Generalization of Finsler metrics on the product of Finsler manifolds

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Abstract

In Riemannian manifolds, there exists a canonical Riemannian metric on the product of them (Lee in Riemannian geometry an introduction to curvature, Springer, New York, 1997). But at the product of Finsler manifolds, the canonical Finsler metric has not been defined. In this paper, we are going to study the product of Finsler manifolds and give a canonical Finsler metric on it.

Keywords Product of Finsler manifolds · Finsler manifolds · Minkowski structure

Mathematics Subject Classification 58B20 · 53C60

Historical remarks

The section brings a survey of Finsler geometry and product manifolds; a Finsler metric on a manifold is a collection of Minkowski norms F_x in tangent space at x such that F_x varies smoothly in x .

In 1854, Reimann introduced the geometry based on the element of arc length

$$ds = F(x^1, x^2, \dots, x^n, dx^1, dx^2, \dots, dx^n).$$

where F is positively homogeneous of degree 1 in dx^i [2, 3]. For more than half a century, there had been no progress until P.Finsler introduced the Riemannian–Finsler geometry (Finsler geometry for short) in his thesis in 1918. A Finsler metric $F = F(x, y)$ is defined on tangent bundle TM , such that gives a for $x \in M$ by Minkowski norm [4–7],

$$F|_{T_x M} = F_x : T_x M \mapsto [0, \infty).$$

Chern and Shen are defined product metric on Finsler spaces [8] as:

(1) For Finsler manifolds (M_1, F_1) and (M_2, F_2) , $x = (x_1, x_2) \in M = M_1 \times M_2$ and $y = (y_1, y_2) \in T_{(x_1, x_2)}(M_1 \times M_2)$, let $F : M = M_1 \times M_2 \mapsto [0, \infty)$ are defined by

$$F(x, y) := \begin{cases} F_1(x_1, y_1) & \text{if } y = y_1 \oplus 0 \\ F_2(x_2, y_2) & \text{if } y = 0 \oplus y_2 \end{cases} \quad (1.1)$$

Then F is Finsler metric on $M_1 \times M_2$, where $T_x M = T_{x_1} M_1 \oplus T_{x_2} M_2$.

(2) For Riemannian manifolds $(M_1, g_1), (M_2, g_2)$ let $f : [0, \infty) \times [0, \infty) \mapsto [0, \infty)$ be a c^∞ function satisfying:

$$f(\lambda s, \lambda t) = \lambda f(s, t), \forall \lambda > 0, \text{ and } f(s, t) > 0, \forall (s, t) \neq (0, 0).$$

Then for all $(x_1, x_2) \in M_1 \times M_2$ and $(y_1, y_2) \in TM_1 \times TM_2$, F defined function by

$$F(x, y) := \sqrt{f\left([g_1(x_1, y_1)]^2, [g_2(x_2, y_2)]^2\right)}$$

is a Finsler metric $M = M_1 \times M_2$.

(3) Kozma et al. have defined a Twisted Products Finsler Manifolds [9] as:

Let $(M_1, F_1), (M_2, F_2)$ be two Finsler manifolds and $M = M_1 \times M_2$, then for all $(x_1, x_2) \in M_1 \times M_2$ and $(y_1, y_2) \in T_{(x_1, x_2)}(M_1 \times M_2) - \{(0, 0)\} \equiv (T_{x_1} M_1 - \{0\}) \times (T_{x_2} M_2 - \{0\})$ and c^∞ function $f : M_1 \times M_2 \mapsto R^+$, the Twisted Products metric is defined by

$$F(y_1, y_2) := \sqrt{F_1^2(x_1, y_1) + f^2(x_1, x_2) F_2^2(x_2, y_2)}.$$

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$$(y_1, y_2) \in TM_1 - \{0\}$$

In this paper, we are going to generalize a new product Finsler metric F on $M = M_1 \times M_2$, then we call (M, F) as canonical product Finsler manifolds.

Introduction and preliminaries

We recall some definitions and fundamental results in Finsler geometry.

Definition 2.1 Let V be a n -dimensional real vector space. A *Minkowski norm* on V is a functional F on V , which is smooth on $V - \{0\}$ and satisfies the following conditions:

- (1) $F(u) \geq 0$, $\forall u \in V$;
- (2) $F(\lambda u) = \lambda F(u)$, $\forall \lambda > 0$, $\forall u \in V$;
- (3) for any basis $e_1, \dots, e_n$ of V , write $F(y) = F(y^1, \dots, y^n)$, $y = y^j e_j$. Then the Hessian matrix

$$(g_{ij}) := \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positive definite at any point of $V - \{0\}$.

