Perfect 2-coloring of the quartic graphs with order at most 8

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Abstract

In this paper, we study perfect 2-coloring of the quartic graphs with at most 8 vertices. The problem of the existence of perfect coloring is a generalization of the concept of completely regular codes, given by Delsarte.

Keywords Perfect coloring · Equitable partition · Quartic graph · Perfect code · Regular graph

Mathematics Subject Classification 05C15 · 05C78

Introduction

Let $X = (V(X), E(X))$ be a connected graph. A vertex coloring with m colors is perfect if there is a matrix $M = (\alpha_{ij})_{i,j=1,2,\dots,m}$ such that the cardinal of vertices of label j connected to a specified vertex of label i is equal to α_{ij} .

In particular, all vertices of the same color in X have the same degree. This type of coloring gives a partition of $V(X)$ that is called **equitable partitions**. These partitions of vertices have two properties as follows: The first property is that the vertices of each part P_i induce a regular graph, and the second is the edges between P_i and P_j induce a half-regular graph. In Section 3, we find the parameter matrices of a perfect 2-coloring. Equitable partitions were previously studied in [7].

In fact, the concept of a perfect m -coloring is bridge of among algebraic combinatorics, graph theory and coding theory (including 1-perfect codes). For the definition of 1-perfect codes and more details, we refer the reader to [1].

This type of coloring is generalized the concept of completely regular codes introduced by P. Delsarte ([8]). The

problem for the Johnson graphs has been studied (see [4], see [3], see [6]).

In subsequent attempts, the problem of existence of perfect coloring was examined on other graphs. For example, Fon-Der-Flass settled perfect coloring with two colors of n -dimensional hypercube graphs Q_n for $n < 24$ (see [5]). In this paper, we will list all parameter matrices of perfect 2-coloring of the quartic graphs up to order 8.

Notations and preliminaries result on the existence of perfect coloring

In this section, some basic definitions that will be used in this paper are given and we declare some conditions for existence of perfect 2-coloring of the quartic graphs.

In this paper, all the graphs are simple. A graph $X = (V(X), E(X))$ is k -regular if for every $v \in V(X)$ we have $\deg(v) = k$, where k is a nonnegative integer number. In particular, graphs that are regular of degrees 3, 4, 5 and 6 are called, respectively, cubic, quartic, quantic and sextic.

In the following remark, we classify the quartic graphs with at most 8 vertices (see [9]).

Remark 1 The quartic graphs are divided into four classes based on their vertices. This classification is shown in Figs. 1, 2, 3 and 4. For simplicity, we name the vertices as shown in these figures.

We now introduce the concept of perfect coloring.

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Fig. 1 Class 1: The quartic graph with 5 vertices

Fig. 2 Class 2: The quartic graph with 6 vertices

Fig. 3 Class 3: The quartic graph with 7 vertices

Definition 1 Let $X = (V(X), E(X))$ be a connected graph. A perfect m -coloring with parameter matrix $M = (\alpha_{ij})_{i,j=1,2,\dots,m}$ is a map C from $V(X)$ to the set of colors $\{1, 2, \dots, m\}$ such that C is surjective, and for a fix vertices v where $C(v) = i$ we have $|\{w \in V(X) \mid w \text{ is adjacent to } v, C(w) = j\}| = \alpha_{ij}$, for all i, j .

In this case, we say C is a $P.m.C$ (or just $P.C$). Here, we study $P.2.C$, where two colors are red and blue.

In this case, we consider the parameter matrix of $P.2.C$ generated by replacing the colors with the primary coloring

is equal. This means the parameter matrix $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ is equal to the parameter matrix $\begin{pmatrix} b' & b \\ a' & a \end{pmatrix}$. We call **equality of parameter matrix**. Some properties of $P.C$ have been studied recently (see [4, 6, 8]). The main result of them is the next proposition that enumerates the cardinal number of blue vertices in a $P.2.C$ with matrix $M = (\alpha_{ij})_{i,j=1,2}$ (see [3]).

Proposition 1 Assume that B is the set of all blue vertices in a $P.2.C$ of a graph $X = (V(X), E(X))$ with matrix $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$. Then we have

$$|B| = \frac{a' |V(X)|}{b + a'}. \quad (1)$$

Now, the following remark provides useful information about calculation $P.2.C$ of the quartic graphs.

