Common fixed point of four maps in S -metric spaces

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Abstract

In this paper, we prove some common fixed point results for four mappings satisfying generalized contractive condition in S -metric space. Our results extend and improve several previous works.

Keywords Common fixed point · S -metric space · Compatible mappings

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Introduction and preliminaries

Banach's contraction principle in metric spaces is one of the most important results in the theory of fixed points and non-linear analysis in general. From 1922, when Stefan Banach formulated the notion of contraction and proved the famous theorem, scientists around the world are publishing new results that are connected either to establish a generalization of metric space or to get an improvement of contractive condition. In the literature most famous improvement of contractive condition is certainly the papers of Edelstein–Nemytskii, Boyd–Wong, Meir–Keeler, Kannan, Chatterje, Zamfirescu, and certainly Ćirić.

In addition to the improvement of Banach's contractive conditions, more and more attention is devoted itself to the generalization of metric spaces such as 2-metric spaces,

D^* -metric spaces, partial metric spaces, cone metric spaces, b -metric spaces, G -metric spaces and others. One of the generalization of metric spaces is given in the paper of Sedghi et al. [1]. They introduced a notion of S -metric spaces and give some of their properties. For more details regarding this spaces we refer [1–12].

For the sake of transparency, we list the basic properties of S -metric spaces that will be used later.

Definition 1.1 [1] Let X be a nonempty set. A function $S : X^3 \rightarrow [0, \infty)$ is said to be an S -metric on X , if for each $x, y, z, a \in X$,

1. $S(x, y, z) = 0$ if and only if $x = y = z$,
2. $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S -metric space.

Example 1.2 [1] We can easily check that the following examples are S -metric spaces.

1. Let $X = \mathbb{R}^n$ and $\|\cdot\|$ be a norm on X . Then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S -metric on X . In general, if X is a vector space over $\mathbb{R}$ and $\|\cdot\|$ is a norm on X . Then it is easy to see that

$$S(x, y, z) = \|\alpha y + \beta z - \lambda x\| + \|y - z\|,$$

where $\alpha + \beta = \lambda$ for every $\alpha, \beta \geq 1$, is an S -metric on X .

2. Let X be a nonempty set and d_1, d_2 be two ordinary metrics on X . Then

$$S(x, y, z) = d_1(x, z) + d_2(y, z),$$

is an S -metric on X .

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Lemma 1.3 [8] Let (X, S) be an S -metric space. Then, we have $S(x, x, y) = S(y, y, x)$, $x, y \in X$.

Definition 1.4 [9] Let (X, S) be an S -metric space and $A \subset X$.

1. A sequence $\{x_n\}$ in X converges to x if $S(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$, that is for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$. This case, we denote by $\lim_{n \rightarrow \infty} x_n = x$ and we say that x is the limit of $\{x_n\}$ in X .
2. A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$.
3. The S -metric space (X, S) is said to be complete if every Cauchy sequence is convergent.

Definition 1.5 [6] Let (X, S) and (X', S') be two S -metric spaces, and let $f : (X, S) \rightarrow (X', S')$ be a function. Then f is said to be continuous at a point $a \in X$ if and only if for every sequence x_n in X , $S(x_n, x_n, a) \rightarrow 0$ implies $S'(f(x_n), f(x_n), f(a)) \rightarrow 0$. A function f is continuous at X if and only if it is continuous at all $a \in X$.

Lemma 1.6 [9] Let (X, S) be an S -metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$, then

$$\lim_{n \rightarrow \infty} S(x_n, x_n, y_n) = S(x, x, y).$$

Definition 1.7 Let (X, S) be an S -metric space. A pair $\{f, g\}$ is said to be compatible if and only if $\lim_{n \rightarrow \infty} S(fgx_n, fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$ for some $t \in X$.

Common fixed point results

We start with the following lemma.

Lemma 2.1 Let (X, S) be an S -metric space. If there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n \rightarrow \infty} S(x_n, x_n, y_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} x_n = t$ for some $t \in X$, then $\lim_{n \rightarrow \infty} y_n = t$.

Proof By the triangle inequality in S -metric space, we have

$$S(y_n, y_n, t) \leq 2S(y_n, y_n, x_n) + S(t, t, x_n).$$

Now, by taking the upper limit when $n \rightarrow \infty$ in above inequality we get

$$\limsup_{n \rightarrow \infty} S(y_n, y_n, t) \leq 2 \limsup_{n \rightarrow \infty} S(y_n, y_n, x_n) + \limsup_{n \rightarrow \infty} S(t, t, x_n) = 0.$$

Hence $\lim_{n \rightarrow \infty} y_n = t$. $\square$

Theorem 2.2 Suppose that f, g, R and T are self maps of a complete S -metric space (X, S) , with $f(X) \subseteq T(X)$, $g(X) \subseteq R(X)$ and that the pairs $\{f, R\}$ and $\{g, T\}$ are compatible. If

$$S(fx, fy, gz) \leq q \max\{S(Rx, Ry, Tz), S(fx, fx, Rx), S(gz, gz, Tz), S(fy, fy, gz)\}, \quad (1)$$

for each $x, y, z \in X$, with $0 < q < 1$.