The pair (V, F) is called Minkowski space.

Definition 2.2 Let M be a (connected) smooth manifold. A Finsler metric on M is a function $F : TM \rightarrow [0, +\infty)$ such that

- (1) F is C^∞ on the slit tangent bundle $TM - \{0\}$;
- (2) The restriction of F to any $T_p M$, $p \in M$ is a Minkowski norm.

The pair (M, F) is called Finsler manifold or Finsler space.

Let (M, F) be a Finsler space and $(x^1, \dots, x^n)$ be a local coordinate system on an open subset U of M . Then $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$ form a basis for the tangent space at any point in U .

Theorem 2.3 (Euler's) Suppose a real-valued H on R^n is differentiable away from the origin of R^n . Then the following two statements are equivalent [5]

- (I) H is positively homogeneous of degree r . That is:

$$H(\lambda y) = \lambda^r H(y); \quad \forall \lambda, \quad \lambda > 0$$

- (II) The radial directional derivative of H is r times H . Namely,

$$y^i H_{y^i}(y) = rH(y).$$

Corollary 2.4 Let F be positively homogeneous of degree 1 on R^n . By using Euler's theorem, we can show that:

- (a) $y^i F_{y^i} = F$.
- (b) $y^i F_{y^i y^j} = 0$.
- (c) $y^k F_{y^i y^j y^k} = -F_{y^i y^j}$.
- (d) $y^l F_{y^i y^j y^k y^l} = -2F_{y^i y^j y^k}$.

Corollary 2.5 Let (M, F) be a Finsler manifold. Then F^2 defined by:

$F_p^2(y) = (F_p(y))^2$; for all $p \in M$ and $y \in T_p M$; is a positively homogeneous function of degree 2 on $T_p M$.

Corollary 2.6 Let (M, F) be a Finsler manifolds. Then

$$y^i \frac{\partial F^2(y)}{\partial y^i} = 2(F(y))^2.$$

For simplicity, we put $F_1(y_1), F_2(y_2)$;

as $F_{p_1}(y_1), F_{p_2}(y_2)$, respectively. (i.e : $F_1 = F_{p_1}, F_2 = F_{p_2}$).

Proposition 2.7 Let (M, F) be a Finsler manifold of dimension n . Then

$$y^i \frac{\partial^2 (F(y))^2}{\partial y^i \partial y^j} = \frac{\partial (F(y))^2}{\partial y^j}.$$

Proof By definition and Corollary 2.6, we have:

$$\begin{aligned} y^i \frac{\partial^2 (F(y))^2}{\partial y^i \partial y^j} &= \sum_{i=1}^n y^i \frac{\partial^2 (F(y))^2}{\partial y^i \partial y^j} = \sum_{i=1}^n y^i \frac{\partial}{\partial y^j} \left(\frac{\partial (F(y))^2}{\partial y^i} \right) \\ &= \sum_{i=1, i \neq j}^n y^i \frac{\partial}{\partial y^j} \left(\frac{\partial (F(y))^2}{\partial y^i} \right) + y^j \frac{\partial}{\partial y^j} \left(\frac{\partial (F(y))^2}{\partial y^j} \right) \\ &= \sum_{i=1, i \neq j}^n \frac{\partial}{\partial y^j} \left(y^i \frac{\partial (F(y))^2}{\partial y^i} \right) + y^j \frac{\partial}{\partial y^j} \left(\frac{\partial (F(y))^2}{\partial y^j} \right) \\ &= \frac{\partial}{\partial y^j} \left(\sum_{i=1, i \neq j}^n y^i \frac{\partial (F(y))^2}{\partial y^i} \right) + y^j \frac{\partial}{\partial y^j} \left(\frac{\partial (F(y))^2}{\partial y^j} \right) \\ &= \frac{\partial}{\partial y^j} \left(\sum_{i=1}^n y^i \frac{\partial (F(y))^2}{\partial y^i} - y^j \frac{\partial (F(y))^2}{\partial y^j} \right) + y^j \frac{\partial}{\partial y^j} \left(\frac{\partial (F(y))^2}{\partial y^j} \right) \\ &= \frac{\partial}{\partial y^j} \left(2(F(y))^2 - y^j \frac{\partial (F(y))^2}{\partial y^j} \right) + y^j \frac{\partial}{\partial y^j} \left(\frac{\partial (F(y))^2}{\partial y^j} \right) = \frac{\partial (F(y))^2}{\partial y^j}. \end{aligned}$$

