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ORIGINAL RESEARCH

On new common fixed points of multivalued (Υ, Λ)-contractions in complete b-metric spaces and related application

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Abstract

The aim of this paper is to introduce the concept of generalized multivalued (Υ, Λ)-contractions and generalized multivalued (Υ, Λ)-Suzuki contractions and introduce some new common fixed point results for such maps in complete b-metric spaces. Our results are an improvement of the Liu et al. fixed point theorem and several comparable results in the existing literature. We set up an example to elucidate our main result. Moreover, we also discuss an application to existence of solution for system of functional equations.

Keywords Fixed point · b-metric space · Generalized multivalued (Υ, Λ)-contraction · Generalized multivalued $(\Upsilon, \Lambda$)-Suzuki contraction

Mathematics Subject Classification Primary 47H10 · Secondary 54H25

Introduction and preliminaries

Fixed point theory plays an important role in functional and nonlinear analysis. Banach [1] proved a significant result for contraction mappings. Afterward, a large number of fixed point results have been established by various authors and they showed different generalizations of the Banach's results, see for example ([2–28]).

On the other hand, Czerwik [26, 27] gave a generalization of the famous Banach fixed point theorem in so-called b-metric spaces. For some results on b-metric spaces, see ([17–25, 28]) and related references therein.

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Definition 1 [18] Let ω be a non-empty set. A function $\check{d} : \omega \times \omega \to [0, \infty)$ is said to be a metric if for all $\zeta, \eta, \upsilon \in \omega$, we have

(1) $\check{d}(\zeta, \eta) = 0$ if and only if $x = y$;
(2) $\check{d}(\zeta, \eta) = \check{d}(\eta, \zeta)$;
(3) $\check{d}(\zeta, \eta) \le \check{d}(\zeta, \upsilon) + \check{d}(\upsilon, \eta)$.

In this case, the pair $(\omega, \check{d})$ is called a metric space (or for short MS).

Definition 2 [27] Let ω be a non-empty set and $\varrho \ge 1 \in (-\infty, \infty)$. A function $\check{d}_b : \omega \times \omega \to [0, \infty)$ is said to be a b-metric if for all $\zeta, \eta, \upsilon \in \omega$, we have

(1) $\check{d}_b(\zeta, \eta) = 0$ if and only if $x = y$;
(2) $\check{d}_b(\zeta, \eta) = \check{d}_b(\eta, \zeta)$;
(3) $\check{d}_b(\zeta, \eta) \le \varrho\left[\check{d}_b(\zeta, \upsilon) + \check{d}_b(\upsilon, \eta)\right]$.

In this case, the pair $(\omega, \check{d}_b)$ is called a b-metric space with constant ϱ (or for short bMS).

Note that the concept of convergence in such spaces is similar to that of the standard metric spaces. The b-metric space $(\omega, \check{d}_b)$ is called complete if every Cauchy sequence of elements from $(\omega, \check{d}_b)$ is convergent. In general, a b-metric is not a continuous functional. If b-metric $\check{d}_b$ is continuous, then every convergent sequence has a unique limit.

Theorem 3 [10] *Let $(\omega, \check{d})$ be a compact MS and let $\hat{S} : \omega \longrightarrow \omega$. Assume that $\forall \zeta, \eta \in \omega$ with $\zeta \neq \eta$,*

$$\frac{1}{2}\check{d}(\zeta, S(\zeta)) < \check{d}(\zeta, \eta) \Longrightarrow \check{d}(S(\zeta), S(\eta)) < \check{d}(\zeta, \eta).$$

Then $\hat{S}$ has a unique fixed point in ω.

Jleli and Samet [3, 4] introduced the notion of θ-contraction.

Definition 4 Let $(\omega, \check{d})$ be a MS. A mapping $\check{T} : \omega \to \omega$ is said to be a θ-contraction, if there exist a constant $k \in (0, 1)$ and $\theta \in \Theta$ such that

$$\zeta, \eta \in \omega, \check{d}(\check{T}(\zeta), \check{T}(\eta)) \neq 0 \\ \Longrightarrow \theta\big(\check{d}(\check{T}(\zeta), \check{T}(\eta))\big) \leq \big[\theta\big(\check{d}(\zeta, \eta)\big)\big]^k,$$

where Θ is the set of functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ satisfying the following conditions:

(Θ1) θ is non-decreasing,
(Θ2) for each sequence $\{t_n\} \subset (0, \infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \text{ if and only if } \lim_{n\to\infty} t_n = 0^+,$$

(Θ3) there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{t\longrightarrow 0^+} \frac{\theta(t)-1}{t^r} = \ell$.
(Θ4) θ is continuous.

Jleli and Samet [3] established the fixed point theorem as follows:

Theorem 5 [3] *Let $(\omega, \check{d})$ be a complete MS and $\check{T} : \omega \to \omega$ be a θ-contraction. Then $\check{T}$ has a unique fixed point.*

Very recently, Liu et al. [6] introduced the notion of (Υ, Λ)-Suzuki contractions.

Definition 6 Let $(\omega, \check{d})$ be a MS. A mapping $\check{T} : \omega \to \omega$ is said to be a (Υ, Λ)-Suzuki contraction, if there exist a comparison function Υ and $\Lambda \in \Phi$ such that, for all, $\zeta, \eta \in \omega$ with $\check{T}(\zeta) \neq \check{T}(\eta)$

$$\frac{1}{2}\check{d}\big(\zeta, \check{T}(\zeta)\big) < \check{d}(\zeta, \eta) \\ \Longrightarrow \Lambda\big(\check{d}\big(\check{T}(\zeta), \check{T}(\eta)\big)\big) \leq \Upsilon[\Lambda(U(\zeta, \eta))],$$

where

$$U(\zeta, \eta) = \max\left\{d(\zeta, \eta), d\big(\zeta, \check{T}(\zeta)\big), d\big(\eta, \check{T}(\eta)\big), \frac{d(\zeta, T(\eta)) + d(\eta, T(\zeta))}{2}\right\},$$

Φ is the set of functions $\Lambda : (0, \infty) \longrightarrow (0, \infty)$ satisfying the following conditions:

(Φ1) Λ is non-decreasing,
(Φ2) for each sequence $\{t_n\} \subset (0, \infty)$,

$$\lim_{n\to\infty} \Lambda(t_n) = 0 \text{ if and only if } \lim_{n\to\infty} t_n = 0,$$

(Φ3) Λ is continuous.

And as in [2], a function $\Upsilon : (0, \infty) \longrightarrow (0, \infty)$ is called a comparison function if it satisfies the following conditions:

(1) ψ is monotone increasing, that is, $t_1 < t_2 \Longrightarrow \Upsilon(t_1) \leq \Upsilon(t_2)$,
(2) $\lim_{n\to\infty}^{n} \Upsilon(t) = 0$ for all $t > 0$, where Υ^n stands for the nth iterate of ψ.

Clearly, if Υ is a comparison function, then $\Upsilon(t) < t$ for each $t > 0 > 0$.

