Weak (α, δ) -skew Armendariz ideals

A. R. Majdabadi Farahani¹ · F. Heydari¹ · H. A. Tavallaee¹ · M. Maghasedi¹

Received: 16 May 2018 / Accepted: 9 September 2018 / Published online: 21 September 2018
 © The Author(s) 2018

Abstract

In this paper, we continue to study properties of skew polynomial ring $R[x; \alpha, \delta]$, where R is an associative ring equipped with an endomorphism α and an α -derivation δ . We first introduce weak (α, δ) -skew Armendariz ideals, a generalization of (α, δ) -compatible ideals have the insertion of factors property (or simply, IFP). We also prove that if J is a weak (α, δ) -skew Armendariz ideal of R , then $J[x]$ is a weak (α, δ) -skew Armendariz ideal of $R[x]$.

Keywords $(\alpha \cdot \delta)$ -Skew Armendariz · Weak Armendariz rings · Weak $(\alpha \cdot \delta)$ -skew

Mathematics Subject Classification 16S36 · 16S50 · 16S99

Introduction

The rings in this paper are associative with identity. Let R be a ring, α an endomorphism and δ an α -derivation of a ring R (i.e., δ is an additive map on R with the property that $\delta(st) = \delta(s)t + \alpha(s)\delta(t)$, for all $s, t \in R$). We denote by $R[x; \alpha, \delta]$ the skew polynomial ring over the coefficient ring R formed by formal polynomials $p(x) = \sum_{i=0}^m s_i x^i$, where x is a variable, $0 \leq i \leq m$ is an integer, and the coefficients s_i of the polynomials p are elements of the ring R . In the ring $R[x; \alpha, \delta]$, addition is defined as usual and multiplication is defined with respect to the relation $xs = \alpha(s)x + \delta(s)$ for any $s \in R$.

For an endomorphism α and an α -derivation δ of a ring R , Alhevaz et al. [1] studied a generalization of the (α, δ) -skew Armendariz rings and the weak Armendariz rings which is called weak (α, δ) -skew Armendariz rings. A ring R is called *weak (α, δ) -skew Armendariz ring*, if whenever

polynomials $p(x) = \sum_{i=0}^m s_i x^i$ and $q(x) = \sum_{j=0}^n t_j x^j \in R[x; \alpha, \delta]$ satisfy $p(x)q(x) = 0$, then $s_i x^i t_j x^j \in \text{nil}(R)[x; \alpha, \delta]$, for each i and j , where $\text{nil}(R)$ denotes the set of all the nilpotent elements of R .

Recall from [4] that a one-sided ideal J of a ring R has the *insertion of factors property* (or simply, IFP) if $st \in J$ implies $sRt \subseteq J$ for $s, t \in R$.

Hashemi [2], for an endomorphism α and an α -derivation δ of a ring R , defined (α, δ) -compatible ideals. An ideal J of R say that J is an α -compatible ideal of R if for each $s, t \in R, st \in J \iff s\alpha(t) \in J$. Moreover, J is said to be a δ -compatible ideal if for each $s, t \in R, st \in J \implies s\delta(t) \in J$. If J is both α -compatible and δ -compatible, we say that J is an (α, δ) -compatible ideal.

In this article, first we introduce *weak (α, δ) -skew Armendariz ideals*. Then, we show that if an (α, δ) -compatible ideal J has the IFP, then J is weak (α, δ) -skew Armendariz ideal; thus, weak (α, δ) -skew Armendariz ideals are a generalization of (α, δ) -compatible ideals which has the IFP. Also, for any positive integer n , we study relationship between ideals of R which are weak (α, δ) -skew Armendariz with some ideals of the n -by- n upper triangular matrices ring with constant main diagonal, the n -by- n upper triangular matrix ring over R and the ring $\frac{R[x]}{(x^n)}$, where (x^n) is the ideal generated by x^n . Also we show that if J is an ideal of R , then J is weak (α, δ) -skew Armendariz ideal if and only if $J[x]$ is a weak $(\overline{\alpha}, \overline{\delta})$ -skew Armendariz ideal of $R[x]$.

✉ F. Heydari
 f-heydari@kiau.ac.ir

A. R. Majdabadi Farahani
 a.majdabadi@kiau.ac.ir

H. A. Tavallaee
 tavallaee@iust.ac.ir

M. Maghasedi
 maghasedi@kiau.ac.ir

¹ Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran

Weak (α, δ) -skew Armendariz ideals

For any $0 \leq u \leq v$ ($u, v \in \mathbb{N}$), $h_u^v \in \text{End}(R, +)$ will denote the map which is the sum of all possible “words” in α, δ built with u letters α and $(v - u)$ letters δ . (e.g., $h_0^n = \delta^n$, $\dots$, $h_{n-1}^n = \alpha^{n-1}\delta + \alpha^{n-2}\delta\alpha + \dots + \delta\alpha^{n-1}$, $h_n^n = \alpha^n$, $n \in \mathbb{N}$) (see [3])

Also for an ideal J of R put

$$\sqrt{J} = \{s \in R \mid s^n \in J \text{ for some nonnegative integer } n \geq 0\}.$$

Definition 2.1 Let α be an endomorphism and δ an α -derivation of a ring R . An ideal J of R is called a *weak (α, δ) -skew Armendariz ideal*, if for each elements $p(x) = \sum_{i=0}^m s_i x^i$ and $q(x) = \sum_{j=0}^n t_j x^j \in R[x; \alpha, \delta]$, $p(x)q(x) \in J[x; \alpha, \delta]$ implies $s_i x^i t_j x^j \in \sqrt{J}[x; \alpha, \delta]$, for each $0 \leq i \leq m$ and $0 \leq j \leq n$.

It is clear that if $J = 0$ is weak (α, δ) -skew Armendariz ideal, then R is a weak (α, δ) -skew Armendariz ring.