Then f, g, R and T have a unique common fixed point in X provided that R and T are continuous.

Proof Let $x_0 \in X$. Since $f(X) \subseteq T(X)$, there exists $x_1 \in X$ such that $fx_0 = Tx_1$, and also as $gx_1 \in R(X)$, we choose $x_2 \in X$ such that $gx_1 = Rx_2$. In general, $x_{2n+1} \in X$ is chosen such that $fx_{2n} = Tx_{2n+1}$ and $x_{2n+2} \in X$ such that $gx_{2n+1} = Rx_{2n+2}$, we obtain a sequence $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = Tx_{2n+1},$$

$$y_{2n+1} = gx_{2n+1} = Rx_{2n+2}, \quad n \geq 0.$$

Now, we show that $\{y_n\}$ is a Cauchy sequence. For this we have

$$\begin{aligned} S(y_{2n}, y_{2n}, y_{2n+1}) &= S(fx_{2n}, fx_{2n}, gx_{2n+1}) \\ &\leq q \max\{S(Rx_{2n}, Rx_{2n}, Tx_{2n+1}), \\ &\quad S(fx_{2n}, fx_{2n}, Rx_{2n}), \\ &\quad S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), \\ &\quad S(fx_{2n}, fx_{2n}, gx_{2n+1})\} \\ &= q \max\{S(y_{2n-1}, y_{2n-1}, y_{2n}), S(y_{2n}, y_{2n}, y_{2n-1}), \\ &\quad S(y_{2n+1}, y_{2n+1}, y_{2n}), \\ &\quad S(y_{2n}, y_{2n}, y_{2n+1})\} \\ &= q \max\{S(y_{2n-1}, y_{2n-1}, y_{2n}), S(y_{2n}, y_{2n}, y_{2n+1})\}. \end{aligned}$$

Now, if $S(y_{2n}, y_{2n}, y_{2n+1}) > S(y_{2n-1}, y_{2n-1}, y_{2n})$, then by above inequality we have

$$S(y_{2n}, y_{2n}, y_{2n+1}) < q S(y_{2n}, y_{2n}, y_{2n+1}),$$

which is a contradiction. Hence, $S(y_{2n}, y_{2n}, y_{2n+1}) \leq S(y_{2n-1}, y_{2n-1}, y_{2n})$, therefore by above inequality we get

$$S(y_{2n}, y_{2n}, y_{2n+1}) \leq q S(y_{2n-1}, y_{2n-1}, y_{2n}). \quad (2)$$

By similar arguments we have:

$$\begin{aligned}
S(y_{2n-1}, y_{2n-1}, y_{2n}) &= S(y_{2n}, y_{2n}, y_{2n-1}) \\
&= S(fx_{2n}, fx_{2n}, gx_{2n-1}) \\
&\leq q \max\{S(Rx_{2n}, Rx_{2n}, Tx_{2n-1}), \\
&\quad S(fx_{2n}, fx_{2n}, Rx_{2n}), \\
&\quad S(gx_{2n-1}, gx_{2n-1}, Tx_{2n-1}), \\
&\quad S(fx_{2n}, fx_{2n}, gx_{2n-1})\} \\
&= q \max\{S(y_{2n-1}, y_{2n-1}, y_{2n-2}), \\
&\quad S(y_{2n}, y_{2n}, y_{2n-1}), \\
&\quad S(y_{2n-1}, y_{2n-1}, y_{2n-2}), \\
&\quad S(y_{2n}, y_{2n}, y_{2n-1})\} \\
&= q \max\{S(y_{2n-2}, y_{2n-2}, y_{2n-1}), \\
&\quad S(y_{2n}, y_{2n}, y_{2n-1})\}.
\end{aligned}$$

Now, if $S(y_{2n}, y_{2n}, y_{2n-1}) > S(y_{2n-2}, y_{2n-2}, y_{2n-1})$, then by above inequality we have

$$S(y_{2n}, y_{2n}, y_{2n-1}) < q S(y_{2n}, y_{2n}, y_{2n-1}),$$

which is a contradiction. Hence, $S(y_{2n-1}, y_{2n-1}, y_{2n}) \leq S(y_{2n-2}, y_{2n-2}, y_{2n-1})$, therefore by above inequality we get

$$S(y_{2n-1}, y_{2n-1}, y_{2n}) \leq q S(y_{2n-2}, y_{2n-2}, y_{2n-1}). \quad (3)$$

Now, from (2) and (3) we have

$$S(y_n, y_n, y_{n-1}) \leq q S(y_{n-1}, y_{n-1}, y_{n-2}), \quad n \geq 2$$

where $0 < q < 1$.

