Mathematical Sciences (2018) 12:25–39
https://doi.org/10.1007/s40096-018-0241-9

ORIGINAL RESEARCH

New oscillation criteria of special type second-order non-linear dynamic equations on time scales

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Received: 29 March 2017 / Accepted: 5 January 2018 / Published online: 24 January 2018

Abstract
By the Riccati transformation technique, we study some new oscillatory properties for the second-order dynamic equation on an arbitrary time scale $\mathbb{T}$. We also establish the Kamenev-type and Philos-type oscillation criteria. At the end, we give examples which illustrate our main results.

Keywords Time scale · Oscillation · Delay dynamic equation · Riccati technique

Mathematical Subject Classification 34N05 · 34K11 · 39A10 · 39A99

Introduction

In [25], Kubyshkin and Moryakova considered a second-order differential–difference equation of delay type

$$\ddot{x}(t) + A\dot{x}(t) + x(t) + \mathcal{K}(x(t-h)) + \mathcal{W}(\dot{x}(t-h)) = 0, \tag{1.1}$$

where the real constants $A, h > 0$, and the functions $\mathcal{K}, \mathcal{W} : \mathbb{R} \to \mathbb{R}$ defined by $\mathcal{K}(x(t)) = k_1 x(t) + k_2 x^2(t)\ldots\ldots$, and $\mathcal{W}(x(t)) = w_1 x(t) + w_2 x^2(t)\ldots\ldots\forall k_i, w_j \in \mathbb{R}$, respectively. The authors have investigated the oscillatory solution of Eq. (1.1). Such Eq. (1.1) occur in the modeling of electronic devices. In this paper, we modify and extend this work by considering the Eq. (1.1) on the arbitrary time scales $\mathbb{T}$,

$$\begin{aligned} y^{\Delta\Delta}(t) + a(t)y^{\Delta}(t) + y(t) + \mathcal{K}(y(t-h)) \\ + f(\mathcal{W}(y^{\Delta}(t-h))) = 0 \quad \forall t \in \mathbb{T}, \end{aligned} \tag{1.2}$$

where the functions $a : \mathbb{T} \to \mathbb{R}, f : \mathbb{R} \to \mathbb{R}$, and time scale $\mathbb{T}$ satisfies $t - h < t$ and $t - h \in \mathbb{T} \forall t \in \mathbb{T}$, for some positive real constant h. One can easily see that for some $h > 0$, Eq. (1.1) can be achieved by taking $\mathbb{T} = \mathbb{R}$, $a(t) = A > 0$ and $f(x) = x \forall x \in \mathbb{R}$ in Eq. (1.2). Equation (1.2) is very general in nature and techniques from time scale calculus to analyze it. Equation (1.2) covers not only differential equations (i.e., $\mathbb{T} = \mathbb{R}$) and difference equations (*i.e.*, $\mathbb{T} = \mathbb{Z}$) but covers more general time scales $h\mathbb{Z} = \{hn : n \in \mathbb{Z}\}$ for $h > 0, \mathbb{T} = \bigcup_{k\in\mathbb{Z}} [k(a+b), k(a+b) + a]$ for $a, b > 0$, and $\mathbb{T} = \bigcup_{m\in\mathbb{Z}} \{m + \frac{1}{n} : n \in \mathbb{N}\} \cup \mathbb{Z}$, etc. Throughout this paper, we obtain the sufficient conditions of oscillation for the dynamic equation (1.2). To the best of our knowledge, no work has been done regarding the oscillatory behavior of (1.2) so far.

Following Stefan's landmark [22], a rapidly expanding body of literature has sought to unify, extend, and generalize ideas from discrete, quantum and continuous calculus to an arbitrary time scale calculus. A time scale is an arbitrary non-empty closed subset of the real numbers which have the topology that inherits from the real numbers with the standard topology. It has applications in electrical engineering, quantum mechanics, population dynamic and economics etc [3, 11]. In particular, a time scale $q^{\mathbb{Z}} \cup \{0\}, q > 1$ is used in quantum physics, see in [6, 7]. Many authors have worked on various aspects of new theory, see in [2, 6, 7, 14, 17, 19, 20, 21, 34] and the

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references therein. These literatures summarize and organize much of time scale calculus.

Stefan's theory has attracted the attention of many researchers on oscillation of second-order linear and nonlinear dynamic equation on time scales. In recent years, many researchers have focused on oscillation and nonoscillation criteria of second-order ordinary dynamic equations on time scales. Several authors have studied the oscillation criteria by employing the Riccati transformation technique as well as established the Kamenev-type and Philos-type oscillation criteria. For more details on such criteria, we refer the reader to the papers [1, 12, 13, 15, 20, 28, 30–33] and reference therein. To establish the oscillation criteria, in [24], Kamenev considered a second-order differential equation

$$(p(t)x'(t))' + q(t)x(t) = 0 \tag{1.3}$$

and investigated the following sufficient conditions of oscillations,

$$\limsup_{t\to\infty} \frac{1}{t^n}\int_0^t (t-s)^n q(s)\mathrm{d}s = \infty, \text{ for } n > 1. \tag{1.4}$$

Several authors (e.g., Sun [31], Philos [27]) have extended the Kamenev's oscillation criteria to more general criteria by taking the kernal function $(t-s)^n$ as the general class of functions $H(t, s)$, which satisfies some assumptions given below follow:

The Kernal function $H : \mathbb{D} = \{(t,s) : t_0 \le s \le t\} \to \mathbb{R}$ defined as continuous function such that

$$H(t,t) = 0 \quad \text{for } t_0 \le t, H(t,s) > 0$$
$$\text{for } t_0 \le s < t \text{ and } \frac{\partial H(t,s)}{\partial s} < 0 \quad \text{on } \mathbb{D},$$

where $\frac{\partial H(t,s)}{\partial s}$ is continuous on D. Furthermore, define a continuous function $h : \mathbb{D} \to \mathbb{R}$ such that

$$\frac{\partial H(t,s)}{\partial s} = -h(t,s)\sqrt{H(t,s)}, \quad \text{for all } (t,s) \in \mathbb{D}.$$

In [27], Philos obtain the following sufficient conditions,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^t \left(H(t,s)q(s) - \frac{1}{4}h^2(t,s)\right) \mathrm{d}s = \infty, \quad \text{for } t_0 \ge 0 \quad \text{and} \quad p(t) = 1. \tag{1.5}$$

From the conditions (1.4) and (1.5), we conclude that the Philos-type is more general to Kamenev-type criteria. A discrete version of differential equation (1.3)

$$\Delta(r_n\Delta x_n) + p_n x_{n+1} = 0 \tag{1.6}$$

has been discussed by Chen and Erbe [9], for oscillation and non-oscillation. To harmonize the differential equation (1.3) and difference equation (1.6), in [12], Del and Kong, and in [16], Erbe, et al. have considered a following self-adjoint second-order scalar dynamic equation on time scale $\mathbb{T}$

$$(p(t)x^\Delta(t))^\Delta + q(t)x(\sigma(t)) = 0, \tag{1.7}$$

and established the oscillation criteria. The problem (1.7) is not only the extension of results of [9, 24] but to the more general results on time scales. In addition, several authors have focused on study of both Kamenev-type and Philos-type oscillation criteria on the time scales. For example, in [4], Agwa et al. studied both Kamenev-type and Philos-type oscillation criteria of the following second-order nonlinear delay dynamic equation

$$(r(t)g(x(t),x^\Delta(t)))^\Delta + p(t)f(x(\tau(t))) = 0, t \in \mathbb{T}, t_0 \le t. \tag{1.8}$$

In [10], Chen et al. considered the second-order dynamic equation with damping on time scales

$$((x^\Delta(t))^\gamma)^\Delta + p(t)(x^\Delta(t))^\gamma + q(t)f(x^\sigma(t)) = 0 \tag{1.9}$$

and by employing Riccati transformation to established the sufficient conditions of Kamenev-type as well as Philos-type oscillation criteria. In [29], Saker have established the sufficient conditions of oscillation of the following second-order nonlinear neutral delay dynamic equations on time scales

$$(r(t)([y(t) + p(t)y(t-\tau)]^\Delta)^\gamma)^\Delta + f(t, y(t-\delta)) \tag{1.10}$$

and discussed both Kamenev-type and Philos-type oscillation criteria. In [8], Bohner and Saker have studied the oscillation criteria for the second-order perturbed dynamic equation on time scales:

$$(r(t)(x^\Delta(t))^\gamma)^\Delta + F(t, x(\sigma(t))) = G(t, x(t), x^\Delta(t)). \tag{1.11}$$

In [1], Agarwal et al. modified the dynamic equation (1.11) and considered the following equation

$$(r(t)(x^\Delta(t))^\gamma)^\Delta + F(t, x(t)) = G(t, x(t), x^\Delta(t)), \tag{1.12}$$

and studied the sufficient conditions for oscillation. They have assumed that there exist two positive rd-continuous functions p and q such that

1. $r : \mathbb{T} \to \mathbb{R}$ is a positive and rd-continuous function and $\gamma \in \mathbb{N}$ is odd ,
2. $p, q : \mathbb{T} \to \mathbb{R}$ are rd-continuous functions such that $p(t) - q(t) > 0$ for $t \in \mathbb{T}$,
3. $F : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ are functions such that $uF(t,u) > 0$ and $uG(t,u,v) > 0$ for all $u \in \mathbb{R} \setminus \{0\}, v \in \mathbb{R}, t \in \mathbb{T}$,
4. $|F(t,u)| \ge p(t)|u|^\gamma, |G(t,u,v)| \le q(t)|u|^\gamma \forall u \in \mathbb{R} \setminus \{0\}, v \in \mathbb{R}, t \in \mathbb{T}$.