It is well known that for a ring R and any positive integer $n \geq 2$, $\frac{R[x]}{(x^n)} \cong T(R, n)$, where (x^n) is the ideal of $R[x]$ generated by x^n and

$$T(R, n) = \left\{ \begin{pmatrix} s_0 & s_1 & \dots & s_{n-1} \\ 0 & s_0 & \dots & s_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s_0 \end{pmatrix} \mid s_i \in R, 0 \leq i \leq n-1 \right\}.$$

Lemma 2.2 Let R be a ring and $n \geq 2$ a positive integer. If $J_0, J_1, \dots, J_{n-1}$ are ideals of R such that $J_i \subseteq J_{i+1}$, $0 \leq i \leq n-2$, then

$$K = \left\{ \begin{pmatrix} s_0 & s_1 & \dots & s_{n-1} \\ 0 & s_0 & \dots & s_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s_0 \end{pmatrix} \mid s_i \in J_i, 0 \leq i \leq n-1 \right\},$$

is an ideal of $\frac{R[x]}{(x^n)}$.

Proof It is clear. $\square$

Let α be an endomorphism and δ an α -derivation of a ring R . The endomorphism α of R is extended to the endomorphism $\bar{\alpha} : T(R, n) \rightarrow T(R, n)$ defined by $\bar{\alpha}((s_{ij})) = (\alpha(s_{ij}))$ and the α -derivation δ of R is also extended to $\bar{\delta} : T(R, n) \rightarrow T(R, n)$ defined by $\bar{\delta}((s_{ij})) = (\delta(s_{ij}))$.

In Proposition 2.3 and Theorem 2.4, J_0 and K are ideals that are expressed in Lemma 2.2.

Proposition 2.3 Let α be an endomorphism and δ an α -derivation of a ring R . If

$$C_i = \begin{pmatrix} s_0^{(i)} & s_1^{(i)} & \dots & s_{n-1}^{(i)} \\ 0 & s_0^{(i)} & \dots & s_{n-2}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s_0^{(i)} \end{pmatrix},$$

$$D_j = \begin{pmatrix} t_0^{(j)} & t_1^{(j)} & \dots & t_{n-1}^{(j)} \\ 0 & t_0^{(j)} & \dots & t_{n-2}^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_0^{(j)} \end{pmatrix} \in \frac{R[x]}{(x^n)},$$

such that $s_0^{(i)} y^i t_0^{(j)} y^j \in \sqrt{J_0}[y; \alpha, \delta]$, then $C_i y^i D_j y^j \in \sqrt{K}[y; \bar{\alpha}, \bar{\delta}]$.

Proof Let $s_0^{(i)} y^i t_0^{(j)} y^j \in \sqrt{J_0}[y; \alpha, \delta]$. Then we have

$$s_0^{(i)} \left(\sum_{l=0}^i h_l^i(t_0^{(j)}) \right) y^{i+j} \in \sqrt{J_0}[y; \alpha, \delta].$$

Taking $s = s_0^{(i)} (\sum_{l=0}^i h_l^i(t_0^{(j)}))$, so there exists $k \in \mathbb{N}$ such that $s^k \in J_0$. We show that $(C_i (\sum_{l=0}^i h_l^i(D_j)))^{nk} \in K$. We work by induction on n . Let $n = 2$. For a positive integer k ,

we have $(C_i (\sum_{l=0}^i h_l^i(D_j)))^k = \begin{pmatrix} s^k & c \\ 0 & s^k \end{pmatrix}$ and $(C_i (\sum_{l=0}^i h_l^i(D_j)))^{2k} = \begin{pmatrix} s^{2k} & u \\ 0 & s^{2k} \end{pmatrix}$, where $u = s^k c + c s^k$.

Hence $(C_i (\sum_{l=0}^i h_l^i(D_j)))^{2k} \in K$, since $s^{2k}, s^k c + c s^k \in J_0$. Now, let

$$C_i = \begin{pmatrix} s_0^{(i)} & s_1^{(i)} & \dots & s_{n-1}^{(i)} \\ 0 & s_0^{(i)} & \dots & s_{n-2}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s_0^{(i)} \end{pmatrix},$$

$$D_j = \begin{pmatrix} t_0^{(j)} & t_1^{(j)} & \dots & t_{n-1}^{(j)} \\ 0 & t_0^{(j)} & \dots & t_{n-2}^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_0^{(j)} \end{pmatrix} \in \frac{R[x]}{(x^n)},$$

such that $s^k = (s_0^{(i)} (\sum_{l=0}^i h_l^i(t_0^{(j)})))^k \in J_0$. Consider

$$\left(C_i \left(\sum_{l=0}^i h_l^i(D_j) \right) \right)^k = \begin{pmatrix} s^k & c_1 & \dots & c_{n-1} \\ 0 & s^k & \dots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s^k \end{pmatrix} \in K$$

and

$$\left(C_i \left(\sum_{l=0}^i h_l^i(D_j) \right) \right)^{(n-1)k} = \begin{pmatrix} s^{(n-1)k} & d_1 & \dots & d_{n-1} \\ 0 & s^{(n-1)k} & \dots & d_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s^{(n-1)k} \end{pmatrix} \in K.$$

By the induction hypothesis all d_i 's, except d_{n-1} , are in J_0 . Let $z = s^k d_{n-1} + c_1 d_{n-2} + \dots + c_{n-1} s^{(n-1)k}$. Hence

$$\left(C_i \left(\sum_{l=0}^i h_l^i(D_j) \right) \right)^{nk} = \begin{pmatrix} s^{nk} & r_1 & \dots & z \\ 0 & s^{nk} & \dots & r_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s^{nk} \end{pmatrix} \in K,$$

since s^{nk}, z and all r_i 's are in J_0 . Thus $C_i(\sum_{l=0}^i h_l^i(D_j))y^{i+j} \in \sqrt{K}[y; \bar{\alpha}, \bar{\delta}]$. Therefore, $C_i y^i D_j y^j \in \sqrt{K}[y; \bar{\alpha}, \bar{\delta}]$. $\square$

Theorem 2.4 Let α be an endomorphism and δ an α -derivation of a ring R . Then J_0 is a weak (α, δ) -skew Armendariz ideal if and only if K is a weak $(\bar{\alpha}, \bar{\delta})$ -skew Armendariz ideal.