Lemma 7 [6] *Let $\Lambda : (0, \infty) \longrightarrow (0, \infty)$ be a non-decreasing and continuous function with $\inf_{t\in(0,\infty)} \phi(t_n) = 0$ and $\{t_k\}_k$ be a sequence in $(0, \infty)$. Then,*

$$\lim_{k\to\infty} \Lambda(t_k) = 0 \text{ if and only if } \lim_{k\to\infty t k} = 0.$$

Theorem 8 [8] *Let $(\omega, \check{d})$ be a complete MS and $\check{S} : \omega \longrightarrow CB(\omega)$ be a multivalued mapping, where $CB(\omega)$ is the family of all non-empty closed and bounded subsets of ω. If $\check{S}$ is a multivalued contraction, that is, if there exists $\lambda \in [0, 1)$ such that*

$$H\big(\check{S}(\zeta), \check{S}(\eta)\big) \leq \lambda\check{d}(\zeta, \eta), \text{ all } \zeta, \eta \in \omega.$$

Then $\check{S}$ has a fixed point $\zeta^* \in \omega$ such that $\zeta^* \in \check{S}(\zeta^*)$.

Definition 9 [9] Let $(\omega, \check{d})$ be a MS. Let $\check{S} : \omega \longrightarrow CB(\omega)$ be a multivalued mapping. Then $\check{S}$ is said to be a generalized multivalued-F-contraction if there exist -F $\in \mathcal{F}$ and $\vartheta > 0$ such that for all $\zeta, \eta \in \omega$,

$$H\big(\check{S}(\zeta), \check{S}(\eta)\big) > 0 \\ \Longrightarrow \vartheta + \text{-F}\big(H\big(\check{S}(\zeta), \check{S}(\eta)\big)\big) \leq \text{-F}(U(\zeta, \eta)),$$

where

$$U(\zeta, \eta) = \max\left\{\check{d}(\zeta, \eta), D\big(\zeta, \check{S}(\zeta)\big), D\big(\eta, \check{S}(\eta)\big), \frac{D\big(\zeta, \check{S}(\eta)\big) + D\big(\eta, \check{S}(\zeta)\big)}{2}\right\}.$$

HanÇer et al. [7] (see also [5]) extended the concept of θ-contraction to multivalued mappings as follows.

Definition 10 [7] Let $(\omega, \check{d})$ be a metric space, $\check{S} : \omega \longrightarrow CB(\omega)$ and $\theta \in \Theta$. Then $\check{S}$ is said to be a multivalued θ- contraction if there exists a constant $k \in [0, 1)$ such that

$$\theta\big(H\big(\check{S}(\zeta), \check{S}(\eta)\big)\big) \leq \big[\theta\big(\check{d}(\zeta, \eta)\big)\big]^k,$$

for all $\zeta, \eta \in \omega$, with $H\big(\check{S}(\zeta), \check{S}(\eta)\big) > 0$.

From now on, let $(\omega, \check{d}_b)$ be a bMS. Let $CB_b(\omega)$ denote the family of all bounded and closed sets in ω. For $\zeta \in \omega$ and $A, B \in CB_b(\omega)$, we define

$$D_b(\zeta, A) = \inf_{I\in A} \check{d}_b(\zeta, I) \text{ and } D_b(A, B) = \sup_{I\in A} D_b(I, B).$$

Define a mapping $H_b : CB_b(\omega) \times CB_b(\omega) \longrightarrow [0, \infty)$ by

$$H_b(A, B) = \max\left\{\sup_{\zeta\in A} D_b(\zeta, B), \sup_{\eta\in B} D_b(\eta, A)\right\},$$

for every $A, B \in CB_b(\omega)$. Then the mapping H_b is a b-metric, and it is called a Hausdorff b-metric induced by a b-metric space $(\omega, \check{d}_b)$.

Lemma 11 [26] *Let $(\omega, \check{d})$ be a bMS. For any $A, B, C \in CB_b(\omega)$ and any $\zeta, \eta \in \omega$, we have the following.*

(1) $D_b(\zeta, B) \leq \check{d}_b(\zeta, b)$ *for any* $b \in B$;
(2) $D_b(\zeta, B) \leq H_b(A, B)$;
(3) $D_b(\zeta, A) \leq s\big[\check{d}_b(\zeta, \eta) + D_b(\eta, B)\big]$;
(4) $D_b(\zeta, A) = 0 \Leftrightarrow \zeta \in A$;
(5) $H_b(A, B) \leq s\big[H_b(A, C) + H_b(C, B)\big]$.

Lemma 12 [26] *Let A and B be non-empty closed and bounded subsets of a bMS $(\omega, \check{d}_b)$ and $q > 1$. Then for all $a \in A$, there exists $b \in B$ such that $\check{d}_b(a, b) \leq qH_b(A, B)$.*

Definition 13 [18] Let $(\omega, \check{d}_b)$ be a bMS, the b-metric–metric d is called $*$-continuous if for every $A \in CB_b(\omega)$, every $\zeta \in \omega$ and every sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ of elements from ω such that $\lim_{n\to\infty} \zeta_n = \zeta$, we have

$$\lim_{n\to\infty} D_b(\zeta_n, A) = D_b(\zeta, A).$$

Now we introduce the following definitions.

Definition 14 Let $(\omega, \check{d}_b)$ be a bMS. Let $\check{S}, \check{T} : \omega \longrightarrow CB_b(\omega)$. Then the pair $(\check{S}, \check{T})$ is called a generalized multivalued (Υ, Λ)-contraction if there exist a comparison function Υ and $\Lambda \in \Phi$ such that for all $\zeta, \eta \in \omega$,

$$H_b\big(\check{S}(\zeta), \check{T}(\eta)\big) > 0 \implies \Lambda\big(s^3 H_b\big(\check{S}(\zeta), \check{T}(\eta)\big)\big) \leq \Upsilon\big(\Lambda\big(U_b(\zeta, \eta)\big)\big), \tag{1}$$

where

$$U_b(\zeta, \eta) = \max\left\{\check{d}_b(\zeta, \eta), D_b\big(\zeta, \check{S}(\zeta)\big), D_b\big(\eta, \check{T}(\eta)\big), \frac{D_b\big(\zeta, \check{T}(\eta)\big) + D_b\big(\eta, \check{S}(\zeta)\big)}{2}\right\}. \tag{2}$$

Definition 15 Let $(\omega, \check{d}_b)$ be a bMS. Let $\check{S}, \check{T} : \omega \longrightarrow CB_b(\omega)$. Then the pair $(\check{S}, \check{T})$ is called a generalized multivalued (Υ, Λ)-Suzuki contraction if there exist a comparison function Υ and $\Lambda \in \Phi$ such that for all $\zeta, \eta \in \omega$, with $\check{S}(\zeta) \neq \check{T}(\eta)$,

$$\frac{1}{2s} \min\big\{D_b\big(\zeta, \check{S}(\zeta)\big), D_b\big(\eta, \check{T}(\eta)\big)\big\} < \check{d}_b(\zeta, \eta) \implies \Lambda\big(s^3 H_b\big(\check{S}(\zeta), \check{T}(\eta)\big)\big) \leq \Upsilon\big(\Lambda\big(U_b(\zeta, \eta)\big)\big), \tag{3}$$

and $U_b(\zeta, \eta)$ is defined as in (2).