Hence, for $n \geq 2$ it follows that

$$S(y_n, y_n, y_{n-1}) \leq \dots \leq q^{n-1} S(y_1, y_1, y_0). \quad (4)$$

By the triangle inequality in S -metric space, for $n > m$ we have

$$\begin{aligned}
S(y_n, y_n, y_m) &\leq 2S(y_m, y_m, y_{m+1}) + 2S(y_{m+1}, y_{m+1}, y_{m+2}) \\
&\quad + \dots + S(y_{n-1}, y_{n-1}, y_n) \\
&< 2S(y_m, y_m, y_{m+1}) + 2S(y_{m+1}, y_{m+1}, y_{m+2}) \\
&\quad + \dots + 2S(y_{n-1}, y_{n-1}, y_n).
\end{aligned}$$

Hence from (4) and as $0 < q < 1$ we have

$$\begin{aligned}
S(y_n, y_n, y_m) &\leq 2(q^m + q^{m+1} + \dots + q^{n-1}) S(y_1, y_1, y_0) \\
&\leq 2q^m [1 + q + q^2 + \dots] S(y_1, y_1, y_0) \\
&\leq 2 \frac{q^m}{1-q} S(y_1, y_1, y_0) \rightarrow 0, \quad \text{as } m \rightarrow \infty.
\end{aligned}$$

It follows that $\{y_n\}$ is a Cauchy sequence. Since X is a complete S -metric space, there is some y in X such that

$$\lim_{n \rightarrow \infty} fx_{2n} = \lim_{n \rightarrow \infty} Tx_{2n+1} = \lim_{n \rightarrow \infty} gx_{2n+1} = \lim_{n \rightarrow \infty} Rx_{2n+2} = y.$$

We show that y is a common fixed point of f , g , R and T .

Since R is continuous it follows that

$$\lim_{n \rightarrow \infty} R^2 x_{2n+2} = Ry, \quad \lim_{n \rightarrow \infty} Rfx_{2n} = Ry.$$

And since f and R are compatible, $\lim_{n \rightarrow \infty} S(fRx_{2n}, fRx_{2n}, Rfx_{2n}) = 0$. So by Lemma 2.1 $\lim_{n \rightarrow \infty} fRx_{2n} = Ry$.

Putting $x = y = Rx_{2n}$ and $z = x_{2n+1}$ in condition (1), we obtain

$$\begin{aligned}
S(fRx_{2n}, fRx_{2n}, gx_{2n+1}) &\leq q \max\{S(R^2 x_{2n}, R^2 x_{2n}, Tx_{2n+1}), \\
&\quad S(fRx_{2n}, fRx_{2n}, R^2 x_{2n}), \\
&\quad S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), \\
&\quad S(fRx_{2n}, fRx_{2n}, gx_{2n+1})\}.
\end{aligned} \quad (5)$$

Now, by taking the upper limit when $n \rightarrow \infty$ in (5) we get

$$\begin{aligned}
S(Ry, Ry, y) &= \lim_{n \rightarrow \infty} S(fRx_{2n}, fRx_{2n}, gx_{2n+1}) \\
&\leq q \max\{\lim_{n \rightarrow \infty} S(R^2 x_{2n}, R^2 x_{2n}, Tx_{2n+1}), \\
&\quad \lim_{n \rightarrow \infty} S(fRx_{2n}, fRx_{2n}, R^2 x_{2n}), \\
&\quad \lim_{n \rightarrow \infty} S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), \\
&\quad \lim_{n \rightarrow \infty} S(fRx_{2n}, fRx_{2n}, gx_{2n+1})\} \\
&\leq q \max\{S(Ry, Ry, y), 0, 0, S(Ry, Ry, y)\} \\
&= q S(Ry, Ry, y).
\end{aligned}$$

Consequently, $S(Ry, Ry, y) \leq q S(Ry, Ry, y)$, as $0 < q < 1$ it follows that $Ry = y$.

In a similar way, since T is continuous, we obtain that

$$\lim_{n \rightarrow \infty} T^2 x_{2n+1} = Ty, \quad \lim_{n \rightarrow \infty} Tgx_{2n+1} = Ty.$$

Since g and T are compatible, $\lim_{n \rightarrow \infty} S(gTx_{2n+1}, gTx_{2n+1}, Tgx_{2n+1}) = 0$. So by Lemma 2.1 $\lim_{n \rightarrow \infty} gTx_{2n+1} = Ty$.