In [18], Graef and Hill both have investigated nonoscillation solutions of higher order nonlinear delay dynamic equation on time scales

$$(a(t)x^{\Delta}(t))^{\Delta^{n-1}} + q(t)f(x(g(t))) = r(t) \quad \forall t \in \mathbb{T}, \tag{1.13}$$

and established the sufficient conditions of non-oscillation. Recently, Negi et al. [26] considered the second-order nonlinear dynamic equations with integro forcing term and deviating argument on time scales

$$\begin{aligned} y^{\Delta\Delta}(t) + \beta y^{\Delta}(t) &= B(t)y(t) \\ &+ H^{\Delta}\left(t, y(t), \int_{-\infty}^{t} \mathcal{J}(t-s)\mathcal{H}(s, y(s))\Delta s\right) \end{aligned} \tag{1.14}$$

and

$$y^{\Delta\Delta}(t) + \beta y^{\Delta}(t) = B(t)y(t) + \mathcal{W}(t, y(t), y(w_1(t, y(t))), \tag{1.15}$$

respectively, where $\beta \geq 0, w_1(t, y(t)) = b_1(t, y(b_2(t, \ldots, y(b_{m_0}(t, y(t)) \ldots))))$. By the Riccati technique, Negi et al. have investigated the Kamenev-type oscillation criteria of both the Eqs. (1.14) and (1.15), respectively.

In this paper, we first deal with two functions $\mathcal{K}(y(t-h))$ and $\mathcal{W}(y^{\Delta}(t-h))$, which play an important role in our analytical findings. As we see in the above assumption (4), the absolute value of functions F and G are related with the absolute value of the unknown function $u(t)$, with the functions $p(t)$ and $q(t)$, respectively. In Eq. (1.10), Saker have assumed that the continuous function $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ such that $uf(t, u) > 0 \forall u \neq 0$ and $|f(t, u)| \geq q(t)|u^{\gamma}|$, where a nonnegative function $q(t)$ defined on $\mathbb{T}$. In Eq. (1.9), Chen have assumed function $f \in C(\mathbb{R}, \mathbb{R})$ such that $xf(x) > 0 \forall x \neq 0$ and $f(x) \geq Lx$, where L is positive real constant. In Eq. (1.13), Graef considered the function $f \in C(\mathbb{R}, \mathbb{R})$ such that for $\gamma > 0$, $|f(x(t))| \leq |x(t)|^{\gamma} + B$ for all $x(t), \forall t \in \mathbb{T}$, where A, B are non-negative real constant. Motivated by the above literature, recently in [26] Negi et al. have assumed $|H^{\Delta}(t, \eta(t), \xi(t))| \geq p(t)|\eta(t)|$ and $\eta(t)H^{\Delta}(t, \eta(t), \xi(t)) < 0$ for $\eta(t) \neq 0$ for the dynamic equation (1.14), and (1.15) by replacing $\mathcal{H}$ to $\mathcal{W}$.

To establish oscillation criteria for (1.2), we need $|\mathcal{K}y(t)| \geq p(t)|y(t)|$ for $y(t) \neq 0$ such that $y(t)\mathcal{K}(y(t)) > 0$. Moreover, there exist a function $f \in C(\mathbb{R}, \mathbb{R})$ such that $|f(\mathcal{W}(y^{\Delta}(t)))| \geq M(t)|y^{\Delta}(t)|$ as well as $y(t)f(\mathcal{W}(y^{\Delta}(t))) > 0 \forall y(t) \neq 0$ in $\mathbb{R}$, where $M(t)$ is a non-negative rd-continuous function defined on $\mathbb{T}$. Now we choose the real coefficients w_j, k_i such that

$$k_i, w_j = \begin{cases} 0 & : \text{ if } i,j \text{ are even natural number }, \\ +ve & : \text{ if } i,j \text{ are odd natural number }, \end{cases}$$

and $\mathcal{W}, \mathcal{K}$ are defined in (1.1), then we obtain the following relation

$$\begin{aligned} |\mathcal{K}(y(t))| &= |k_1 y(t) + k_3 y^3(t) + \cdots| \\ &= k_1|y(t)||1 + \frac{k_3}{k_1}y^2(t), \ldots| \geq k_1|y(t)|, \end{aligned} \tag{1.16}$$

such that $y(t)\mathcal{K}(y(t)) > 0 \forall y(t) \in \mathbb{R}, t \in \mathbb{T}$. Similarly, we immediately obtain an inequality

$$|\mathcal{W}(y^{\Delta}(t))| \geq w_1|y^{\Delta}(t)| \quad \forall t \in \mathbb{T}. \tag{1.17}$$

Now let us consider a function $f \in C(\mathbb{R}, \mathbb{R})$ for $y(t) \in \mathbb{R}$ such that $f(\mathcal{W}y^{\Delta}(t)) = q(t)\text{sgn}(y(t))|\mathcal{W}(y^{\Delta}(t))|, \forall t \in \mathbb{T}$, then, from (1.17), we obtain,

$$\begin{aligned} &|f(\mathcal{W}y^{\Delta}(t))| = |q(t)\text{sgn}(y(t))|\mathcal{W}(y^{\Delta}(t))|| \\ &\quad \geq q(t)w_1|y^{\Delta}(t)| \quad \forall y(t) \in \mathbb{R}, \quad \forall t \in \mathbb{T}, \end{aligned}$$

where $q(t)$ is non-negative rd-continuous function defined on $\mathbb{T}$. Thus, we can find such function $f \in C(\mathbb{R}, \mathbb{R})$ which satisfy

$$|f(\mathcal{W}(y^{\Delta}(t)))| \geq q(t)w_1|y^{\Delta}(t)|, \tag{1.18}$$

and $y(t)f(\mathcal{W}(y^{\Delta}(t))) > 0$, for $y(t) \neq 0 \forall t \in \mathbb{T}$ and $q(t)$ is rd-continuous function defined on $\mathbb{T}$. In Eq. (1.18), absolute value of f related to the absolute value of $y^{\Delta}(t) \forall t \in \mathbb{T}$.

For simplicity, throughout this paper, we denote $[a\infty)_{\mathbb{T}} = [a\infty) \bigcap \mathbb{T}$. In addition, we also need the following assumptions, as follows:

- (O_1) Assume $a, p : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ are positive rd-continuous functions such that $0 < p(t) \leq k_1 < \infty$ and $q(t) := p(t)\frac{\mu(t)}{w_1}$.
- (O_2) $\displaystyle\int_{t_0}^{\infty} \frac{1}{e_{z(t)}(t, s_0)}\Delta t = \infty,$ where $z(t) := \frac{a(t)}{(1-a(t)\mu(t))} > 0 \quad \forall t \in \mathbb{T}$.

Let us recall that, a solution $y(t)$ of (1.2) is a non-trivial or $y(t) \neq 0$, such that $y(t) \in \mathbb{C}^{\Delta^2}_{rd}([t_y, \infty)_{\mathbb{T}})$ for certain $t_y \geq t_0$. If it is eventually positive or eventually negative, then it must be non-oscillatory, otherwise oscillatory, i.e., it is oscillatory if there exists a real sequence, say $\{a_n\}$ such that $a_n \to \infty$ as $n \to \infty$ and $y(a_n) = 0 \forall n \in \mathbb{N}$. Our attention is restricted to those solutions of (1.2) which exist on the half-line $[t_y, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|y(t)| : t > t_*\} \neq 0$ for any $t_y \leq t_*$ and $\sup \mathbb{T} = \infty$.

This paper is organized as follows: In Sect. 2, we give basic definitions and present some necessary Lemmas. In the next section, we establish the sufficient conditions of oscillation of our Eq. (1.2). We further establish the Kamenev-type and Philos-type oscillation criteria. Some remarks for the particular case are also discussed. At the end in Sect. 4, to validation our results, we give an example. We also discuss the cases when the time scale is of a particular form.

Preliminaries

In this section, we present some basic definitions, useful Theorems and basic facts of time scales.

Definition 2.1 [6]. For $t \in \mathbb{T}$, forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

respectively. The classification of points of time scale $\mathbb{T}$. For $t \in \mathbb{T}$, t is called right-scattered if $t < \sigma(t)$, and right dense if for all $t < \sup \mathbb{T}$ such that $t = \sigma(t)$. Similarly, t is left-scattered if $t > \rho(t)$, and left dense if for all $t > \inf \mathbb{T}$ such that $t = \rho(t)$. The graininess operator $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$.

Remark 2.2 We put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t), $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\emptyset$ is an empty set.

Definition 2.3 [6] A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limit exist (finite) at all left−dense points in $\mathbb{T}$, which is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

We define $\mathbb{T}^{\kappa} = \mathbb{T} - \{\xi\}$, if $\mathbb{T}$ has a left-scattered maximum ξ, and $\mathbb{T}^{\kappa} = \mathbb{T}$, otherwise.

Definition 2.4 [6] For a function $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$, we define $f^{\Delta}(t)$, to be a number (provided it exists) with the property that for any given $\epsilon > 0$, there exists a neighborhood $\mathcal{Z} = (t - \delta, t + \delta) \bigcap \mathbb{T}$ for some $\delta > 0$ such that

$$\left|[f(\sigma(t)) - f(r)] - f^{\Delta}(t)[\sigma(t) - r]\right| \leq \epsilon|\sigma(t) - r| \qquad \forall r \in \mathcal{Z}.$$

Thus, we call $f^{\Delta}(t)$ the Δ or Hilger derivative of f at t.

Theorem 2.5 [6] *For the functions $g, f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$. The following statements are true*:

1. *If f is differentiable at t, then f is continuous at t*;
2. *If f is continuous at t and t is right-scattered, then f is Δ-derivative at t and* $f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$;
3. *If t is right-dense, then f is differentiable at t iff* $f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$ *exists and finite value*;
4. *If f is differentiable at t, then* $f^{\sigma} = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$;
5. *If f and g both are differentiable at t, then a product* $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at t and*
$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t)\\ &= f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)),\end{aligned}$$
hence, for $t \in \mathbb{T}$ *such that* $a \leq t \leq b, \forall a, b \in \mathbb{T}$, *we have the following facts*
$$\int_a^b f^{\sigma}(s)g^{\Delta}(s)\Delta s = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(s)g(s)\Delta s \tag{2.1}$$
$$\int_a^b f(s)g^{\Delta}(s)\Delta s = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(s)g^{\sigma}(s)\Delta s; \tag{2.2}$$
6. If $g(t)g(\sigma(t)) \neq 0$, *then* $\frac{f(t)}{g(t)}$ *is differentiable at t and*
$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Definition 2.6 [6] A function $w : \mathbb{T} \to \mathbb{R}$ is regressive if $1 + \mu(t)q(t) \neq 0, \forall t \in \mathbb{T}$. Denote the collection of all rd-continuous functions $w : \mathbb{T} \to \mathbb{R}$ by $\mathcal{R}$, and $\mathcal{R}^{+} = \{w \in \mathcal{R} : 1 + \mu(t)w(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Definition 2.7 [6] A function $F : \mathbb{T} \to \mathbb{R}$ is called an anti−derivative of $f : \mathbb{T} \to \mathbb{R}$, provided $F^{\Delta}(t) = f(t) \forall t \in \mathbb{T}$. Then $\forall a, b \in \mathbb{T}$ such that $a \leq b$, Cauchy integral is defined by

$$\int_a^b f(s)\Delta(s) = F(b) - F(a). \tag{2.3}$$

Definition 2.8 [23] Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, we have the relation $f^{\Delta}(t) = D_q f(t)$, where

$$D_q f(t) = \begin{cases} \dfrac{f(qt) - f(t)}{t(q-1)} & t \neq 0 \\ \lim\limits_{n \to \infty} \dfrac{f(q^n) - f(0)}{q^n} & t = 0, \end{cases} \tag{2.4}$$

is the q-difference operator.