□

Minkowski structure on product of Finsler manifolds

Theorem 3.1 *Let (M_1, F_1) , (M_2, F_2) be Finsler Manifolds of dimensions n and m , respectively. Suppose that $(p_1, p_2) \in M_1 \times M_2$ and $F_{(p_1, p_2)} : T_{(p_1, p_2)}(M_1 \times M_2) \rightarrow [0, +\infty)$ be a function defined by: $F_{(p_1, p_2)}(y_1, y_2) := (F_1 \oplus F_2)_{(p_1, p_2)}(y_1, y_2) := F_1(y_1) + F_2(y_2)$, $\forall (y_1, y_2) \in T_{(p_1, p_2)}(M_1 \times M_2)$. Then,*

- (I) $F_{(p_1, p_2)}(y_1, y_2) \geq 0$, $\forall (y_1, y_2) \in T_{(p_1, p_2)}(M_1, M_2)$.
- (II) $F_{(p_1, p_2)}(\lambda y_1, \lambda y_2) = \lambda F_{(p_1, p_2)}(y_1, y_2)$; $\forall \lambda > 0$, $\forall (y_1, y_2) \in T_{(p_1, p_2)}(M_1, M_2)$.

Proof

- (I) Since for all $y_1 \in T_{p_1}M_1$; $F_1(y_1) \geq 0$ and for all $y_2 \in T_{p_2}M_2$; $F_2(y_2) \geq 0$, it follows that $F_{(p_1, p_2)}(y_1, y_2) \geq 0$.
- (II) By the definition of Finsler metric, we have $\forall \lambda > 0$, $F_1(\lambda y_1) = \lambda F_1(y_1)$, $F_2(\lambda y_2) = \lambda F_2(y_2)$. It follows that $F_{(p_1, p_2)}(\lambda y_1, \lambda y_2) = \lambda F_{(p_1, p_2)}(y_1, y_2)$; $\forall \lambda > 0$.

□

Let us denote by

$$y^i := \begin{cases} y_1^i & 1 \leq i \leq n \\ y_2^{i-n} & n+1 \leq i \leq n+m \end{cases} \quad (3.1)$$

and

$$\begin{cases} y_1^i = 0 & n+1 \leq i \leq n+m \\ y_2^i = 0 & 1 \leq i \leq n \end{cases}. \quad (3.2)$$

Theorem 3.2 *Let (M_1, F_1) , (M_2, F_2) be Finsler Manifolds of dimensions n and m , respectively. Suppose that $(p_1, p_2) \in M_1 \times M_2$ and $F_{(p_1, p_2)} : T_{(p_1, p_2)}(M_1 \times M_2) \rightarrow [0, +\infty)$ be a function defined by:*

$$F_{(p_1, p_2)}(y_1, y_2) := (F_1 \oplus F_2)_{(p_1, p_2)}(y_1, y_2) := F_1(y_1) + F_2(y_2), \\ \forall (y_1, y_2) \in T_{(p_1, p_2)}(M_1, M_2). \text{ Then the Hessian matrix } (g_{ij}) := \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right) (y \neq 0) \text{ is positively definite.}$$

Proof By the definition of positive definiteness in linear algebra, it is sufficient to show that:

$$\forall y = (y_1, y_2) \in T_{(p_1, p_2)}(M_1 \times M_2) - \{0\}, \text{ we have}$$

$$[y]^t [g_{ij}] [y] > 0.$$

where $y_1 = (y_1^1, \dots, y_1^n) \in T_{p_1}M_1 - \{0\}$ and $y_2 = (y_2^1, \dots, y_2^m) \in T_{p_2}M_2 - \{0\}$. $[y]^t$ is transpose of matrix $[y]$. According to the definition of F , we have

$$F^2(y) = [F_1(y_1) + F_2(y_2)]^2 = F_1^2(y_1) + F_2^2(y_2) + 2F_1(y_1)F_2(y_2).$$

