load for different values of μ

$K_u^{(s)} = K_\phi^{(s)} = K_w^{(s)} = 10^7, 5 \times 10^7, 10^8, 5 \times 10^8, 10^9, 5 \times 10^9, 10^{10}$ are considered and variations of w when the boundary condition tends from the $C-F$ to the $C-F$ edge are shown. It is seen from Fig. 7 that increasing $K_u^{(s)}$, $K_\phi^{(s)}$ and $K_w^{(s)}$ from zero to infinity reduces the lateral deflections. In Table 3, the lateral deflections at the $C-F$ edge of the mentioned

sandwich beams based on the CTM are compared with those extracted from the ABAQUS software based on the 3D-elasticity. It is seen from Table 3, there are good agreement between present results and results of the FEM for $C-F$ boundary condition.

Example 3 For analysis of sandwich beams regarded to $\mu = 0$, $V = 0$ and the non-uniformly normal load $q = T = (1 + x + 2x^2 - 3x^3)10^7$:

1. The values of in-plane, rotation and transverse translational stiffness parameters are assumed to be equal ($K_u^{(s)} = K_\phi^{(s)} = K_w^{(s)}$). Five different values $K_u^{(s)} = K_\phi^{(s)} = K_w^{(s)} = 0$, $K_u^{(s)} = K_\phi^{(s)} = K_w^{(s)} = \infty(10^{100})$, $K_u^{(s)} = K_\phi^{(s)} = K_w^{(s)} = 10^8, 10^9, 10^{10}$ are considered, and variations of the in-plane displacement when the boundary condition tends from the $C-F$ to the $C-C$ edge are plotted at $x = 0.2$ and $x = 0.9$ in Figs. 8 and 9, respectively. It may be seen from Figs. 8 and 9, increasing $K_u^{(s)}$, $K_\phi^{(s)}$ and $K_w^{(s)}$ from zero to infinity reduces the in-plane displacements.
2. The values of in-plane and rotation stiffness parameters are supposed to be equal ($K_u^{(s)} = K_\phi^{(s)}$), and transverse translational stiffness parameter tends infinity $K_w^{(s)} = \infty(10^{100})$. Six different values $K_u^{(s)} = K_\phi^{(s)} = 0$, $K_u^{(s)} = K_\phi^{(s)} = \infty(10^{100})$, $K_u^{(s)} = K_\phi^{(s)} = 10^9, 10^{10}, 10^{11}, 10^{12}$ are considered, and variations of the in-plane displacement when the boundary condition tends from the $C-S$ to the $C-C$ edge are shown at $x = 0.2$, $x = 0.5$ and $x = 0.9$ in Figs. 10, 11 and 12, respectively.
3. The values of in-plane translational and rotation stiffness parameters are supposed to be ($K_u^{(s)} = K_\phi^{(s)} = 0$). Five different values $K_w^{(s)} = 0$, $K_w^{(s)} = \infty(10^{100})$, $K_w^{(s)} = 10^8, 10^9, 10^{10}$ are considered, and variations of the in-plane displacement when the boundary condition tends from the $C-F$ to the $C-S$ edge are depicted at $x = 0.2$ and $x = 0.9$ in Figs. 13 and 14, respectively. It is seen from Figs. 13 and 14, increasing $K_w^{(s)}$ from zero to infinity reduces the in-plane displacements.

Example 4 For analysis of FG micro/nano-sandwich beams subjected to $V = 0$, the non-uniformly normal load $q = (1 + x + 2x^2 - 3x^3)10^7$ and $T = (1 + x + 2x^2 - 3x^3)10^8$ and six different values, $\mu = 0, 0.1, 0.2, 0.3, 0.4, 0.5$ are considered and variations of w when the boundary conditions are the $C-C$, $C-F$ and $C-S$ edges, are shown in Figs. 15, 16 and 17, respectively. When μ increases from zero to 0.5, the lateral deflections of $C-C$ and $C-S$ sandwich beams decrease first and then increase in the positive direction of the coordinate axis, as it is seen from Figs. 15 and 17. Also the lateral deflections of $C-F$ sandwich beams increase as shown in Fig. 16.

Conclusion

In this paper, Chebyshev tau method is employed for solution of governing equations of functionally graded micro/nano-sandwich beams regarded to non-uniform normal and shear loads. The method leads to solving a system of linear algebraic equations. The obtained results reveal the accuracy, validity and applicability of the technique, even for sandwich beams under relatively complicated normal and shear loads. The comparisons reveal that there are good agreement between present results and results of the FEM for various edge conditions $C-C$, $C-F$ and $C-S$. In this work, MAPLE software is used to calculate the solutions obtained from the CT method.

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