Remark 2.9 In Definition 2.7, Eq. (2.3) does not hold for all time scale, for example, in q-calculus (i.e., $\mathbb{T} = \overline{q^{\mathbb{Z}}}$) the following relation is not correct always : $\int_a^b D_q f(t) d_q t = f(b) - f(a)$. For more detail see page 12 in [5].

Definition 2.10 [6] If $w \in \mathcal{R}$, then we define a exponential function by

$$e_w(t, s) = exp\left(\int_s^t \eta_{\mu(\tau)}(w(\tau))\Delta\tau\right), \quad \forall t \in \mathbb{T}, \quad s \in \mathbb{T}^{\kappa},$$

where $\eta_h(z)$ is the cylinder transformation, which is defined by

$$\eta_h(z) = \begin{cases} \dfrac{\log(1 + hz)}{h}, & : h \neq 0, \\ z, & : h = 0. \end{cases}$$

Before going to our main section, we first introduce some necessary lemmas which are crucial for our proofs.

Lemma 2.11 Let $y(t)$ be a non-oscillate solution of (1.2) and assume that O_1, O_2, relations (1.16) and (1.18) hold, then there exists $s_0 \geq 0$; $s_0 > t_0$ such that

$$y(t) > 0, y^{\Delta}(t) > 0 \text{ and } y^{\Delta\Delta}(t) < 0 \tag{2.5}$$

and

$$y(t-h) > 0, y^{\Delta}(t-h) > 0 \text{ and } y^{\Delta\Delta}(t-h) < 0, \text{ on } [s_0, \infty)_{\mathbb{T}}. \tag{2.6}$$

Proof Let $y(t)$ be an eventually positive solution of (1.2). Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Let us take $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $t_2 \geq t_1 + h$, then we must have $y(t) > 0$ and $y(t-h) > 0$ on $[t_2, \infty)_{\mathbb{T}}$. From equation (1.2) and $O_1 - O_2$, (1.16) and (1.18), we obtain

$$\begin{aligned} y^{\Delta\Delta}(t) + a(t)y^{\Delta}(t) &\leq -(y(t) + p(t)y(t-h) + p(t)\mu(t)|y^{\Delta}(t-h)|) \\ &\leq -(y(t) + p(t)y(t-h) + p(t)\mu(t)y^{\Delta}(t-h)) \\ &\leq -(y(t) + p(t)y^{\sigma}(t-h)) < 0. \end{aligned} \tag{2.7}$$

Since $a(t)\mu(t) < 1$ and using the above relation, we obtain

$$\frac{y^{\Delta\Delta}(t)}{1 - a(t)\mu(t)} + \frac{a(t)}{1 - a(t)\mu(t)} y^{\Delta}(t) < 0, \quad \text{on } [t_2, \infty)_{\mathbb{T}}, \tag{2.8}$$

which is equivalent to

$$\left(e_{\frac{a(t)}{1-a(t)\mu(t)}}(t, s_0)y^{\Delta}(t)\right)^{\Delta} < 0, \quad \forall t \in [t_2, \infty)_{\mathbb{T}}. \tag{2.9}$$

Then $e_{\frac{a(t)}{1-a(t)\mu(t)}}(t, s_0)y^{\Delta}(t)$ is an eventually decreasing function and thus it is eventually of one sign. We claim that it is eventually non-negative. Let us assume that it is eventually negative, i.e., $y^{\Delta}(t)$ is eventually negative because $e_w(t, s_0) > 0$ as $w > 0$, for all $w \in \mathcal{R}^+$, then there exist $t_2 \leq t_3$ and a constant $C > 0$ such that

$$e_{\frac{a(t)}{1-a(t)\mu(t)}}(t, s_0)y^{\Delta}(t) \leq C < 0, \quad \text{for all } t_3 + h \leq t_4 \leq t. \tag{2.10}$$

Integrating Eq. (2.10) from t_4 to t, we obtain

$$y(t) \leq y(t_4) + C\int_{t_4}^{t} \frac{1}{e_{\frac{a(s)}{1-a(s)\mu(s)}}(s, s_0)} \Delta s.$$

For sufficiently large t, we obtain a contradiction because of O_2. Therefore, we have

$$y(t) > 0, y^{\Delta}(t) > 0 \text{ and } y^{\Delta\Delta}(t) < 0,$$

and

$$y(t-h) > 0, y^{\Delta}(t-h) > 0 \text{ and } y^{\Delta\Delta}(t-h) < 0, \text{ on } [s_0, \infty)_{\mathbb{T}}.$$

□

Lemma 2.12 If (2.5) holds, then for $t \neq s_0$, we have

$$0 < \mathcal{G}(t) \leq \frac{y(t)}{y^{\sigma}(t)} \leq 1, \tag{2.11}$$

where $\mathcal{G}(t) := \frac{t - s_0}{t - s_0 + \mu(t)}$.

Proof For $t \neq s_0$, and from (2.5), we obtain

$$y(t) > y(t) - y(s_0) = \int_{s_0}^{t} y^{\Delta}(s)\Delta s > y^{\Delta}(t)(t - s_0). \tag{2.12}$$

From (2.12) and using (4) of Theorem 2.5, we obtain

$$0 < \frac{t - s_0}{t - s_0 + \mu(t)} := \mathcal{G}(t) \leq \frac{y(t)}{y^{\sigma}(t)} \leq 1.$$

Now with the help of Lemmas 2.11 and 2.12, we obtain a new Lemma as follows, □

Lemma 2.13 If Eqs. (2.5) and (2.11) hold, then for $2s_0 \leq t$, we have

$$\frac{t}{2}\frac{\mathcal{G}(t)w(t)}{\delta(t)} \leq (t - s_0)\frac{\mathcal{G}(t)w(t)}{\delta(t)} \leq \frac{y(t-h)}{y^{\sigma}(t)} \leq 1, \tag{2.13}$$

where $w(t) = \delta(t)\frac{y^{\Delta}(t)}{y(t)}$ is Riccati transformation function.

Proof The relation $2s_0 \leq t$ can be rewritten as $t - s_0 \geq \frac{t}{2}$. Using (2.6), we obtain

$$\begin{aligned} y(t-h) - y(s_0 - h) &= \int_{s_0-h}^{t-h} y^{\Delta}(s)\Delta s > y^{\Delta}(t-h)(t - s_0) \\ &> y^{\Delta}(t)(t - s_0), \end{aligned}$$

which implies

$$\frac{t}{2}y^{\Delta}(t) < y(t-h). \tag{2.14}$$

From Eqs. (2.14), (2.11) and $w(t)$. Hence, we obtain

$$\frac{t}{2}\frac{\mathcal{G}(t)w(t)}{\delta(t)} \leq \frac{y(t-h)}{y^{\sigma}(t)} \leq 1.$$

□

Oscillation criteria

In this section, we establish some sufficient conditions of oscillation for Eq. (1.2). For $s_0 \in \mathbb{T}$.

Theorem 3.1 *Assume that O_1, O_2, relations (1.16) and (1.18) hold. Moreover, if there exist a Δ-derivative function*

$\delta(t) > 0$ and $s_0 \geq 0; t_0 \leq 2s_0 < t$ respectively and satisfying the following

$$\limsup_{t\to\infty} \int_{2s_0}^{t} \mathcal{F}(s)\Delta s = \infty, \tag{3.1}$$

where

$$\mathcal{F}(t) := \left(\delta^\sigma(t)\mathcal{G}(t) - \frac{\left(\delta^\Delta(t) - \mathcal{G}(t)\delta^\sigma(t)(a(t) + p(t)\frac{t}{2})\right)^2}{4\delta^\sigma(t)\mathcal{G}(t)} \right), \tag{3.2}$$

then, Eq. (1.2) oscillates on $[t_0\infty)_{\mathbb{T}}$.

Proof Assume to the contrary that (1.2) has a non-oscillatory solution. Let $y(t)$ be a nonoscillatory solution of (1.2). Then, without loss of generality, we assume that $y(t)$ is an eventually positive solution, i.e., there exists $s_0 \geq t_0$ such that $y(t) > 0 \forall t \in [s_0\infty)_{\mathbb{T}}$. A similar argument holds for the case when $y(t)$ is an eventually negative solution. We define a Riccati transformation function such that

$$w(t) = \delta(t)\frac{y^\Delta(t)}{y(t)}, \quad t_0 \leq 2s_0 < t. \tag{3.3}$$

Δ-derivative of Eq. (3.3) w.r.t t, we have

$$w^\Delta(t) = \delta^\Delta(t)\frac{y^\Delta(t)}{y(t)} + \delta^\sigma(t)\left(\frac{y^{\Delta\Delta}(t)y(t) - (y^\Delta(t))^2}{y(t)y^\sigma(t)}\right)$$

$$= w(t)\frac{\delta^\Delta(t)}{\delta(t)} - w^2(t)\frac{\delta^\sigma(t)y(t)}{y^\sigma(t)\delta^2(t)} + \frac{\delta^\sigma(t)}{y^\sigma(t)}y^{\Delta\Delta}(t). \tag{3.4}$$

From (2.11) and (3.4), we obtain

$$w^\Delta(t) \leq w(t)\frac{\delta^\Delta(t)}{\delta(t)} - w^2(t)\mathcal{G}(t)\frac{\delta^\sigma(t)}{\delta^2(t)} + \frac{\delta^\sigma(t)}{y^\sigma(t)}y^{\Delta\Delta}(t). \tag{3.5}$$