Main results

Theorem 16 *Let $(\omega, \check{d}_b)$ be a complete bMS and $\check{S}, \check{T} : \omega \longrightarrow CB_b(\omega)$ be a generalized multivalued (Υ, Λ)-Suzuki contraction. Suppose that*

(1) *Υ is continuous*
(2) *$\check{d}_b$ is $*$-continuous.*

Then $\check{S}$ and $\check{T}$ have a common fixed point $\zeta^* \in \omega$.

Proof Let $\zeta_0 \in \omega$. Choose $\zeta_1 \in \check{S}(\zeta_0)$. Assume that $D_b\big(\zeta_0, \check{S}(\zeta_0)\big), D_b\big(\zeta_1, T(\zeta_1)\big) > 0$, therefore,

$$\frac{1}{2s} \min\big\{D_b\big(\zeta_0, \check{S}(\zeta_0)\big), D\big(\zeta_1, \check{T}(\zeta_1)\big)\big\} < \check{d}_b\big(\zeta_0, \zeta_1\big) \tag{4}$$

By Lemma 18,

$$0 < D_b\big(\zeta_1, \check{T}(\zeta_1)\big) \leq H_b\big(\check{S}(\zeta_0), \check{T}(\zeta_1)\big) \leq \big(s^3 H_b\big(\check{S}(\zeta_0), \check{T}(\zeta_1)\big)\big).$$

Hence, there exists $\zeta_2 \in \check{T}(\zeta_1)$,

$$0 < \check{d}_b\big(\zeta_1, \zeta_2\big) \leq H_b\big(\check{S}(\zeta_0), \check{T}(\zeta_1)\big) \leq \big(s^3 H_b\big(\check{S}(\zeta_0), \check{T}(\zeta_1)\big)\big). \tag{5}$$

Since Λ is non-decreasing, we have

$$\begin{aligned} \Lambda(\check{d}_b(\zeta_1,\zeta_2)) &\leq \Lambda(H_b(\check{S}(\zeta_0),\check{T}(\zeta_1))) \\ &\leq \Lambda(s^3 H_b(\check{S}(\zeta_0),\check{T}(\zeta_1))). \end{aligned} \tag{6}$$

Hence from (3)

$$\begin{aligned} 0 &\leq \Lambda(\check{d}_b(\zeta_1,\zeta_2)) \leq \Lambda(s^3 H_b(\check{S}(\zeta_0),\check{T}(\zeta_1))) \\ &\leq \Upsilon(\Lambda(U_b(\zeta_0,\zeta_1))), \end{aligned} \tag{7}$$

where

$$\begin{aligned} &U_b(\zeta_0,\zeta_1) \\ &= \max\left\{ \begin{array}{c} \check{d}_b(\zeta_0,\zeta_1), D_b(\zeta_0,\check{S}(\zeta_0)), D_b(\zeta_1,\check{T}(\zeta_1)), \\ \frac{D_b(\zeta_0,\check{T}(\zeta_1))+D(\zeta_1,\check{S}(\zeta_0))}{2s} \end{array} \right\} \\ &\leq \max\left\{ \check{d}_b(\zeta_0,\zeta_1), D_b(\zeta_1,\check{T}(\zeta_1)), \frac{D_b(\zeta_0,\check{T}(\zeta_1))}{2s} \right\} \\ &\leq \max\left\{ \check{d}_b(\zeta_0,\zeta_1), D_b(\zeta_1,\check{T}(\zeta_1)) \right\}. \end{aligned}$$

If $\max\{\check{d}_b(\zeta_0,\zeta_1), D_b(\zeta_1,\check{T}(\zeta_1))\} = D_b(\zeta_1,\check{T}(\zeta_1))$, then from (7), we have

$$\Lambda(\check{d}_b(\zeta_1,\zeta_2)) \leq \Upsilon(\Lambda(\check{d}_b(\zeta_1,\zeta_2))) < \Lambda(\check{d}_b(\zeta_1,\zeta_2)),$$

a contradiction. Thus, $\max\{\check{d}_b(\zeta_0,\zeta_1), D_b(\zeta_1,\check{T}(\zeta_1))\} = \check{d}_b(\zeta_0,\zeta_1)$. By (7), we get that

$$\Lambda(\check{d}_b(\zeta_1,\zeta_2)) \leq \Upsilon(\Lambda(\check{d}_b(\zeta_0,\zeta_1))).$$

Similarly, for $\zeta_2 \in \check{T}(\zeta_1)$ and $\zeta_3 \in \check{S}(\zeta_2)$. We have

$$\begin{aligned} \Lambda(\check{d}_b(\zeta_2,\zeta_3)) &= \Lambda(D_b(\zeta_2,\check{S}(\zeta_2))) \\ &\leq \Lambda(H_b(\check{T}(\zeta_1),\check{S}(\zeta_2))) \\ &\leq \Lambda(s^3 H_b(\check{T}(\zeta_1),\check{S}(\zeta_2))) \\ &\leq \Upsilon(\Lambda(U_b(\zeta_1,\zeta_2))) \\ &\leq \Upsilon(\Lambda(\check{d}_b(\zeta_1,\zeta_2))), \end{aligned}$$

which implies

$$\Lambda(\check{d}_b(\zeta_2,\zeta_3)) \leq \Upsilon(\Lambda(\check{d}_b(\zeta_1,\zeta_2))). \tag{8}$$

By continuing this manner, we construct a sequence $\{\zeta_n\}$ in ω such that $\zeta_{2i+1} \in \check{S}(\zeta_{2i})$ and $\zeta_{2i+2} \in \check{T}(\zeta_{2i+1}), i = 0, 1, 2, \ldots$,

$$\begin{aligned} &\frac{1}{2s}\min\left\{ D_b(\zeta_{2i},\check{S}(\zeta_{2i})), D(\zeta_{2i+1},\check{T}(\zeta_{2i+1})) \right\} \\ &< \check{d}_b(\zeta_{2i},\zeta_{2i+1}), \end{aligned}$$

Hence from (3), we have

$$\begin{aligned} 0 &< \Lambda(\check{d}_b(\zeta_{2i+1},\zeta_{2i+2})) \\ &\leq \Lambda(s^3 H_b(\check{S}(\zeta_{2i}),\check{T}(\zeta_{2i+1}))) \\ &\leq \Upsilon(\Lambda(U_b(\zeta_{2i},\zeta_{2i+1}))) \end{aligned} \tag{9}$$

where

$$\begin{aligned} &U_b(\zeta_{2i},\zeta_{2i+1}) \\ &= \max\left\{ \begin{array}{c} \check{d}_b(\zeta_{2i},\zeta_{2i+1}), D_b(\zeta_{2i},\check{S}(\zeta_{2i})), D_b(\zeta_{2i+1},\check{T}(\zeta_{2i+1})), \\ \frac{D_b(\zeta_{2i},\check{T}(\zeta_{2i+1}))+D_b(\zeta_{2i+1},\check{S}(\zeta_{2i}))}{2s} \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} \check{d}_b(\zeta_{2i},\zeta_{2i+1}), \check{d}_b(\zeta_{2i+1},\zeta_{2i+2}), \\ \frac{\check{d}_b(\zeta_{2i},\zeta_{2i+2})}{2s} \end{array} \right\} \\ &\leq \max\left\{ \check{d}_b(\zeta_{2i},\zeta_{2i+1}), \check{d}_b(\zeta_{2i+1},\zeta_{2i+2}) \right\}. \end{aligned}$$