Putting $x = y = x_{2n}$ and $z = Tx_{2n+1}$ in condition (1), we obtain

$$\begin{aligned}
S(fx_{2n}, fx_{2n}, gTx_{2n+1}) &\leq q \max\{S(Rx_{2n}, Rx_{2n}, T^2 x_{2n+1}), \\
&\quad S(fx_{2n}, fx_{2n}, Rx_{2n}), \\
&\quad S(gTx_{2n+1}, gTx_{2n+1}, T^2 x_{2n+1}), \\
&\quad S(fx_{2n}, fx_{2n}, gTx_{2n+1})\}.
\end{aligned} \quad (6)$$

Similarly by taking the upper limit when $n \rightarrow \infty$ in (6), we obtain

$$\begin{aligned}
S(y, y, Ty) &= \lim_{n \rightarrow \infty} S(fx_{2n}, fx_{2n}, gTx_{2n+1}) \\
&\leq q \max\{S(y, y, Ty), 0, 0, S(y, y, Ty)\},
\end{aligned}$$

that is, again it follows that $Ty = y$. Also, we can apply condition (1) to obtain

$$S(fy, fy, gx_{2n+1}) \leq q \max\{S(Ry, Ry, Tx_{2n+1}), S(fy, fy, Ry), \\ S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), S(fy, fy, gx_{2n+1})\}. \quad (7)$$

And by taking the upper limit when $n \rightarrow \infty$ in (7), as $Ry = Ty = y$, we have

$$S(fy, fy, y) \leq q \max\{S(Ry, Ry, y), S(fy, fy, y), S(y, y, y), S(fy, fy, y)\} \\ = qS(fy, fy, y).$$

Since $0 < q < 1$, it follows that $S(fy, fy, y) = 0$ and $fy = y$.

Finally by using of condition (1) and as, $Ry = Ty = fy = y$, we obtain

$$S(y, y, gy) = S(fy, fy, gy) \\ \leq q \max\{S(Ry, Ry, Ty), S(fy, fy, Ry), \\ S(gy, gy, Ty), S(fy, fy, gy)\} \\ = qS(y, y, gy),$$

which implies that $S(y, y, gy) = 0$ and $gy = y$.

Thus we proved that

$$Ry = Ty = fy = gy = y.$$

If there exists another common fixed point x in X of all f, g, R and T , then

$$S(x, x, y) = S(fx, fx, gy) \\ \leq q \max\{S(Rx, Rx, Ty), S(fx, fx, Rx), \\ S(gy, gy, Ty), S(fx, fx, gy)\} \\ = q \max\{S(x, x, y), S(x, x, x), \\ S(y, y, y), S(x, x, y)\} \\ = qS(x, x, y),$$

which implies that $S(x, x, y) = 0$ and $x = y$. Thus y is a unique common fixed point of f, g, R and T . The proof of the theorem is completed. $\square$

Now we give an example to support our result.

Example 2.3 Let $X = [0, 1]$ be endowed with S -metric $S(x, y, z) = |x - z| + |y - z|$. Define f, g, R and T on X by

$$f(x) = \left(\frac{x}{2}\right)^8, \quad g(x) = \left(\frac{x}{2}\right)^4,$$

$$R(x) = \left(\frac{x}{2}\right)^2, \quad T(x) = \frac{x}{2}.$$

Obviously $f(X) \subseteq T(X)$ and $g(X) \subseteq R(X)$. Furthermore, the pairs $\{f, R\}$, and $\{g, T\}$ are compatible mappings.

Also for each $x, y, z \in X$, we have

$$S(fx, fy, gz) = |fx - gz| + |fy - gz| \\ = \left|\left(\frac{x}{2}\right)^8 - \left(\frac{z}{2}\right)^4\right| + \left|\left(\frac{y}{2}\right)^8 - \left(\frac{z}{2}\right)^4\right| \\ = \left|\left(\frac{x}{2}\right)^4 - \left(\frac{z}{2}\right)^2\right| \left|\left(\frac{x}{2}\right)^4 + \left(\frac{z}{2}\right)^2\right| \\ + \left|\left(\frac{y}{2}\right)^4 - \left(\frac{z}{2}\right)^2\right| \left|\left(\frac{y}{2}\right)^4 + \left(\frac{z}{2}\right)^2\right| \\ \leq \frac{5}{16} \left|\left(\frac{x}{2}\right)^2 - \frac{z}{2}\right| \left|\left(\frac{x}{2}\right)^2 + \frac{z}{2}\right| + \frac{5}{16} \left|\left(\frac{y}{2}\right)^2 - \frac{z}{2}\right| \left|\left(\frac{y}{2}\right)^2 + \frac{z}{2}\right| \\ \leq \frac{15}{64} |Rx - Tz| + \frac{15}{64} |Ry - Tz| = \frac{15}{64} S(Rx, Ry, Tz) \\ \leq \frac{15}{64} \max\{S(Rx, Ry, Tz), S(fx, fx, Rx), \\ S(gz, gz, Tz), S(fy, fy, gz)\},$$

where $\frac{15}{64} \leq q < 1$. Thus f, g, R and T satisfy the conditions given in Theorem 2.2 and 0 is the unique common fixed point of f, g, R and T .