We now compute the Hessian matrix of component at slit tangent space

$$T_{(p_1, p_2)}(M_1 \times M_2) - \{0\} \text{ that is:}$$

$$g_{ij} := \frac{1}{2} \frac{\partial^2 F^2(y)}{\partial y^i \partial y^j} = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} [F_1^2(y_1) + F_2^2(y_2) + 2F_1(y_1)F_2(y_2)].$$

For simplicity, we write (g_{ij}) as, $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where

A_{11}, A_{12}, A_{21} and A_{22} are matrices, where.

$$(A_{11})_{ij} = \frac{1}{2} \frac{\partial^2 F^2(y)}{\partial y_1^i \partial y_1^j}; \quad 1 \leq i, j \leq n,$$

$$(A_{12})_{ij} = \frac{1}{2} \frac{\partial^2 F^2(y)}{\partial y_1^i \partial y_2^j}; \quad 1 \leq i \leq n, \quad 1 \leq j \leq m$$

$$(A_{21})_{ij} = \frac{1}{2} \frac{\partial^2 F^2(y)}{\partial y_2^i \partial y_1^j}; \quad 1 \leq i \leq m, \quad 1 \leq j \leq n,$$

$$(A_{22})_{ij} = \frac{1}{2} \frac{\partial^2 F^2(y)}{\partial y_2^i \partial y_2^j}; \quad 1 \leq i, j \leq m.$$

By definition of F and its partial derivatives, we have:

$$(A_{11})_{ij} = \frac{1}{2} \left(\frac{\partial^2 F_1^2(y_1)}{\partial y_1^i \partial y_1^j} + 2F_2(y_2) \frac{\partial^2 F_1(y_1)}{\partial y_1^i \partial y_1^j} \right); \quad 1 \leq i, j \leq n,$$

$$(A_{12})_{ij} = \frac{1}{2} \left(2 \frac{\partial F_1(y_1)}{\partial y_1^i} \times \frac{\partial F_2(y_2)}{\partial y_2^j} \right); \quad 1 \leq i \leq n, \quad 1 \leq j \leq m,$$

$$(A_{21})_{ij} = \frac{1}{2} \left(2 \frac{\partial F_1(y_1)}{\partial y_1^j} \times \frac{\partial F_2(y_2)}{\partial y_2^i} \right); \quad 1 \leq j \leq n, \quad 1 \leq i \leq m,$$

$$(A_{22})_{ij} = \frac{1}{2} \left(\frac{\partial^2 F_2^2(y_2)}{\partial y_2^i \partial y_2^j} + 2F_1(y_1) \frac{\partial^2 F_2(y_2)}{\partial y_2^i \partial y_2^j} \right); \quad 1 \leq i, j \leq m.$$

It can be seen that:

$$(A_{11})_{n \times n} = \frac{1}{2} \begin{bmatrix} \left(\frac{\partial^2 F_1(y_1)}{\partial y_1^1 \partial y_1^1} + 2F_2(y_2) \frac{\partial^2 F_1(y_1)}{\partial y_1^1 \partial y_1^1} \right) & \dots & \left(\frac{\partial^2 F_1(y_1)}{\partial y_1^1 \partial y_1^n} + 2F_2(y_2) \frac{\partial^2 F_1(y_1)}{\partial y_1^1 \partial y_1^n} \right) \\ \left(\frac{\partial^2 F_1(y_1)}{\partial y_1^2 \partial y_1^1} + 2F_2(y_2) \frac{\partial^2 F_1(y_1)}{\partial y_1^2 \partial y_1^1} \right) & \dots & \left(\frac{\partial^2 F_1(y_1)}{\partial y_1^2 \partial y_1^n} + 2F_2(y_2) \frac{\partial^2 F_1(y_1)}{\partial y_1^2 \partial y_1^n} \right) \\ \vdots & & \vdots \\ \left(\frac{\partial^2 F_1(y_1)}{\partial y_1^n \partial y_1^1} + 2F_2(y_2) \frac{\partial^2 F_1(y_1)}{\partial y_1^n \partial y_1^1} \right) & \dots & \left(\frac{\partial^2 F_1(y_1)}{\partial y_1^n \partial y_1^n} + 2F_2(y_2) \frac{\partial^2 F_1(y_1)}{\partial y_1^n \partial y_1^n} \right) \end{bmatrix}$$