If $\max\{\check{d}_b(\zeta_{2i},\zeta_{2i+1}), \check{d}_b(\zeta_{2i+1},\zeta_{2i+2})\} = \check{d}_b(\zeta_{2i+1},\zeta_{2i+2})$, then from (9) we have

$$\begin{aligned} \Lambda(\check{d}_b(\zeta_{2i+1},\zeta_{2i+2})) &\leq \Upsilon(\Lambda(\check{d}_b(\zeta_{2i+1},\zeta_{2i+2}))) \\ &< \Lambda(\check{d}_b(\zeta_{2i+1},\zeta_{2i+2})), \end{aligned}$$

which is a contradiction. Thus,

$$\max\{\check{d}_b(\zeta_{2i},\zeta_{2i+1}), \check{d}_b(\zeta_{2i+1},\zeta_{2i+2})\} = \check{d}_b(\zeta_{2i},\zeta_{2i+1}).$$

By (9), we get that

$$\Lambda(\check{d}_b(\zeta_{2i},\zeta_{2i+1})) < \Upsilon(\Lambda(\check{d}_b(\zeta_{2i},\zeta_{2i+1}))).$$

This implies that

$$\begin{aligned} &\frac{1}{2s}\min\left\{ D_b(\zeta_n,\check{S}(\zeta_n)), D_b(\zeta_{n+1},\check{T}(\zeta_{n+1})) \right\} \\ &< \check{d}_b(\zeta_n,\zeta_{n+1}), \end{aligned}$$

Hence,

$$\begin{aligned} &\Lambda(\check{d}_b(\zeta_{2n+1},\zeta_{2n+2})) \\ &< \Upsilon(\Lambda(\check{d}_b(\zeta_{2n},\zeta_{2n+1}))), \text{ for all } n \in \mathbb{N}, \end{aligned}$$

which implies

$$\begin{aligned} \Lambda(\check{d}_b(\zeta_{2n+1},\zeta_{2n+2})) &\leq \Upsilon(\Lambda(\check{d}_b(\zeta_{2n+1},\zeta_{2n+2}))) \\ &\leq \Upsilon^2(\Lambda(\check{d}_b(\zeta_{2n-1},\zeta_{2n}))) \\ &\leq \cdots \leq \Upsilon^n(\Lambda(\check{d}_b(\zeta_0,\zeta_1))) \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality, we get

$$\begin{aligned} 0 &\leq \lim_{n\longrightarrow\infty} \Lambda(\check{d}_b(\zeta_{2n+1},\zeta_{2n+2})) \\ &\leq \lim_{n\longrightarrow\infty} \Upsilon^n(\Lambda(\check{d}_b(\zeta_0,\zeta_1))) = 0, \end{aligned}$$

which implies

$$\lim_{n\longrightarrow\infty} \Lambda(\check{d}_b(\zeta_{2n+1},\zeta_{2n+2})) = 0.$$

From (Φ2) and Lemma (10), we get

$$\lim_{n\longrightarrow\infty} \check{d}_b\left(\zeta_{2n+1}, \zeta_{2n+2}\right) = 0. \tag{10}$$

Now, we will prove that the sequence $\{\zeta_n\}$ is a Cauchy. Arguing by contradiction, we assume that there exist $\varepsilon > 0$ and sequence $\{\hat{h}_n\}_{n=1}^{\infty}$ and $\{\hat{j}_n\}_{n=1}^{\infty}$ of natural numbers such that for all $n \in \mathbb{N}$, $\hat{h}_n > \hat{j}_n > n$ with $\check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right) \geq \varepsilon$, $\check{d}_b\left(\zeta_{\hat{h}(n)-1}, \zeta_{\hat{j}(n)}\right) < \varepsilon$. Therefore,

$$\begin{aligned}\varepsilon &\leq \check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right) \\ &\leq s\left[\check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{h}(n)-1}\right) + \check{d}_b\left(\zeta_{\hat{h}(n)-1}, \zeta_{\hat{j}(n)}\right)\right] \\ &< s\varepsilon + s\check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{h}(n)-1}\right).\end{aligned} \tag{11}$$

By setting $n \to \infty$ in (11) , we get

$$\varepsilon < \lim_{n\to\infty} \check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right) < s\varepsilon. \tag{12}$$

From triangular inequality, we have

$$\begin{aligned}\check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right) &\leq [\check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{h}(n)+1}\right) \\ &\quad + \check{d}_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right)],\end{aligned} \tag{13}$$

and

$$\begin{aligned}\check{d}_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right) &\leq s[\check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{h}(n)+1}\right) \\ &\quad + \check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right)].\end{aligned} \tag{14}$$

By taking upper limit as $n \to \infty$ in (13) and applying (10), (12) ,

$$\begin{aligned}\varepsilon &\leq \lim_{n\to\infty} \sup \check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right) \\ &\leq s\left(\lim_{n\to\infty} \sup \check{d}_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right)\right).\end{aligned}$$

Again, by taking the upper limit as $n \to \infty$ in (14), we get

$$\begin{aligned}\lim_{n\to\infty} \sup \check{d}_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right) &\leq s\left(\lim_{n\to\infty} \sup \check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right)\right) \\ &\leq s.s\varepsilon = s^2\varepsilon.\end{aligned}$$

Thus

$$\frac{\varepsilon}{s} \leq \lim_{n\to\infty} \sup \check{d}_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right) \leq s^2\varepsilon. \tag{15}$$

Similarly

$$\frac{\varepsilon}{s} \leq \lim_{n\to\infty} \sup \check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)+1}\right) \leq s^2\varepsilon. \tag{16}$$

By triangular inequality, we have

$$\begin{aligned}\check{d}_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right) &\leq s[\check{d}_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right) \\ &\quad + \check{d}_b\left(\zeta_{\hat{j}(n)+1}, \zeta_{\hat{j}(n)}\right)].\end{aligned} \tag{17}$$

On letting $n \to \infty$ in (17) and using the inequalities (10), (15), we get

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} \sup \check{d}_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right). \tag{18}$$

Following the above process, we find

$$\lim_{n\to\infty} \sup \check{d}_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right) \leq s^3\varepsilon. \tag{19}$$

From (18) and (19), we get

$$\frac{\varepsilon}{s^2} \leq \lim_{n\to\infty} \sup \check{d}_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right) \leq s^3\varepsilon. \tag{20}$$