Remark 2 Suppose that $X = (V(X), E(X))$ be a simple connected graph. Then necessary conditions for existence of $P.2.C$ of X with parameter matrix $M = \begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ are listed as follows:

1. Degree condition: This simple condition that results from the regularity of X tells us:

$$a + b = a' + b' = 4.$$

2. Connectedness condition: Since the graph X is connected, another condition is induced as follows:

$$b, a' \neq 0.$$

We will define a new definition for the third condition. The number μ is called an **eigenvalue of a graph X** , if μ is an eigenvalue of the adjacency matrix of X . The number μ is called an **eigenvalue of a $P.2.C$** with the matrix M , if μ is an eigenvalue of M .

The next theorem gives us the third condition and the connection between two types of eigenvalues (see [4]):

Theorem 1 Let C be a $P.m.C$ of a graph X . Thus any eigenvalues of C are an eigenvalues of X .

It has been proved that a $P.2.C$ of a k -regular graph X has just two eigenvalues (see [4]).

Proposition 2 If C be a $P.2.C$ with parameters matrix $M = (\alpha_{ij})_{i,j=1,2}$ of a regular graph X of valency k , then the numbers $\alpha_{11} - \alpha_{21}$ and k are eigenvalues of C and therefore are eigenvalues of X .

Lemma 1 Given an arbitrarily connected quartic graph X , the following ten matrices are the only matrices that can be selected as **acceptable parameter matrices** of a perfect 2-coloring C of X :

Fig. 4 Class 4: The quartic graph with 8 vertices

Table 1 Acceptable parameter matrices of quartic graph with order 5

Matrices	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}
$ B $	$5/2$	$15/7$	$5/3$	1	$5/2$	2	$5/4$	$5/2$	$5/3$	$5/2$
Acceptability	$\times$	$\times$	$\times$	$\checkmark$	$\times$	$\checkmark$	$\times$	$\times$	$\times$	$\times$

Table 2 Acceptable parameter matrices of quartic graph with order 6

Matrices	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}
$ B $	3	$18/7$	2	$6/5$	3	$12/5$	$3/2$	3	2	3
Acceptability	$\checkmark$	$\times$	$\checkmark$	$\times$	$\checkmark$	$\times$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$

Table 3 Acceptable parameter matrices of quartic graph with order 7

Matrices	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}
$ B $	$7/2$	3	$7/3$	$7/5$	$7/2$	$14/5$	$7/4$	$7/2$	$7/3$	$7/2$
Acceptability	$\times$	$\checkmark$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

$$\begin{aligned}
 M_1 &= \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 4 \\ 3 & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}, M_4 = \begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}, \\
 M_5 &= \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, M_6 = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}, M_7 = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}, M_8 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, M_9 = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}, \\
 M_{10} &= \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.
 \end{aligned}$$

Proof By Remark 2, Theorem 1, Proposition 2 and equality of parameter matrix, proof is clear. $\square$

Note that the word **acceptable** in the previous lemma states that some of the listed matrices may not be a parameter matrix of a P.2.C of X . Therefore, we reduce them by Proposition 1 in the next lemma (Tables 1, 2, 3, 4):

Table 4 Acceptable parameter matrices of quartic graph with order 8

Matrices	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}
$ B $	4	24 / 7	8 / 3	8 / 5	4	16 / 5	2	4	8 / 3	4
Acceptability	✓	×	×	×	✓	×	✓	✓	×	✓

Lemma 2 Let X be the quartic graph with at most 8 vertices. According to order of X , we can reduce the listed parameter matrices in Lemma 1 as follows:

Proof Since the number of blue vertices ($|B|$) is not an integer, there is not exit a $P.2.C$ of X with the given parameter matrix M_i . $\square$

Corollary 1 If X be a connected quartic graph with at most 8 vertices, then only the matrices listed in the following table are acceptable as parameter matrices of a $P.2.C$ of X (Table 5):

Perfect 2-coloring of the quartic graphs with order at most 8

In the previous section, we determined all acceptable parameter matrices of a $P.2.C$ of a connected quartic graph with at most 8 vertices. In this section, we show which of them have a **structure**. In the other words, we introduce a perfect 2-coloring C for them.

Lemma 3 Assume that X be a connected k -regular graph. Then there are no $P.2.C$ with parameter matrix $M = \begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ such that $a = |B| \leq k$.