and

$$(A_{12})_{n \times m} = \frac{1}{2} \begin{bmatrix} \left(2 \frac{\partial F_1(y_1)}{\partial y_1^1} \times \frac{\partial F_2(y_2)}{\partial y_2^1} \right) & \dots & \left(2 \frac{\partial F_1(y_1)}{\partial y_1^1} \times \frac{\partial F_2(y_2)}{\partial y_2^m} \right) \\ \left(2 \frac{\partial F_1(y_1)}{\partial y_1^2} \times \frac{\partial F_2(y_2)}{\partial y_2^1} \right) & \dots & \left(2 \frac{\partial F_1(y_1)}{\partial y_1^2} \times \frac{\partial F_2(y_2)}{\partial y_2^m} \right) \\ \vdots & & \vdots \\ \left(2 \frac{\partial F_1(y_1)}{\partial y_1^n} \times \frac{\partial F_2(y_2)}{\partial y_2^1} \right) & \dots & \left(2 \frac{\partial F_1(y_1)}{\partial y_1^n} \times \frac{\partial F_2(y_2)}{\partial y_2^m} \right) \end{bmatrix}$$

$$(A_{21})_{m \times n} = \frac{1}{2} \begin{bmatrix} \left(2 \frac{\partial F_2(y_2)}{\partial y_2^1} \times \frac{\partial F_1(y_1)}{\partial y_1^1} \right) & \dots & \left(2 \frac{\partial F_2(y_2)}{\partial y_2^1} \times \frac{\partial F_1(y_1)}{\partial y_1^n} \right) \\ \left(2 \frac{\partial F_2(y_2)}{\partial y_2^2} \times \frac{\partial F_1(y_1)}{\partial y_1^1} \right) & \dots & \left(2 \frac{\partial F_2(y_2)}{\partial y_2^2} \times \frac{\partial F_1(y_1)}{\partial y_1^n} \right) \\ \vdots & & \vdots \\ \left(2 \frac{\partial F_2(y_2)}{\partial y_2^m} \times \frac{\partial F_1(y_1)}{\partial y_1^1} \right) & \dots & \left(2 \frac{\partial F_2(y_2)}{\partial y_2^m} \times \frac{\partial F_1(y_1)}{\partial y_1^n} \right) \end{bmatrix}$$

and

$$(A_{22})_{m \times m} = \frac{1}{2} \begin{bmatrix} \left(\frac{\partial^2 F_2(y_2)}{\partial y_2^1 \partial y_2^1} + 2F_1(y_1) \frac{\partial^2 F_2(y_2)}{\partial y_2^1 \partial y_2^1} \right) & \dots & \left(\frac{\partial^2 F_2(y_2)}{\partial y_2^1 \partial y_2^m} + 2F_1(y_1) \frac{\partial^2 F_2(y_2)}{\partial y_2^1 \partial y_2^m} \right) \\ \left(\frac{\partial^2 F_2(y_2)}{\partial y_2^2 \partial y_2^1} + 2F_1(y_1) \frac{\partial^2 F_2(y_2)}{\partial y_2^2 \partial y_2^1} \right) & \dots & \left(\frac{\partial^2 F_2(y_2)}{\partial y_2^2 \partial y_2^m} + 2F_1(y_1) \frac{\partial^2 F_2(y_2)}{\partial y_2^2 \partial y_2^m} \right) \\ \vdots & & \vdots \\ \left(\frac{\partial^2 F_2(y_2)}{\partial y_2^m \partial y_2^1} + 2F_1(y_1) \frac{\partial^2 F_2(y_2)}{\partial y_2^m \partial y_2^1} \right) & \dots & \left(\frac{\partial^2 F_2(y_2)}{\partial y_2^m \partial y_2^m} + 2F_1(y_1) \frac{\partial^2 F_2(y_2)}{\partial y_2^m \partial y_2^m} \right) \end{bmatrix}$$

This give the Hessian matrix $(g_{ij}) := \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right) =$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix}.$$