From (10) and (12), we can choose a positive integer $n_0 \geq 1$ such that

$$\begin{aligned}&\frac{1}{2s} \min\left\{D_b\left(\zeta_{\hat{h}(n)}, \check{S}\left(\zeta_{\hat{h}(n)}\right)\right), D_b\left(\zeta_{\hat{j}(n)}, \check{T}\left(\zeta_{\hat{j}(n)}\right)\right)\right\} \\ &\quad < \frac{1}{2s}\varepsilon < \check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right),\end{aligned}$$

for all $n \geq n_0$, from (3), we get

$$\begin{aligned}0 &< \Lambda\left(s^3\check{d}_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right)\right) \\ &\leq \Lambda\left(s^3 H_b\left(\check{S}\left(\zeta_{\hat{h}(n)}\right), \check{T}\left(\zeta_{\hat{j}(n)}\right)\right)\right) \\ &\leq \psi\left(\phi\left(U_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right)\right)\right), \text{ for all } n \geq n_0,\end{aligned}$$

where

$$\begin{aligned}&U_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right) \\ &= \max\left\{\begin{array}{c}\check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right), D_b\left(\zeta_{\hat{h}(n)}, \check{S}\left(\zeta_{\hat{h}(n)}\right)\right), D_b\left(\zeta_{\hat{j}(n)}, \check{T}\left(\zeta_{\hat{j}(n)}\right)\right), \\ \frac{D_b\left(\zeta_{\hat{h}(n)}, \check{T}\left(\zeta_{\hat{j}(n)}\right)\right) + D_b\left(\zeta_{\hat{j}(n)}, \check{S}\left(\zeta_{p(n)}\right)\right)}{2s}\end{array}\right\} \\ &\leq \max\left\{\begin{array}{c}\check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right), \check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{h}(n)+1}\right), \check{d}_b\left(\zeta_{\hat{j}(n)}, \zeta_{\hat{j}(n)+1}\right), \\ \frac{\check{d}_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)+1}\right) + \check{d}_b\left(\zeta_{\hat{j}(n)}, \zeta_{\hat{h}(n)+1}\right)}{2s}\end{array}\right\}.\end{aligned}$$

Taking the limit as $n \to \infty$ and using (10), (12), (15) and (16), we get

$$\begin{aligned}\varepsilon &= \max\left\{\varepsilon, \frac{\frac{\varepsilon}{s} + \frac{\varepsilon}{s}}{2s}\right\} \\ &\leq \lim_{n\to\infty} \sup U_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right) \\ &\leq \max\left\{s\varepsilon, \frac{s^2\varepsilon + s^2\varepsilon}{2s}\right\} = s\varepsilon.\end{aligned}$$

From (18) , and (Φ2), we get

$$\begin{aligned}\Lambda(s\varepsilon) = &\ \Lambda\Big(s^3(\frac{\varepsilon}{s^2})\Big)\\ &\le \Lambda\Big(s^3 \lim_{n\to\infty}\sup \check{d}_b\big(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\big)\Big)\\ &\le \lim_{n\to\infty} \Upsilon\big(\Lambda\big(U_b\big(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\big)\big)\big)\\ &= \Upsilon(\Lambda(s\varepsilon)) < \Lambda(s\varepsilon).\end{aligned}$$

This is a contradiction. Therefore $\{\zeta_n\}$ is a Cauchy. Since X is a complete, we can assume that $\{x_n\}$ converges to some point $\zeta^* \in \omega$, that is, $\lim_{n\to\infty} \check{d}_b(\zeta_n, \zeta^*) = 0$ and so

$$\begin{aligned}\lim_{n\to\infty} \check{d}_b(\zeta_n, \zeta^*) = &\ \lim_{n\to\infty} \check{d}_b(\zeta_{2n}, \zeta^*)\\ = &\ \lim_{n\to\infty} \check{d}_b(\zeta_{2n+1}, \zeta^*) = 0\end{aligned} \tag{21}$$

Now, we claim that

$$\begin{aligned}&\frac{1}{2s}\min\{D_b(\zeta_n, \check{S}(\zeta_n)), D_b(\zeta^*, \check{T}(\zeta^*))\}\\ &< \check{d}_b(\zeta_n, \zeta^*),\end{aligned} \tag{22}$$

or

$$\begin{aligned}&\frac{1}{2s}\min\{D_b(\zeta^*, \check{S}(\zeta^*)), D_b(\zeta_{n+1}, \check{T}(\zeta_{n+1}))\}\\ &< \check{d}_b(\zeta_{n+1}, \zeta^*),\ \forall n \in \mathbb{N}.\end{aligned}$$

Assume that it does not hold, there exists $m \in \mathbb{N}$ such that

$$\begin{aligned}&\frac{1}{2s}\min\{D_b(\zeta_m, \check{S}(\zeta_m)), D_b(\zeta^*, \check{T}(\zeta^*))\}\\ &\ge \check{d}_b(\zeta_m, \zeta^*),\end{aligned} \tag{23}$$

and

$$\begin{aligned}&\frac{1}{2s}\min\Big\{D_b(\zeta^*, \check{S}(\zeta^*)), D_b\Big(x_{m+1}, Tx_{m+1}\Big)\Big\}\\ &\ge \check{d}_b(x_{m+1}, x^*).\end{aligned} \tag{24}$$

Therefore,

$$\begin{aligned}2s\check{d}_b(\zeta_m, \zeta^*) &\le \min\{D_b(\zeta_m, \check{S}(\zeta_m)), D_b(\zeta^*, \check{T}(\zeta^*))\}\\ &\le \min\{s[\check{d}_b(\zeta_m, \zeta^*) + D_b(\zeta^*, \check{S}(\zeta_m))],\\ &\quad D_b(\zeta^*, \check{T}(\zeta^*))\}\\ &\le s[\check{d}_b(\zeta_m, \zeta^*) + D_b(\zeta^*, \check{S}(\zeta_m))]\\ &\le s\Big[\check{d}_b(\zeta_m, \zeta^*) + \check{d}_b\Big(\zeta^*, \zeta_{m+1}\Big)\Big],\end{aligned}$$

which implies

$$\check{d}_b(\zeta_m, \zeta^*) \le \check{d}_b\Big(\zeta^*, \zeta_{m+1}\Big).$$

This together with (23) shows that

$$\begin{aligned}\check{d}_b(\zeta_m, \zeta^*) &\le \check{d}_b\Big(\zeta^*, \zeta_{m+1}\Big)\\ &\le \frac{1}{2s}\min\{D_b(\zeta^*, \check{S}(\zeta_m)),\\ &\quad D_b\Big(\zeta_{m+1}, \check{T}\Big(\zeta_{m+1}\Big)\Big)\Big\}.\end{aligned} \tag{25}$$

