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ORIGINAL RESEARCH

Exponential Jacobi spectral method for hyperbolic partial differential equations

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Abstract
Herein, we have proposed a scheme for numerically solving hyperbolic partial differential equations (HPDEs) with given initial conditions. The operational matrix of differentiation for exponential Jacobi functions was derived, and then a collocation method was used to transform the given HPDE into a linear system of equations. The preferences of using the exponential Jacobi spectral collocation method over other techniques were discussed. The convergence and error analyses were discussed in detail. The validity and accuracy of the proposed method are investigated and checked through numerical experiments.

Keywords First-order partial differential equations · Exponential Jacobi functions · Operational matrix of differentiation · Heisenberg matrix · Convergence analysis

Mathematics Subject Classification 35L02 · 65M70

Introduction

Hyperbolic partial differential equations (HPDEs) constitute an important subclass of partial differential equations. The HPDEs are used in many disciplines of science and engineering, such as studying the transmission and propagation of electrical signals [1], wave propagation [2], hypoelastic solids [3], astrophysics [4], process engineering [5], acoustic transmission [6] and random walk theory [7]. The HPDEs are used in shaping the vibrational motion of structures (e.g., beams, machines and buildings) and represent basis for fundamental equations of atomic physics [8, 9]. Recently, the study of exact and numerical solutions of either hyperbolic or parabolic PDEs has received increasing attention [10–15].

Spectral techniques have been successfully applied for approximating the solution of differential problems defined in unbounded domains. For problems with sufficient smooth analytic solutions, they exhibit exponential rates of convergence, high accuracy and low computational cost. Doha et al. [16] used a Jacobi rational spectral technique for solving Lane–Emden initial value problems, in astrophysics, on a semi-infinite interval. Hafez et al. [17] applied a new collocation scheme for solving hyperbolic equations of second order in a semi-infinite domain. Doha et al. [18] proposed a new spectral Jacobi rational-Gauss collocation method for solving the multi-pantograph delay differential equations on the half line. Bhrawy et al. [19] solved some higher order ordinary differential equations using a new exponential Jacobi pseudospectral method.

In this study, we used exponential Jacobi functions for numerically solving the HPDEs. The operational matrices of derivatives and products of exponential Jacobi functions were derived. These matrices were jointly implemented with the collocation approach to evaluate the solutions of the HPDEs. Collocation method [20–24] is an effective technique for numerically approximating different kinds of equations.

The workflow of this paper encompass: In the next section, we present some notations and other mathematical facts. "Operational matrix of differentiation for exponential Jacobi" section is devoted to the operational matrix of differentiation for exponential Jacobi functions. In "Implementation of the method" section, the operational matrix of differentiation for exponential Jacobi was used in a combination with the exponential Jacobi collocation method to solve the HPDEs. The

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error analysis was executed in "Error analysis" section. Two numerical examples are given in "Numerical results" section. Finally, some concluding remarks are mentioned in "Conclusion" section.

Mathematical preliminaries

Here, we list some useful mathematical relations and identities needed in the construction of the exponential Jacobi operational matrix.

Exponential Jacobi functions

Consider the standard classical Jacobi polynomials $J_k^{(\rho,\sigma)}(z)$ on the interval $[-1,1]$ with the weight function $\omega^{(\rho,\sigma)}(z)=(1-z)^{\rho}(1+z)^{\sigma}, \rho,\sigma>-1,$

$$J_0^{(\rho,\sigma)}(z)=1, \quad J_1^{(\rho,\sigma)}(z)=\frac{1}{2}(\rho-\sigma+z(\rho+\sigma+2)),$$

the set $\{J_k^{(\rho,\sigma)}(z) : k=0,1,\dots\}$ forms a complete orthogonal system in the weighted Hilbert space $L^2_{\omega^{\rho,\sigma}(x)}[-1,1]$ equipped with the inner product

$$(f,g)_{\omega^{(\rho,\sigma)}(x)} := \int_{-1}^{1} f(x)g(x)\omega^{(\rho,\sigma)}(x)dx,$$

and the norm

$$\|f\|_{\omega^{(\rho,\sigma)}(x)}=(f,f)^{\frac{1}{2}}_{\omega^{(\rho,\sigma)}(x)}.$$

Let us define the exponential Jacobi functions by replacing z by $1-2e^{-\frac{x}{L}}$. Denoting the exponential Jacobi functions $J_i^{(\rho,\sigma)}(1-2e^{-\frac{x}{L}})$ by $\Upsilon_i^{(\rho,\sigma)}(x)$, $x\in[0,\infty)$. Therefore, $\Upsilon_i^{(\rho,\sigma)}(x)$ may be generated by the following recurrence relation:

$$\begin{aligned}\Upsilon_{k+1}^{(\rho,\sigma)}(x)=&\frac{(2k+\rho+\sigma+1)(2k+\rho+\sigma+2)}{(k+1)(k+\rho+\sigma+1)}\\ &\left[\left(\frac{((\rho+1)(\rho+\sigma)+2k^2+2k(\rho+\sigma+1))}{(2k+\rho+\sigma)(2k+\rho+\sigma+2)}-e^{-\frac{x}{L}}\right)\Upsilon_k^{(\rho,\sigma)}(x)\right.\\ &\left.-\frac{(k+\rho)(k+\sigma)}{(2k+\rho+\sigma)(2k+\rho+\sigma+1)}\Upsilon_{k-1}^{(\rho,\sigma)}(x)\right], \quad k\geq 1,\end{aligned} \tag{1}$$

where

$$\Upsilon_0^{(\rho,\sigma)}(x)=1, \quad \Upsilon_1^{(\rho,\sigma)}(x)=(\rho+1)-(\rho+\sigma+2)e^{-\frac{x}{L}},$$

and

$$(k+\rho+\sigma)\Upsilon_i^{(\rho,\sigma)}(x)=(k+\sigma)\Upsilon_i^{(\rho,\sigma-1)}(x)+(k+\rho)\Upsilon_i^{(\rho-1,\sigma)}(x).$$

The exponential Jacobi functions $\Upsilon_i^{(\rho,\sigma)}(x)$ of degree i can be written as

$$\Upsilon_i^{(\rho,\sigma)}(x)=\sum_{k=0}^{i}(-1)^k\frac{\Gamma(i+\rho+1)\Gamma(i+k+\rho+\sigma+1)}{\Gamma(\rho+k+1)\Gamma(i+\rho+\sigma+1)(i-k)!k!}exp(-kx/L),$$

where

$$\Upsilon_i^{(\rho,\sigma)}(0)=\frac{(-1)^i\,\Gamma(\sigma+i+1)}{i!\,\Gamma(\sigma+1)}. \tag{2}$$

The set $\{\Upsilon_i^{(\rho,\sigma)}(x) : i=0,1,\dots\}$, satisfy the following orthogonality relation:

$$\int_0^{\infty}\Upsilon_i^{(\rho,\sigma)}(x)\,\Upsilon_j^{(\rho,\sigma)}(x)\,w^{(\rho,\sigma)}\,dx=h_i^{(\rho,\sigma)}\,\delta_{ij}, \tag{3}$$

where

$$w^{(\rho,\sigma)}=e^{-\frac{\rho+1}{L}x}(1-e^{-\frac{x}{L}})^{\sigma},$$
$$h_i^{(\rho,\sigma)}=\frac{L\,\Gamma(i+\rho+1)\,\Gamma(i+\sigma+1)}{i!\,(2i+\rho+\sigma+1)\,\Gamma(i+\rho+\sigma+1)},$$

and δ_{ij} is the well-known kronecker delta.

Function approximation

Now, approximation of $u(x)$ by $N+1$ terms of exponential Jacobi functions yields

$$u(x)\simeq\sum_{j=0}^{N}c_j\Upsilon_j^{(\rho,\sigma)}(x)=\mathbf{C}^T\phi(x), \tag{4}$$

where $\mathbf{C}$ and $\phi(x)$ are the unknown coefficients vector and the exponential Jacobi function vector, respectively, and are given by:

$$\mathbf{C}=[c_0,c_1,\dots,c_N]^T, \tag{5}$$

$$c_i=\frac{1}{h_i^{(\rho,\sigma)}}\int_0^{\infty}u(x)\,\Upsilon_i^{(\rho,\sigma)}(x)\,w^{(\rho,\sigma)}\,dx, \tag{6}$$

and

$$\phi(x)=[\Upsilon_0^{(\rho,\sigma)}(x),\Upsilon_1^{(\rho,\sigma)}(x),\dots,\Upsilon_N^{(\rho,\sigma)}(x)]^T. \tag{7}$$

Operational matrix of differentiation for exponential Jacobi

Here, we report the derivation of the operational matrix of derivatives of the exponential Jacobi functions, which is of important use to our numerical scheme.

Theorem 1 *Let $\phi(x)$ be the exponential Jacobi vector defined in (7). The derivative of the vector $\phi(x)$ can be expressed by*

$$\phi'(x) = \frac{d\phi(x)}{dx} \simeq \mathbf{D}\phi(x), \tag{8}$$

where $\mathbf{D}$ is $(N+1)\times(N+1)$ operational matrix of the derivative. Then, the nonzero elements $d_{k\ell}$ for $0 \le k, \ell \le N$ are given as follows:

$$d_{k+1,k} = \frac{(\rho+k+1)(\rho+\sigma+2k+1)}{L(\rho+\sigma+k+1)}, \quad d_{kk} = -\frac{k}{L},$$
$$d_{k\ell} = \frac{(-1)^{k+\ell+1}(2\ell+\rho+\sigma+1)}{L}\prod_{r=1}^{k-\ell}\frac{(\rho+k-r+1)}{(\rho+\sigma+k-r+1)}, \quad \ell < k-1.$$

It easily noted that $\mathbf{D}$ is a lower-Heisenberg matrix.