It is sufficient to show that: $\forall (y_1, y_2) \in T_{(p_1, p_2)}(M_1 \times M_2) - \{0\}$ we have:

$$[y]^t [g_{ij}] [y] > 0. [y_1^1, \dots, y_1^n, y_2^1, \dots, y_2^m] [g_{ij}] \begin{bmatrix} y_1^1 \\ \vdots \\ y_1^n \\ y_2^1 \\ \vdots \\ y_2^m \end{bmatrix} = [y_1^1, \dots, y_1^n, y_2^1, \dots, y_2^m] \times \frac{1}{2}$$

$$\begin{bmatrix} \left(y_1^i \frac{\partial^2 (F_1(y_1))^2}{\partial y_1^i \partial y_1^i} \right) + 2F_2(y_2) \left(y_1^i \frac{\partial^2 F_1(y_1)}{\partial y_1^i \partial y_1^i} \right) + 2 \frac{\partial F_1(y_1)}{\partial y_1^i} \left(y_2^j \frac{\partial F_2(y_2)}{\partial y_2^j} \right) \\ \left(y_1^i \frac{\partial^2 (F_1(y_1))^2}{\partial y_1^2 \partial y_1^i} \right) + 2F_2(y_2) \left(y_1^i \frac{\partial^2 F_1(y_1)}{\partial y_1^2 \partial y_1^i} \right) + 2 \frac{\partial F_1(y_1)}{\partial y_1^2} \left(y_2^j \frac{\partial F_2(y_2)}{\partial y_2^j} \right) \\ \vdots \\ \left(y_1^i \frac{\partial^2 (F_1(y_1))^2}{\partial y_1^n \partial y_1^i} \right) + 2F_2(y_2) \left(y_1^i \frac{\partial^2 F_1(y_1)}{\partial y_1^n \partial y_1^i} \right) + 2 \frac{\partial F_1(y_1)}{\partial y_1^n} \left(y_2^j \frac{\partial F_2(y_2)}{\partial y_2^j} \right) \\ 2 \frac{\partial F_2(y_2)}{\partial y_2^1} \left(y_1^i \frac{\partial F_1(y_1)}{\partial y_1^i} \right) + \left(y_2^j \frac{\partial^2 (F_2(y_2))^2}{\partial y_2^1 \partial y_2^j} \right) + 2F_1(y_1) \left(y_2^j \frac{\partial^2 F_2(y_2)}{\partial y_2^1 \partial y_2^j} \right) \\ 2 \frac{\partial F_2(y_2)}{\partial y_2^2} \left(y_1^i \frac{\partial F_1(y_1)}{\partial y_1^i} \right) + \left(y_2^j \frac{\partial^2 (F_2(y_2))^2}{\partial y_2^2 \partial y_2^j} \right) + 2F_1(y_1) \left(y_2^j \frac{\partial^2 F_2(y_2)}{\partial y_2^2 \partial y_2^j} \right) \\ \vdots \\ 2 \frac{\partial F_2(y_2)}{\partial y_2^m} \left(y_1^i \frac{\partial F_1(y_1)}{\partial y_1^i} \right) + \left(y_2^j \frac{\partial^2 (F_2(y_2))^2}{\partial y_2^m \partial y_2^j} \right) + 2F_1(y_1) \left(y_2^j \frac{\partial^2 F_2(y_2)}{\partial y_2^m \partial y_2^j} \right) \end{bmatrix}$$

