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Laguerre approach for solving system of linear Fredholm integro-differential equations

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Abstract

A numerical scheme has been developed for solving the system of linear Fredholm integro-differential equations subject to the mixed conditions using Laguerre polynomials. Using collocation method, the system of Fredholm integro-differential equations has been transformed to the system of linear equations in unknown Laguerre coefficients, which leads to the solution in terms of Laguerre polynomials. Moreover, the accuracy and applicability of the scheme have been compared with Tau method and Adomian decomposition method that reveals the proposed scheme to be more efficient.

Keywords System of Fredholm integro-differential equations · Laguerre polynomials · Collocation method

Mathematics Subject Classification 45A05 · 45J05

Introduction

There are many branches of science, such as control theory and financial mathematics, which leads to integro-differential equations (IDEs). In modern mathematics, IDEs mostly occur in many applied areas including engineering, physics and biology [1–6]. The resolution of many problems in physics and engineering leads to differential and integral equations in bounded or unbounded domains. For example, problems occur in coastal hydrodynamics and in meteorology. The integrals appear in many physical contexts, containing the product of orthogonal polynomials or special functions. For example, the wave functions of the hydrogen as well as 2-, 3- and, in general, n-dimensional harmonic oscillator encompassing Laguerre polynomials

and the evaluation of integrals having the product of these polynomials are essential [7].

In the fields of applied mathematics and scientific computing, spectral methods [8–10] became popular among researchers as a robust numerical tool. The remarkable results are obtained, using the spectral methods, to solve the problems [11–13] in different fields of natural sciences. Moreover, system of IDEs found in the field of science and engineering, such as nano-hydrodynamics [14], glass-forming process [15], dropwise condensation [16], wind ripples in the desert [17], modeling the competition between tumor cell and the immune system [18] and examining the noise term phenomenon [19, 20]. Since analytical solutions of such type of problems are hard to determine, therefore the numerical methods are required. Many researchers presented numerical methods for system of IDEs, for instance the Tau method [21], Fibonacci matrix method [22], Bessel matrix method [23, 24], Adomian decomposition method (ADM) [25], modified decomposition method [26], Galerkin methods with hybrid functions [27–30, 38], differential transform method [31] and the block pulse functions method [32].

The main objective of this paper is to study the concept of the system of IDEs and manipulate the Laguerre matrix method for solving the system of linear Fredholm IDEs.

The following system of linear Fredholm IDEs has been considered

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$$\sum_{k=0}^{n}\sum_{q=1}^{r}\rho_{p,q}^{k}(s)u_{q}^{(k)}(s)=f_{p}(s)+\int_{a}^{b}\sum_{q=1}^{r}K_{p,q}(s,t)u_{q}(t)\mathrm{d}t,$$
$$p=1,2,\ldots,r,\quad 0\le a\le s\le b, \tag{1}$$

subject to

$$\mathbf{H}=\begin{pmatrix} \dfrac{(-1)^0}{0!}\dbinom{0}{0} & 0 & 0 & \ldots & 0 \\[2ex] \dfrac{(-1)^0}{0!}\dbinom{1}{0} & \dfrac{(-1)^1}{1!}\dbinom{1}{1} & 0 & \ldots & 0 \\[2ex] \dfrac{(-1)^0}{0!}\dbinom{2}{0} & \dfrac{(-1)^1}{1!}\dbinom{2}{1} & \dfrac{(-1)^2}{2!}\dbinom{2}{2} & \ldots & 0 \\[1ex] \vdots & \vdots & \vdots & \ddots & \vdots \\[1ex] \dfrac{(-1)^0}{0!}\dbinom{N}{0} & \dfrac{(-1)^1}{1!}\dbinom{N}{1} & \dfrac{(-1)^2}{2!}\dbinom{N}{2} & \ldots & \dfrac{(-1)^N}{N!}\dbinom{N}{N} \end{pmatrix}.$$

$$\sum_{q=0}^{n-1}(a_{p,q}^{k}u_{k}^{(q)}(a)+b_{p,q}^{k}u_{k}^{(q)}(b))=\mu_{k,p},\quad p=0,1,\ldots,n-1,$$
$$k=1,2,\ldots,r, \tag{2}$$

where $u_q(s)$ is the unknown and $\rho_{p,q}^{k}$, $K_{p,q}(s,t)$, $f_p(s)$ are the known functions defined in the interval $[a, b]$, the kernel function $K_{p,q}(s,t)$ can be expanded using Maclaurin series and also $a_{p,q}^{k}$, $b_{p,q}^{k}$, $\mu_{k,p}$ are real constants.

Taking $u_p(s)$ to be the approximate solution of Eq. (1) in terms of truncated Laguerre series yields

$$u_p(s)=\sum_{n=0}^{N}\alpha_{p,n}L_n(s),\quad p=1,2,\ldots,r,\quad a\le s\le b, \tag{3}$$

where $\alpha_{p,n}$ are the unknown Laguerre coefficients, to be determined for $n=0,1,2,\ldots,N$, and $L_n(s)$ be the Laguerre polynomial defined by

$$L_n(s)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n}{n-k}s^k,\quad 0\le s\le b<\infty.$$

Matrix relation

The matrix form of Laguerre polynomials $L_n(s)$ is as follows

$$\mathbf{L}^{\mathrm{T}}(s)=\mathbf{H}\,\mathbf{S}^{\mathrm{T}}(s)\Leftrightarrow\mathbf{L}(s)=\mathbf{S}(s)\,\mathbf{H}^{\mathrm{T}}, \tag{4}$$

where

$$\mathbf{L}(s)=(\,L_0(s)\quad L_1(s)\quad L_2(s)\quad\ldots\quad L_N(s)\,),$$
$$\mathbf{S}(s)=(\,1\quad s\quad s^2\quad\ldots\quad s^N\,),$$

and

The relation defined in Eq. (3) can be written in matrix form, as

$$(u_q(s))=\mathbf{L}(s)\,\mathbf{A_q},\quad \mathbf{q}=1,2,\ldots,\mathbf{r},$$

where

$$\mathbf{A}_q=(\alpha_{q,0},\alpha_{q,1},\ldots,\alpha_{q,N})^{\mathrm{T}}.$$

or from Eq. (4)

$$(u_q(s))=\mathbf{S}(s)\mathbf{H}^{\mathrm{T}}\mathbf{A_q},\quad \mathbf{q}=1,2,\ldots,\mathbf{r}. \tag{5}$$