Since $\frac{1}{2s}\min\{D_b(\zeta_m, \check{S}(\zeta_m)), D_b(\zeta^*, \check{T}(\zeta^*))\} < \check{d}_b\Big(\zeta_m, \zeta_{m+1}\Big)$, by (3), we have

$$\begin{aligned}0 &< \Lambda(\check{d}_b(\zeta_{m+1}, \zeta_{m+2}))\\ &\le \Lambda\Big(s^3 H_b\Big(\check{S}(\zeta_m), \check{T}\Big(\zeta_{m+1}\Big)\Big)\Big)\\ &\le \Upsilon\big(\Lambda\big(U_b(\zeta_m, \zeta_{m+1})\big)\big)\end{aligned} \tag{26}$$

where

$$\begin{aligned}&U_b(\zeta_m, \zeta_{m+1})\\ &= \max\left\{\begin{array}{c}\check{d}_b\Big(\zeta_m, \zeta_{m+1}\Big), D_b(\zeta_m, \check{S}(\zeta_m)), D_b\Big(\zeta_{m+1}, \check{T}\Big(\zeta_{m+1}\Big)\Big),\\ \frac{D_b\left(\zeta_m, \check{T}\left(\zeta_{m+1}\right)\right) + D_b(\zeta_{m+1}, \check{S}(\zeta_m))}{2s}\end{array}\right\}\\ &\le \max\left\{\begin{array}{c}\check{d}_b\Big(\zeta_m, \zeta_{m+1}\Big), \check{d}_b(\zeta_{m+1}, \zeta_{m+2}),\\ \frac{\check{d}_b(\zeta_m, \zeta_{m+2})}{2s}\end{array}\right\}\\ &\le \max\Big\{\check{d}_b\Big(\zeta_m, \zeta_{m+1}\Big), \check{d}_b(\zeta_{m+1}, \zeta_{m+2})\Big\}.\end{aligned}$$

If $\max\Big\{\check{d}_b\Big(\zeta_m, \zeta_{m+1}\Big), \check{d}_b(\zeta_{m+1}, \zeta_{m+2})\Big\} = \check{d}_b(\zeta_{m+1}, \zeta_{m+2})$, then from (26) we have

$$\begin{aligned}\Lambda(\check{d}_b(\zeta_{m+1}, \zeta_{m+2})) &\le \Upsilon(\Lambda(\check{d}_b(\zeta_{m+1}, \zeta_{m+2})))\\ &< \Lambda(\check{d}_b(\zeta_{m+1}, \zeta_{m+2})),\end{aligned}$$

a contradiction. Thus,

$$\max\Big\{\check{d}_b\Big(\zeta_m, \zeta_{m+1}\Big), \check{d}_b(\zeta_{m+1}, \zeta_{m+2})\Big\} = \check{d}_b\Big(\zeta_m, \zeta_{m+1}\Big).$$

By (25), we get that

$$\begin{aligned}\Lambda(\check{d}_b(\zeta_{m+1}, \zeta_{m+2})) &\le \Upsilon\Big(\Lambda\Big(\check{d}_b\Big(\zeta_m, \zeta_{m+1}\Big)\Big)\Big)\\ &< \Lambda\Big(\check{d}_b\Big(\zeta_m, \zeta_{m+1}\Big)\Big).\end{aligned}$$

It follows from conditions (Φ1)

$$\check{d}_b(\zeta_{m+1}, \zeta_{m+2}) < \check{d}_b\Big(\zeta_m, \zeta_{m+1}\Big). \tag{27}$$

From (24), (25), and (27), we get

$$\begin{aligned}\check{d}_b(\zeta_{m+1},\zeta_{m+2}) < \check{d}_b\left(\zeta_m,\zeta_{m+1}\right) &\le s\left[\check{d}_b(\zeta_m,\zeta^*) + \check{d}_b(\zeta^*,\zeta_{m+1})\right]\\ &\le \frac{1}{2}\min\left\{D_b(\zeta^*,\check{S}(\zeta^*)), D_b\left(\zeta_{m+1},\check{T}\left(\zeta_{m+1}\right)\right)\right\}\\ &+\frac{1}{2}\min\left\{D_b(\zeta^*,\check{S}(\zeta^*)), D_b\left(\zeta_{m+1},\check{T}\left(\zeta_{m+1}\right)\right)\right\}\\ &= \min\left\{D_b(\zeta^*, \check{S}(\zeta^*)), \check{d}_b(\zeta_{m+1},\zeta_{m+2})\right\}\\ &\le \check{d}_b(\zeta_{m+1},\zeta_{m+2}),\end{aligned}$$

a contradiction. Hence (22) holds, that is, $\forall n \ge 2$

$$\frac{1}{2s}\min\left\{D_b(\zeta_n,\check{S}(\zeta_n)), D_b(\zeta^*,\check{T}(\zeta^*))\right\} < \check{d}_b(\zeta_n,\zeta^*), \tag{28}$$

holds. By (3), it follows that for every $n \ge 2$

$$\begin{aligned}0 < \Lambda\left(D_b(\zeta_{n+1},\check{T}(\zeta^*))\right) &\le \Lambda\left(s^3 H_b(\check{S}(\zeta_n),\check{T}(\zeta^*))\right)\\ &\le \Upsilon\left(\Lambda\left(U_b(\zeta_n,\zeta^*)\right)\right)\end{aligned} \tag{29}$$

where

$$U_b(\zeta_n,\zeta^*) = \max\left\{\begin{array}{c}\check{d}_b(\zeta_n,\zeta^*), \check{d}_b(\zeta_n,\zeta_{n+1}), D_b(\zeta^*,\check{T}(\zeta^*)),\\ \frac{D_b(\zeta_n,\check{T}(\zeta^*)) + \check{d}_b(\zeta_{n+1},\zeta_{n+1})}{2s}\end{array}\right\}.$$

Now, we show that $\zeta^* \in \check{T}(\zeta^*)$. Suppose on the contrary, $D_b(\zeta^*,\check{T}(\zeta^*)) > 0$. Since d is $*$-continuous,

$$\lim_{n\to\infty} D_b(\zeta_n,\check{T}(\zeta^*)) = D_b(\zeta^*,\check{T}(\zeta^*)). \tag{30}$$

Letting $n \longrightarrow \infty$ in (29) and by using (21), (30), (Φ3), we obtain

$$\begin{aligned}\Lambda\left(D_b(\zeta^*,\check{T}(\zeta^*))\right) &= \lim_{n\to\infty}\Lambda\left(D_b(\zeta_{n+1},\check{T}(\zeta^*))\right)\\ &\le \lim_{n\to\infty}\Upsilon\left(\Lambda\left(U_b(\zeta_n,\zeta^*)\right)\right)\\ &= \Upsilon\left(\Lambda\left(D_b(\zeta^*,\check{T}(\zeta^*))\right)\right)\\ &< \Lambda\left(D_b(\zeta^*,\check{T}(\zeta^*))\right),\end{aligned}$$

which is a contradiction. Therefore, $D_b(\zeta^*,\check{T}(\zeta^*)) = 0$ and from Lemma 18, we obtain $\zeta^* \in \check{T}(\zeta^*)$. Similarly we can show that $\zeta^* \in \check{S}(\zeta^*)$. Thus $\check{S}$ and $\check{T}$ have a common fixed point. □

Corollary 17 *Let $(\omega,\check{d})$ be a complete bMS and $\check{S},\check{T} : \omega \longrightarrow CB_b(\omega)$ be a generalized multivalued (Υ,Λ)-contraction. Suppose that*

(1) *Υ is continuous*
(2) *d is $*$-continuous.*

Then $\check{S}$ and $\check{T}$ have a common fixed point $\zeta^* \in \omega$.