Proof See, Bhrawy et al. [19].

Studying the class of exponential Jacobi functions yields many special orthogonal functions as a direct special cases, and these cases are reported in the following corollaries:

Corollary 1 (Legendre Case) *If $\rho = \sigma = 0$, then the nonzero elements, of the operational matrix of the exponential Legendre functions, $d_{k\ell}$* for *$0 \le k, \ell \le N$ are given as follows:*

$$d_{k+1,k} = \frac{2k+1}{L}, \quad d_{kk} = -\frac{k}{L},$$
$$d_{k\ell} = (-1)^{k+\ell+1}\frac{(2\ell+1)}{L}, \quad \ell < k-1.$$

Corollary 2 (ChebyshevT Case) *If $\rho = \sigma = -\frac{1}{2}$, then the nonzero elements, of the operational matrix of the exponential Chebyshev functions of the first kind, $d_{k\ell}$ for $0 \le k, \ell \le N$ are given as follows*:

$$d_{k+1,k} = \frac{2k+1}{L}, \quad d_{kk} = -\frac{k}{L},$$
$$d_{k\ell} = \frac{2(-1)^{k+\ell+1}\ell(k-\frac{1}{2})_{k-\ell}}{L(1-k)_{k-\ell}}, \quad \ell < k-1.$$

Corollary 3 (ChebyshevU Case) *If $\rho = \sigma = \frac{1}{2}$, then the nonzero elements, of the operational matrix of the exponential Chebyshev functions of the second kind, $d_{k\ell}$* for *$0 \le k, \ell \le N$ are given as follows:*

$$d_{k+1,k} = \frac{(2k+3)(k+1)}{L(k+2)}, \quad d_{kk} = -\frac{k}{L},$$
$$d_{k\ell} = \frac{2(-1)^{k+\ell+1}(\ell+1)\left(k-\frac{1}{2}\right)_{k-\ell}}{L(-k-1)_{k-\ell}}, \quad \ell < k-1.$$

Corollary 4 (ChebyshevV Case) *If $\rho = -\frac{1}{2}$, $\sigma = \frac{1}{2}$, then the nonzero elements $d_{k\ell}$* for *$0 \le k, \ell \le N$ are given as follows:*

$$d_{k+1,k} = \frac{(2k+1)^2}{2L(k+1)}, \quad d_{kk} = -\frac{k}{L},$$
$$d_{k\ell} = \frac{(2\ell+1)(-1)^{k+\ell+1}\Gamma(-k)\left(\frac{1}{2}-k\right)_{k-\ell}}{L\Gamma(-\ell)}, \quad \ell < k-1.$$

Corollary 5 (ChebyshevW Case) *If $\rho = \frac{1}{2}$, $\sigma = -\frac{1}{2}$, then the nonzero elements $d_{k\ell}$ for $0 \le k, \ell \le N$ are given as follows:*

$$d_{k+1,k} = \frac{(2k+3)(2k+1)}{2L(k+1)}, \quad d_{kk} = -\frac{k}{L},$$
$$d_{k,\ell} = \frac{2(-1)^{k+\ell}\Gamma(-k)\Gamma\left(\frac{1}{2}-\ell\right)}{L\Gamma\left(-k-\frac{1}{2}\right)\Gamma(-\ell)}, \quad \ell < k-1,$$

Remark 1 The operational matrix for r-th derivative can be derived as

$$\frac{d^r\phi(x)}{dx^r} = (\mathbf{D}^{(1)})^r\phi(x), \tag{9}$$

where $r \in N$ and the superscript in $\mathbf{D}^{(1)}$ denote matrix powers. Thus,

$$\mathbf{D}^{(r)} = (\mathbf{D}^{(1)})^r, \quad r = 1, 2, \dots . \tag{10}$$

Implementation of the method

The target of this part is to derive a scheme for the exponential Jacobi spectral collocation method based on the operational matrix of derivative of exponential Jacobi function to numerically solve the HPDEs on the half line. Let us consider the HPDEs of the form [25]

$$\frac{\partial v(x,t)}{\partial t} = \xi_1\frac{\partial v(x,t)}{\partial x} + \xi_2 v(x,t) + \mathcal{S}(x,t), \ (x,t) \in [0,\infty)\times[0,\infty), \tag{11}$$

subject to the initial conditions

$$v(x,0) = k_0(x), \quad x \in [0,\infty), \tag{12}$$

$$v(0,t) = k_1(t), \quad t \in [0,\infty). \tag{13}$$

We approximate $v(x,t)$, $\frac{\partial v(x,t)}{\partial t}$ and $\frac{\partial v(x,t)}{\partial x}$ by the double exponential Jacobi functions as

$$\begin{aligned} v(x,t) \approx v_{N,M}(x,t) &= \sum_{i=0}^{M}\sum_{j=0}^{N} c_{ij}\Upsilon_i^{(\rho_1,\sigma_1)}(x)\Upsilon_j^{(\rho_2,\sigma_2)}(t) \\ &= \phi_N(t)\mathbf{C}^T\phi_M(x), \end{aligned} \tag{14}$$