To solve the right-hand side of Eq. (3.5), we use Eqs. (2.7), (2.11) and a relation $y^\Delta(t) > 0$, we obtain

$$\frac{\delta^\sigma(t)}{y^\sigma(t)}y^{\Delta\Delta}(t) \leq -\delta^\sigma(t)\left(\frac{y(t)}{y^\sigma(t)} + p(t)\frac{y^\sigma(t-h)}{y^\sigma(t)} + a(t)\frac{y^\Delta(t)}{y^\sigma(t)}\right)$$

$$\leq -\delta^\sigma(t)\mathcal{G}(t) - \delta^\sigma(t)a(t)\frac{\mathcal{G}(t)}{\delta(t)}w(t) - \delta^\sigma(t)p(t)\frac{y(t-h)}{y^\sigma(t)}. \tag{3.6}$$

From (3.6) and (2.13), we obtain

$$\frac{\delta^\sigma(t)}{y^\sigma(t)}y^{\Delta\Delta}(t) \leq -\delta^\sigma(t)\mathcal{G}(t) - \delta^\sigma(t)a(t)\frac{\mathcal{G}(t)}{\delta(t)}w(t)$$
$$- p(t)\delta^\sigma(t)\frac{t}{2}\frac{\mathcal{G}(t)}{\delta(t)}w(t). \tag{3.7}$$

Substituting (3.7) into (3.5), we arrive at

$$w^\Delta(t) \leq w(t)\frac{\delta^\Delta(t)}{\delta(t)} - w^2(t)\mathcal{G}(t)\frac{\delta^\sigma(t)}{\delta^2(t)} - \delta^\sigma(t)\mathcal{G}(t)$$
$$- \delta^\sigma(t)a(t)\frac{\mathcal{G}(t)}{\delta(t)}w(t) - p(t)\delta^\sigma(t)\frac{t}{2}\frac{\mathcal{G}(t)}{\delta(t)}w(t).$$

$$= -\delta^\sigma(t)\mathcal{G}(t) + \frac{1}{\delta(t)}\left(\delta^\Delta(t) - \mathcal{G}(t)\delta^\sigma(t)\left(a(t) + p(t)\frac{t}{2}\right)\right)$$
$$w(t) - \frac{\delta^\sigma(t)}{\delta^2(t)}\mathcal{G}(t)w^2(t)$$

$$= -\delta^\sigma(t)\mathcal{G}(t)$$
$$- \left(w(t)\frac{\sqrt{(\delta^\sigma(t)\mathcal{G}(t))}}{\delta(t)} - \frac{\left(\delta^\Delta(t) - \mathcal{G}(t)\delta^\sigma(t)(a(t) + p(t)\frac{t}{2})\right)}{2\sqrt{(\delta^\sigma(t)\mathcal{G}(t))}} \right)^2$$
$$+ \frac{\left(\delta^\Delta(t) - \mathcal{G}(t)\delta^\sigma(t)(a(t) + p(t)\frac{t}{2})\right)^2}{4\delta^\sigma(s)\mathcal{G}(s)}. \tag{3.8}$$

From Eqs. (3.2) and (3.8), we arrive at

$$w^\Delta(t) \leq -\mathcal{F}(t), \quad \text{for } 2s_0 < t. \tag{3.9}$$

Integrating Eq. (3.9) from $2s_0$ to t, we have

$$\int_{2s_0}^{t} \mathcal{F}(s)\Delta s \leq w(2s_0) < \infty. \tag{3.10}$$

For sufficient large t, we derive a contradiction to (3.1), as the left-hand side of (3.10) finite, which completes the proof of our theorem. □

From Theorem 3.1, we may also obtain some results concerning the oscillation behavior of solutions of Eq. (1.2).

Corollary 3.2 *Assume that O_1, O_2, relations (1.16) and (1.18) hold. Moreover, if there exist a Δ-derivative function $\delta(t) > 0$ and $s_0 \geq 0, t_0 \leq 2s_0 < t$, respectively, and satisfying the following conditions as follows,*

$$\limsup_{t\to\infty} \int_{2s_0}^{t} \delta^\sigma(s)\mathcal{G}(s)\Delta s = \infty \tag{3.11}$$

and

$$\limsup_{t\to\infty} \int_{2s_0}^{t} \frac{\left(\delta^\Delta(s) - \mathcal{G}(s)\delta^\sigma(s)(a(s) + p(s)\frac{s}{2})\right)^2}{4\delta^\sigma(s)\mathcal{G}(s)}\Delta s < \infty, \tag{3.12}$$

then, Eq. (1.2) oscillates on $[t_0\infty)_{\mathbb{T}}$.

The next result immediately follows from Theorem 3.1 by different choices of $\delta(t)$. In particular, we take $\delta(t)$ as positive constant (say $C > 0$), we establish the following corollary.

Corollary 3.3 *Assume that O_1, O_2, relations (1.16) and (1.18) hold. Moreover, if there exists $s_0 \geq 0$ such that $t_0 \leq 2s_0 < t$, and satisfying the following condition below*

$$\limsup_{t\to\infty} \int_{2s_0}^{t} \frac{\mathcal{G}(s)}{4}\left(4-\left(a(s)+p(s)\frac{s}{2}\right)^2\right)\Delta s=\infty, \tag{3.13}$$

then, Eq. (1.2) oscillates on $[t_0\infty)_{\mathbb{T}}$.

We introduce one more condition $\mu(t)\frac{\delta^\Delta(t)}{\delta^\sigma(t)}<1$ to obtain a new oscillations criteria for (1.2).

Theorem 3.4 *Assume that O_1, O_2, relations (1.16) and (1.18) hold. Moreover, if there exist a Δ-derivative function $\delta(t)>0$ and $s_0 \geq 0$ such that $\mu(t)\frac{\delta^\Delta(t)}{\delta^\sigma(t)}<1$ and $t_0 \leq 2s_0 < t$, respectively, and satisfying the condition*

$$\limsup_{t\to\infty} \int_{2s_0}^{t} \frac{1}{\left(1-\frac{\delta^\Delta(s)}{\delta^\sigma(s)}\mu(s)\right)}\left[\mathcal{G}(s)\delta(s)-\frac{\delta(s)A^2(s)}{4\mathcal{G}(s)}\right]\Delta s=\infty, \tag{3.14}$$

where

$$A(t):=\frac{\delta^\Delta(t)}{\delta^\sigma(t)}-\mathcal{G}(t)\left(a(t)+p(t)\frac{t}{2}\right), \tag{3.15}$$

then, Eq. (1.2) oscillates on $[t_0\infty)_{\mathbb{T}}$.

Proof Assume to the contrary that (1.2) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that $y(t)$ is an eventually positive solution of (1.2), i.e., there exists $t_0 \leq s_0$ such that $y(t)>0 \forall t \in [s_0\infty)_{\mathbb{T}}$. A similar argument holds for the case when $y(t)$ is eventually negative. Now Δ-differentiate equation (3.3) w.r.t t, we have

$$w^\Delta(t)=\delta(t)\left(\frac{y^\Delta(t)}{y(t)}\right)^\Delta+\delta^\Delta(t)\left(\frac{y^\Delta(t)}{y(t)}\right)^\sigma. \tag{3.16}$$

From Eqs. (3.3), (2.11) and (3.16), we obtain

$$\begin{aligned} w^\Delta(t) &\leq \frac{\delta^\Delta(t)}{\delta^\sigma(t)}w^\sigma(t)-\frac{\mathcal{G}(t)}{\delta(t)}w^2(t)+\delta(t)\frac{y^{\Delta\Delta}(t)}{y^\sigma(t)} \\ &= \frac{\delta^\Delta(t)}{\delta^\sigma(t)}(w(t)+\mu(t)w^\Delta(t))-\frac{\mathcal{G}(t)}{\delta(t)}w^2(t)+\delta(t)\frac{y^{\Delta\Delta}(t)}{y^\sigma(t)}. \end{aligned} \tag{3.17}$$

To solve the right-hand side of Eq. (3.17), we replace $\delta^\sigma(t)$ by $\delta(t)$ in (3.7). We obtain

$$\delta(t)\frac{y^{\Delta\Delta}}{y^\sigma(t)} \leq -\mathcal{G}(t)\delta(t)-a(t)\mathcal{G}(t)w(t)-p(t)\frac{t}{2}\mathcal{G}(t)w(t). \tag{3.18}$$

Substituting (3.18) into (3.17), we arrive at

$$\begin{aligned} w^\Delta(t) \leq{}& \frac{\delta^\Delta(t)}{\delta^\sigma(t)}(w(t)+\mu(t)w^\Delta(t))-\frac{\mathcal{G}(t)}{\delta(t)}w^2(t) \\ &-\mathcal{G}(t)\delta(t)-a(t)\mathcal{G}(t)w(t)-p(t)\frac{t}{2}\mathcal{G}(t)w(t), \end{aligned}$$

which is equivalent to

$$\begin{aligned} \left(1-\frac{\delta^\Delta(t)}{\delta^\sigma(t)}\mu(t)\right)w^\Delta(t) &\leq -\mathcal{G}(t)\delta(t)+A(t)w(t)-\frac{\mathcal{G}(t)}{\delta(t)}w^2(t) \\ &= -\mathcal{G}(t)\delta(t)+\frac{\delta(t)A^2(t)}{4\mathcal{G}(t)}-\left(\sqrt{\frac{\mathcal{G}(t)}{\delta(t)}}w(t)-\frac{A(t)}{2}\sqrt{\frac{\delta(t)}{\mathcal{G}(t)}}\right)^2 \\ &\leq -\mathcal{G}(t)\delta(t)+\frac{\delta(t)A^2(t)}{4\mathcal{G}(t)}. \end{aligned} \tag{3.19}$$

Since $\frac{\delta^\Delta(t)}{\delta^\sigma(t)}\mu(t)<1$. So dividing (3.19) by $1-\frac{\delta^\Delta(t)}{\delta^\sigma(t)}\mu(t)$, we arrive at

$$w^\Delta(t) \leq -\frac{1}{\left(1-\frac{\delta^\Delta(t)}{\delta^\sigma(t)}\mu(t)\right)}\left[\mathcal{G}(t)\delta(t)-\frac{\delta(t)A^2(t)}{4\mathcal{G}(t)}\right]. \tag{3.20}$$