Now we get the special cases of Theorem 2.2 as follows.

Corollary 2.4 Let (X, S) be a complete S -metric space and let $f, g : X \rightarrow X$ be two mappings such that

$$S(fx, fy, gz) \leq q \max\{S(x, y, z), S(fx, fx, x), S(gz, gz, z), \\ S(fy, fy, gz)\}, \text{ for all } x, y, z \in X,$$

with $0 < q < 1$.

Then there exists a unique point $y \in X$ such that $fy = gy = y$.

Proof If we take R and T as identity maps on X , then Theorem 2.2 follows that f and g have a unique common fixed point. $\square$

Corollary 2.5 Let (X, S) be a complete S -metric space and let $R, T : X \rightarrow X$ be continuous mappings onto X , such that

$$S(x, y, z) \leq q \max\{S(Rx, Ry, Tz), S(x, x, Rx), S(z, z, Tz), \\ S(y, y, gz)\}, \text{ for all } x, y, z \in X,$$

with $0 < q < 1$.

Then R and T have a unique common fixed point.

Proof If we take f and g as identity maps on X , then from Theorem 2.2 follows that S and T have a unique common fixed point. $\square$

Corollary 2.6 Let (X, S) be a complete S -metric space and let $f : X \rightarrow X$ be a mapping such that

$$S(fx, fy, fz) \leq q \max\{S(x, y, z), S(fx, fx, x), S(fz, fz, z), \\ S(fy, fy, fz)\}, \text{ for all } x, y, z \in X,$$

with $0 < q < 1$.

Then f has a unique fixed point in X .

Proof If we take S and T as identity maps on X and $f = g$, then from Theorem 2.2 follows that f have a unique common fixed point.

Theorem 2.7 Let $\{f, R\}$ and $\{g, T\}$ be compatible self mappings on a complete S -metric space (X, S) and for all $x, y, z \in X$, satisfying

$$S(fx, fy, gz) \leq a_1 S(Rx, Ry, Tz) + a_2 S(fx, fx, Tz) + a_3 S(Rx, Ry, gz) + a_4 S(fy, fy, Tz) + a_5 S(gz, gz, Tz), \quad (8)$$

where $a_i \geq 0$ ($i = 1, 2, 3, 4, 5$) are real constants with $a_1 + 3a_2 + 3a_3 + 3a_4 + a_5 < 1$. If $f(X) \subseteq T(X)$ and $g(X) \subseteq R(X)$ and R and T are continuous, then all f, g, R and T have a unique common fixed point.

Proof Let x_0 in X . Since $f(X) \subseteq T(X)$, let $x_1 \in X$ be such that $Tx_1 = fx_0$, and also, as $gx_1 \in R(X)$, let $x_2 \in X$ be such that $Rx_2 = gx_1$. In general, $x_{2n+1} \in X$ is chosen such that $Tx_{2n+1} = fx_{2n}$ and $x_{2n+2} \in X$ such that $Rx_{2n+2} = gx_{2n+1}$; $n = 0, 1, 2, \dots$. Denote

$$y_{2n} = Tx_{2n+1} = fx_{2n}, \\ y_{2n+1} = Rx_{2n+2} = gx_{2n+1}, \quad n \geq 0.$$

Now, we show that $\{y_n\}$ is a Cauchy sequence. For this we have

$$\begin{aligned} S(y_{2n}, y_{2n}, y_{2n+1}) &= S(fx_{2n}, fx_{2n}, gx_{2n+1}) \\ &\leq a_1 S(Rx_{2n}, Rx_{2n}, Tx_{2n+1}) \\ &\quad + a_2 S(fx_{2n}, fx_{2n}, Tx_{2n+1}) \\ &\quad + a_3 S(Rx_{2n}, Rx_{2n}, gx_{2n+1}) \\ &\quad + a_4 S(fx_{2n}, fx_{2n}, Tx_{2n+1}) \\ &\quad + a_5 S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}) \\ &= a_1 S(y_{2n-1}, y_{2n-1}, y_{2n}) + a_2 S(y_{2n}, y_{2n}, y_{2n}) \\ &\quad + a_3 S(y_{2n-1}, y_{2n-1}, y_{2n+1}) \\ &\quad + a_4 S(y_{2n}, y_{2n}, y_{2n}) + a_5 S(y_{2n+1}, y_{2n+1}, y_{2n}) \\ &\leq a_1 S(y_{2n-1}, y_{2n-1}, y_{2n}) + a_3 [2S(y_{2n-1}, y_{2n-1}, y_{2n}) \\ &\quad + S(y_{2n+1}, y_{2n+1}, y_{2n})] + a_5 S(y_{2n}, y_{2n}, y_{2n+1}). \end{aligned}$$