Also, the relation between the matrix $\mathbf{S}(s)$ and its first derivative is

$$\mathbf{S}^{(1)}(s)=S(s)\mathbf{B},\quad \mathbf{S}^{(0)}(s)=\mathbf{S}(s), \tag{6}$$

where

$$\mathbf{B}=\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & N \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

It follows from Eqs. (4) and (6) that

$$\mathbf{S}^{(p)}(s) = \mathbf{S}^{(p-1)}(s)\mathbf{B} = \mathbf{S}(s)(\mathbf{B})^p, \quad p = 0, 1, 2, \ldots, n, \tag{7}$$

and thus

$$\mathbf{L}^{(p)}(s) = \mathbf{S}^{(p)}(s)\mathbf{H}^{\mathrm{T}} = \mathbf{S}(s)(\mathbf{B})^p\mathbf{H}^{\mathrm{T}}, \quad p = 0, 1, 2, \ldots, n. \tag{8}$$

The following recurrence relation is obtained using Eqs. (5), (7) and (8)

$$
\begin{aligned}
u_q^{(p)}(s) &= \mathbf{L}^{(p)}(s)\mathbf{A}_q \\
&= \mathbf{S}^{(p)}(s)\mathbf{H}^{\mathrm{T}}\mathbf{A}_q \\
&= \mathbf{S}(s)(\mathbf{B})^p\mathbf{H}^{\mathrm{T}}\mathbf{A}_q, \quad p = 0, 1, 2, \ldots, n, \; q = 1, 2, \ldots, r.
\end{aligned}
\tag{9}
$$

Hence, the matrices $\mathbf{u}^{(p)}(s)$ can be expressed as

$$\mathbf{u}^{(p)}(s) = \overline{\mathbf{S}}(s)(\overline{\mathbf{B}})^p\overline{\mathbf{H}}\mathbf{A}, \quad p = 0, 1, 2, \ldots, n, \tag{10}$$

where

$$
\mathbf{u}^{(p)}(s) = \begin{pmatrix} u_1^{(p)}(s) \\ u_2^{(p)}(s) \\ \vdots \\ u_r^{(p)}(s) \end{pmatrix}, \quad
\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_r \end{pmatrix},
$$

$$
\overline{\mathbf{S}}(s) = \begin{pmatrix} \mathbf{S}(s) & 0 & \ldots & 0 \\ 0 & \mathbf{S}(s) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{S}(s) \end{pmatrix}_{r \times r},
$$

$$
\overline{\mathbf{B}} = \begin{pmatrix} \mathbf{B} & 0 & \ldots & 0 \\ 0 & \mathbf{B} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{B} \end{pmatrix}_{r \times r} \quad \text{and}
$$

$$
\overline{\mathbf{H}} = \begin{pmatrix} \mathbf{H}^{\mathrm{T}} & 0 & \ldots & 0 \\ 0 & \mathbf{H}^{\mathrm{T}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{H}^{\mathrm{T}} \end{pmatrix}.
$$

Method of solution

The matrix form of the system defined in Eq. (1) is as follows

$$\sum_{p=0}^{n} \mathbf{P}_p(s)\mathbf{u}^{(p)}(s) = \boldsymbol{f}(s) + \mathbf{I}(s), \tag{11}$$

where

$$
\mathbf{u}^{(p)}(s) = \begin{pmatrix} u_1^{(p)}(s) \\ u_2^{(p)}(s) \\ \vdots \\ u_r^{(p)}(s) \end{pmatrix}, \quad
\boldsymbol{f} = \begin{pmatrix} f_1(s) \\ f_2(s) \\ \vdots \\ f_r(s) \end{pmatrix},
$$

$$
\mathbf{P}_p(s) = \begin{pmatrix} \rho_{1,1}^p(s) & \rho_{1,2}^p(s) & \cdots & \rho_{1,r}^p(s) \\ \rho_{2,1}^p(s) & \rho_{2,2}^p(s) & \cdots & \rho_{2,r}^p(s) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{r,1}^p(s) & \rho_{r,2}^p(s) & \cdots & \rho_{r,r}^p(s) \end{pmatrix},
$$

$$\mathbf{I}(s) = \int_a^b \mathbf{K}(s,t)u(t)\mathrm{d}t,$$

$$
\mathbf{K}(s,t) = \begin{pmatrix} K_{1,1}(s,t) & K_{1,2}(s,t) & \ldots & K_{1,r}(s,t) \\ K_{2,1}(s,t) & K_{2,2}(s,t) & \ldots & K_{2,r}(s,t) \\ \vdots & \vdots & \ddots & \vdots \\ K_{r,1}(s,t) & K_{r,2}(s,t) & \ldots & K_{r,r}(s,t) \end{pmatrix},
$$

$$
\mathbf{I} = \begin{pmatrix} I_1(s) \\ I_2(s) \\ \vdots \\ I_r(s) \end{pmatrix}
$$

and

$$I_p(s) = \int_a^b \sum_{q=1}^{r} K_{p,q}(s,t)u_q(t). \tag{12}$$

Using truncated Taylor and Laguerre series to approximate the kernel $K_{p,q}(s,t)$ yields

$$
\begin{aligned}
K_{p,q}(s,t) &= \sum_{m=0}^{N}\sum_{n=0}^{N} k_{pq}^t s^m t^n \quad \text{and} \\
K_{p,q}(s,t) &= \sum_{m=0}^{N}\sum_{n=0}^{N} k_{pq}^l L_m(s)L_n(t),
\end{aligned}
\tag{13}
$$

where

$$
k_{pq}^t = \frac{1}{m!n!}\frac{\partial^{m+n}K(0,0)}{\partial s^m \partial t^n}, \quad m, n = 0, 1, 2, \ldots, N, \\
p, q = 0, 1, 2, \ldots, r.
$$