Proof Let J_0 be a weak (α, δ) -skew Armendariz ideal. Suppose that $P(y) = \sum_{i=0}^m C_i y^i$ and $Q(y) = \sum_{j=0}^r D_j y^j \in R[x](x^n)[y; \bar{\alpha}, \bar{\delta}]$ satisfy $P(y)Q(y) \in K[y; \bar{\alpha}, \bar{\delta}]$. Let

$$C_i = \begin{pmatrix} s_0^{(i)} & s_1^{(i)} & \dots & s_{n-1}^{(i)} \\ 0 & s_0^{(i)} & \dots & s_{n-2}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s_0^{(i)} \end{pmatrix},$$

$$D_j = \begin{pmatrix} t_0^{(j)} & t_1^{(j)} & \dots & t_{n-1}^{(j)} \\ 0 & t_0^{(j)} & \dots & t_{n-2}^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_0^{(j)} \end{pmatrix} \in \frac{R[x]}{(x^n)},$$

for $0 \leq i \leq m$ and $0 \leq j \leq r$. Let $p_0(y) = s_0^{(0)} + s_0^{(1)}y + \dots + s_0^{(m)}y^m$ and $q_0(y) = t_0^{(0)} + t_0^{(1)}y + \dots + t_0^{(r)}y^r$. Then $p_0(y)q_0(y) \in J_0[y; \alpha, \delta]$. Since J_0 is a weak (α, δ) -skew Armendariz ideal, $s_0^{(i)}y^i t_0^{(j)}y^j \in \sqrt{J_0}[y; \alpha, \delta]$ for each i, j . Then $C_i y^i D_j y^j \in \sqrt{K}[y; \bar{\alpha}, \bar{\delta}]$ for each i, j , by Propositions 2.3. Therefore, K is a weak $(\bar{\alpha}, \bar{\delta})$ -skew Armendariz ideal.

Conversely, it is clear. $\square$

For an endomorphism α and an α -derivation δ of R , the natural extension $\bar{\alpha}: R[x] \rightarrow R[x]$ defined by $\bar{\alpha}(p(x)) = \sum_{i=k}^n \alpha(s_i)x^i$ for $p(x) = \sum_{i=k}^n s_i x^i \in R[x]$ is an endomorphism of $R[x]$ and $\bar{\delta}: R[x] \rightarrow R[x]$ defined by $\bar{\delta}(p(x)) = \sum_{i=k}^n \delta(s_i)x^i$ for $p(x) = \sum_{i=k}^n s_i x^i \in R[x]$ is an $\bar{\alpha}$ -derivation of $R[x]$.

Theorem 2.5 Let α be an endomorphism, δ an α -derivation and J an ideal of a ring R . Then J is a weak (α, δ) -skew Armendariz ideal if and only if $J[x]$ is a weak $(\bar{\alpha}, \bar{\delta})$ -skew Armendariz ideal.

Proof Let J be a weak (α, δ) -skew Armendariz ideal. Suppose that $P(y) = \sum_{i=0}^m p_i(x)y^i$ and $Q(y) = \sum_{j=0}^r q_j(x)y^j \in R[x][y; \bar{\alpha}, \bar{\delta}]$ satisfy $P(y)Q(y) \in J[x][y; \bar{\alpha}, \bar{\delta}]$. Let

$$n = \max\{\deg(p_i), \deg(q_j) \mid 0 \leq i \leq m, 0 \leq j \leq r\}.$$

Then we can assume $p_i(x) = s_0^{(i)} + s_1^{(i)}x + \dots + s_n^{(i)}x^n$ and $q_j(x) = t_0^{(j)} + t_1^{(j)}x + \dots + t_n^{(j)}x^n$ for any $0 \leq i \leq m$ and $0 \leq j \leq r$, where

$$s_0^{(i)}, s_1^{(i)}, \dots, s_n^{(i)}, t_0^{(j)}, t_1^{(j)}, \dots, t_n^{(j)} \in R.$$

Let

$$K = \left\{ \begin{pmatrix} s_0 & s_1 & \dots & s_n \\ 0 & s_0 & \dots & s_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s_0 \end{pmatrix} \mid s_i \in J, 0 \leq i \leq n \right\}.$$

By Lemma 2.2, K is an ideal of $\frac{R[x]}{(x^{n+1})}$. Let

$$C_i = \begin{pmatrix} s_0^{(i)} & s_1^{(i)} & \dots & s_n^{(i)} \\ 0 & s_0^{(i)} & \dots & s_{n-1}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s_0^{(i)} \end{pmatrix},$$

$$D_j = \begin{pmatrix} t_0^{(j)} & t_1^{(j)} & \dots & t_n^{(j)} \\ 0 & t_0^{(j)} & \dots & t_{n-1}^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_0^{(j)} \end{pmatrix} \in \frac{R[x]}{(x^{n+1})},$$

for $0 \leq i \leq m$ and $0 \leq j \leq r$. Since $P(y)Q(y) \in J[x][y; \bar{\alpha}, \bar{\delta}]$, we have $(C_0 + C_1y + \dots + C_my^m)(D_0 + D_1y + \dots + D_ry^r) \in K[y; \bar{\alpha}, \bar{\delta}]$. By Theorem 2.4, K is weak $(\bar{\alpha}, \bar{\delta})$ -skew Armendariz ideal. Hence $C_i y^i D_j y^j \in \sqrt{K}[y; \bar{\alpha}, \bar{\delta}]$ for each i and j . Thus $p_i(x)y^i q_j(x)y^j \in \sqrt{J[x]}[y; \bar{\alpha}, \bar{\delta}]$. Therefore, $J[x]$ is a weak $(\bar{\alpha}, \bar{\delta})$ -skew Armendariz ideal.

Conversely, it is clear. $\square$

The subring $S(R, n)$ of $T_n(R)$ is defined as in

$$S(R, n) = \left\{ \begin{pmatrix} s & s_{12} & \dots & s_{1n} \\ 0 & s & \dots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s \end{pmatrix} \mid s, s_{ij} \in R \text{ for all } i, j \right\},$$

In fact, $S(R, n)$ is the subring of the triangular matrix ring $T_n(R)$ such that all entries on its main diagonal are the same. Also, the endomorphism α of R is extended to the endomorphism $\bar{\alpha} : S(R, n) \rightarrow S(R, n)$ defined by $\bar{\alpha}((s_{ij})) = (\alpha(s_{ij}))$ and the α -derivation δ of R is also extended to $\bar{\delta} : S(R, n) \rightarrow S(R, n)$ defined by $\bar{\delta}((s_{ij})) = (\delta(s_{ij}))$.