Hence,

$$S(y_{2n}, y_{2n}, y_{2n+1}) \leq a_1 S(y_{2n-1}, y_{2n-1}, y_{2n}) + 2a_3 S(y_{2n-1}, y_{2n-1}, y_{2n}) + (a_3 + a_5) S(y_{2n}, y_{2n}, y_{2n+1}). \quad (9)$$

Now we prove that $S(y_{2n}, y_{2n}, y_{2n+1}) \leq S(y_{2n-1}, y_{2n-1}, y_{2n})$, for each $n \in \mathbb{N}$.

If $S(y_{2n-1}, y_{2n-1}, y_{2n}) < S(y_{2n}, y_{2n}, y_{2n+1})$ for some $n \in \mathbb{N}$, then from (9) we have

$$\begin{aligned} S(y_{2n}, y_{2n}, y_{2n+1}) &< a_1 S(y_{2n}, y_{2n}, y_{2n+1}) + 2a_3 S(y_{2n}, y_{2n}, y_{2n+1}) \\ &\quad + (a_3 + a_5) S(y_{2n}, y_{2n}, y_{2n+1}) \\ &= (a_1 + 3a_3 + a_5) S(y_{2n}, y_{2n}, y_{2n+1}) \\ &< S(y_{2n}, y_{2n}, y_{2n+1}), \end{aligned}$$

which is a contradiction.

So we have $S(y_{2n}, y_{2n}, y_{2n+1}) \leq S(y_{2n-1}, y_{2n-1}, y_{2n})$, for each $n \in \mathbb{N}$ and from (9) we get

$$S(y_{2n}, y_{2n}, y_{2n+1}) \leq (a_1 + 3a_3 + a_5) S(y_{2n-1}, y_{2n-1}, y_{2n}). \quad (10)$$

Also we have

$$\begin{aligned} S(y_{2n-1}, y_{2n-1}, y_{2n}) &= S(y_{2n}, y_{2n}, y_{2n-1}) \\ &= S(fx_{2n}, fx_{2n}, gx_{2n-1}) \\ &\leq a_1 S(Rx_{2n}, Rx_{2n}, Tx_{2n-1}) \\ &\quad + a_2 S(fx_{2n}, fx_{2n}, Tx_{2n-1}) \\ &\quad + a_3 S(Rx_{2n}, Rx_{2n}, gx_{2n-1}) \\ &\quad + a_4 S(fx_{2n}, fx_{2n}, Tx_{2n-1}) \\ &\quad + a_5 S(gx_{2n-1}, gx_{2n-1}, Tx_{2n-1}) \\ &= a_1 S(y_{2n-1}, y_{2n-1}, y_{2n-2}) \\ &\quad + a_2 S(y_{2n}, y_{2n}, y_{2n-2}) \\ &\quad + a_3 S(y_{2n-1}, y_{2n-1}, y_{2n-1}) \\ &\quad + a_4 S(y_{2n}, y_{2n}, y_{2n-2}) \\ &\quad + a_5 S(y_{2n-1}, y_{2n-1}, y_{2n-2}) \\ &\leq a_1 S(y_{2n-1}, y_{2n-1}, y_{2n-2}) \\ &\quad + (2a_2 + 2a_4) S(y_{2n-1}, y_{2n-1}, y_{2n}) \\ &\quad + (a_2 + a_4) S(y_{2n-1}, y_{2n-1}, y_{2n-2}) \\ &\quad + a_5 S(y_{2n-1}, y_{2n-1}, y_{2n-2}). \end{aligned}$$

Hence,

$$\begin{aligned} S(y_{2n}, y_{2n}, y_{2n-1}) &\leq a_1 S(y_{2n-1}, y_{2n-1}, y_{2n-2}) \\ &\quad + (2a_2 + 2a_4) S(y_{2n}, y_{2n}, y_{2n-1}) \\ &\quad + (a_2 + a_4 + a_5) S(y_{2n-1}, y_{2n-1}, y_{2n-2}). \end{aligned} \quad (11)$$

Similarly, if $S(y_{2n-1}, y_{2n-1}, y_{2n-2}) < S(y_{2n}, y_{2n}, y_{2n-1})$ for some $n \in \mathbb{N}$ then from (11), we obtain

$$S(y_{2n}, y_{2n}, y_{2n-1}) \leq (a_1 + 3a_2 + 3a_4 + a_5) S(y_{2n}, y_{2n}, y_{2n-1}) < S(y_{2n}, y_{2n}, y_{2n-1}),$$

which is a contradiction.