$$\begin{aligned} \frac{\partial v_{N,M}(x,t)}{\partial t} &= \sum_{i=0}^{M}\sum_{j=0}^{N} c_{ij}\Upsilon_i^{(\rho_1,\sigma_1)}(x)\frac{\partial \Upsilon_j^{(\rho_2,\sigma_2)}(t)}{\partial t} \\ &= \phi_N^{'}(t)\mathbf{C}^T\phi_M(x), \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{\partial v_{N,M}(x,t)}{\partial x} &= \sum_{i=0}^{M}\sum_{j=0}^{N} c_{ij}\frac{\partial \Upsilon_i^{(\rho_1,\sigma_1)}(x)}{\partial x}\Upsilon_j^{(\rho_2,\sigma_2)}(t) \\ &= \phi_N(t)\mathbf{C}^T\phi_M^{'}(x), \end{aligned} \tag{16}$$

where $\mathbf{C}^T$ is $(N+1)\times(M+1)$ unknown matrix. Now, using Eqs. (14), (15) and (16), then it is easy to write

$$\phi_N^{'}(t)\mathbf{C}^T\phi_M(x) = \xi_1\phi_N(t)\mathbf{C}^T\phi_M^{'}(x) + \xi_2\phi_N(t)\mathbf{C}^T\phi_M(x) + \mathcal{S}(x,t), \tag{17}$$

$$\phi_N(0)\mathbf{C}^T\phi_M(x) = k_0(x), \tag{18}$$

$$\phi_N(t)\mathbf{C}^T\phi_M(0) = k_1(t), \tag{19}$$

Now, we tame the collocation procedure for solving Eqs. (17)–(19). Suppose $x_i^{(\rho_1,\sigma_1)}$ ($0 \leqslant i \leqslant M$) are the exponential Jacobi collocation points of $\Upsilon_i^{(\rho_1,\sigma_1)}(x)$ and $t_j^{(\rho_2,\sigma_2)}$ $(0 \leqslant j \leqslant N-1)$ are the exponential Jacobi collocation points of $\Upsilon_j^{(\rho_2,\sigma_2)}(t)$. We substitute these collocation points in (17)–(19); therefore, the collocation scheme can be written as:

$$\begin{aligned} &\phi_N^{'}(t_j^{(\rho_2,\sigma_2)})\mathbf{C}^T\phi_M(x_i^{(\rho_1,\sigma_1)}) = \xi_1\phi_N(t_j^{(\rho_2,\sigma_2)})\mathbf{C}^T\phi_M^{'}(x_i^{(\rho_1,\sigma_1)}) \\ &\quad + \xi_2\phi_N(t_j^{(\rho_2,\sigma_2)})\mathbf{C}^T\phi_M(x_i^{(\rho_1,\sigma_1)}) + \mathcal{S}(x_i^{(\rho_1,\sigma_1)}, t_j^{(\rho_2,\sigma_2)}), \\ &\quad 1 \le i \le M,\ 0 \le j \le N-1. \end{aligned} \tag{20}$$

$$\phi_N(0)\mathbf{C}^T\phi_M(x_i^{(\rho_1,\sigma_1)}) = k_0(x_i^{(\rho_1,\sigma_1)}),\ 0 \le i \le M, \tag{21}$$

$$\phi_N(t_j^{(\rho_2,\sigma_2)})\mathbf{C}^T\phi_M(0) = k_1(t_j^{(\rho_2,\sigma_2)}), \quad 0 \le j \le N-1. \tag{22}$$

This yields a algebraic system of $(N+1)\times(M+1)$ equations in the required double exponential Jacobi coefficients $c_{ij}, i = 0,1,\dots,M; j = 0,1,\dots,N$, which can be solved by using any standard iteration technique, like Newton's iteration solver. Consequently, the approximate solution $v_{N,M}(x,t)$ can be evaluated.

Error analysis

Here, we discuss the convergence rate of the suggested double basis expansion, for this target, the following lemmas are needed:

Lemma 1 *The following definite integral is valid:*

$$\begin{aligned} &\int_0^{\infty} \Upsilon_i^{(\rho,\sigma)}(x)\, w^{(\rho+\mu+1,\sigma)} \\ &dx = \frac{L\,\Gamma(i+\sigma+1)\,\Gamma(\rho+\mu+1)\,(-\mu)_i}{i!\,\Gamma(i+\rho+\sigma+\mu+2)}; \\ &\quad \rho+\mu > -1, \quad \sigma > -1, \end{aligned}$$

where $(a)_i$ denote the Pochhammer notation, i.e., $(a)_i = \Gamma(a+i)/\Gamma(a)$.