Equation (13) can be put into the matrix form, as

$$K_{p,q}(s,t) = \mathbf{S}(s)\mathbf{K}_{pq}^t\mathbf{S}^{\mathrm{T}}(t), \quad \mathbf{K}_{pq}^t = \left(k_{pq}^t\right), \tag{14}$$

and

$$K_{p,q}(s,t) = \mathbf{L}(s)\mathbf{K}_{pq}^l\mathbf{L}^{\mathrm{T}}(t), \quad \mathbf{K}_{pq}^l = \left(k_{pq}^l\right). \tag{15}$$

Using Eqs. (5) and (15) in Eq. (12) to get the matrix form of the integral part yields

$$(I_p(s)) = \int_a^b \sum_{q=0}^r \mathbf{L}(s)\mathbf{K}_{pq}^l \mathbf{L}^{\mathrm{T}}(t)\mathbf{S}(t)\mathbf{H}^{\mathrm{T}}\mathbf{A}_q \mathrm{d}t$$

$$= \sum_{q=0}^r \int_a^b \mathbf{L}(s)\mathbf{K}_{pq}^l \mathbf{L}^{(T)}(t)\mathbf{S}(t)\mathbf{H}^{\mathrm{T}}\mathbf{A}_q \mathrm{d}t \qquad (16)$$

$$= \sum_{q=0}^r \mathbf{L}(s)\mathbf{K}_{pq}^l \mathbf{Q}\mathbf{A}_q,$$

such that

$$\mathbf{Q} = \int_a^b \mathbf{L}^{\mathrm{T}}(t)\mathbf{S}(t)\mathbf{H}^{\mathrm{T}}\mathrm{d}t$$

$$= \int_a^b \mathbf{H}\mathbf{S}^{\mathrm{T}}(t)\mathbf{S}(t)\mathbf{H}^{\mathrm{T}}\mathrm{d}t = \mathbf{H}\mathbf{M}\mathbf{H}^{\mathrm{T}},$$

where

$$\mathbf{M} = \int_a^b \mathbf{S}^{\mathrm{T}}(t)\mathbf{S}(t)\mathrm{d}t = (m_{ij}), \quad m_{ij} = \frac{b^{i+j+1} - a^{i+j+1}}{i+j+1},$$
$$i,j = 0,1,2,\ldots,N.$$

Using Eq. (4) in Eq. (16) leads to

$$(I_p(s)) = \sum_{q=0}^r \mathbf{S}(s)\mathbf{H}^{\mathrm{T}}\mathbf{K}_{pq}^l \mathbf{Q}\mathbf{A}_q. \qquad (17)$$

In order to determine the system of equations in matrix form, replacing the collocation points defined by

$$s_t = a + \frac{b-a}{N}t, \quad t = 0,1,2,\ldots,N, \qquad (18)$$

in Eq. (11), leads to

$$\sum_{p=0}^n \mathbf{P}_p(s_t)\mathbf{u}^{(p)}(s_t) = \boldsymbol{f}(s_t) + \mathbf{I}(s_t)$$

or

$$\sum_{p=0}^n \mathbf{P}_p\mathbf{U}^{(p)} = \boldsymbol{F} + \mathbf{I}, \qquad (19)$$

where

$$\mathbf{P}_p = \begin{pmatrix} \mathbf{P}_p(s_0) & 0 & \ldots & 0 \\ 0 & \mathbf{P}_p(s_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{P}_p(s_N) \end{pmatrix},$$

$$\mathbf{U}^{(p)} = \begin{pmatrix} \mathbf{u}^{(p)}(s_0) \\ \mathbf{u}^{(p)}(s_1) \\ \vdots \\ \mathbf{u}^{(p)}(s_N) \end{pmatrix}, \quad \boldsymbol{F} = \begin{pmatrix} \boldsymbol{f}(s_0) \\ \boldsymbol{f}(s_1) \\ \vdots \\ \boldsymbol{f}(s_N) \end{pmatrix}$$

$$\text{and} \quad \mathbf{I} = \begin{pmatrix} \mathbf{I}(s_0) \\ \mathbf{I}(s_1) \\ \vdots \\ \mathbf{I}(s_N) \end{pmatrix}.$$

Using the collocation points in Eq. (10) yields

$$\mathbf{u}^{(p)}(s_t) = \overline{\mathbf{S}}(s_t)(\overline{\mathbf{B}})^p \overline{\mathbf{H}}\mathbf{A}, \quad t = 0,1,2,\ldots,N,$$
$$p = 0,1,2,\ldots,r, \qquad (20)$$

which can be written as

$$\mathbf{U}^{(p)} = \mathbf{S}(\overline{\mathbf{B}})^p \overline{\mathbf{H}}\mathbf{A},$$

where

$$\mathbf{S} = \begin{pmatrix} \overline{\mathbf{S}}(s_0) \\ \overline{\mathbf{S}}(s_1) \\ \vdots \\ \overline{\mathbf{S}}(s_N) \end{pmatrix} \quad \text{and}$$

$$\overline{\mathbf{S}}(s_t) = \begin{pmatrix} \mathbf{S}(s_t) & 0 & \ldots & 0 \\ 0 & \mathbf{S}(s_t) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{S}(s_t) \end{pmatrix}_{r \times r}, \quad \begin{matrix} t = 0,1,2,\ldots,N, \\ p = 0,1,2,\ldots,r. \end{matrix}$$