So we have $S(y_{2n}, y_{2n}, y_{2n-1}) \leq S(y_{2n-1}, y_{2n-1}, y_{2n-2})$, for each $n \in \mathbb{N}$ and from (11) we get

$$\begin{aligned} S(y_{2n}, y_{2n}, y_{2n-1}) &\leq (a_1 + 3a_2 + 3a_4 + a_5) \\ S(y_{2n-1}, y_{2n-1}, y_{2n-2}). \end{aligned} \quad (12)$$

Now, from (10) and (12), we have

$$S(y_n, y_n, y_{n-1}) \leq \lambda S(y_{n-1}, y_{n-1}, y_{n-2}), \quad n \geq 2,$$

where $\lambda = \min\{a_1 + 3a_3 + a_5, a_1 + 3a_2 + 3a_4 + a_5\}$. We know that $\lambda \in (0, 1)$.

Hence, for $n \geq 2$ it follows that

$$S(y_n, y_n, y_{n-1}) \leq \dots \leq \lambda^{n-1} S(y_1, y_1, y_0). \quad (13)$$

By the triangle inequality in S -metric space, for $n > m$ we have

$$\begin{aligned} S(y_n, y_n, y_m) &\leq 2S(y_m, y_m, y_{m+1}) + 2S(y_{m+1}, y_{m+1}, y_{m+2}) \\ &+ \dots + 2S(y_{n-1}, y_{n-1}, y_n). \end{aligned}$$

Hence from (13) and as $\lambda < 1$, we have

$$\begin{aligned} S(y_n, y_n, y_m) &\leq 2(\lambda^m + \lambda^{m+1} + \dots + \lambda^{n-1}) S(y_1, y_1, y_0) \\ &\leq 2\lambda^m [1 + \lambda + (\lambda)^2 + \dots] S(y_1, y_1, y_0) \\ &\leq \frac{2\lambda^m}{1 - \lambda} S(y_1, y_1, y_0) \\ &= \frac{2\lambda^m}{1 - \lambda} S(y_1, y_1, y_0) \rightarrow 0 \quad \text{as } m \rightarrow \infty. \end{aligned}$$

It follows that $\{y_n\}$ is a Cauchy sequence. Let $y \in X$ be such that

$$\lim_{n \rightarrow \infty} fx_{2n} = \lim_{n \rightarrow \infty} Tx_{2n+1} = \lim_{n \rightarrow \infty} gx_{2n+1} = \lim_{n \rightarrow \infty} Rx_{2n+2} = y.$$

Since R is continuous it follows that

$$\lim_{n \rightarrow \infty} R^2x_{2n+2} = Ry, \quad \lim_{n \rightarrow \infty} Rfx_{2n} = Ry.$$

And since f and R are compatible, $\lim_{n \rightarrow \infty} S(fRx_{2n}, fRx_{2n}, Rfx_{2n}) = 0$. So by Lemma 2.1 $\lim_{n \rightarrow \infty} fRx_{2n} = Ry$.

From (8) it follows that

$$\begin{aligned} S(fRx_{2n}, fRx_{2n}, gx_{2n+1}) &\leq a_1 S(R^2x_{2n}, R^2x_{2n}, Tx_{2n+1}) \\ &+ a_2 S(fRx_{2n}, fRx_{2n}, Tx_{2n+1}) \\ &+ a_3 S(R^2x_{2n}, R^2x_{2n}, gx_{2n+1}) \\ &+ a_4 S(fRx_{2n}, fRx_{2n}, Tx_{2n+1}) \\ &+ a_5 S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}). \end{aligned}$$

Taking the upper limit as $n \rightarrow \infty$, we get

$$\begin{aligned} S(Ry, Ry, y) &\leq a_1 S(Ry, Ry, y) + a_2 S(Ry, Ry, y) + a_3 S(Ry, Ry, y) \\ &+ a_4 S(Ry, Ry, y) + a_5 S(y, y, y) \\ &\leq a_1 S(Ry, Ry, y) + a_2 S(Ry, Ry, y) + a_3 S(Ry, Ry, y) \\ &+ a_4 S(Ry, Ry, y) \\ &= (a_1 + a_2 + a_3 + a_4) S(Ry, Ry, y) \\ &\leq (a_1 + a_2 + a_3 + a_4 + a_5) S(Ry, Ry, y). \end{aligned}$$

Therefore $S(Ry, Ry, y) \leq (a_1 + 3a_2 + 3a_3 + 3a_4 + a_5) S(Ry, Ry, y)$, as $a_1 + 3a_2 + 3a_3 + 3a_4 + a_5 < 1$, we know that $Ry = y$.