Integrating Eq. (3.20) from $2s_0$ to t. Therefore, we have

$$\begin{aligned} &\int_{2s_0}^{t} \frac{1}{\left(1-\frac{\delta^\Delta(s)}{\delta^\sigma(s)}\mu(s)\right)}\left[\mathcal{G}(s)\delta(s)-\frac{\delta(s)A^2(s)}{4\mathcal{G}(s)}\right]\Delta s \leq w(2s_0) \\ &\quad -w(t)<w(2s_0)<\infty. \end{aligned}$$

For sufficiently large t, we derive a contradiction to (3.14), as the left-hand side of above relation finite, which completes the proof of our theorem. □

Theorem 3.5 *Assume that O_1, O_2, relations (1.16) and (1.18) hold. Moreover, if there exist a Δ-derivative function $\delta(t)>0$ and $s_0 \geq 0, t_0 \leq 2s_0 < t$, respectively, and satisfying the following condition,*

$$\begin{aligned} &\limsup_{t\to\infty} \int_{2s_0}^{t} \\ &\left(\delta(s)\mathcal{G}(s)-\frac{\left(\delta^\Delta(s)-\delta(s)\mathcal{G}(s)(a(s)+p(s)\frac{s}{2})\right)^2}{4\delta(s)}\right)\Delta s=\infty, \end{aligned} \tag{3.21}$$

then, Eq. (1.2) oscillates on $[t_0\infty)_{\mathbb{T}}$.

Proof Assume to the contrary that (1.2) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that $y(t)$ is an eventually positive solution of (1.2) i.e., there exists $t_0 \leq s_0$ such that $y(t)>0 \forall t \in [s_0\infty)_{\mathbb{T}}$. A similar argument holds also for the case when $y(t)$ is eventually negative. Now from equations (2.7), (3.3) and (3.16), we obtain

$$w^{\Delta}(t) \le \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)} w^{\sigma}(t) - \frac{\delta(t)}{(\delta^{\sigma}(t))^2}(w^{\sigma}(t))^2 + \delta(t)\frac{y^{\Delta\Delta}(t)}{y^{\sigma}(t)}. \tag{3.22}$$

From (3.18), we have

$$\delta(t)\frac{y^{\Delta\Delta}}{y^{\sigma}(t)} \le -\mathcal{G}(t)\delta(t) - a(t)\mathcal{G}(t)w(t) - p(t)\frac{t}{2}\mathcal{G}(t)w(t). \tag{3.23}$$

Substituting (3.23) into (3.22), we arrive at

$$w^{\Delta}(t) \le -\delta(t)\mathcal{G}(t) + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)} w^{\sigma}(t) - \frac{\delta(t)}{(\delta^{\sigma}(t))^2}(w^{\sigma}(t))^2 - \mathcal{G}(t)\Big(a(t) + p(t)\frac{t}{2}\Big)w(t). \tag{3.24}$$

From (3.3), (2.11) and $y^{\Delta\Delta}(t) < 0$, we obtain a relation below,

$$w(t) \ge \frac{\delta(t)}{\delta^{\sigma}(t)} w^{\sigma}(t). \tag{3.25}$$

Substituting (3.25) into (3.24), which yields

$$w^{\Delta}(t) \le -\delta(t)\mathcal{G}(t) + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)} w^{\sigma}(t) - \frac{\delta(t)}{(\delta^{\sigma}(t))^2}(w^{\sigma}(t))^2 - \frac{\delta(t)}{\delta^{\sigma}(t)}\mathcal{G}(t)\Big(a(t) + p(t)\frac{t}{2}\Big)w^{\sigma}(t), \tag{3.26}$$

which is equivalent to

$$w^{\Delta}(t) \le -\delta(t)\mathcal{G}(t) + \frac{\big(\delta^{\Delta}(t) - \delta(t)\mathcal{G}(t)\big(a(t) + p(t)\frac{t}{2}\big)\big)}{\delta^{\sigma}(t)} w^{\sigma}(t) - \frac{\delta(t)}{(\delta^{\sigma}(t))^2}(w^{\sigma}(t))^2. \tag{3.27}$$

By following the similar steps of Eqs. (3.8) and (3.9), Eq. (3.27) become

$$w^{\Delta}(t) \le -\delta(t)\mathcal{G}(t) + \frac{\big(\delta^{\Delta}(t) - \delta(t)\mathcal{G}(t)\big(a(t) + p(t)\frac{t}{2}\big)\big)^2}{4\delta(t)}. \tag{3.28}$$

Integrating Eq. (3.28) from $2s_0$ to t. Therefore, we have

$$\int_{2s_0}^{t}\left[\delta(s)\mathcal{G}(s) - \frac{\big(\delta^{\Delta}(s) - \delta(s)\mathcal{G}(s)\big(a(s) + p(s)\frac{s}{2}\big)\big)^2}{4\delta(s)}\right]\Delta s \le w(2s_0) < \infty. \tag{3.29}$$

For sufficiently large t, we derive a contradiction to (3.21), as the left-hand side finite,which completes the proof of our theorem. □

In view of Theorem 3.5, we immediately obtain following corollary.

Corollary 3.6 *Assume that O_1, O_2, relations (1.16) and (1.18) hold. Moreover, if there exist Δ-derivative function $\delta(t) > 0$ and $s_0 \ge 0, t_0 \le 2s_0 < t$, respectively, and satisfying the following conditions,*

$$\limsup_{t\to\infty}\int_{2s_0}^{t}\delta(s)\mathcal{G}(s)\Delta s = \infty \tag{3.30}$$

and

$$\limsup_{t\to\infty}\int_{2s_0}^{t}\frac{\big(\delta^{\Delta}(s) - \delta(s)\mathcal{G}(s)\big(a(s) + p(s)\frac{s}{2}\big)\big)^2}{4\delta(s)}\Delta s < \infty, \tag{3.31}$$

then, Eq. (1.2) oscillates on $[t_0\infty)_{\mathbb{T}}$.

Remark 3.7 In view of Theorems 3.4 and 3.5, we obtain various sufficient conditions of oscillation of Eq. (1.2) by taking different choice of $\delta(t)$.

To present our next theorems, we first introduce Saker's result [29] as follows

$$((t-s)^{\mathcal{N}})^{\Delta_s} \le -\mathcal{N}(t-\sigma(s))^{\mathcal{N}-1} \le 0 \quad \text{for} \quad \mathcal{N} > 1 \quad \text{and} \quad \sigma(s) \le t. \tag{3.32}$$

By using an integral averaging technique of Kamenev-type, we present some new oscillation criteria of (1.2).

Theorem 3.8 *Assume that O_1, O_2, relations (1.16) and (1.18) hold. Moreover, if there exist Δ-derivative function $\delta(t) > 0$ and there exists $\mathcal{N} > 1$ and all sufficiently large $s_0 \ge 0$, $t_0 \le 2s_0 < t$ such that*

$$\limsup_{t\to\infty}\frac{1}{t^{\mathcal{N}}}\int_{2s_0}^{t}(t-s)^{\mathcal{N}}\mathcal{F}(s)\Delta s = \infty, \tag{3.33}$$

where

$$\mathcal{F}(t) := \left(\delta^{\sigma}(t)\mathcal{G}(t) - \frac{\big(\delta^{\Delta}(t) - \mathcal{G}(t)\delta^{\sigma}(t)\big(a(t) + p(t)\frac{t}{2}\big)\big)^2}{4\delta^{\sigma}(t)\mathcal{G}(t)}\right). \tag{3.34}$$

Then, Eq. (1.2) oscillates on $[t_0\infty)_{\mathbb{T}}$.

Proof Assume to the contrary that (1.2) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that $y(t)$ is an eventually positive function, i.e., there exists t_0 such that $y(t) > 0 \forall t \in [t_0\infty)_{\mathbb{T}}$. A similar argument holds also for the case when $y(t)$ is eventually negative. From Eq. (3.9), we have

$$\mathcal{F}(t) \leq -w^{\Delta}(t), \quad \text{for } 2s_0 \leq t.$$

Multiplying above relation by $(t-s)^{\mathcal{N}}$ and then integrating from $2s_0$ to t. we obtain

$$\int_{2s_0}^{t} (t-s)^{\mathcal{N}} \mathcal{F}(s)\Delta s \leq -\int_{2s_0}^{t} (t-s)^{\mathcal{N}} w^{\Delta}(s)\Delta s. \tag{3.35}$$

By comparing the right-hand side of (3.35) with Eq. (2.2), we have

$$\begin{aligned} &-\int_{2s_0}^{t} (t-s)^{\mathcal{N}} w^{\Delta}(s)\Delta s = (t-2s_0)^{\mathcal{N}} w(2s_0) \\ &\quad + \int_{2s_0}^{t} ((t-s)^{\mathcal{N}})^{\Delta_s} w^{\sigma}(s)\Delta s. \end{aligned} \tag{3.36}$$

From Eqs. (3.32), (3.35) and (3.36), we arrive at

$$\int_{2s_0}^{t} (t-s)^{\mathcal{N}} \mathcal{F}(s)\Delta s \leq (t-2s_0)^{\mathcal{N}} w(2s_0). \tag{3.37}$$

Thus

$$\begin{aligned} &\frac{1}{t^{\mathcal{N}}} \int_{2s_0}^{t} (t-s)^{\mathcal{N}} \mathcal{F}(s)\Delta s \leq \frac{(t-2s_0)^{\mathcal{N}}}{t^{\mathcal{N}}} w(2s_0), \\ &\text{for} \quad 2s_0 \leq t. \end{aligned} \tag{3.38}$$

Taking lim sup as $t \to \infty$ both sides of Eq. (3.38), we have

$$\limsup_{t\to\infty} \frac{1}{t^{\mathcal{N}}} \int_{2s_0}^{t} (t-s)^{\mathcal{N}} \mathcal{F}(s)\Delta s < \infty. \tag{3.39}$$

Thus, we derive a contradiction to (3.33), which completes the proof of our theorem. □

Corollary 3.9 *Assume that O_1, O_2, relations (1.16) and (1.18) hold. Moreover, if there exists Δ-derivative function $\delta(t) > 0$, and for $\mathcal{N} > 1$, and all sufficiently large $s_0 \geq 0$, such that $t_0 \leq 2s_0 < t$ and satisfying the conditions below,*

$$\limsup_{t\to\infty} \frac{1}{t^{\mathcal{N}}} \int_{2s_0}^{t} (t-s)^{\mathcal{N}} \delta^{\sigma}(t)\mathcal{G}(t)\Delta s = \infty, \tag{3.40}$$

and

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t^{\mathcal{N}}} \int_{2s_0}^{t} (t-s)^{\mathcal{N}} \frac{(\delta^{\Delta}(t) - \mathcal{G}(t)\delta^{\sigma}(t)(a(t) + p(t)\frac{t}{2}))^2}{4\delta^{\sigma}(t)\mathcal{G}(t)} \\ &\quad \Delta s < \infty, \end{aligned} \tag{3.41}$$

then, Eq. (1.2) oscillates on $[t_0\infty)_{\mathbb{T}}$.