Lemma 2.6 Let J and J_{ij} be ideals of R such that $J \subseteq J_{ij} \subseteq J_{is}$ for $1 \leq i < j \leq s \leq n$, and $J_{pq} \subseteq J_{lq}$ for $3 \leq q \leq n$, $2 \leq l \leq p \leq n$. Then

$$K = \left\{ \begin{pmatrix} s & s_{12} & \dots & s_{1n} \\ 0 & s & \dots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s \end{pmatrix} \mid s \in J, s_{ij} \in J_{ij} \right\},$$

is an ideal of $S(R, n)$.

Proof It is clear. $\square$

In Proposition 2.7 and Theorem 2.8, J and K are ideals that are expressed in Lemma 2.6.

Proposition 2.7 Let α be an endomorphism and δ an α -derivation of a ring R . If

$$C_i = \begin{pmatrix} s^{(i)} & s_{12}^{(i)} & \dots & s_{1n}^{(i)} \\ 0 & s^{(i)} & \dots & s_{2n}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s^{(i)} \end{pmatrix}, D_j = \begin{pmatrix} t^{(j)} & t_{12}^{(j)} & \dots & t_{1n}^{(j)} \\ 0 & t^{(j)} & \dots & t_{2n}^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{(j)} \end{pmatrix} \in S(R, n),$$

such that $s^{(i)}y^i t^{(j)}y^j \in \sqrt{J}[y; \alpha, \delta]$, then $C_i y^i D_j y^j \in \sqrt{K}[y; \bar{\alpha}, \bar{\delta}]$.

Proof Let $s^{(i)}y^i t^{(j)}y^j \in \sqrt{J}[y; \alpha, \delta]$. Then, we have

$$s^{(i)} \left(\sum_{l=0}^i h_l^i(t^{(j)}) \right) y^{i+j} \in \sqrt{J}[y; \alpha, \delta].$$

Taking $s = s^{(i)}(\sum_{l=0}^i h_l^i(t^{(j)}))$, so there exists $k \in \mathbb{N}$ such that $s^k \in J$. We show that $(C_i(\sum_{l=0}^i h_l^i(D_j)))^{nk} \in K$. We work by induction on n . Let $n = 2$. For a positive integer k ,

$$\text{we have } (C_i(\sum_{l=0}^i h_l^i(D_j)))^k = \begin{pmatrix} s^k & c \\ 0 & s^k \end{pmatrix} \text{ and } (C_i(\sum_{l=0}^i h_l^i(D_j)))^{2k} = \begin{pmatrix} s^{2k} & u \\ 0 & s^{2k} \end{pmatrix}, \text{ where } u = s^k c + c s^k.$$

Hence $(C_i(\sum_{l=0}^i h_l^i(D_j)))^{2k} \in K$, since $s^{2k}, s^k c + c s^k \in J$. Now, let

$$C_i = \begin{pmatrix} s^{(i)} & s_{12}^{(i)} & \dots & s_{1n}^{(i)} \\ 0 & s^{(i)} & \dots & s_{2n}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s^{(i)} \end{pmatrix}, D_j = \begin{pmatrix} t^{(j)} & t_{12}^{(j)} & \dots & t_{1n}^{(j)} \\ 0 & t^{(j)} & \dots & t_{2n}^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{(j)} \end{pmatrix} \in S(R, n),$$

such that $s^k = (s^{(i)}(\sum_{l=0}^i h_l^i(t^{(j)})))^k \in J$. Consider

$$\left(C_i \left(\sum_{l=0}^i h_l^i(D_j) \right) \right)^k = \begin{pmatrix} s^k & c_1 & \dots & c_{n-1} \\ 0 & s^k & \dots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s^k \end{pmatrix} \in K$$

and

$$\left(C_i \left(\sum_{l=0}^i h_l^i(D_j) \right) \right)^{(n-1)k} = \begin{pmatrix} s^{(n-1)k} & d_1 & \dots & d_{n-1} \\ 0 & s^{(n-1)k} & \dots & d_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s^{(n-1)k} \end{pmatrix} \in K.$$

By the induction hypothesis all d_i 's, except d_{n-1} , are in J . Let $z = s^k d_{n-1} + c_1 d_{n-2} + \dots + c_{n-1} s^{(n-1)k}$. Hence

$$\left(C_i \left(\sum_{l=0}^i h_l^i(D_j) \right) \right)^{nk} = \begin{pmatrix} s^{nk} & r_1 & \dots & z \\ 0 & s^{nk} & \dots & r_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s^{nk} \end{pmatrix} \in K,$$

since s^{nk}, z and all r_i 's are in J . Thus $C_i(\sum_{l=0}^i h_l^i(D_j))y^{i+j} \in \sqrt{K}[y; \bar{\alpha}, \bar{\delta}]$. Therefore, $C_i y^i D_j y^j \in \sqrt{K}[y; \bar{\alpha}, \bar{\delta}]$. $\square$

Theorem 2.8 Let α be an endomorphism and δ an α -derivation of a ring R . Then J is a weak (α, δ) -skew Armendariz ideal if and only if K is a weak $(\bar{\alpha}, \bar{\delta})$ -skew Armendariz ideal.

Proof Let J be a weak (α, δ) -skew Armendariz ideal. Suppose that $P(y) = \sum_{i=0}^m C_i y^i$ and $Q(y) = \sum_{j=0}^r D_j y^j \in S(R, n)[y; \bar{\alpha}, \bar{\delta}]$ satisfy $P(y)Q(y) \in K[y; \bar{\alpha}, \bar{\delta}]$. Let

$$C_i = \begin{pmatrix} s^{(i)} & s_{12}^{(i)} & \dots & s_{1n}^{(i)} \\ 0 & s^{(i)} & \dots & s_{2n}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s^{(i)} \end{pmatrix}, D_j = \begin{pmatrix} t^{(j)} & t_{12}^{(j)} & \dots & t_{1n}^{(j)} \\ 0 & t^{(j)} & \dots & t_{2n}^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{(j)} \end{pmatrix} \in S(R, n),$$

for $0 \leq i \leq m$ and $0 \leq j \leq r$. Let $p_0(y) = s^{(0)} + s^{(1)}y + \cdots + s^{(m)}y^m$ and $q_0(y) = t^{(0)} + t^{(1)}y + \cdots + t^{(r)}y^r$. Then $p_0(y)q_0(y) \in J[y; \alpha, \delta]$. Since J is a weak (α, δ) -skew Armendariz ideal, $s^{(i)}y^i t^{(j)}y^j \in \sqrt{J}[y; \alpha, \delta]$ for each i, j . Then $C_i y^i D_j y^j \in \sqrt{K}[y; \bar{\alpha}, \bar{\delta}]$ for each i, j , by Propositions 2.7. Therefore, K is a weak $(\bar{\alpha}, \bar{\delta})$ -skew Armendariz ideal.