$$\begin{aligned}
&= \frac{1}{2} y_1^1 \left[\left(y_1^i \frac{\partial^2 (F_1(y_1))^2}{\partial y_1^1 \partial y_1^i} \right) + 2F_2(y_2) \left(y_1^i \frac{\partial^2 F_1(y_1)}{\partial y_1^1 \partial y_1^i} \right) + 2 \frac{\partial F_1(y_1)}{\partial y_1^1} \left(y_2^j \frac{\partial F_2(y_2)}{\partial y_2^j} \right) \right] \\
&+ \frac{1}{2} y_1^2 \left[\left(y_1^i \frac{\partial^2 (F_1(y_1))^2}{\partial y_1^2 \partial y_1^i} \right) + 2F_2(y_2) \left(y_1^i \frac{\partial^2 F_1(y_1)}{\partial y_1^2 \partial y_1^i} \right) + 2 \frac{\partial F_1(y_1)}{\partial y_1^2} \left(y_2^j \frac{\partial F_2(y_2)}{\partial y_2^j} \right) \right] \\
&+ \vdots \\
&+ \frac{1}{2} y_1^n \left[\left(y_1^i \frac{\partial^2 (F_1(y_1))^2}{\partial y_1^n \partial y_1^i} \right) + 2F_2(y_2) \left(y_1^i \frac{\partial^2 F_1(y_1)}{\partial y_1^n \partial y_1^i} \right) + 2 \frac{\partial F_1(y_1)}{\partial y_1^n} \left(y_2^j \frac{\partial F_2(y_2)}{\partial y_2^j} \right) \right] \\
&+ \frac{1}{2} y_2^1 \left[2 \frac{\partial F_2(y_2)}{\partial y_2^1} \left(y_1^i \frac{\partial F_1(y_1)}{\partial y_1^i} \right) + \left(y_2^j \frac{\partial^2 (F_2(y_2))^2}{\partial y_2^1 \partial y_2^j} \right) + 2F_1(y_1) \left(y_2^j \frac{\partial^2 F_2(y_2)}{\partial y_2^1 \partial y_2^j} \right) \right] \\
&+ \frac{1}{2} y_2^2 \left[2 \frac{\partial F_2(y_2)}{\partial y_2^2} \left(y_1^i \frac{\partial F_1(y_1)}{\partial y_1^i} \right) + \left(y_2^j \frac{\partial^2 (F_2(y_2))^2}{\partial y_2^2 \partial y_2^j} \right) + 2F_1(y_1) \left(y_2^j \frac{\partial^2 F_2(y_2)}{\partial y_2^2 \partial y_2^j} \right) \right] \\
&+ \vdots \\
&+ \frac{1}{2} y_2^m \left[2 \frac{\partial F_2(y_2)}{\partial y_2^m} \left(y_1^i \frac{\partial F_1(y_1)}{\partial y_1^i} \right) + \left(y_2^j \frac{\partial^2 (F_2(y_2))^2}{\partial y_2^m \partial y_2^j} \right) + 2F_1(y_1) \left(y_2^j \frac{\partial^2 F_2(y_2)}{\partial y_2^m \partial y_2^j} \right) \right] \\
&= \frac{1}{2} y_1^k \left[\left(y_1^i \frac{\partial^2 (F_1(y_1))^2}{\partial y_1^k \partial y_1^i} \right) + 2F_2(y_2) \left(y_1^i \frac{\partial^2 F_1(y_1)}{\partial y_1^k \partial y_1^i} \right) + 2 \frac{\partial F_1(y_1)}{\partial y_1^k} \left(y_2^j \frac{\partial F_2(y_2)}{\partial y_2^j} \right) \right] \\
&+ \frac{1}{2} y_2^h \left[2 \frac{\partial F_2(y_2)}{\partial y_2^h} \left(y_1^i \frac{\partial F_1(y_1)}{\partial y_1^i} \right) + \left(y_2^j \frac{\partial^2 (F_2(y_2))^2}{\partial y_2^h \partial y_2^j} \right) + 2F_1(y_1) \left(y_2^j \frac{\partial^2 F_2(y_2)}{\partial y_2^h \partial y_2^j} \right) \right].
\end{aligned}$$

In the first expression by Corollaries 2.4 (b), and 2.5, 2.6 and Proposition 2.7, it follows that

$$y_1^i \frac{\partial^2 F_1(y_1)}{\partial y_1^k \partial y_1^i} = 0, \quad y_2^j \frac{\partial F_2(y_2)}{\partial y_2^j} = F_2(y_2), \quad y_1^i \frac{\partial^2 (F_1(y_1))^2}{\partial y_1^k \partial y_1^i} = \frac{\partial (F_1(y_1))^2}{\partial y_1^k}.$$

In the second expression,

$$\begin{aligned}
y_2^j \frac{\partial^2 F_2(y_2)}{\partial y_2^h \partial y_2^j} &= 0, \quad y_1^i \frac{\partial F_1(y_1)}{\partial y_1^i} = F_1(y_1), \\
y_2^j \frac{\partial^2 (F_2(y_2))^2}{\partial y_2^h \partial y_2^j} &= \frac{\partial (F_2(y_2))^2}{\partial y_2^h}.
\end{aligned}$$

Then we will have

$$\begin{aligned}
[y]^t [g_{ij}] [y] &= \frac{1}{2} y_1^k \left[\frac{\partial (F_1(y_1))^2}{\partial y_1^k} + 2 \frac{\partial F_1(y_1)}{\partial y_1^k} F_2(y_2) \right] \\
&+ \frac{1}{2} y_2^h \left[2 \frac{\partial F_2(y_2)}{\partial y_2^h} F_1(y_1) + \frac{\partial (F_2(y_2))^2}{\partial y_2^h} \right].
\end{aligned}$$

Therefore $[y]^t [g_{ij}] [y] = [F_1^2(y_1) + 2F_1(y_1)F_2(y_2) + F_2^2(y_2)] = [F_1(y_1) + F_2(y_2)]^2$. It is clear that $\forall y = (y_1, y_2) \in T_{(p_1, p_2)}$