Substituting the collocation points in Eq. (17) leads to

$$(I_p(s_t)) = \sum_{q=0}^r \mathbf{S}(s_t)\mathbf{H}^{\mathrm{T}}\mathbf{K}_{pq}^l \mathbf{Q}\mathbf{A}_q, \quad t = 0,1,2,\ldots,N,$$
$$p = 0,1,2,\ldots,r. \qquad (21)$$

Similarly, substituting the collocation points into the matrix $\mathbf{I}(s)$ of Eq. (11) and using Eq. (21) yields

$$\mathbf{I}(s_t) = \overline{\mathbf{S}}(s_t)\overline{\mathbf{H}}\mathbf{K}_l\overline{\mathbf{Q}}\mathbf{A}, \ t = 0,1,2,\ldots,N, \ p = 0,1,2,\ldots,r, \qquad (22)$$

where

$$\mathbf{I}(s_t) = \begin{pmatrix} I_1(s_t) \\ I_2(s_t) \\ \vdots \\ I_r(s_t) \end{pmatrix},$$

$$\overline{\mathbf{S}}(s_t) = \begin{pmatrix} \mathbf{S}(s_t) & 0 & \ldots & 0 \\ 0 & \mathbf{S}(s_t) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{S}(s_t) \end{pmatrix}_{r \times r},$$

$$\overline{\mathbf{H}} = \begin{pmatrix} \mathbf{H}^{\mathrm{T}} & 0 & \ldots & 0 \\ 0 & \mathbf{H}^{\mathrm{T}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{H}^{\mathrm{T}} \end{pmatrix},$$

$$\mathbf{K}_l = \begin{pmatrix} \mathbf{K}_{11}^l & \mathbf{K}_{12}^l & \ldots & \mathbf{K}_{1r}^l \\ \mathbf{K}_{21}^l & \mathbf{K}_{22}^l & \ldots & \mathbf{K}_{2r}^l \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}_{r1}^l & \mathbf{K}_{r2}^l & \ldots & \mathbf{K}_{rr}^l \end{pmatrix},$$

$$\overline{\mathbf{Q}} = \begin{pmatrix} \mathbf{Q} & 0 & \ldots & 0 \\ 0 & \mathbf{Q} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{Q} \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_r \end{pmatrix}.$$

Thus, using Eq. (22) in **I** defined in Eq. (19) gives

$$\mathbf{I} = \begin{pmatrix} \mathbf{I}(s_0) \\ \mathbf{I}(s_1) \\ \vdots \\ \mathbf{I}(s_N) \end{pmatrix} = \mathbf{S}\overline{\mathbf{H}}\mathbf{K}_l\overline{\mathbf{Q}}\mathbf{A}. \tag{23}$$

Hence, from Eq. (19) the fundamental matrix equation of the system defined in Eq. (1), using Eqs. (20) and (23), is obtained as under

$$\left[\sum_{p=0}^{n} \mathbf{P}_p \mathbf{S}(\overline{\mathbf{B}})^p \overline{\mathbf{H}} - \mathbf{S}\overline{\mathbf{H}}\mathbf{K}_l\overline{\mathbf{Q}} \right] \mathbf{A} = \boldsymbol{F}. \tag{24}$$

$r(N+1) \times r(N+1)$ and $r(N+1) \times 1$ are the dimensions of the respective matrices $\mathbf{P}_p$, $\mathbf{S}$, $(\overline{\mathbf{B}})^p$, $\overline{\mathbf{H}}$, $\mathbf{K}_l$, $\overline{\mathbf{Q}}$ and $\mathbf{A}$, $\boldsymbol{F}$. Moreover, Eq. (24) is written in more instructive form, as

$$\mathbf{W}\mathbf{A} = \boldsymbol{F}, \quad \text{or} \quad (\mathbf{W}; \boldsymbol{F}), \tag{25}$$

where

$$\mathbf{W} = (w_{m,n}) = \sum_{p=0}^{n} \mathbf{P}_p \mathbf{S}(\overline{\mathbf{B}})^p \overline{\mathbf{H}} - \mathbf{S}\overline{\mathbf{H}}\mathbf{K}_l\overline{\mathbf{Q}}, \quad m, n = 1, 2, \ldots, r(N+1).$$

The conditions defined in Eq. (2) can be expressed in the following matrix form

$$\sum_{q=0}^{n-1} \left[a_{p,q}^1 u_1^{(q)}(a) + b_{p,q}^1 u_1^{(q)}(b) \right] = \mu_{1,p},$$

$$\sum_{q=0}^{n-1} \left[a_{p,q}^2 u_2^{(q)}(a) + b_{p,q}^2 u_2^{(q)}(b) \right] = \mu_{2,p},$$

$$\vdots$$

$$\sum_{q=0}^{n-1} \left[a_{p,q}^r u_r^{(q)}(a) + b_{p,q}^r u_r^{(q)}(b) \right] = \mu_{r,p}$$

or

$$\sum_{q=0}^{n-1} \left[a_q^1 u_1^{(q)}(a) + b_q^1 u_1^{(q)}(b) \right] = \mu_1,$$

$$\sum_{q=0}^{n-1} \left[a_q^2 u_2^{(q)}(a) + b_q^2 u_2^{(q)}(b) \right] = \mu_2,$$

$$\vdots$$

$$\sum_{q=0}^{n-1} \left[a_q^r u_r^{(q)}(a) + b_q^r u_r^{(q)}(b) \right] = \mu_r,$$