Conversely, it is clear. $\square$

Let α be an endomorphism and δ an α -derivation of a ring R . Recall that for an ideal J of R , if $\alpha(J) \subseteq J$, then $\bar{\alpha} : R/J \rightarrow R/J$ defined by $\bar{\alpha}(s + J) = \alpha(s) + J$ for $s \in R$ is an endomorphism of a factor ring R/J and if $\delta(J) \subseteq J$, then $\bar{\delta} : R/J \rightarrow R/J$ defined by $\bar{\delta}(s + J) = \delta(s) + J$ for $s \in R$ is an $\bar{\alpha}$ -derivation of a factor ring R/J . Also, for each $p(x) = \sum_{i=0}^m s_i x^i \in R[x; \alpha, \delta]$, denote $\bar{p}(x) = \sum_{i=0}^m \bar{s}_i x^i \in (R/J)[x; \bar{\alpha}, \bar{\delta}]$, where $\bar{s}_i = s_i + J$, for each i .

Theorem 2.9 Let J, K be ideals of a ring R . If $J \subseteq \sqrt{K}$ and $\frac{J+K}{J}$ is a weak $(\bar{\alpha}, \bar{\delta})$ -skew Armendariz ideal, then K is a weak (α, δ) -skew Armendariz ideal.

Proof Let $\frac{J+K}{J}$ be a weak $(\bar{\alpha}, \bar{\delta})$ -skew Armendariz ideal. Suppose that $p(x) = \sum_{i=0}^m s_i x^i$ and $q(x) = \sum_{j=0}^r t_j x^j \in R[x; \alpha, \delta]$ satisfy $p(x)q(x) \in K[x; \alpha, \delta]$. Then

$$\left(\sum_{i=0}^m \bar{s}_i x^i \right) \left(\sum_{j=0}^r \bar{t}_j x^j \right) \in \frac{J+K}{J}[x; \bar{\alpha}, \bar{\delta}].$$

Thus $(\bar{s}_i (\sum_{l=0}^i h_l^i(\bar{t}_j)))^{n_{ij}} \in \frac{J+K}{J}$ for some positive integer n_{ij} , since $\frac{J+K}{J}$ is a weak $(\bar{\alpha}, \bar{\delta})$ -skew Armendariz ideal. Hence $(s_i (\sum_{l=0}^i h_l^i(t_j)))^{n_{ij}} \in J + K$, and so $(s_i (\sum_{l=0}^i h_l^i(t_j)))^{n_{ij}} \in K$, since $J \subseteq \sqrt{K}$. Thus $s_i x^i t_j x^j \in \sqrt{K}[x; \alpha, \delta]$. Therefore, K is a weak (α, δ) -skew Armendariz ideal. $\square$

Let R be a ring and let $M_n(R)$ be the $n \times n$ full matrix ring over R . For an endomorphism α and an α -derivation δ of R , the natural extension $\bar{\alpha} : M_n(R) \rightarrow M_n(R)$ defined by $\bar{\alpha}((s_{ij})) = (\alpha(s_{ij}))$ is an endomorphism of $M_n(R)$ and $\bar{\delta} : M_n(R) \rightarrow M_n(R)$ defined by $\bar{\delta}((s_{ij})) = (\delta(s_{ij}))$ is an $\bar{\alpha}$ -derivation of $M_n(R)$. Clearly, $\bar{\alpha}|_{T_n(R)}$, the restriction of $\bar{\alpha}$ to $T_n(R)$, is an endomorphism of $T_n(R)$ and $\bar{\delta}|_{T_n(R)}$, the restriction of $\bar{\delta}$ to $T_n(R)$, is an $\bar{\alpha}$ -derivation of $T_n(R)$, where $T_n(R)$ is the $n \times n$ upper triangular matrix ring over R . We also denote $\bar{\alpha}|_{T_n(R)}$ by $\bar{\alpha}$ and $\bar{\delta}|_{T_n(R)}$ by $\bar{\delta}$.

Lemma 2.10 Let J_{ij} be ideals of R such that $J_{ij} \subseteq J_{is}$ for $1 \leq i \leq j \leq s \leq n$, and $J_{pq} \subseteq J_{lq}$ for $2 \leq q \leq n$, $1 \leq l \leq p \leq n$. Then

$$K = \left\{ \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ 0 & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{nn} \end{pmatrix} \mid s_{ij} \in J_{ij}, 1 \leq i, j \leq n \right\},$$

is an ideal of $T_n(R)$.

Proof It is clear. $\square$

In Proposition 2.11 and Theorem 2.12, J_{ij} 's and K are ideals that are expressed in Lemma 2.10. By a similar way as used in the proof of Proposition 2.3 and Theorem 2.4, one can prove Proposition 2.11 and Theorem 2.12.

Proposition 2.11 Let α be an endomorphism and δ an α -derivation of a ring R . If

$$C_i = \begin{pmatrix} s_{11}^{(i)} & s_{12}^{(i)} & \cdots & s_{1n}^{(i)} \\ 0 & s_{22}^{(i)} & \cdots & s_{2n}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{nn}^{(i)} \end{pmatrix}, D_j = \begin{pmatrix} t_{11}^{(j)} & t_{12}^{(j)} & \cdots & t_{1n}^{(j)} \\ 0 & t_{22}^{(j)} & \cdots & t_{2n}^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{nn}^{(j)} \end{pmatrix} \in T_n(R),$$

such that $s_{rr}^{(i)} y^i t_{rr}^{(j)} y^j \in \sqrt{J}_{ij}[y; \alpha, \delta]$ for $1 \leq r \leq n$, then $C_i y^i D_j y^j \in \sqrt{K}[y; \bar{\alpha}, \bar{\delta}]$.