Lemma 2 *For all $\rho > -1$, there exist two generic constants $0 < \kappa_1 < \kappa_2$ such that:*

$$\kappa_1\, n^{\rho-1}\, n! \le \Gamma(n+\rho) \le \kappa_2\, n^{\rho-1}\, n!; \quad \forall n \in \mathbb{N}.$$

Lemma 3 *If $\rho,\sigma > -1$ then $|\ \Upsilon_i^{(\rho,\sigma)}(x)\ | \le J/i^q$ where $q = \max(\rho,\sigma,-\frac{1}{2})$, where J is a generic positive constant.*

In this theorem, we ascertain the vanishing rate of the unknown expansion coefficients of the approximate solution, under certain constrains on the exact smooth solution of the solved problem.

Theorem 2 *If $v(x,t)$ is separable, i.e., $v(x,t) = v_1(x)\,v_2(t)$ and v_1, v_2 are of exponential order, in the sense that, there exist A_1, A_2, μ_1 and μ_2 positive constants, such that $|v_1(x)| \le A_1\, e^{-\mu_1 x}$ and $|v_2(t)| \le A_2\, e^{-\mu_2 t}$, then the expansion coefficients in (14) satisfy the following estimate*:

$$|c_{ij}| \le \frac{C}{i^{\rho_1+2\mu_1+1}\, j^{\rho_2+2\mu_2+1}}.$$

Proof By the hypothesis of theorem, we have,

$$v(x,t) = v_1(x)\,v_2(t) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} c_{km}\,\Upsilon_k^{(\rho_1,\sigma_1)}(x)\,\Upsilon_m^{(\rho_2,\sigma_2)}(t),$$

applying the inner product, and by the orthogonality relation (3), we get,

$$\left(v_1(x)\,v_2(t), \Upsilon_i^{(\rho_1,\sigma_1)}(x)\,\Upsilon_j^{(\rho_2,\sigma_2)}(t)\right)_{w^{(\rho_1,\sigma_1)}\,w^{(\rho_2,\sigma_2)}} = c_{ij}\,h_i^{(\rho_1,\sigma_1)}\ h_j^{(\rho_2,\sigma_2)},$$

i.e.,

$$c_{ij} = \frac{1}{h_i^{(\rho_1,\sigma_1)}\, h_j^{(\rho_2,\sigma_2)}} \int_0^\infty \int_0^\infty v(x,t) \Upsilon_i^{(\rho_1,\sigma_1)}(x)\, \Upsilon_j^{(\rho_2,\sigma_2)}(t)\, w^{(\rho_1,\sigma_1)}\, w^{(\rho_2,\sigma_2)}\, dx\, dt$$
$$= \frac{1}{h_i^{(\rho_1,\sigma_1)}\, h_j^{(\rho_2,\sigma_2)}} \left(\int_0^\infty v_1(x) \Upsilon_i^{(\rho_1,\sigma_1)}(x)\, w^{(\rho_1,\sigma_1)}\, dx \right) \left(\int_0^\infty v_2(t) \Upsilon_j^{(\rho_2,\sigma_2)}(t)\, w^{(\rho_2,\sigma_2)}\, dt \right)$$
$$= I_1^{(\rho_1,\sigma_1)}(i)\, I_2^{(\rho_2,\sigma_2)}(j),$$

where,

$$I_r^{(\rho_r,\sigma_r)}(k) = \frac{1}{h_k^{(\rho_r,\sigma_r)}} \int_0^\infty v_r(z) \Upsilon_k^{(\rho_r,\sigma_r)}(z)\, w^{(\rho_r,\sigma_r)}\, dz, \qquad r = 1, 2.$$

Now by application of integration by parts on $I_1^{(\rho_1,\sigma_1)}(i)$ and $I_2^{(\rho_2,\sigma_2)}(j)$, since v_1 and v_2 are of exponential order, by the integral formula in Lemma 1, repeated use of the estimate in Lemma 2 on $I_1^{(\rho_1,\sigma_1)}(i)$ and $I_2^{(\rho_2,\sigma_2)}(j)$, the theorem is proved. □

In this theorem, based on the result of the previous theorem, we ascertain the convergence of the approximate solution as the number of retained modes increases.

Theorem 3 *If* $\min(\rho_1 + 2\mu_1, \rho_2 + 2\mu_2) > \frac{1}{2}$ *and* $-1 < \max(\rho_1, \rho_2, \sigma_1, \sigma_2) < -\frac{1}{2}$, *then series in* (14) *converges absolutely.*

Proof We show that the series $|\sum_0^\infty \sum_0^\infty c_{ij}\, \Upsilon_i^{(\rho_1,\sigma_1)}(x)\, \Upsilon_j^{(\rho_2,\sigma_2)}(t)|$ converges absolutely.

By the estimate in Theorem 2, using Lemma 3, then

$$|c_{ij}\, \Upsilon_i^{(\rho_1,\sigma_1)}(x)\, \Upsilon_j^{(\rho_2,\sigma_2)}(t)| \leq \frac{A}{i^{\rho_1+2\mu_1+\frac{1}{2}}\, j^{\rho_2+2\mu_2+\frac{1}{2}}},$$

which completes the proof of the theorem. □

In this theorem, we control the estimate of two consecutive approximate solutions, to ascertain the stability when the number of retained modes increases.