Proof Suppose, contrary to our claim, there is a $P.2.C$ with presumed conditions. Let V be an arbitrary vertex with blue color. So, since $a = k$, we conclude that all k vertices adjacent to v must be blue. This is in contradiction with $|B| \leq k$. $\square$

Corollary 2 There are no $P.2.C$ of Q_2 with the matrices M_9 and M_{10} .

Proof From Lemma 2, we conclude that graph Q_2 has the conditions of Lemma 3, So the proof is straightforward. $\square$

We will discuss about acceptability of matrix M_7 is the following lemma:

Lemma 4 There are no $P.2.C$ of graphs Q_6 , Q_7 and Q_8 with matrix M_7 .

Proof Suppose that there is a $P.2.C$ of the graphs with M_7 . Note that both adjacent vertices of the graph Q_6 , Q_7 and Q_8 have a common vertex. On the other hand, from $\alpha_{11} = 1$ in M_7 , we obtain that there are two adjacent vertices with blue color. So, these two blue vertices have common red vertex. This is a contradiction with $\alpha_{21} = 1$. $\square$

Lemma 5 There are no $P.2.C$ of graphs Q_5 , Q_6 , Q_7 , Q_8 and Q_9 with the matrices M_1 and M_{10} .

Proof The proof of this lemma is similar to the proof of Lemma 4. $\square$

In the following, we will state the main result of this article. For this, we determine which class of the quartic graphs have a structure.

Theorem 2 Only parameter matrices of $P.2.C$ of Q_1 are M_4 and M_6 .

Proof By Corollary 1, only matrices M_4 and M_6 are acceptable as parameter matrices of a $P.2.C$ of Q_1 . Two mapping C_1 and C_2 determine two structures for Q_1 with matrices M_4 and M_6 , respectively, as follows:

$$\begin{aligned} C_1(v_1) = 1, C_1(v_2) = C_1(v_3) = C_1(v_4) = C_1(v_5) = 2, \\ C_2(v_1) = C_2(v_2) = 1, C_2(v_3) = C_2(v_4) = C_2(v_5) = 2. \end{aligned}$$

It is clear that C_1 and C_2 are $P.2.C$ with matrices M_4 and M_6 , respectively. $\square$

Theorem 3 The parameter matrices of $P.2.C$ of Q_2 are M_3 and M_8 .

Proof From Corollary 1, we deduce that acceptable parameter matrices of Q_2 are M_1, M_3, M_5, M_8, M_9 and M_{10} . We next show that only M_3 and M_8 have a structure. For this, let C_1 and C_2 be given by

Table 5 Acceptable parameter matrices of quartic graph with at most order 8

Graphs	Order 5	Order 6	Order 7	Order 8
Matrices	M_4, M_6	$M_1, M_3, M_5, M_8, M_9, M_{10}$	M_2	$M_1, M_5, M_7, M_8, M_{10}$

$$C_1(v_1) = C_1(v_6) = 1, C_1(v_2) = C_1(v_3) = C_1(v_4) = C_1(v_5) = 2,$$

$$C_2(v_1) = C_2(v_3) = C_2(v_4) = 1, C_2(v_2) = C_2(v_5) = C_2(v_6) = 2.$$

It is obvious that the function C_1 and C_2 stats a $P.2.C$ for Q_2 with matrices M_3 and M_8 , respectively. In the following, we prove that other matrices can not be as parameter matrices of Q_2 . First, we show that there is no $P.2.C$ with the matrix M_1 for the graph Q_2 . Contrary to our claim, suppose that C is a $P.2.C$ of Q_2 with the matrix M_1 . There is no loss of generality in assuming $C(v_1) = 1$. From M_1 , we conclude that $C(v_2) = C(v_3) = C(v_4) = C(v_5) = 2$. Hence only one vertex remains, even if it is blue, in contradiction with the number of blue vertices ($|B| = 3$).

We have shown in Corollary 2 that there is no $P.2.C$ of Q_2 with the matrices M_9 and M_{10} .

Now we prove that there is no $P.2.C$ of Q_2 with the matrix M_5 . Suppose that there is a $P.2.C$ of Q_2 with the matrix M_5 , say C . From $\alpha_{11} = 1$, without loss of generality, we assume that $C(v_1) = C(v_2) = 1$. Thus, from $\alpha_{12} = 3$, we have $C(v_3) = C(v_4) = C(v_5) = C(v_6) = 2$. In this case, the red vertex v_6 is connected to the other three red vertices, which leads to a contradiction. $\square$

Theorem 4 Only the graph Q_4 has a $P.2.C$ between the quartic graphs with 7 vertices.