Theorem 2.12 Let α be an endomorphism and δ an α -derivation of a ring R . Then, K is a weak $(\bar{\alpha}, \bar{\delta})$ -skew Armendariz ideal if and only if all J_{ii} are weak (α, δ) -skew Armendariz ideals for $1 \leq i \leq n$.

(α, δ) -Compatible ideals have the insertion of factors property

In this section, our focus of study will be on (α, δ) -compatible ideals have the insertion of factors property (or simply, IFP) of a ring R .

The next lemma appears in [2, Proposition 2.3] and will be helpful in the sequel.

Lemma 3.1 Let J be an (α, δ) -compatible ideal of R and $s, t \in R$.

- (i) If $st \in J$, then $s\alpha^n(t) \in J$ and $\alpha^n(s)t \in J$ for every positive integer n .
Conversely, if $s\alpha^k(t)$ or $\alpha^k(s)t \in J$ for some positive integer k , then $st \in J$.
- (ii) If $st \in J$, then $\alpha^m(s)\delta^n(t), \delta^n(s)\alpha^m(t) \in J$ for any nonnegative integers m, n .

Lemma 3.2 If J is an (α, δ) -compatible ideal that has the IFP of a ring R , then $\sqrt{J}$ is an (α, δ) -compatible ideal that has the IFP of a ring R .

Proof First, we prove that $\sqrt{J}$ is an ideal that has the IFP of a ring R . Let $s, t \in \sqrt{J}$. Then $s^n, t^m \in J$ for some integers $m, n \geq 0$. Hence

$$(s+t)^{m+n+1} = \sum (s^{i_1} t^{j_1}) \cdots (s^{i_{m+n+1}} t^{j_{m+n+1}}),$$

where $i_k + j_k = 1$, $0 \leq i_k \leq 1$, $0 \leq j_k \leq 1$. It can be easily checked that s more than n or t more than m appears in $(s^{i_1} t^{j_1}) \cdots (s^{i_{m+n+1}} t^{j_{m+n+1}})$. Since $s^n, t^m \in J$ and J has the IFP, we have $(s^{i_1} t^{j_1}) \cdots (s^{i_{m+n+1}} t^{j_{m+n+1}}) \in J$. Thus $(s+t)^{m+n+1} \in J$ and $s+t \in \sqrt{J}$. Now suppose that $s^m \in J$ and $u \in R$. Then $(us)^m, (su)^n \in J$, since J has the IFP. Therefore, $\sqrt{J}$ is an ideal of R . Clearly $\sqrt{J}$ has the IFP.

Now we show that $\sqrt{J}$ is an (α, δ) -compatible ideal of a ring R . Let $s, t \in R$ and $st \in \sqrt{J}$. Then

$$\begin{aligned} st \in \sqrt{J} &\Leftrightarrow st \cdots st \in J \\ &\Rightarrow \alpha(st)st \cdots st \in J \\ &\Rightarrow \alpha(s)\alpha(t)st \cdots st \in J \\ &\Rightarrow s\alpha(t)st \cdots st \in J \\ &\Rightarrow \alpha(s\alpha(t)st)st \cdots st \in J \\ &\Rightarrow \alpha(s\alpha(t)s)\alpha(t)st \cdots st \in J \\ &\Rightarrow s\alpha(t)s\alpha(t)st \cdots st \in J \\ &\vdots \\ &\Rightarrow s\alpha(t)s\alpha(t) \cdots s\alpha(t) \in J \\ &\Rightarrow s\alpha(t) \in \sqrt{J}. \end{aligned}$$

By a similar argument as above we can see that if $s, t \in R$ and $s\alpha(t) \in \sqrt{J}$, then $st \in \sqrt{J}$. Now, let $s, t \in R$ and $st \in \sqrt{J}$. Thus $st \cdots st \in J$. From δ -compatibility of J , we have $st \cdots s\delta(t) \in J$ and so $st \cdots s\delta(ts\delta(t)) \in J$, then $st \cdots s\delta(t)s\delta(t) + st \cdots s\alpha(t)\delta(s\delta(t)) \in J$. Since $st \cdots s\alpha(t)\delta(s\delta(t)) \in J$ and recent equality, $st \cdots s\delta(t)s\delta(t) \in J$. By repeating this method, we obtain $s\delta(t) \cdots s\delta(t)s\delta(t) \in J$. Thus $s\delta(t) \in \sqrt{J}$. Therefore, $\sqrt{J}$ is an (α, δ) -compatible ideal that has the IFP of a ring R . $\square$

Proposition 3.3 Let J be an (α, δ) -compatible ideal of a ring R . Then

- (1) If $s_1 \cdots s_n \in \sqrt{J}$, then $\alpha^{k_1}(s_1) \cdots \alpha^{k_n}(s_n) \in \sqrt{J}$, for all positive integers $k_1, \dots, k_n$.
- (2) If $\alpha^{k_1}(s_1) \cdots \alpha^{k_n}(s_n) \in \sqrt{J}$, for some positive integers $k_1, \dots, k_n$, then $s_1 \cdots s_n \in \sqrt{J}$.
- (3) If $s_1 \cdots s_n \in \sqrt{J}$, then we have $\delta^{k_1}(s_1) \cdots \delta^{k_n}(s_n) \in \sqrt{J}$, for all positive integers $k_1, \dots, k_n$.

- (4) If $s_1 \cdots s_n \in \sqrt{J}$, then $h_{u_1}^{v_1}(s_1) \cdots h_{u_n}^{v_n}(s_n) \in \sqrt{J}$, for all positive integers $k_1, \dots, k_n$ and for all $v_i \geq u_i \geq 0$.