Example 18 Let $X = [0,1]$. Define $\check{d} : \omega\times\omega \to [0,+\infty)$ by $\check{d}(\zeta,\eta) = |\zeta - \zeta|^2$, for all $\zeta,\eta \in \omega$. Clearly, $(\omega,\check{d})$ is a complete bMS with $s = 2$, but $(\omega,\check{d})$ is not a metric space. For $\zeta = 0, \eta = 1$ and $\upsilon = \frac{1}{2}$, we have

$$\check{d}(\zeta,\eta) = 1 > \frac{1}{4} + \frac{1}{4} = \check{d}(\zeta,\upsilon) + \check{d}(\upsilon,\eta).$$

Define $\Lambda : (0,\infty) \longrightarrow (0,\infty)$ by $\Lambda(t) = te^t$, for all $t > 0$. Then $\Lambda \in \Phi$. Also, define $\Upsilon : (0,\infty) \longrightarrow (0,\infty)$ by $\Upsilon(t) = \frac{198t}{200}$, for all $t > 0$. Then Υ is a continuos comparison function. Define the mappings $\check{S},\check{T} : \omega \longrightarrow CB_b(\omega)$ by

$$\check{S}(\zeta) = \left[0,\frac{\zeta}{6}\right] \text{ and } \check{T}(\zeta) = \left[0,\frac{\zeta}{4}\right].$$

Suppose, without any loss of generality, that all ζ,η are nonzero and $\zeta < \eta$. Then

$$\begin{aligned}\Lambda\left(s^3 H_b(\check{S}(\zeta),\check{T}(\eta))\right) &= \Lambda\left(s^3 H_b\left(\left[0,\frac{\zeta}{6}\right],\left[0,\frac{\eta}{4}\right]\right)\right)\\ &= \Lambda\left(8\left|\frac{\zeta}{6}-\frac{\eta}{4}\right|^2\right)\\ &= 8\left|\frac{\gamma}{6}-\frac{\eta}{4}\right|^2 e^{8\left|\frac{\zeta}{6}-\frac{\eta}{4}\right|^2}\\ &\le \frac{198}{200}|\zeta-\eta|^2 e^{|\zeta-\eta|^2}\\ &\le \frac{198}{200}U_b(\zeta,\eta)e^{U_b(\zeta,\eta)}\\ &= \frac{198}{200}\Lambda\left(U_b(\zeta,\eta)\right)\\ &= \Upsilon\left(\Lambda\left(U_b(\zeta,\eta)\right)\right).\end{aligned}$$

Hence all the hypotheses of Corollary 17 are satisfied, and thus, $\check{S}$ and $\check{T}$ have a common fixed point.

Corollary 19 *Let $(\omega,\check{d})$ be a complete bMS such that d is a continuous function and $\check{S},\check{T} : \omega \longrightarrow \omega$ be a generalized (Υ,Λ)-type contraction, that is, if there exist a comparison function Υ and $\Lambda \in \Phi$ such that, for all $\zeta,\eta \in \omega$ with $\check{S}(\zeta) \ne \check{T}(\eta)$,*

$$\Lambda\left(s^3\check{d}(\check{S}(\zeta),\check{T}(\eta))\right) \le \Upsilon\left[\Lambda\left(U_b(\zeta,\eta)\right)\right],$$

where

$$U_b(\zeta,\eta) = \max\left\{\check{d}_b(\zeta,\eta), \check{d}_b(\zeta,\check{S}(\zeta)), \check{d}_b(\eta,\check{T}(\eta)), \frac{\check{d}_b(\zeta,\check{T}(\eta)) + \check{d}_b(\eta,\check{S}(\zeta))}{2}\right\}.$$

If ψ is continuous, then S and T have a unique common fixed point $x^* \in X$.

Corollary 20 *Let $(\omega,\check{d})$ be a complete bMS and $\check{S} : \omega \longrightarrow CB_b(\omega)$ be a generalized multivalued (Υ,Λ)-Suzuki contraction, that is, if there exist a comparison function Υ and $\Lambda \in \Phi$ such that, for all, $\zeta,\eta \in \omega$ with $\check{S}(\zeta) \neq \check{S}(\eta)$,*

$$\frac{1}{2s}D_b(\zeta,\check{S}(\zeta)) < \check{d}_b(\zeta,\eta) \Longrightarrow \Lambda(s^3 H_b(\check{S}(\zeta),\check{S}(\eta))) \leq \Upsilon[\Lambda(U_b(\zeta,\eta))],$$

where

$$U_b(\zeta,\eta) = \max\left\{\check{d}_b(\zeta,\eta), D_b(\zeta,\check{S}(\zeta)), D_b(\eta,\check{S}(\eta)), \frac{D_b(\zeta,\check{S}(\eta)) + D_b(\eta,\check{S}(\zeta))}{2}\right\}.$$

Suppose that

(1) *Υ is continuous*
(2) *d is $*$-continuous.*

Then $\check{S}$ has a fixed point $\zeta^* \in \omega$.

Corollary 21 *Let $(\omega,\check{d})$ be a complete bMS such that $\check{d}$ is a continuous function and $\check{S} : \omega \longrightarrow \omega$ be a generalized (Υ,Λ)-type Suzuki contraction. If Υ is continuous. Then $\check{S}$ has a unique fixed point $\zeta^* \in \omega$.*

Corollary 22 [6] *Let $(\omega,\check{d})$ be a complete MS such that $\check{d}$ is a continuous function and $\check{S} : \omega \longrightarrow \omega$ be a generalized (Υ,Λ)-type Suzuki contraction. If Υ is continuous. Then $\check{S}$ has a unique fixed point $\zeta^* \in \omega$.*

Remark 23 Theorem 16 is a generalization of the main results in Suzuki [10] and the recent result in Liu [6].

Remark 24 Corollary 17 is a generalization of Nadler [8] and the recent results in Jleli et al. [3, 4], HanÇer et al. [7] and Vetro [5].

Application

In this section, we present an application of our result in solving functional equations arising in dynamic programming.