In a similar way, since T is continuous, we obtain that

$$\lim_{n \rightarrow \infty} T^2x_{2n+1} = Ty, \quad \lim_{n \rightarrow \infty} Tgx_{2n+1} = Ty.$$

Since g and T are compatible, $\lim_{n \rightarrow \infty} S(gTx_{2n+1}, gTx_{2n+1}, Tgx_{2n+1}) = 0$. So by Lemma 2.1 $\lim_{n \rightarrow \infty} gTx_{2n+1} = Ty$.

From (8), it follows that

$$\begin{aligned} S(fx_{2n}, fx_{2n}, gTx_{2n+1}) &\leq a_1 S(Rx_{2n}, Rx_{2n}, T^2x_{2n+1}) \\ &+ a_2 S(fx_{2n}, fx_{2n}, T^2x_{2n+1}) \\ &+ a_3 S(Rx_{2n}, Rx_{2n}, gTx_{2n+1}) \\ &+ a_4 S(fx_{2n}, fx_{2n}, T^2x_{2n+1}) \\ &+ a_5 S(gTx_{2n+1}, gTx_{2n+1}, T^2x_{2n+1}). \end{aligned}$$

Taking the upper limit as $n \rightarrow \infty$, we get

$$\begin{aligned} S(y, y, Ty) &\leq a_1 S(y, y, Ty) + a_2 S(y, y, Ty) + a_3 S(y, y, Ty) \\ &+ a_4 S(y, y, Ty) + a_5 S(Ty, Ty, Ty) \\ &= (a_1 + a_2 + a_3 + a_4) S(y, y, Ty) \\ &\leq (a_1 + 3a_2 + 3a_3 + 3a_4 + a_5) S(y, y, Ty), \end{aligned}$$

that is,

$$S(y, y, Ty) \leq (a_1 + 3a_2 + 3a_3 + 3a_4 + a_5) S(y, y, Ty).$$

Therefore, by $a_1 + 3a_2 + 3a_3 + 3a_4 + a_5 < 1$, we know that $Ty = y$.

Again from (8), it follows that

$$\begin{aligned} S(fy, fy, gx_{2n+1}) &\leq a_1 S(Ry, Ry, Tx_{2n+1}) + a_2 S(fy, fy, Tx_{2n+1}) \\ &+ a_3 S(Ry, Ry, gx_{2n+1}) + a_4 S(fy, fy, Tx_{2n+1}) \\ &+ a_5 S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}). \end{aligned}$$

And by taking the upper limit as $n \rightarrow \infty$, as $Ry = y$, $Ty = y$, we get

$$\begin{aligned} S(fy, fy, y) &\leq a_1 S(y, y, y) + a_2 S(fy, fy, y) + a_3 S(y, y, y) \\ &+ a_4 S(fy, fy, y) + a_5 S(y, y, y) \\ &\leq (a_2 + a_4) S(fy, fy, y). \end{aligned}$$

Therefore, $S(fy, fy, y) \leq (a_1 + 3a_2 + 3a_3 + 3a_4 + a_5) S(fy, fy, y)$ and by $a_1 + 3a_2 + 3a_3 + 3a_4 + a_5 < 1$, we know that $fy = y$. Again from (8) we have $S(fy, fy, gy) = 0$; hence $fy = gy$. Thus we prove that $fy = gy = Ry = Ty = y$. If there exists another common fixed point x in X of all f, g, R and T , then

$$\begin{aligned}
S(x, x, y) &= S(fx, fx, gy) \\
&\leq a_1 S(Rx, Rx, Ty) + a_2 S(fx, fx, Ty) + a_3 S(Rx, Rx, gy) \\
&\quad + a_4 S(fx, fx, Ty) + a_5 S(gy, gy, Ty) \\
&= (a_1 + a_2 + a_3 + a_4) S(x, x, y) \\
&\leq (a_1 + 3a_2 + 3a_3 + 3a_4 + a_5) S(x, x, y).
\end{aligned}$$

From which it follows

$$S(x, x, y) \leq (a_1 + 3a_2 + 3a_3 + 3a_4 + a_5) S(x, x, y).$$

Since $a_1 + 3a_2 + 3a_3 + 3a_4 + a_5 < 1$, it follows that $S(x, x, y) = 0$, i.e., $x = y$. Therefore y is a unique common fixed point of all f, g, R and T . The proof of the theorem is completed. $\square$

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