Proof (1) The proof is by induction on n . If $s_1 s_2 \in \sqrt{J}$, then $(s_1 s_2)^t \in J$, for some $t \in \mathbb{N}$. Thus

$$\begin{aligned} s_1 s_2 \in \sqrt{J} &\Leftrightarrow s_1 s_2 \cdots s_1 s_2 \in J \\ &\Rightarrow \alpha^{k_1}(s_1) s_2 s_1 s_2 \cdots s_1 s_2 \in J \\ &\Rightarrow \alpha^{k_2}(\alpha^{k_1}(s_1) s_2) s_1 s_2 \cdots s_1 s_2 \in J \\ &\Rightarrow \alpha^{k_1+k_2}(s_1) \alpha^{k_2}(s_2) s_1 s_2 \cdots s_1 s_2 \in J \\ &\Rightarrow s_1 \alpha^{k_2}(s_2) s_1 s_2 \cdots s_1 s_2 \in J \\ &\Rightarrow \alpha^{k_1}(s_1) \alpha^{k_2}(s_2) s_1 s_2 \cdots s_1 s_2 \in J \\ &\Rightarrow \alpha^{k_1}(\alpha^{k_1}(s_1) \alpha^{k_2}(s_2) s_1) s_2 \cdots s_1 s_2 \in J \\ &\vdots \\ &\Rightarrow \alpha^{k_1}(s_1) \alpha^{k_2}(s_2) \alpha^{k_1}(s_1) \alpha^{k_2}(s_2) \cdots \alpha^{k_1}(s_1) \alpha^{k_2}(s_2) \in J \\ &\Rightarrow \alpha^{k_1}(s_1) \alpha^{k_2}(s_2) \in \sqrt{J} \end{aligned}$$

Now, suppose by induction that $\alpha^{k_1}(s_1) \alpha^{k_2}(s_2) \cdots \alpha^{k_l}(s_l) \in \sqrt{J}$, for all $l < n$ and all positive integers $k_1, k_2, \dots, k_l$, when $s_1 s_2 \cdots s_l \in \sqrt{J}$. Now, let $s_1 s_2 \cdots s_n \in \sqrt{J}$. Thus

$$\begin{aligned} s_1 \cdots s_{n-2} (s_{n-1} s_n) &\in \sqrt{J} \\ &\Rightarrow \alpha^{k_1}(s_1) \cdots \alpha^{k_{n-2}}(s_{n-2}) \alpha^{k_{n-1}}(s_{n-1} s_n) \in \sqrt{J} \\ &\Rightarrow \alpha^{k_1}(s_1) \cdots \alpha^{k_{n-2}}(s_{n-2}) \alpha^{k_{n-1}}(s_{n-1}) \alpha^{k_{n-1}}(s_n) \in \sqrt{J} \\ &\Rightarrow \alpha^{k_1}(s_1) \cdots \alpha^{k_{n-2}}(s_{n-2}) \alpha^{k_{n-1}}(s_{n-1}) s_n \in \sqrt{J} \\ &\Rightarrow \alpha^{k_1}(s_1) \cdots \alpha^{k_{n-2}}(s_{n-2}) \alpha^{k_{n-1}}(s_{n-1}) \alpha^{k_n}(s_n) \in \sqrt{J}. \end{aligned}$$

This finishes our inductive step and the proof.

(2) The proof is similar to that of (1).

(3) The proof is by induction on n . Let $s_1 s_2 \in \sqrt{J}$. Therefore, $s_1 s_2 s_1 s_2 \cdots s_1 s_2 \in J$. From δ -compatibility of J , we have $s_1 s_2 s_1 s_2 \cdots \delta(s_1 s_2) \in J$, then $s_1 s_2 s_1 s_2 \cdots \delta(s_1) s_2 + s_1 s_2 s_1 s_2 \cdots \alpha(s_1) \delta(s_2) \in J$. Since $s_1 s_2 s_1 s_2 \cdots \alpha(s_1) \delta(s_2) \in J$ and recent equality,

$$s_1 s_2 s_1 s_2 \cdots \delta(s_1) s_2 \in J.$$

Now $s_1 s_2 s_1 s_2 \cdots \delta^2(s_1 s_2) \in J$, then

$$\begin{aligned} s_1 s_2 s_1 s_2 \cdots \delta^2(s_1) s_2 + s_1 s_2 s_1 s_2 \cdots \alpha(\delta(s_1)) \delta(s_2) \\ + s_1 s_2 s_1 s_2 \cdots \delta(\alpha(s_1)) \delta(s_2) + s_1 s_2 s_1 s_2 \cdots \alpha^2(s_1) \delta^2(s_2) \in J. \end{aligned}$$

So $s_1 s_2 s_1 s_2 \cdots \delta^2(s_1) s_2 \in J$. This process yields that $s_1 s_2 s_1 s_2 \cdots \delta^{k_1}(s_1) s_2 \in J$. Now from $s_1 s_2 s_1$

$s_2 \cdots \delta(s_1 s_2 s_1 s_2) \in J$ we have $s_1 s_2 s_1 s_2 \cdots \delta(s_1) s_2 s_1 s_2 \in J$ and

$$s_1 s_2 s_1 s_2 \cdots \delta^2(s_1 s_2 s_1 s_2) \in J$$

implies that $s_1 s_2 s_1 s_2 \cdots \delta^2(s_1) s_2 s_1 s_2 \in J$. Therefore, $\delta^{k_1}(s_1) s_2 \delta^{k_1}(s_1) s_2 \cdots \delta^{k_1}(s_1) s_2 \in J$. By repeating this method for s_2 , we obtain $\delta^{k_1}(s_1) \delta^{k_2}(s_2) \delta^{k_1}(s_1) \delta^{k_2}(s_2) \cdots \delta^{k_1}(s_1) \delta^{k_2}(s_2) \in J$. Hence $\delta^{k_1}(s_1) \delta^{k_2}(s_2) \in \sqrt{J}$. Now, suppose by induction that $\delta^{k_1}(s_1) \delta^{k_2}(s_2) \cdots \delta^{k_l}(s_l) \in \sqrt{J}$, for all $l < n$ and all positive integers $k_1, k_2, \dots, k_l$, when $s_1 s_2 \cdots s_l \in \sqrt{J}$. Now, let $s_1 s_2 \cdots s_n \in \sqrt{J}$. Thus $\delta^{k_1}(s_1) \cdots \delta^{k_{n-2}}(s_{n-2}) \delta(s_{n-1} s_n) \in \sqrt{J}$ and also

$$\delta^{k_1}(s_1) \cdots \delta^{k_{n-2}}(s_{n-2}) s_{n-1} s_n \in \sqrt{J}.$$

Then $\delta^{k_1}(s_1) \cdots \delta^{k_{n-2}}(s_{n-2}) \delta(s_{n-1} s_n) \in \sqrt{J}$, since $\sqrt{J}$ is an ideal. By continuing this way, (k_{n-1}) -times, we get $\delta^{k_1}(s_1) \cdots \delta^{k_{n-1}}(s_{n-1}) s_n \in \sqrt{J}$. So $\delta^{k_1}(s_1) \cdots \delta^{k_{n-1}}(s_{n-1}) \delta^{k_n}(s_n) \in \sqrt{J}$. This finishes our inductive step and the proof.