where

$$\mu_p = \begin{pmatrix} \mu_{p,0} \\ \mu_{p,1} \\ \vdots \\ \mu_{p,n-1} \end{pmatrix}_{n \times 1}, \quad a_q^p = \begin{pmatrix} a_{0,q}^p \\ a_{1,q}^p \\ \vdots \\ a_{n-1,q}^p \end{pmatrix}_{n \times 1},$$

$$b_q^p = \begin{pmatrix} b_{0,q}^p \\ b_{1,q}^p \\ \vdots \\ b_{n-1,q}^p \end{pmatrix}_{n \times 1}, \quad p = 1, 2, \ldots, r.$$

or briefly

$$\sum_{q=0}^{n-1} \left[a_q u^{(q)}(a) + b_q u^{(q)}(b) \right] = \mu, \tag{26}$$

where

$$a_q = \begin{pmatrix} a_q^1 & 0 & \cdots & 0 \\ 0 & a_q^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_q^r \end{pmatrix}_{r \times r},$$

$$b_q = \begin{pmatrix} b_q^1 & 0 & \cdots & 0 \\ 0 & b_q^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_q^r \end{pmatrix}_{r \times r} \quad \text{and} \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_r \end{pmatrix}_{r \times 1}.$$

Putting the values of $u^{(q)}(a)$ and $u^{(q)}(b)$ from Eq. (10) into Eq. (26) yields

$$\sum_{q=0}^{n-1} \left[a_q \overline{\mathbf{S}}(a) + b_q \overline{\mathbf{S}}(b) \right] (\overline{\mathbf{B}})^q \overline{\mathbf{H}} \mathbf{A} = \mu. \tag{27}$$

or briefly

$$\mathbf{V}\mathbf{A} = \mu \quad \text{or} \quad (\mathbf{V}; \mu), \tag{28}$$

where

$$\mathbf{V} = \sum_{q=0}^{n-1} \left[a_q \overline{\mathbf{S}}(a) + b_q \overline{\mathbf{S}}(b) \right] (\overline{\mathbf{B}})^q \overline{\mathbf{H}}$$

Thus, Eq. (28) is the matrix form of the conditions defined in Eq. (2). Replacing the augmented matrix in Eq. (28) with the augmented matrix in Eq. (25) yields

$$\tilde{\mathbf{W}}\mathbf{A} = \tilde{\boldsymbol{F}}. \tag{29}$$

By replacing the nr-rows of the matrix $\mathbf{W}$, the augmented matrix of the above system can be obtained as under [23, 33–36]

It is not necessary to replace the last rows of $\mathbf{W}$. For instance, if the matrix $\mathbf{W}$ is singular, then the rows that are linearly dependent (or have same factors) or all zeros are replaced. If rank $\tilde{\mathbf{W}}$=rank $(\tilde{\mathbf{W}}; \tilde{F}) = r(N + 1)$, then $\mathbf{A} = (\tilde{\mathbf{W}})^{-1}\tilde{F}$. Thus, $u_p(s)$ can be approximated by Eq. (3). However, if $|\tilde{W}| = 0$, then a particular solution may be found; otherwise, the solution will not be possible.

Error analysis

Since Eq. (3) approximates the system defined in Eq. (1), therefore substituting $u_q(s)$ by $u_{p,N}(s)$, $p = 1, 2, \ldots, r$ in Eq. (1), the resulting equation must be satisfied approximately, i.e., for $s = s_i \in [a, b]$, $i = 0, 1, 2, \ldots$

$$E_p(s_i) = \left| \sum_{k=0}^{n} \sum_{q=1}^{r} \rho_{p,q}^k(s_i) u_{p,N}^{(k)}(s_i) - f_p(s_i) \right.$$

$$\left. - \int_a^b \sum_{q=1}^{r} K_{p,q}(s_i, t) u_{p,N}(t) \mathrm{d}t \right| \cong 0, p = 1, 2, \ldots, r \tag{31}$$

and $E_p(s_i) \leq 10^{-k_i}$, where k_i is a nonnegative integer. If max $10^{-k_i} = 10^{-k}$ (k positive integer) is prescribed, then the truncation limit N is increased till the difference $E_p(s_i)$ at each of the points becomes smaller than the prescribed 10^{-k}[23, 36, 37]. For max $10^{-k_i} \neq 10^{-k}$, the error can be estimated by the following function

$$(\tilde{\mathbf{W}}; \tilde{\boldsymbol{F}}) = \begin{pmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,r(N+1)} & ; & f_1(s_0) \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ w_{r,1} & w_{r,2} & \cdots & w_{r,r(N+1)} & ; & f_r(s_0) \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ w_{r(N-n+1),1} & w_{r(N-n+1),2} & \cdots & w_{r(N-n+1),r(N+1)} & ; & f_r(s_{N-n}) \\ v_{1,1} & v_{1,2} & \cdots & v_{1,r(N+1)} & ; & \mu_{1,0} \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ v_{r,1} & v_{r,2} & \vdots & v_{r,r(N+1)} & ; & \mu_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ v_{nr,1} & v_{nr,2} & \cdots & v_{nr,r(N+1)} & ; & \mu_{r,n-1} \end{pmatrix}. \tag{30}$$

$$E_{p,N}(s) = \sum_{k=0}^{n} \sum_{q=1}^{r} \rho_{p,q}^{k}(s) u_{q}^{(k)}(s) - f_p(s)$$
$$- \int_a^b \sum_{q=1}^{r} K_{p,q}(s,t) u_q(t) dt \cong 0, \quad p = 1, 2, \ldots, r.$$

The error will be decreasing, if $E_{p,N}(s) \to 0$, for sufficiently large N.

Numerical examples

Following examples have been considered to examine the reliability and efficiency of the proposed technique.