Theorem 4 *If* $\min(\rho_1 + 2\mu_1, \rho_2 + 2\mu_2) > \frac{1}{4}$ *and* $-1 < \max(\rho_1, \rho_2, \sigma_1, \sigma_2) < -\frac{1}{2}$, *then*

$$\lim_{N,M\to\infty} \|u_{N+1,M+1} - u_{N,M}\|_2 = 0.$$

Proof By the triangle inequality, we have,

$$\|u_{N+1,M+1} - u_{N,M}\|_2 = \|u_{N+1,M+1} - u_{N,M+1} + u_{N,M+1} - u_{N,M}\|_2$$
$$\leq \|u_{N+1,M+1} - u_{N,M+1}\|_2 + \|u_{N,M+1} - u_{N,M}\|_2$$
$$= \left\| \sum_{j=0}^{M+1} c_{N+1,j}\, \Upsilon_{N+1}^{(\rho_1,\sigma_1)}(x)\, \Upsilon_j^{(\rho_2,\sigma_2)}(t) \right\|_2 + \left\| \sum_{i=0}^{N} c_{i,M+1}\, \Upsilon_i^{(\rho_1,\sigma_1)}(x)\, \Upsilon_{M+1}^{(\rho_2,\sigma_2)}(t) \right\|_2.$$

Now, application of Lemma 2, Lemma 3 to the two norms of the R.H.S of the later inequality, respectively, and by the result of Theorem 3, we get

$$\|u_{N+1,M+1} - u_{N,M}\|_2 < \frac{B}{M^{2\rho_1+4\mu_1-\frac{1}{2}}\, N^{2\rho_2+4\mu_2-\frac{1}{2}}},$$

which completes the proof of the theorem. □

Numerical results

In this section, we test our algorithm by exibiting two numerical experiments to check the applicability and accuracy of the proposed scheme. Comparison of the numerical results obtained by the suggested technique with those obtained by generalized Laguerre–Gauss–Radau collocation approach [25] confirms that the presented scheme is very effective and convenient. Thereby, we assert that the proposed scheme is more appropriate for solving these kinds of problems.

The absolute errors in the given tables are

$$E(x,t) = |v(x,t) - v_{N,M}(x,t)|, \tag{23}$$

where $v(x,\ t)$ and $v_{N,M}(x,t)$ are the exact solution and the numerical solution, respectively, at the point $(x,\ t)$, respectively. Moreover, the maximum absolute errors are given by

$$L^\infty = \text{Max}\{E(x,t) : \forall (x,t) \in [0,\infty) \times [0,\infty)\}. \tag{24}$$

Example 1 [25] Consider the hyperbolic equation of first-order of the form

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial v(x,t)}{\partial x} + v(x,t) + \mathcal{S}(x,t), \quad ,x \in [0,\infty), \quad t \in [0,\infty), \tag{25}$$

subject to initial conditions,

$$v(x,0) = e^{-x}, \quad x \in [0,\infty), \qquad v(0,t) = e^{-\sqrt{2}t}, \quad t \in [0,\infty),$$

Table 1 Comparison of the absolute errors for Example 1 at $t = 0.1$ and $N = M = 16$

x	Bhrawy et al. [25]	Our method		
		$\rho_1 = \sigma_1 = -\frac{1}{2}$, $\rho_2 = \sigma_2 = -\frac{1}{2}$	$\rho_1 = \sigma_1 = 0$ $\rho_2 = \sigma_2 = 0$	$\rho_1 = \sigma_1 = \frac{1}{2}$, $\rho_2 = \sigma_2 = \frac{1}{2}$
0.1	2.84×10^{-7}	2.29×10^{-8}	1.04×10^{-8}	2.97×10^{-8}
0.2	8.79×10^{-6}	1.84×10^{-8}	$9 \times 10 \times 10^{-9}$	2.64×10^{-9}
0.3	1.20×10^{-5}	1.84×10^{-8}	8.57×10^{-9}	2.45×10^{-9}
0.4	1.12×10^{-5}	1.48×10^{-8}	7.47×10^{-9}	2.15×10^{-9}
0.5	7.95×10^{-6}	1.51×10^{-8}	6.98×10^{-9}	2.02×10^{-9}
0.6	3.29×10^{-6}	1.33×10^{-8}	6.29×10^{-9}	1.79×10^{-9}
0.7	1.78×10^{-6}	1.08×10^{-8}	5.51×10^{-9}	1.58×10^{-9}
0.8	6.53×10^{-6}	1.05×10^{-8}	5.08×10^{-9}	1.48×10^{-9}
0.9	1.04×10^{-5}	1.06×10^{-8}	4.75×10^{-9}	1.37×10^{-9}
1	1.32×10^{-5}	9.25×10^{-9}	4.24×10^{-9}	1.20×10^{-9}