Proof According to Corollary 1, we see that only parameter matrix of Q_3 and Q_4 is M_2 . First, we give a structure for Q_4 with matrix M_2 by defining the function C as follows:

$$C(v_1) = C(v_2) = C(v_3) = C(v_4) = 2, C(v_5) = C(v_6) = C(v_7) = 1.$$

It is easy to check that C is a $P.2.C$ of Q_4 with matrix M_2 . Now we prove that there is no $P.2.C$ with the matrix M_2 for the graph Q_3 . Let C be a perfect 2-coloring of Q_3 with the matrix M_2 . Let $C(v_1) = 1$. From $\alpha_{12} = 4$, we deduce that all vertices are red except for adjacent vertices v_4 and v_5 . On the other hand, we know that $|B| = 3$. Thus $C(v_4) = C(v_5) = 1$. This is a contradiction with $\alpha_{11} = 0$. $\square$

Finally, we get the following theorem (Table 6):

Theorem 5 The parameter matrices of a $P.2.C$ of a quartic graph of order 8 are listed in the following table:

Proof From Corollary 1, we conclude that acceptable parameter matrices of the quartic graphs with order 8 are M_1, M_5, M_7, M_8 and M_{10} . First, we determine structures for the quartic graphs with order 8 with their listed parameter matrices in Table 6 as follows:

Structure for Q_5 with matrix M_7 :

$$C(v_1) = C(v_2) = 1, C(v_j) = 2 \text{ for } j = 3, 4, \dots, 8.$$

Structure for Q_5 with matrix M_8 :

$$C(v_1) = C(v_2) = C(v_3) = C(v_4) = 1, C(v_j) = 2 \text{ for } j = 5, 6, \dots, 8.$$

Structure for Q_7 with matrix M_7 :

Table 6 Parameter matrices of the quartic graphs of order 8

Matrices graphs	M_1	M_5	M_7	M_8	M_{10}
Q_5	$\times$	$\times$	$\sqrt$	$\sqrt$	$\times$
Q_6	$\times$	$\times$	$\times$	$\times$	$\times$
Q_7	$\times$	$\times$	$\sqrt$	$\sqrt$	$\times$
Q_8	$\times$	$\sqrt$	$\times$	$\sqrt$	$\times$
Q_9	$\times$	$\sqrt$	$\sqrt$	$\sqrt$	$\times$
Q_{10}	$\sqrt$	$\times$	$\sqrt$	$\sqrt$	$\times$

$$C(v_1) = C(v_2) = 1, C(v_j) = 2 \text{ for } j = 3, 4, \dots, 8.$$

Structure for Q_7 with matrix M_8 :

$$C(v_1) = C(v_2) = C(v_3) = C(v_4) = 1, C(v_j) = 2 \text{ for } j = 5, 6, \dots, 8.$$

Structure for Q_8 with matrix M_5 :

$$C(v_1) = C(v_3) = C(v_5) = C(v_7) = 1, C(v_2) = C(v_4) = C(v_6) = C(v_8) = 2.$$

Structure for Q_8 with matrix M_8 :

$$C(v_1) = C(v_2) = C(v_3) = C(v_4) = 1, C(v_j) = 2 \text{ for } j = 5, 6, \dots, 8.$$

Structure for Q_9 with matrix M_5 :

$$C(v_1) = C(v_3) = C(v_5) = C(v_7) = 1, C(v_2) = C(v_4) = C(v_6) = C(v_8) = 2.$$

Structure for Q_9 with matrix M_7 :

$$C(v_1) = C(v_2) = 1, C(v_j) = 2 \text{ for } j = 3, 4, \dots, 8.$$

Structure for Q_9 with matrix M_8 :

$$C(v_1) = C(v_2) = C(v_3) = C(v_4) = 1, C(v_5) = C(v_6) = C(v_7) = C(v_8) = 2.$$

Structure for Q_{10} with matrix M_1 :

$$C(v_1) = C(v_3) = C(v_5) = C(v_7) = 1, C(v_2) = C(v_4) = C(v_6) = C(v_8) = 2.$$

Structure for Q_{10} with matrix M_7 :

$$C(v_1) = C(v_2) = 1, C(v_j) = 2 \text{ for } j = 3, 4, \dots, 8.$$

Structure for Q_{10} with matrix M_8 :

$$C(v_1) = C(v_2) = C(v_3) = C(v_4) = 1, C(v_5) = C(v_6) = C(v_7) = C(v_8) = 2.$$

It is easily to see that the above functions are perfect 2-coloring with their expressed parameter matrices.