Example 1 Consider the system of Fredholm IDEs, as

$$u_1^{(2)}(s) + u_2^{(1)}(s) = 3s^2 + \frac{3s}{10} + 8 - \int_0^1 2st\{u_1(t) - 3u_2(t)\}dt$$
$$u_1^{(1)}(s) + u_2^{(2)}(s) = 21s + \frac{4}{5} - \int_0^1 3(2s + t^2)\{u_1(t) - 2u_2(t)\}dt,$$
$$0 \le s \le 1,$$
(32)

subject to the following mixed conditions

$$u_1(0) + u_1^{(1)}(0) = 1, \quad u_1(1) + u_1^{(1)}(1) = 10,$$
$$u_2(0) + u_2^{(1)}(0) = 1, \quad u_2(1) + u_2^{(1)}(1) = 7.$$
(33)

The analytical solutions are $u_1(s) = 3s^2 + 1$ and $u_2(s) = s^3 + 2s - 1$.

Solving the system of equations for $N = 3$ by following the procedure stated above yields the approximate

solutions $u_{1,3}(s) = 3s^2 + 1$ and $u_{2,3}(s) = s^3 + 2s - 1$ which are exactly the same as the analytical one. Table 1 shows the numerical results obtained by the proposed technique and their comparison with Tau method [21], whereas Figs. 1 and 2 depict the absolute errors $e_{1,3}$ and $e_{2,3}$ at $N = 3$ for Example 1.

Example 2 Consider the system of Fredholm IDEs, as

$$u_1^{(2)}(s) = \frac{8}{9} + \int_0^1 \left\{\frac{1}{3}u_1(t) + \frac{1}{4}u_2(t)\right\}dt$$
$$u_2^{(2)}(s) = 6s - \frac{s^2}{18} + \int_0^1 \left\{\frac{s^2}{6}u_1(t) - \frac{s^2}{3}u_2(t)\right\}dt, \quad 0 \le s \le 1,$$
(34)

subject to the following initial conditions

$$u_1(0) = 0, \quad u_1^{(1)}(0) = \frac{1}{3}, \quad u_2(0) = 0, \quad u_2^{(1)}(0) = -\frac{1}{2}.$$
(35)

The analytical solution is $u_1(s) = \frac{s^2}{2} + \frac{s}{3}$ and $u_2(s) = s^3 - \frac{s}{2}$.

Solving the system of equations for $N = 3$ by following the procedure stated above yields the approximate solutions $u_{1,3}(s) = \frac{s^2}{2} + \frac{s}{3}$ and $u_{2,3}(s) = s^3 - \frac{s}{2}$ which are exactly the same as the analytical one. Numerical results obtained by the proposed technique are shown in Table 2, while the comparison of the maximum absolute errors of the proposed technique with Adomian decomposition method [25] is $\|e(u_{1,3})\|_\infty = 0$, $\|e(u_{2,3})\|_\infty = 0$ and $\|e(u_{1,6})\|_\infty = 0.2 \times 10^{-6}$, $\|e(u_{2,6})\|_\infty = 0$, respectively.

Example 3 Consider the system of Fredholm IDEs, as

Table 1 Numerical results for Example 1

s	Exact $u_1(s)$	Proposed $u_{1,3}(s)$	Absolute errors $e_{1,3}(s)$	Absolute errors Tau method [21]
0	1	1	0	3×10^{-14}
0.2	1.12	1.12	0	2×10^{-14}
0.4	1.48	1.48	0	1×10^{-14}
0.6	2.08	2.08	0	2×10^{-14}
0.8	2.92	2.92	0	1×10^{-14}
1.0	4	4	0	1×10^{-14}
s	Exact $u_2(s)$	Proposed $u_{2,3}(s)$	Absolute errors $e_{2,3}(s)$	Absolute errors Tau method [21]
0	-1	-1	0	3.1×10^{-14}
0.2	-0.592	-0.592	0	2.7×10^{-14}
0.4	-0.136	-0.136	5.55112×10^{-17}	2.4×10^{-14}
0.6	0.416	0.416	1.11022×10^{-16}	2.3×10^{-14}
0.8	1.112	1.112	0	2.0×10^{-14}
1.0	2	2	0	2.0×10^{-14}

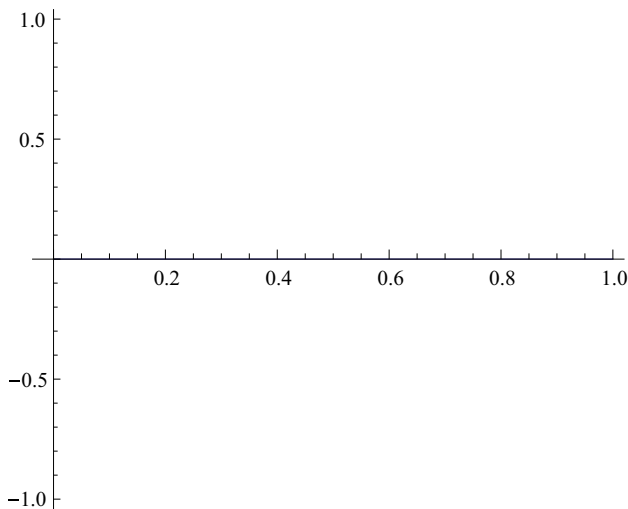**Fig. 1** Absolute errors $e_{1,3}(s)$ for $N = 3$