$(M_1 \times M_2) - \{0\}$ we will have

$$[y]^t [g_{ij}] [y] > 0.$$

And since the Hessian matrix $(g_{ij}) := \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right); (y \neq 0)$

is positive definite. $\square$

Corollary 3.3 Let (M_1, F_1) , (M_2, F_2) be Finsler Manifolds of dimensions n and m , respectively. Suppose that $(p_1, p_2) \in M_1 \times M_2$ and $F_{(p_1, p_2)} : T_{(p_1, p_2)}(M_1 \times M_2) \rightarrow [0, +\infty)$ be a function defined by:

$$\begin{aligned}
F_{(p_1, p_2)}(y_1, y_2) &= (F_1 \oplus F_2)_{(p_1, p_2)}(y_1, y_2) = F_1(y_1) + F_2(y_2), \\
\forall (y_1, y_2) &\in T_{(p_1, p_2)}(M_1 \times M_2)
\end{aligned}$$

Then $F_{(p_1, p_2)} : T_{(p_1, p_2)}(M_1 \times M_2) \rightarrow [0, +\infty)$ is a Minkowski norm.

Proof The proof is result of Theorems 3.1 and 3.2. $\square$

Theorem 3.4 *Let $(M_1, F_1), (M_2, F_2)$ be Finsler manifolds of dimensions n and m , respectively. Then the function $F : T(M_1 \times M_2) \rightarrow [0, +\infty)$ defined by:*

$$F_{(p_1, p_2)}(y_1, y_2) := (F_1 \oplus F_2)_{(p_1, p_2)}(y_1, y_2) := F_1(y_1) + F_2(y_2) \\ \forall (p_1, p_2) \in M_1 \times M_2, \forall (y_1, y_2) \in T_{(p_1, p_2)}(M_1 \times M_2) = T_{p_1}M_1 \oplus T_{p_2}M_2$$

is a c^∞ -function on the slit tangent bundle $T(M_1 \times M_2) - \{0\}$.

Proof Since F_1 and F_2 are c^∞ -functions, it follows that vector function $\theta : T_{p_1}M_1 \oplus T_{p_2}M_2 \rightarrow [0, +\infty) \times [0, +\infty)$ defined by: $\theta(y_1, y_2) = (F_1(y_1), F_2(y_2))$ is a c^∞ -function, and so $\lambda : R \times R \rightarrow R$ defined by $\lambda(s, t) = s + t$ is a c^∞ -function. It follows that $F_{(p_1, p_2)} = \lambda \circ \theta$ is a c^∞ -function on $T(M_1 \times M_2) - \{0\}$.

As the restriction of F to any $T_{(p_1, p_2)}(M_1 \times M_2) - \{0\}$ is c^∞ , we have F is a c^∞ function on the slit tangent bundle $T(M_1 \times M_2) - \{0\}$. $\square$

Finsler structure on product of Finsler manifolds

Theorem 4.1 *Let $(M_1, F_1), (M_2, F_2)$ be Finsler manifolds of dimensions n and m , respectively. Then the function $F : T(M_1 \times M_2) \rightarrow [0, +\infty)$ defined by:*

$$\forall (p_1, p_2) \in M_1 \times M_2, \forall (y_1, y_2) \in T_{(p_1, p_2)}(M_1 \times M_2) = T_{p_1}M_1 \oplus T_{p_2}M_2 \\ F_{(p_1, p_2)}(y_1, y_2) := (F_1 \oplus F_2)_{(p_1, p_2)}(y_1, y_2) := F_1(y_1) + F_2(y_2)$$

is a Finsler metric on $M_1 \times M_2$.

Proof By the Corollary 3.3, the restriction of F to $T_{(p_1, p_2)}(M_1 \times M_2), \forall (p_1, p_2) \in M_1 \times M_2$ is a Minkowski norm, and according to the Theorem 3.4 F is c^∞ function on the slit tangent bundle $T(M_1 \times M_2) - \{0\}$. Therefore F is a Finsler metric on $M_1 \times M_2$. $\square$

Definition 4.2 Let $(M_1, F_1), (M_2, F_2)$ be Finsler manifolds of dimensions n and m , respectively. Then Finsler metric $F = F_1 + F_2$ on $M_1 \times M_2$ is called canonical product Finsler metric.

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