Theorem 3.10 *Assume that O_1, O_2, relations (1.16) and (1.18) hold. Moreover, if there exists Δ-derivative function $\delta(t) > 0$ such that $\mu(t)\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)} < 1$. For $\mathcal{N} > 1$ and all sufficiently large $s_0 \geq 0$, $t_0 \leq 2s_0 < t$ and satisfying the condition below,*

$$\limsup_{t\to\infty} \frac{1}{t^{\mathcal{N}}} \int_{2s_0}^{t} \frac{(t-s)^{\mathcal{N}}}{\left(1 - \frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\mu(s)\right)} \left[\mathcal{G}(s)\delta(s) - \frac{\delta(s)A^2(s)}{4\mathcal{G}(s)}\right]\Delta s = \infty, \tag{3.42}$$

where $A(t)$ is given by (3.15). Then, Eq. (1.2) oscillates on $[t_0\infty)_{\mathbb{T}}$.

Proof Assume to the contrary that (1.2) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that $y(t)$ is an eventually positive function, i.e., there exists t_0 such that $y(t) > 0 \forall t \in [t_0\infty)_{\mathbb{T}}$. A similar argument holds for the case when $y(t)$ is eventually negative. From Eq. (3.20), we have

$$\frac{1}{\left(1 - \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}\mu(t)\right)} \left[\mathcal{G}(t)\delta(t) - \frac{\delta(t)A^2(t)}{4\mathcal{G}(t)}\right] \leq -w^{\Delta}(t) \text{ for } 2s_0 \leq t.$$

Multiplying above relation by $(t-s)^{\mathcal{N}}$ and then integrating from $2s_0$ to t, we obtain

$$\begin{aligned} &\int_{2s_0}^{t} \frac{(t-s)^{\mathcal{N}}}{\left(1 - \frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\mu(s)\right)} \left[\mathcal{G}(s)\delta(s) - \frac{\delta(s)A^2(s)}{4\mathcal{G}(s)}\right]\Delta s \leq \\ &\quad -\int_{2s_0}^{t} (t-s)^{\mathcal{N}} w^{\Delta}(s)\Delta s. \end{aligned} \tag{3.43}$$

By following the similar steps of Eqs. (3.35)–(3.37), Eq. (3.43) become

$$\begin{aligned} &\int_{2s_0}^{t} \frac{(t-s)^{\mathcal{N}}}{\left(1 - \frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\mu(s)\right)} \left[\mathcal{G}(s)\delta(s) - \frac{\delta(s)A^2(s)}{4\mathcal{G}(s)}\right] \\ &\quad \Delta s \leq (t-2s_0)^{\mathcal{N}} w(2s_0). \end{aligned} \tag{3.44}$$

Thus, we have

$$\begin{aligned} &\frac{1}{t^{\mathcal{N}}} \int_{2s_0}^{t} \frac{(t-s)^{\mathcal{N}}}{\left(1 - \frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\mu(s)\right)} \\ &\quad \left[\mathcal{G}(s)\delta(s) - \frac{\delta(s)A^2(s)}{4\mathcal{G}(s)}\right]\Delta s \leq \frac{(t-2s_0)^{\mathcal{N}}}{t^{\mathcal{N}}} w(2s_0). \end{aligned} \tag{3.45}$$

Taking lim sup as $t \to \infty$ in Eq. (3.45), we obtain

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t^{\mathcal{N}}} \int_{2s_0}^{t} \frac{(t-s)^{\mathcal{N}}}{\left(1 - \frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\mu(s)\right)} \\ &\quad \left[\mathcal{G}(s)\delta(s) - \frac{\delta(s)A^2(s)}{4\mathcal{G}(s)}\right]\Delta s < \infty. \end{aligned} \tag{3.46}$$

Thus, we derive a contradiction to (3.42), which completes the proof of our theorem. □

Theorem 3.11 *Assume that O_1, O_2, relations (1.16) and (1.18) hold. Moreover, if there exists a Δ-derivative function $\delta(t) > 0$. For $\mathcal{N} > 1$ and all sufficiently large $s_0 \geq 0$, $t_0 \leq 2s_0 < t$, and satisfying the condition below*

$$\limsup_{t\to\infty} \frac{1}{t^{\mathcal{N}}} \int_{2s_0}^{t} (t-s)^{\mathcal{N}} \left(\delta(s)\mathcal{G}(s) - \frac{\left(\delta^{\Delta}(s) - \delta(s)\mathcal{G}(s)\left(a(s)+p(s)\frac{s}{2}\right)\right)^2}{4\delta(s)} \right) \Delta s = \infty, \tag{3.47}$$

Then, Eq. (1.2) oscillates on $[t_0 \infty)_{\mathbb{T}}$.

Proof Assume to the contrary that (1.2) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that $y(t)$ is an eventually positive function, i.e., there exists t_0 such that $y(t) > 0 \forall t \in [t_0 \infty)_{\mathbb{T}}$. From Eq. (3.28) and following the similar steps of Eqs. (3.35)–(3.39), we easily obtain a following relation

$$\frac{1}{t^{\mathcal{N}}} \int_{2s_0}^{t} (t-s)^{\mathcal{N}} \left(\delta(s)\mathcal{G}(s) - \frac{\left(\delta^{\Delta}(s) - \delta(s)\mathcal{G}(s)\left(a(s)+p(s)\frac{s}{2}\right)\right)^2}{4\delta(s)} \right) \Delta s \leq \left(1 - \frac{2s_0}{t}\right)^{\mathcal{N}} w(2s_0) < \infty. \tag{3.48}$$

For all sufficiently large t, we derive a contradiction to (3.47). □

Corollary 3.12 *Assume that O_1, O_2, relations (1.16) and (1.18) hold. Moreover, if there exists a Δ-derivative function $\delta(t) > 0$. For $\mathcal{N} > 1$ and all sufficiently large $s_0 \geq 0$, $t_0 \leq 2s_0 < t$ and satisfying the following conditions below,*

$$\limsup_{t\to\infty} \frac{1}{t^{\mathcal{N}}} \int_{2s_0}^{t} (t-s)^{\mathcal{N}} \delta(s)\mathcal{G}(s) \Delta s = \infty, \tag{3.49}$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\mathcal{N}}} \int_{2s_0}^{t} (t-s)^{\mathcal{N}} \frac{\left(\delta^{\Delta}(s) - \delta(s)\mathcal{G}(s)\left(a(s)+p(s)\frac{s}{2}\right)\right)^2}{4\delta(s)} \Delta s < \infty. \tag{3.50}$$

Then, Eq. (1.2) oscillates on $[t_0 \infty)_{\mathbb{T}}$.

Our next aim to establish the Philos-type oscillation criteria for (1.2). We define some elementary assumptions as follows:

For any number $\eta \in \mathbb{R}$, we define positive and negative parts, η_+ and η_-, respectively, of η by

$$\eta_+ := \max\{0, \eta\} \quad \text{and} \quad \eta_- := \max\{0, -\eta\}.$$

Assume that the rd-continuous functions $H, h : \mathbb{D} \to \mathbb{R}$, where $\mathbb{D} = \{(t,s); t_0 \leq s_0 \leq t\}$ such that

$$H(t,t) = 0, H(t,s) > 0 \text{ and } H^{\Delta_s}(t,s) < 0, \quad t_0 \leq s < t \tag{3.51}$$

and $H^{\Delta_s}(t,s)$ (Δ-derivative w.r.t second variable) is rd-continuous function.

Theorem 3.13 *Assume that O_1, O_2, relations (1.16), (1.18) and Eq. (3.51) hold. Moreover, if there exist Δ-derivative function $\delta(t) > 0$ and $s_0 \geq 0$, $t_0 \leq 2s_0 < t$, respectively, and satisfying the following conditions below,*

$$H^{\Delta_s}(\sigma(t),s) + \frac{H^{\sigma}(\sigma(t),s)}{\delta(t)} \left(\delta^{\Delta}(t) - \mathcal{G}(t)\delta^{\sigma}(t)\left(a(t)+p(t)\frac{t}{2}\right)\right) = -\frac{h(t,s)}{\delta(t)}\sqrt{H^{\sigma}(\sigma(t),s)} \tag{3.52}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),2s_0)} \int_{2s_0}^{\sigma(t)} \left(H^{\sigma}(\sigma(t),s)\delta^{\sigma}(s)\mathcal{G}(s) - \frac{(h_-(t,s))^2}{4\delta^{\sigma}(s)\mathcal{G}(s)} \right) \Delta s = \infty, \tag{3.53}$$

then, Eq. (1.2) oscillates on $[t_0 \infty)_{\mathbb{T}}$.