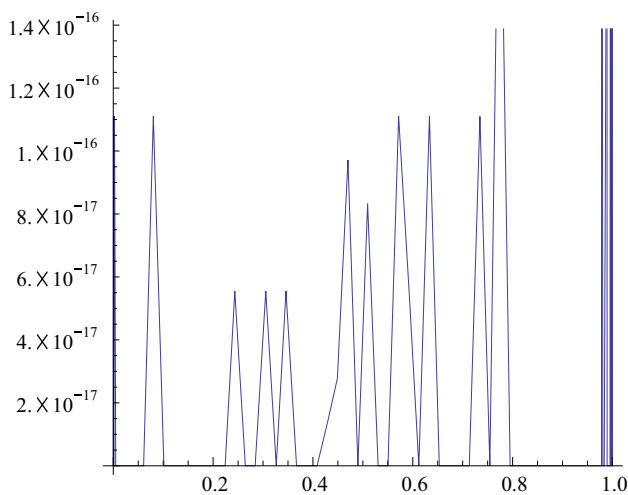**Fig. 2** Absolute errors $e_{2,3}(s)$ for $N = 3$

$$u_1^{(1)}(s) = \frac{5s}{3} + \int_0^1 \left\{\frac{s}{2}u_1(t) + \frac{s}{3}u_2(t)\right\}dt$$

$$u_2^{(1)}(s) = \frac{7}{6} + \int_0^1 \{u_1(t) - u_2(t)\}dt, \ 0 \leq s \leq 1, \tag{36}$$

subject to the following initial conditions

$$u_1(0) = 0, \quad u_2(0) = 0. \tag{37}$$

The analytical solution is $u_1(s) = s^2$ and $u_2(s) = s$.

Solving the system of equations for $N = 2$ by following the procedure stated above yields the approximate solutions $u_{1,2}(s) = s^2$ and $u_{2,2}(s) = s$ which are exactly the same as the analytical one. Numerical results obtained by the proposed technique are shown in Table 3, while the comparison of the maximum absolute errors of the proposed technique with Adomian decomposition method [25] is $\|e(u_{1,2})\|_\infty = 0$, $\|e(u_{2,2})\|_\infty = 0$ and $\|e(u_{1,20})\|_\infty = 0.18 \times 10^{-6}$, $\|e(u_{2,20})\|_\infty = 0.123 \times 10^{-5}$, respectively.

Example 4 Consider the system of Fredholm IDEs, as

$$u_1^{(2)}(s) - su_2^{(1)}(s) + 2su_1(s) = 2s^3 - \frac{37s^2}{12} + \frac{320s}{60} + 2$$

$$+ \int_0^1 \{s^2tu_1(t) - st^2u_2(t)\}dt - 2su_1^{(1)}(s) + u_2^{(2)}(s) + u_2(s)$$

$$= -5s^2 - \frac{109s}{30} - 1 - \int_0^1 \{stu_1(t) + st^3u_2(t)\}dt,$$

$$0 \leq s \leq 1, \tag{38}$$

subject to the following boundary conditions

$$u_1(0) = 3, \quad u_1(1) = 2, \quad u_2(0) = 1, \quad u_2(1) = 1. \tag{39}$$

The analytical solution is $u_1(s) = s^2 - 2s + 3$ and $u_2(s) = -s^2 + s + 1$.

Solving the system of equations for $N = 2$ by following the procedure stated above yields the approximate solutions $u_{1,2}(s) = s^2 - 2s + 3$ and $u_{2,2}(s) = -s^2 + s + 1$ which are exactly the same as the analytical one. Numerical results obtained by the proposed technique are shown in Table 4.

Example 5 Consider the system of Fredholm IDEs, as

Table 2 Numerical results for Example 2	s	Exact $u_1(s)$	Proposed $u_{1,3}(s)$	Absolute errors $e_{1,3}(s)$	Exact $u_2(s)$	Proposed $u_{2,3}(s)$	Absolute errors $e_{2,3}(s)$
	0	0	0	0	0	0	0
	0.2	0.086667	0.086667	0	-0.092	-0.092	0
	0.4	0.213333	0.213333	0	-0.136	-0.136	0
	0.6	0.38	0.38	0	-0.084	-0.084	0
	0.8	0.586667	0.586667	0	0.112	0.112	0
	1.0	0.833333	0.833333	0	0.5	0.5	0

Table 3 Numerical results for Example 3

s	Exact $u_1(s)$	Proposed $u_{1,2}(s)$	Absolute errors $e_{1,2}(s)$	Exact $u_2(s)$	Proposed $u_{2,2}(s)$	Absolute errors $e_{2,2}(s)$
0	0	0	0	0	0	0
0.2	0.04	0.04	0	0.2	0.2	0
0.4	0.16	0.16	0	0.4	0.4	0
0.6	0.36	0.36	0	0.6	0.6	0
0.8	0.64	0.64	0	0.8	0.8	0
1.0	1	1	0	1	1	0

Table 4 Numerical results for Example 4

s	Exact $u_1(s)$	Proposed $u_{1,2}(s)$	Proposed errors $e_{1,2}(s)$	Exact $u_2(s)$	Proposed $u_{2,2}(s)$	Proposed errors $e_{2,2}(s)$
0	3	3	0	1	1	0
0.2	2.64	2.64	0	1.16	1.16	0
0.4	2.36	2.36	4.44089×10^{-16}	1.24	1.24	0
0.6	2.16	2.16	0	1.24	1.24	0
0.8	2.04	2.04	0	1.16	1.16	0
1.0	2	2	0	1	1	0

Table 5 Numerical results for Example 5

s	Exact $u_1(s)$	Proposed $u_{1,3}(s)$	Proposed $u_{1,4}(s)$	Proposed $u_{1,5}(s)$
0	0	-1.9984×10^{-15}	-1.33227×10^{-15}	2.3892×10^{-13}
0.2	0.198669	0.19869	0.198517	0.198669
0.4	0.389418	0.389516	0.388638	0.389419
0.6	0.564642	0.564618	0.563349	0.564646
0.8	0.717356	0.716132	0.717141	0.717369
1.0	0.841471	0.836195	0.846011	0.841529

s	Exact $u_2(s)$	Proposed $u_{2,3}(s)$	Proposed $u_{2,4}(s)$	Proposed $u_{2,5}(s)$
0	1	1	1	1
0.2	0.980067	0.980214	0.978639	0.980066
0.4	0.921061	0.921711	0.917059	0.92106
0.6	0.825336	0.825775	0.83689	0.825337
0.8	0.696707	0.693689	0.783602	0.696719
1.0	0.540302	0.526737	0.8265	0.540404