Note that, from Lemma 5, we see that there is no perfect 2-coloring for Q_5, Q_6, Q_7, Q_8 and Q_9 with the matrix M_1 .

Also, there is no perfect 2-coloring of Q_5 with the matrix M_5 . On the contrary, suppose that C is a perfect 2-coloring of Q_5 with the matrix M_5 . There is no loss of generality in assuming $C(v_1) = C(v_2) = 1$. From $\alpha_{21} = 3$ and $C(v_1) = 1$, we obtain $C(v_4) = C(v_5) = 2$. On the other hand, from $\alpha_{21} = 3$ and $C(v_2) = 1$, we have $C(v_7) = C(v_3) = 2$. Thus vertex v_4 with red color is adjacent to the other two red vertices (v_3 and v_5), contrary to $\alpha_{22} = 1$. The proof for Q_5 with matrix M_{10} is similar.

Now, we show there are no $P.2.C$ of Q_6 with any matrices M_1, M_5, M_7, M_8 and M_{10} . From Lemmas 4 and 5, we see that there is no $P.2.C$ of Q_6 with matrices M_7, M_1 and M_{10} , respectively. It suffices to show that there is no

Table 7 All parameter matrices of the quartic graphs up to order 8

Graphs	Parameter matrices of a perfect 2-coloring
Q_1	M_4, M_6
Q_2	M_3, M_8
Q_3	The Q_3 has no any perfect 2-coloring
Q_4	M_2
Q_5	M_7, M_8
Q_6	The Q_6 has no any perfect 2-coloring
Q_7	M_7, M_8
Q_8	M_5, M_8
Q_9	M_5, M_7, M_8
Q_{10}	M_1, M_7, M_8

$P.2.C$ of Q_6 with matrices M_5 and M_8 . Suppose that C is a $P.2.C$ of Q_6 with matrix M_5 . Let $C(v_1) = 1$. From $\alpha_{11} = 1$ we deduce that each of v_2, v_5, v_6 or v_7 can be the only blue vertex adjacent to v_1 . Without restriction of generality, we can assume that $C(v_2) = 1$. Thus, from M_5 we have $C(v_3) = C(v_4) = C(v_5) = C(v_6) = C(v_7) = 2$, which is a contradiction with $\alpha_{22} = 1$. The same conclusion can be drawn for M_8 .

For graph Q_7 , we only need to show that there is no $P.2.C$ of Q_7 with matrix M_5 . Suppose there is a $P.2.C$ of Q_7 with matrix M_5 . Without loss of generality, we can suppose that $C(v_1) = C(v_2) = 1$. Since $\alpha_{12} = 3$, we have $C(v_6) = C(v_7) = C(v_8) = 2$, contrary to $\alpha_{22} = 1$. It remains only to prove that there is no $P.2.C$ of Q_{10} with matrix M_5 . On the contrary, suppose that there is a $P.2.C$ of Q_{10} with matrix M_5 . From $\alpha_{11} = 1$, we conclude that the two adjacent vertices in Q_{10} , all other vertices will be red. This gets to a contradiction with $|B| = 4$, and the proof is complete. $\square$

Corollary 3 *The only $P.2.C$ of the quartic graphs with at most 8 vertices is perfect coloring with listed matrices in the following table: (Table 7).*

Conclusion

The perfect coloring of graphs is closely related to coding theory, algebraic theory, graph theory and combinatorics, including designs. We can consider perfect m -coloring as

a generalization of the concept of completely regular codes presented by P. Delsarte for the first time. This class of codes has been of interest to coding theorists and graph theorists alike. In this way, the problem of existence of perfect coloring of the Johnson graphs, the generalized Petersen graphs and the n -dimensional hypercube graphs Q_n for $n < 24$ has been settled in [2–4, 6] and [5].

In this paper, we have listed all parameters of existing perfect 2-coloring in the quartic graphs with order at most 8. The question of existence of perfect 3-coloring of the quartic graphs remains open.

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