Table 2 Comparison of the absolute errors for Example 1 at $t = 0.5$ and $N = M = 16$

x	Bhrawy et al. [25]	Our method		
		$\rho_1 = \sigma_1 = -\frac{1}{2}$, $\rho_2 = \sigma_2 = -\frac{1}{2}$	$\rho_1 = \sigma_1 = 0$ $\rho_2 = \sigma_2 = 0$	$\rho_1 = \sigma_1 = \frac{1}{2}$, $\rho_2 = \sigma_2 = \frac{1}{2}$
0.1	8.95×10^{-6}	7.65×10^{-8}	3.54×10^{-8}	1.50×10^{-8}
0.2	$4 \times 10 \times 10^{-6}$	6.22×10^{-8}	3.09×10^{-8}	1.34×10^{-8}
0.3	1.39×10^{-5}	6.16×10^{-8}	2.90×10^{-8}	1.23×10^{-8}
0.4	2.07×10^{-5}	5.04×10^{-8}	2.54×10^{-8}	$1 \times 10 \times 10^{-8}$
0.5	2.47×10^{-5}	5.05×10^{-8}	2.36×10^{-8}	1.00×10^{-8}
0.6	2.62×10^{-6}	4.48×10^{-8}	2.13×10^{-8}	9.13×10^{-9}
0.7	2.55×10^{-5}	3.68×10^{-8}	1.87×10^{-8}	8.19×10^{-9}
0.8	2.31×10^{-5}	3.53×10^{-8}	1.72×10^{-8}	7.45×10^{-9}
0.9	1.93×10^{-5}	3.51×10^{-8}	1.61×10^{-8}	6.80×10^{-9}
1	1.45×10^{-5}	3.08×10^{-8}	1.43×10^{-8}	6.11×10^{-9}

where

$$\mathcal{S}(x,t) = -\sqrt{2}e^{-\sqrt{2}t-x}.$$

The exact solution is given by

$$v(x,t) = e^{-(\sqrt{2}t+x)}.$$

In Tables 1, 2 and 3, we give the absolute errors with $\rho_1 = \sigma_1 = \rho_2 = \sigma_2 = -\frac{1}{2}$ (first kind exponential Chebyshev functions), $\rho_1 = \sigma_1 = \rho_2 = \sigma_2 = 0$ (exponential Legendre functions) and $\rho_1 = \sigma_1 = \rho_2 = \sigma_2 = \frac{1}{2}$ (second kind exponential Chebyshev functions), respectively, at $N = M = 16$.

Table 3 Comparison of the absolute errors for Example 1 at $t = 1$ and $N = M = 16$

x	Bhrawy et al. [25]	Our method		
		$\rho_1 = \sigma_1 = -\frac{1}{2}$, $\rho_2 = \sigma_2 = -\frac{1}{2}$	$\rho_1 = \sigma_1 = 0$ $\rho_2 = \sigma_2 = 0$	$\rho_1 = \sigma_1 = \frac{1}{2}$, $\rho_2 = \sigma_2 = \frac{1}{2}$
0.1	4.87×10^{-5}	1.71×10^{-8}	2.47×10^{-8}	2.19×10^{-8}
0.2	4.89×10^{-5}	2.37×10^{-8}	2.34×10^{-8}	1.99×10^{-8}
0.3	4.17×10^{-5}	1.51×10^{-8}	2.02×10^{-8}	1.79×10^{-8}
0.4	3.03×10^{-5}	1.99×10^{-8}	1.91×10^{-8}	1.63×10^{-8}
0.5	1.75×10^{-5}	1.24×10^{-8}	1.66×10^{-8}	1.47×10^{-8}
0.6	4.74×10^{-6}	1.21×10^{-8}	1.51×10^{-8}	1.33×10^{-8}
0.7	6.68×10^{-6}	1.53×10^{-8}	1.42×10^{-8}	1.21×10^{-8}
0.8	1.61×10^{-5}	1.16×10^{-8}	1.26×10^{-8}	1.09×10^{-8}
0.9	2.32×10^{-5}	6.90×10^{-9}	1.09×10^{-8}	9.79×10^{-9}
1	2.79×10^{-5}	7.26×10^{-9}	1.00×10^{-8}	8.87×10^{-9}

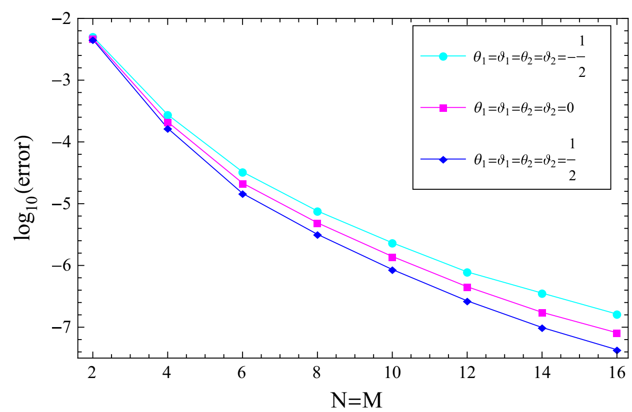

Fig. 1 L^∞ error for Example 1 versus $N = M$ and $\rho_1 = \sigma_1 = \rho_2 = \sigma_2$