(4) This is just a direct use of (1) and (3). $\square$

Theorem 3.4 If J is an (α, δ) -compatible ideal that has the IFP of a ring R , then J and $\sqrt{J}$ are weak (α, δ) -skew Armendariz ideals.

Proof Suppose that $p(x) = \sum_{i=0}^m s_i x^i$ and $q(x) = \sum_{j=0}^n t_j x^j \in R[x; \alpha, \delta]$ satisfy $p(x)q(x) \in J[x; \alpha, \delta]$. Hence

$$\begin{aligned} p(x)q(x) &= \left(\sum_{i=0}^m s_i x^i \right) \left(\sum_{j=0}^n t_j x^j \right) \\ &= \left(\sum_{i=0}^m s_i x^i \right) t_0 + \left(\sum_{i=0}^m s_i x^i \right) t_1 x + \cdots + \left(\sum_{i=0}^m s_i x^i \right) t_n x^n \\ &= \sum_{i=0}^m s_i h_0^i(t_0) + \left(\sum_{i=1}^m s_i h_1^i(t_0) \right) x + \cdots \\ &\quad + \left(\sum_{i=r}^m s_i h_r^i(t_0) \right) x^r + \cdots + s_m \alpha^m(t_0) x^m \\ &\quad + \left(\sum_{i=0}^m s_i h_0^i(t_1) \right) \left(\sum_{i=1}^m s_i h_1^i(t_1) \right) x + \cdots \\ &\quad + \left(\sum_{i=r}^m s_i h_r^i(t_1) \right) x^r + \cdots + s_m \alpha^m(t_1) x^m \Big) x + \cdots \\ &\quad + \left(\sum_{i=0}^m s_i h_0^i(t_n) \right) \left(\sum_{i=1}^m s_i h_1^i(t_n) \right) x + \cdots + s_m \alpha^m(t_n) x^m \Big) x^n \\ &= \sum_{i=0}^m s_i h_0^i(t_0) + \left(\sum_{i=1}^m s_i h_1^i(t_0) + \sum_{i=0}^m s_i h_0^i(t_1) \right) x + \cdots \\ &\quad + \left(\sum_{r+u=k}^m \left(\sum_{i=r}^m s_i h_r^i(t_u) \right) \right) x^k + \cdots + s_m \alpha^m(t_n) x^{m+n} \in J[x; \alpha, \delta]. \end{aligned}$$

Then we have the following system

$$\Delta_{m+n} = s_m \alpha^m(t_n) \in J;$$

$$\Delta_{m+n-1} = s_m \alpha^m(t_{n-1}) + s_{m-1} \alpha^{m-1}(t_n) + s_m h_{m-1}^m(t_n) \in J;$$

$\vdots$

$$\Delta_k = \sum_{r+u=k} \left(\sum_{i=r}^m s_i h_r^i(t_u) \right) \in J;$$

$\vdots$

$$\Delta_0 = \sum_{i=0}^m s_i h_0^i(t_0) \in J.$$

From equation Δ_{m+n} , we have $s_m \alpha^m(t_n) \in \sqrt{J}$. Using (2) of Proposition 3.3, we obtain $s_m t_n \in \sqrt{J}$ and so $t_n s_m \in \sqrt{J}$. Now we show that $s_i t_n \in \sqrt{J}$, for all $0 \leq i \leq m$. If we multiply equation Δ_{m+n-1} on the left side by t_n , then $t_n s_{m-1} \alpha^{m-1}(t_n) = t_n \Delta_{m+n-1} - t_n s_m \alpha^m(t_{n-1}) - t_n s_m h_{m-1}^m(t_n) \in \sqrt{J}$, since $\sqrt{J}$ is an ideal of R , by Lemma 3.2. By similar discussion as above, we have $t_n s_{m-1} t_n \in \sqrt{J}$, and so $t_n s_{m-1} \in \sqrt{J}$, $s_{m-1} t_n \in \sqrt{J}$. Continuing this process yields that $s_i t_n \in \sqrt{J}$, for all $0 \leq i \leq m$. Thus $s_i h_u^v(t_n) \in \sqrt{J}$, for any $v \geq u \geq 0$ and $0 \leq i \leq m$, by (4) of Proposition 3.3. By a similar way as above, we can prove that $s_i t_{n-1} \in \sqrt{J}$, for all $0 \leq i \leq m$. Then by continuing this process, we have $s_i t_j \in \sqrt{J}$, for all $0 \leq i \leq m$ and $0 \leq j \leq n$. So for each $0 \leq l \leq i$, $s_i h_l^i(t_j) \in \sqrt{J}$, by (4) of Proposition 3.3, and then $s_i x^i t_j x^j = \sum_{l=0}^i s_i h_l^i(t_j) x^{l+j} \in \sqrt{J}[x; \alpha, \delta]$. Therefore, J is weak (α, δ) -skew Armendariz ideal.

Since J is an (α, δ) -compatible ideal that has the IFP of a ring R , hence by Lemma 3.2, $\sqrt{J}$ is an (α, δ) -compatible ideal that has the IFP of a ring R . Thus, $\sqrt{J}$ is weak (α, δ) -skew Armendariz ideal. $\square$

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

- Alhevaz, A., Moussavi, A., Habibi, M.: On rings having McCoy-like conditions. *Commun. Algebra* **40**(4), 1195–1221 (2012)
- Hashemi, E.: Compatible ideals and radicals of Ore extensions. *N. Y. J. Math.* **12**, 349–356 (2006)

3. Lam, T.Y., Leroy, A., Matczuk, J.: Primeness, semiprimeness and the prime radical of ore extensions. *Commun. Algebra* **25**(8), 2459–2516 (1997)
4. Mason, G.: Reflexive ideals. *Commun. Algebra* **9**(17), 1709–1724 (1981)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.