Table 6 Maximum absolute errors for Example 5

N	3	4	5
$e_{1,N}$	5.27579×10^{-3}	4.53959×10^{-3}	5.80152×10^{-5}
$e_{2,N}$	1.35651×10^{-2}	2.86198×10^{-1}	1.01691×10^{-4}

$$u_1^{(2)}(s) - s\,u_2^{(1)}(s) - u_1(s) = (s-2)\sin(s)$$
$$+ \int_0^1 \{s\cos(t)u_1(t) - s\sin(t)u_2(t)\}\mathrm{d}t$$
$$- 2su_1^{(1)}(s) + u_2^{(2)}(s) + u_2(s) = -2s\cos(s) \qquad (40)$$
$$+ \int_0^1 \{\sin(s)\cos(t)u_1(t) - \sin(s)\sin(t)u_2(t)\}\mathrm{d}t,$$
$$0 \le s \le 1,$$
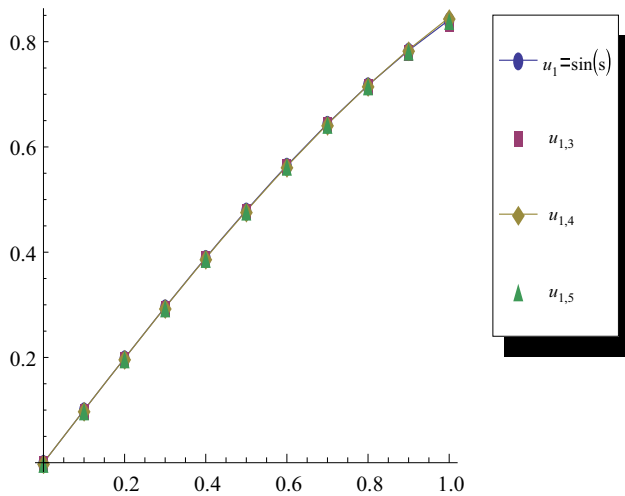
subject to the following initial conditions

Fig. 3 Comparison of $u_1(s)$ and $u_{1,N}$ for $N = 3, 4, 5$
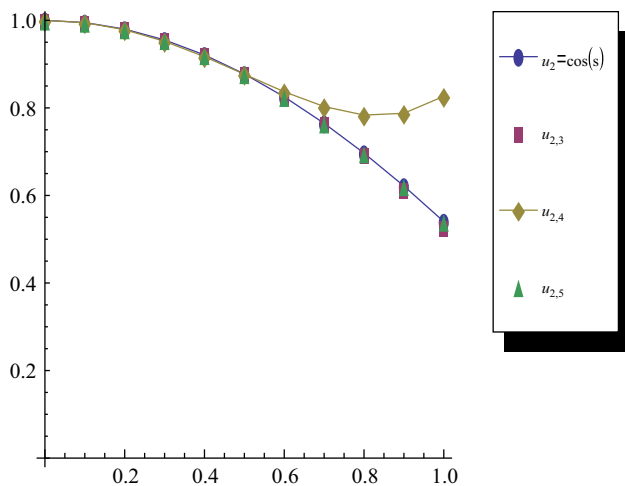

Fig. 4 Comparison of $u_2(s)$ and $u_{2,N}$ for $N = 3, 4, 5$

$$u_1(0) = 0, \quad u_1^{(1)}(0) = 1, \quad u_2(0) = 1, \quad u_2^{(1)}(0) = 0. \tag{41}$$

The analytical solution is $u_1(s) = \sin(s)$ and $u_2(s) = \cos(s)$.

Solving the system of equations for $N = 3, 4, 5$ by following the procedure stated above, yields the approximate solutions:

$$u_{1,3}(s) = -1.9984 \times 10^{-15} + 1s - 8.88178 \times 10^{-16}s^2 \\ - 0.163805s^3,$$
$$u_{2,3}(s) = 1 - 3.33067 \times 10^{-16}s - 0.5s^2 + 0.0267372s^3,$$
and
$$u_{1,4}(s) = -1.33227 \times 10^{-15} + 1s - 1.33227 \times 10^{-14}s^2 \\ - 0.193214s^3 + 0.0392248s^4,$$

$$u_{2,4}(s) = 1 + 8.52651 \times 10^{-14}s - 0.5s^2 - 0.294265s^3 \\ + 0.620765s^4,$$
also
$$u_{1,5}(s) = 2.3892 \times 10^{-13} + 1s + 4.01457 \times 10^{-13}s^2 \\ - 0.166728s^3 + 0.000337574s^4 \\ + 0.00791926s^5,$$
$$u_{2,5}(s) = 1. - 8.08242 \times 10^{-13}s - 0.5s^2 - 0.00030213s^3 \\ + 0.0431559s^4 - 0.00244974s^5.$$

Numerical results are summarized in Tables 5, while 6 shows the maximum absolute errors for Example 5. Comparison of exact and proposed solutions is shown in Figs. 3 and 4 for $N = 3, 4$ and 5, respectively.

Conclusion

In this paper, Laguerre operational matrix approach has been manipulated to solve the system of linear Fredholm IDEs. The scheme converted the system of IDEs, using Laguerre operational matrices, to a matrix equation that can be solved by any suitable method. Comparison of the results with other methods such as Tau method [21] and Adomian decomposition method (ADM) [25] reveals that the Laguerre approach has more accuracy. In addition, to get the best approximating solution of the system, the truncation limit N must be chosen large enough. It is also to be mentioned that the method is efficient to determine the solution in closed form, as well.

Compliance with ethical standards

Conflict of interest All the authors declare that they have no conflict of interest.

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