Decision space and a state space are two basic components of dynamic programming problem. State space is a set of states including initial states, action states and transitional states. So a state space is set of parameters representing different states. A decision space is the set of possible actions that can be taken to solve the problem. We assume that U and V are Banach spaces, $W \subseteq U$, $D \subseteq V$ and

$$\begin{aligned} \xi &: \quad W \times D \longrightarrow W \\ g,u &: \quad W \times D \longrightarrow \mathbb{R} \\ \Gamma,\Psi &: \quad W \times D \times \mathbb{R} \longrightarrow \mathbb{R}, \end{aligned}$$

and for more details on dynamic programming we refer to ([29–32]). Suppose that W and D are the state and decision spaces, respectively, and the problem of dynamic programming related reduces to the problem of solving the functional equations

$$p(\zeta) = \sup_{\eta\in D}\{g(\zeta,\eta) + \Gamma(\zeta,\eta,p(\xi(\zeta,\eta)))\}, \text{ for } \zeta \in W \tag{31}$$

$$q(\zeta) = \sup_{\eta\in D}\{u(\zeta,\eta) + \Psi(\zeta,\eta,q(\xi(x,y)))\}, \text{ for } \zeta \in W, \tag{32}$$

We aim to give the existence and uniqueness of common and bounded solution of functional equations given in (31) and (32). Let $B(W)$ denote the set of all bounded real-valued functions on W. For $h,k \in B(W)$, define

$$\check{d}(h,k) = \left\|(h-k)^2\right\|_\infty = \sup_{x\in W}|h\zeta - k\zeta|^2. \tag{33}$$

Suppose that the following conditions hold:

$(B1)$: Γ,Ψ,g, and u are bounded and continuous.

$(B2)$: For $\zeta \in W$, $h \in B(W)$ and $b > 0$, define $E,A : B(W) \longrightarrow B(W)$ by

$$Eh(\zeta) = \sup_{\eta\in D}\{g(\zeta,\eta) + \Gamma(\zeta,\eta,h(\xi(\zeta,\eta)))\}, \tag{34}$$

$$Ah(\zeta) = \sup_{\eta\in D}\{u(\zeta,\eta) + \Psi(\zeta,\eta,h(\xi(\zeta,\eta)))\}. \tag{35}$$

Moreover, for every $(\zeta,\eta) \in W \times D, h,k \in B(W)$ and $t \in W$ we have

$$|\Gamma(\zeta,\eta,h(t)) - \Psi(\zeta,\eta,k(t))| \leq \sqrt{\frac{U_b(h(t),k(t))}{s^3\left(U_b(h(t),k(t)) + 1\right)}} \tag{36}$$

where

$$U_b((h(t),k(t)) = \max\left\{\check{d}(h(t),k(t)), \check{d}(h(t),Eh(t)), \check{d}(k(t),Ak(t)), \frac{\check{d}(h(t),Ak(t)) + \check{d}(k(t),Eh(t))}{2s}\right\}.$$

Theorem 25 *Assume that the conditions* $(B1)-(B2)$ *are satisfied. Then the system of functional equations* (31) *and* (32) *has a unique common and bounded solution in* $B(W)$.

Proof Note that $(B(W), d)$ is a complete bMS with constant $s=2$. By $(B1)$, E, A are self-maps of $B(W)$. Let λ be an arbitrary positive number and $h_1, h_2 \in B(W)$. Choose $\zeta \in W$ and $\eta_1, \eta_2 \in D$ such that

$$Eh_1 < g(\zeta,\eta_1) + \Gamma(\zeta,\eta_1,h_1(\xi(\zeta,\eta_1))) + \lambda \tag{37}$$

$$Ah_2 < g(\zeta,\eta_2) + \Psi(\zeta,\eta_2,h_2(\xi(\zeta,\eta_2))) + \lambda \tag{38}$$

Further from (37) and (38), we have

$$Eh_1 \geq g(\zeta,\eta_2) + \Gamma(\zeta,\eta_2,h_1(\xi(\zeta,\eta_2)) \tag{39}$$

$$Ah_2 \geq g(\zeta,\eta_1) + \Psi(\zeta,\eta_1,h_2(\xi(\zeta,\eta_1)). \tag{40}$$

Then (37) and (40) together with (36) imply

$$\begin{aligned} Eh_1(\zeta) - Ah_2(\zeta) &< \Gamma(\zeta,\eta_1,h_1(\xi(\zeta,\eta_1))) \\ &\quad - \Psi(\zeta,\eta_1,h_2(\xi(\zeta,\eta_1))) + \lambda \\ &\leq \big|\Gamma(\zeta,\eta_1,h_1(\xi(\zeta,\eta_1))) \\ &\quad -\Psi(\zeta,\eta_1,h_2(\xi(\zeta,\eta_1)))\big| + \lambda \\ &\leq \sqrt{\frac{U_b(h_1(\zeta),h_2(\zeta))}{s^3(U_b(h_1(\zeta),h_2(\zeta)+1)}} + \lambda. \end{aligned} \tag{41}$$

Then (38) and (39) together with (36) imply

$$\begin{aligned} Ah_2(\zeta) - Eh_1(\zeta) &\leq \Gamma(\zeta,\eta_2,h_1(\xi(\zeta,\eta_2))) \\ &\quad - \Psi(\zeta,\eta_2,h_2(\xi(\zeta,\eta_2))) + \lambda \\ &\leq \big|\Gamma(\zeta,\eta_2,h_1(\xi(\zeta,\eta_2))) \\ &\quad -\Psi(\zeta,\eta_2,h_2(\xi(\zeta,\eta_2))\big| + \lambda \\ &\leq \sqrt{\frac{U_b(h_1(\zeta),h_2(\zeta))}{s^3\big(U_b(h_1(\zeta),h_2(\zeta)+1\big)}} + \lambda, \end{aligned} \tag{42}$$

where

$$\begin{aligned} &U_b((h_1(\zeta),h_2(\zeta)) \\ &\quad = \max\Big\{\check{d}(h_1(\zeta),h_2(\zeta)),\check{d}(h_1(\zeta), \\ &\quad Eh_1(\zeta)),\check{d}(h_2(\zeta),Ah_2(\zeta)), \\ &\quad \frac{\check{d}(h_1(\zeta),Ah_2(\zeta)) + \check{d}(h_2(\zeta),Eh_1(\zeta))}{2s}\Big\}. \end{aligned}$$

From (41), (42), and since $\lambda > 0$ was taken as an arbitrary number, we obtain

$$\big|Eh_1(\zeta) - Ah_2(\zeta)\big| \leq \sqrt{\frac{U_b(h_1(\zeta),h_2(\zeta))}{s^3\big(U_b(h_1(x),h_2(x)+1\big)}}.$$

Thus,

$$\big|Eh_1(\zeta) - Ah_2(\zeta)\big|^2 \leq \frac{U_b(h_1(\zeta),h_2(\zeta))}{s^3\big(U_b(h_1(x),h_2(x)+1\big)}. \tag{43}$$

The inequality (43) implies

$$d(Eh_1(\zeta),Ah_2(\zeta)) \leq \frac{U_b(h_1(\zeta),h_2(\zeta))}{s^3\big(U_b(h_1(x),h_2(x)+1\big)}. \tag{44}$$

Taking $\Lambda(t) = t, t>0$ and $\Upsilon(t) = \frac{t}{t+1}, t>0$, we get

$$\Lambda\big(s^3 d(Eh_1(\zeta),Ah_2(\zeta))\big) \leq \Upsilon\big(\Lambda\big(U_b(h_1(\zeta),h_2(\zeta))\big)\big). \tag{45}$$

Therefore, all the conditions of Corollary 17 immediately hold. Thus, E and A have a common fixed point $h^* \in B(W)$, that is, $h^*(\zeta)$ is a unique, bounded and common solution of the system of functional equations (31) and (32). □

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