Proof Assume to the contrary that (1.2) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that $y(t)$ is an eventually positive function, i.e., there exists t_0 such that $y(t) > 0 \forall t \in [t_0 \infty)_{\mathbb{T}}$. A similar argument holds for the case when $y(t)$ is eventually negative. We have defined Riccati transformation function in (3.3). Now from (3.8), we have

$$w^{\Delta}(t) \leq -\delta^{\sigma}(t)\mathcal{G}(t) + \frac{1}{\delta(t)} \left(\delta^{\Delta}(t) - \mathcal{G}(t)\delta^{\sigma}(t)\left(a(t)+p(t)\frac{t}{2}\right)\right) w(t) - \frac{\delta^{\sigma}(t)}{\delta^2(t)}\mathcal{G}(t)w^2(t) \text{ for } 2s_0 \leq t. \tag{3.54}$$

Multiplying Eq. (3.54) by $H^{\sigma}(\sigma(t),s)$ i.e., $H(\sigma(t),\sigma(s))$, and then integrating from $2s_0$ to $\sigma(t)$, we obtain

$$\int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t),s)\delta^\sigma(s)\mathcal{G}(s)\Delta s \leq - \int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t),s)w^\Delta(s)\Delta s + \int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t),s) \left(\frac{1}{\delta(s)}\left(\delta^\Delta(s) - \mathcal{G}(s)\delta^\sigma(s)\left(a(s)+p(s)\frac{s}{2}\right)\right)w(s) - \frac{\delta^\sigma(s)}{\delta^2(s)}\mathcal{G}(s)w^2(s)\right)\Delta s. \tag{3.55}$$

From (2.1), we obtain the right-hand side of (3.55) as follows

$$\leq H(\sigma(t),2s_0)w(2s_0) + \int_{2s_0}^{\sigma(t)} \left[H^{\Delta_s}(\sigma(t),s) + \frac{H^\sigma(\sigma(t),s)}{\delta(s)}\left(\delta^\Delta(s) - \mathcal{G}(s)\delta^\sigma(s)\left(a(s)+p(s)\frac{s}{2}\right)\right)\right] w(s)\Delta s - \int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t),s)\frac{\delta^\sigma(s)}{\delta^2(s)}\mathcal{G}(s)w^2(s)\Delta s. \tag{3.56}$$

Substituting (3.52) into (3.56), we arrive at

$$\int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t),s)\delta^\sigma(s)\mathcal{G}(s)\Delta s \leq H(\sigma(t),2s_0)w(2s_0) + \int_{2s_0}^{\sigma(t)} \left(\frac{h_-(t,s)\sqrt{H^\sigma(\sigma(t),s)}}{\delta(s)}w(s) - H^\sigma(\sigma(t),s)\frac{\delta^\sigma(s)}{\delta^2(s)}\mathcal{G}(s)w^2(s)\right)\Delta s. \tag{3.57}$$

which is an equivalent to

$$\int_{2s_0}^{\sigma(s)} H^\sigma(\sigma(t),s)\delta^\sigma(s)\mathcal{G}(s)\Delta s \leq H(\sigma(t),2s_0)w(2s_0) + \int_{2s_0}^{\sigma(t)} \frac{(h_-(t,s))^2}{4\delta^\sigma(s)\mathcal{G}(s)}\Delta s - \int_{2s_0}^{\sigma(t)} \left(\frac{\sqrt{H^\sigma(\sigma(t),s)\delta^\sigma(s)\mathcal{G}(s)}}{\delta(s)}w(s) - \frac{h_-(t,s)}{2\sqrt{\delta^\sigma(s)\mathcal{G}(s)}}\right)^2\Delta s. \tag{3.58}$$

Implies that

$$\int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t),s)\delta^\sigma(s)\mathcal{G}(s)\Delta s \leq H(\sigma(t),2s_0)w(2s_0) + \int_{2s_0}^{\sigma(t)} \frac{(h_-(t,s))^2}{4\delta^\sigma(s)\mathcal{G}(s)}\Delta s. \tag{3.59}$$

Dividing (3.59) by $H(\sigma(t),2s_0)$, we obtain

$$\frac{1}{H(\sigma(t),2s_0)}\int_{2s_0}^{\sigma(t)} \left(H^\sigma(\sigma(t),s)\delta^\sigma(s)\mathcal{G}(s) - \frac{(h_-(t,s))^2}{4\delta^\sigma(s)\mathcal{G}(s)}\right)\Delta s \leq w(2s_0) < \infty. \tag{3.60}$$

for sufficiently large t. Thus, we derive a contradiction to (3.53). □

Corollary 3.14 *Assume that O_1, O_2, relations (1.16), (1.18) and Eq. (3.51) hold. Moreover, if there exist Δ-derivative function $\delta(t) > 0$ and $s_0 \geq 0$, $t_0 \leq 2s_0 < t$, respectively, and satisfy the following conditions below*

$$H^{\Delta_s}(\sigma(t),s) + \frac{H^\sigma(\sigma(t),s)}{\delta(t)}\left(\delta^\Delta(t) - \mathcal{G}(t)\delta^\sigma(t)\left(a(t)+p(t)\frac{t}{2}\right)\right) = -\frac{h(t,s)}{\delta(t)}\sqrt{H^\sigma(\sigma(t),s)}, \tag{3.61}$$

$$\limsup_{t\to\infty}\frac{1}{H(\sigma(t),2s_0)}\int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t),s)\delta^\sigma(s)\mathcal{G}(s)\Delta s = \infty \text{ and } \limsup_{t\to\infty}\frac{1}{H(\sigma(t),2s_0)}\int_{2s_0}^{\sigma(t)} \frac{(h_-(t,s))^2}{4\delta(s)\mathcal{G}(s)}\Delta s < \infty. \tag{3.62}$$

Then, Eq. (1.2) oscillates on $[t_0\infty)_{\mathbb{T}}$.

Theorem 3.15 *Assume that O_1, O_2, relations (1.16), (1.18) and Eq. (3.51) hold. Moreover, if there exist Δ-derivative function $\delta(t) > 0$ and $s_0 \geq 0$, such that $\mu(t)\frac{\delta^\Delta(t)}{\delta^\sigma(t)} < 1$ and $t_0 \leq 2s_0 < t$, respectively, and satisfy the conditions below*

$$H^{\Delta_s}(\sigma(t),s) + \frac{H^\sigma(\sigma(t),s)}{\left(1 - \frac{\delta^\Delta(t)}{\delta^\sigma(t)}\mu(t)\right)}A(t) = -\frac{h(t,s)}{\left(1 - \frac{\delta^\Delta(t)}{\delta^\sigma(t)}\mu(t)\right)}\sqrt{H^\sigma(\sigma(t),s)}, \tag{3.63}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(\sigma(t),2s_0)}\int_{2s_0}^{\sigma(t)}\left(H^{\sigma}(\sigma(t),s)\mathcal{G}(s)-\frac{(h_-(t,s))^2}{4\mathcal{G}(s)}\right)\frac{\delta(s)}{\left(1-\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\mu(s)\right)}\Delta s=\infty,\tag{3.64}$$

where $A(t)$ is given by (3.15). Then, Eq. (1.2) oscillates on $[t_0\infty)_{\mathbb{T}}$.

Proof Assume to the contrary that (1.2) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that $y(t)$ is an eventually positive function, i.e., there exists t_0 such that $y(t)>0\forall t\in[t_0\infty)_{\mathbb{T}}$. A similar argument holds for the case when $y(t)$ is eventually negative. We have defined Riccati transformation function in (3.3). Now from (3.19), we have

$$\left(1-\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}\mu(t)\right)w^{\Delta}(t)\leq-\mathcal{G}(t)\delta(t)+A(t)w(t)-\frac{\mathcal{G}(t)}{\delta(t)}w^2(t),$$

which can be written as

$$\frac{\mathcal{G}(t)\delta(t)}{\left(1-\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}\mu(t)\right)}\leq-w^{\Delta}(t)+\frac{A(t)}{\left(1-\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}\mu(t)\right)}w(t)-\frac{\mathcal{G}(t)}{\delta(t)\left(1-\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}\mu(t)\right)}w^2(t).\tag{3.65}$$

Multiplying Eq. (3.65) by $H^{\sigma}(\sigma(t),s)$ and then integrating from $2s_0$ to $\sigma(t)$, we have

$$\int_{2s_0}^{\sigma(t)}\frac{H^{\sigma}(\sigma(t),s)\mathcal{G}(s)\delta(s)}{\left(1-\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\mu(s)\right)}\Delta s\leq-\int_{2s_0}^{\sigma(t)}H^{\sigma}(\sigma(t),s)w^{\Delta}(s)\Delta s+\int_{2s_0}^{\sigma(t)}\frac{H^{\sigma}(\sigma(t),s)A(s)}{\left(1-\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\mu(s)\right)}w(s)\Delta s-\int_{2s_0}^{\sigma(t)}\frac{H^{\sigma}(\sigma(t),s)\mathcal{G}(s)}{\delta(s)\left(1-\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\mu(s)\right)}w^2(s)\Delta s.\tag{3.66}$$

From (3.63), (3.66) and by following the similar steps of Eqs. (3.56)–(3.59), we obtain

$$\int_{2s_0}^{\sigma(t)}\frac{H^{\sigma}(\sigma(t),s)\mathcal{G}(s)\delta(s)}{\left(1-\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\mu(s)\right)}\Delta s\leq H(\sigma(t),2s_0)w(2s_0)+\int_{2s_0}^{\sigma(t)}\frac{\delta(s)(h_-(t,s))^2}{4\left(1-\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\mu(s)\right)\mathcal{G}(s)}\Delta s.\tag{3.67}$$

Dividing Eq. (3.67) by $H(\sigma(t),2s_0)$, we obtain

$$\frac{1}{H(\sigma(t),2s_0)}\int_{2s_0}^{\sigma(t)}\left(H^{\sigma}(\sigma(t),s)\mathcal{G}(s)-\frac{(h_-(t,s))^2}{4\mathcal{G}(s)}\right)\frac{\delta(s)}{\left(1-\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\mu(s)\right)}\Delta s<w(2s_0)<\infty.\tag{3.68}$$

For sufficiently large t, we derive a contradiction to (3.64), which completes the proof of our theorem. □

Theorem 3.16 *Assume that O_1, O_2, relations (1.16), (1.18) and Eq. (3.51) hold. Moreover, if there exist Δ-derivative function $\delta(t)>0$ and $s_0\geq0$, $t_0\leq2s_0<t$, respectively, and satisfying the following conditions below*

$$H^{\Delta_s}(t,s)+\frac{H(t,s)}{\delta^{\sigma}(t)}\left(\delta^{\Delta}(t)-\mathcal{G}(t)\delta(t)(a(t)+p(t)\frac{t}{2})\right)=-\frac{h(t,s)}{\delta^{\sigma}(t)}\sqrt{H(t,s)},\tag{3.69}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,2s_0)}\int_{2s_0}^{t}\left(H(t,s)\delta(s)\mathcal{G}(s)-\frac{(h_-(t,s))^2}{4\delta(s)}\right)\Delta s=\infty,\tag{3.70}$$

then, Eq. (1.2) oscillates on $[t_0\infty)_{\mathbb{T}}$.