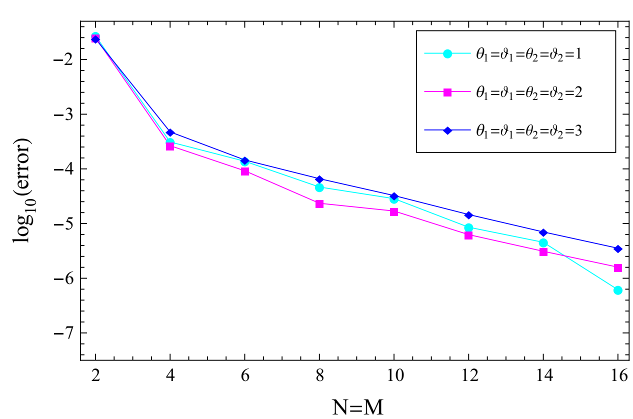

Fig. 2 L^∞ error for Example 2 versus $N = M$ and $\rho_1 = \sigma_1 = \rho_2 = \sigma_2$

Table 4 The absolute errors for Example 2 at $N = M = 16$

(x, t)	$\rho_1 = \sigma_1 = -\frac{1}{2}$, $\rho_2 = \sigma_2 = -\frac{1}{2}$	$\rho_1 = \sigma_1 = 0$, $\rho_2 = \sigma_2 = 0$	$\rho_1 = \sigma_1 = \frac{1}{2}$, $\rho_2 = \sigma_2 = \frac{1}{2}$
(0.1, 0.1)	9.61×10^{-7}	4.22×10^{-7}	1.95×10^{-7}
(0.2, 0.2)	6.37×10^{-7}	5.92×10^{-7}	7.08×10^{-7}
(0.3, 0.3)	9.63×10^{-7}	6.47×10^{-7}	1.67×10^{-7}
(0.4, 0.4)	1.44×10^{-6}	8.90×10^{-7}	7.52×10^{-7}
(0.5, 0.5)	2.04×10^{-6}	9.36×10^{-7}	2.47×10^{-7}
(0.6, 0.6)	1.28×10^{-8}	4.81×10^{-8}	1.35×10^{-7}
(0.7, 0.7)	2.23×10^{-6}	1.23×10^{-6}	8.48×10^{-7}
(0.8, 0.8)	7.97×10^{-7}	4.56×10^{-7}	1.16×10^{-7}
(0.9, 0.9)	3.33×10^{-6}	1.55×10^{-6}	6.54×10^{-7}

Table 5 Comparison of the maximum absolute errors for Example 2 versus $\rho_1 = \sigma_1,\ \rho_2 = \sigma_2$ at $N = M = 16$

$\rho_1 = \sigma_1$	$\rho_2 = \sigma_2$	Bhrawy et al. [25]	Our method
1	1	6.00×10^{-5}	6.16×10^{-7}
2	2	2.56×10^{-4}	1.59×10^{-6}
3	3	3.51×10^{-4}	3.53×10^{-6}

Moreover, the results obtained by our method are compared with these obtained by generalized Laguerre–Gauss–Radau collocation method [25]. Figure 1 shows L^∞ error versus $N = M$ and $\rho_1 = \sigma_1 = \rho_2 = \sigma_2$.

Example 2 [25] Consider the following hyperbolic equation of first-order

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial v(x,t)}{\partial x} + v(x,t) + e^{-t-x}(\cos(t) - \sin(t)),\ x \in [0,\infty),\quad t \in [0,\infty), \tag{26}$$

subject to initial conditions,

$$v(x,0) = 0, \quad x \in [0,\infty), \quad v(0,t) = e^{-t}\sin(t), \quad t \in [0,\infty).$$

The exact solution is given by

$$v(x,t) = e^{-(t+x)}\sin(t).$$

Table 4 lists the results obtained by the our method in terms of absolute errors at $N = M = 16$ for different values of $\rho_1, \sigma_1, \rho_2, \sigma_2, x$ and t. Figure 2 shows the L^∞ error versus $\rho_1 = \sigma_1 = \rho_2 = \sigma_2$ and $N = M$. Moreover, the results in Table 5 are more accurate if compared with these obtained by generalized Laguerre–Gauss–Radau collocation method [25].

Conclusion

We developed an accurate numerical technique and applied it to solve hyperbolic partial differential equations. The proposed operational matrix in combination with the exponential Jacobi spectral-collocation approach was elaborated for reducing the solution of hyperbolic first-order partial differential equations on the semi-infinite domain to an algebraic system of equations, which can be solved more easily. The operational matrices of derivatives of exponential Legendre, ChebyshevT, U, V, W functions can be obtained as direct special cases of the operational matrix of exponential Jacobi functions. The numerical results evince the high efficiency and accuracy of our approach.

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