Proof Assume to the contrary that (1.2) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that $y(t)$ is an eventually positive function, i.e., there exists t_0 such that $y(t)>0\forall t\in[t_0\infty)_{\mathbb{T}}$. We have defined Riccati transformation function in (3.3). Now multiplying Eq. (3.28) by $H(t,s)$ and integrating from $2s_0$ to t, we have following relation

$$\int_{2s_0}^{t}H(t,s)\delta(t)\mathcal{G}(s)\Delta s\leq-\int_{2s_0}^{t}H(t,s)w^{\Delta}(s)\Delta s+\int_{2s_0}^{t}H(t,s)\frac{(\delta^{\Delta}(s)-\delta(s)\mathcal{G}(s)(a(s)+p(s)\frac{s}{2}))}{\delta^{\sigma}(s)}w^{\sigma}(s)\Delta s$$

$$-\int_{2s_0}^{t}\frac{H(t,s)\delta(s)}{(\delta^{\sigma}(s))^2}(w^{\sigma}(s))^2\Delta s. \tag{3.71}$$

From (2.2) and (3.51), we obtain

$$\int_{2s_0}^{t}H(t,s)\delta(s)\mathcal{G}(s)\Delta s\leq H(t,2s_0)w(2s_0)$$
$$+\int_{2s_0}^{t}\left(H^{\Delta}(t,s)+H(t,s)\frac{(\delta^{\Delta}(s)-\delta(s)\mathcal{G}(s)(a(s)+p(s)\frac{s}{2}))}{\delta^{\sigma}(s)}\right)$$
$$w^{\sigma}(s)\Delta s$$
$$-\int_{2s_0}^{t}\frac{H(t,s)\delta(s)}{(\delta^{\sigma}(s))^2}(w^{\sigma}(s))^2\Delta s. \tag{3.72}$$

From Eqs. (3.69), (3.72) and by following the similar steps of Eqs. (3.35)–(3.39), we obtain a new relation

$$\frac{1}{H(t,2s_0)}\int_{2s_0}^{t}\left(H(t,s)\delta(s)\mathcal{G}(s)-\frac{(h_-(t,s))^2}{4\delta(s)}\right)\Delta s\leq w(2s_0)<\infty, \tag{3.73}$$

for sufficiently large t. Thus, we derive a contradiction to (3.70), which completes the proof of our theorem. □

Corollary 3.17 *Assume that O_1, O_2, relations (1.16), (1.18) and Eq. (3.51) hold. Moreover, if there exist Δ-derivative function $\delta(t)>0$ and $s_0\geq 0$, $t_0\leq 2s_0<t$, respectively, and satisfying the following conditions below,*

$$H^{\Delta_s}(t,s)+\frac{H(t,s)}{\delta^{\sigma}(t)}\left(\delta^{\Delta}(t)-\mathcal{G}(t)\delta(t)\left(a(t)+p(t)\frac{t}{2}\right)\right)=-\frac{h(t,s)}{\delta^{\sigma}(t)}\sqrt{H(t,s)} \tag{3.74}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,2s_0)}\int_{2s_0}^{t}H(t,s)\delta(s)\mathcal{G}(s)\Delta s=\infty \text{ and } \limsup_{t\to\infty}\frac{1}{H(t,2s_0)}\int_{2s_0}^{t}\frac{(h_-(t,s))^2}{4\delta(s)}\Delta s<\infty, \tag{3.75}$$

then, Eq. (1.2) oscillates on $[t_0\infty)_{\mathbb{T}}$. We can easily see that Eq. (1.2) in its general form involves differential equation and different types of difference equations depending on the choice of the time scale $\mathbb{T}$.

Remark 3.18 When $\mathbb{T}=\mathbb{R}$, Eq. (1.2) become second-order linear delay difference equation:

$$y''(t)+a(t)y'(t)+y(t)+\mathcal{K}(y(t-h))+f(\mathcal{W}(y'(t-h)=0, \tag{3.76}$$

where we have $\sigma(t)=t, \mu(t)=0, g^{\Delta}(t)=g'(t)$ and $\int_a^b g(t)\Delta t=\int_a^b g(t)\mathrm{d}t$.

Remark 3.19 When $\mathbb{T}=\mathbb{Z}$, Eq. (1.2) become second-order linear delay difference equation:

$$\Delta^2y(t)+a(t)\Delta y(t)+y(t)+\mathcal{K}(y(t-h))+f(\mathcal{W}(\Delta y(t-h)))=0, \tag{3.77}$$

where we have $\sigma(t)=t+1, \mu(t)=1, g^{\Delta}(t)=\Delta g(t)$ and $\int_a^b g(t)\Delta t=\sum_{t=a}^{b-1}g(t)\mu(t)=\sum_{t=a}^{b-1}g(t)$.

Remark 3.20 When $\mathbb{T}=h'\mathbb{Z}, h'>0$, Eq. (1.2) become second-order linear delay difference equation with step size h'

$$\Delta_{h'}^2y(t)+a(t)\Delta_{h'}y(t)+y(t)+\mathcal{K}(y(t-h))+f(\mathcal{W}(\Delta_{h'}y(t-h)))=0, \tag{3.78}$$

where we have $\sigma(t)=t+h', \mu(t)=h', f^{\Delta}(t)=\Delta_{h'}f(t)$ and $\int_a^b f(t)\Delta t=\sum_{l=\frac{a}{h'}}^{\frac{b}{h'}-1}f(lh')h'$.

For a function

$$f(\mathcal{W}(x^{\Delta}(t)))=q(t)\mathrm{sgn}(x(t))|\mathcal{W}(x^{\Delta}(t))| \quad \forall x\in\mathbb{R},$$

where $q(t)=\frac{\mu(t)}{w_1p(t)}\forall t\in\mathbb{T}$, we would like to illustrate our results by means of the following examples.

Example

Let us consider a second-order non-linear dynamic equation on time scale $\mathbb{T}$.

$$y^{\Delta\Delta}(t)+\frac{1}{t}y^{\Delta}(t)+y(t)+\sum_{n=1}^{\infty}k_{(2n-1)}(y(t-h))^{(2n-1)}+2\frac{\mu(t)}{w_1t^2}\mathrm{sgn}(y(t))\left|\sum_{m=1}^{\infty}w_{(2m-1)}(y^{\Delta}(t-h))^{(2m-1)}\right|=0. \tag{4.1}$$

In view of Corollary 3.3, we choose the parameters as follows: $\delta(t):=1, a(t):=\frac{1}{t}, p(t):=\frac{2}{t^2}, \forall k_n, w_m\in\mathbb{R}^+$ such that $0<p(t)\leq k_1<\infty, k_{2n+1}>0, k_{2n}:=0$ and $w_{2m}:=0, w_{(2m-1)}>0\forall n,m\in\mathbb{N}$ respectively. Also $0<\mu(t)<t-\ s_0<\ t$ for all $t>2s_0,\ s_0\geq 2$. Thus, we have $\frac{1}{2}\leq\mathcal{G}(t)<1$. Therefore, it is easy to verify that the conditions O_1, O_2, (1.16), (1.18) and (3.13) hold. The parameters also satisfying the following conditions below:

$$\int_{2s_0}^{t}\left(\mathcal{G}(s)-\mathcal{G}(s)\frac{(a(s)+p(s)\frac{s}{2})^2}{4}\right)\Delta s\geq\int_{2s_0}^{t}\left(\frac{1}{2}-\frac{1}{s^2}\right)\Delta s>\frac{7}{16}(t-2s_0)\to\infty \text{ as } t\to\infty. \tag{4.2}$$

When $\mu(t)=0$, we have

$$\int_{2s_0}^{t}\left(\mathcal{G}(s)-\mathcal{G}(s)\frac{(a(s)+p(s)\frac{s}{2})^2}{4}\right)\mathrm{d}s=\int_{2s_0}^{t}\left(1-\frac{1}{s^2}\right)\mathrm{d}s$$
$$>\frac{3}{4}(t-2s_0)\to\infty \text{ as } t\to\infty. \tag{4.3}$$

Now by the above parameters and conditions (4.2) and (4.3), we establish some examples on different time scales as follows.

Example 4.1 For $\mathbb{T}=\mathbb{Z}$, a second-order non-linear difference equation

$$\Delta^2 y(t)+\frac{1}{t}\Delta y(t)+y(t)+\sum_{n=1}^{\infty}k_{(2n-1)}(y(t-h))^{(2n-1)}$$
$$+2\frac{\operatorname{sgn}(y(t))}{w_1t^2}\left|\sum_{m=1}^{\infty}w_{(2m-1)}(\Delta y(t-h))^{(2m-1)}\right|=0. \tag{4.4}$$

Where $h=1$ $s_0=t_0=2$ such that $[2,\infty)_{\mathbb{T}}=\{2,3,4,\ldots\}, 0.5<k_1=1,\ k_{2n+1}=\frac{1}{(2n+1)^n}$ and $w_{(2m-1)}=\frac{1}{(2m-1)!}\forall n,m\in\mathbb{N}$. We have $0<\mu(t)=1<t-2$ for all $4<t\in\{5,6,8,\ldots\}$ and $0.5<\mathcal{G}(t)<1$. Thus O_1-O_2, (1.16) and (1.18) hold. It is easy to calculate that for each $t\in[2,\infty)_{\mathbb{Z}}$, we have

$$\int_2^{\infty}\frac{1}{e_{\xi(t)}(t,2)}\Delta t=\int_2^{\infty}e_{-\frac{1}{t}}(t,2)\Delta t=\int_2^{\infty}$$
$$\exp\left(\int_2^t\log\left(1-\frac{1}{s}\right)\Delta s\right)\Delta t=\int_2^{\infty}\prod_{l=2}^{t-1}\left(1-\frac{1}{l}\right)\Delta t$$
$$=\int_2^{\infty}\frac{1}{t-1}\Delta t=\infty.$$

Therefore O_2 and Eq. (4.2) hold. Hence (4.4) oscillates on $[2,\infty)_{\mathbb{T}}=\{2,3,4,\ldots\}$.

Remark 4.2 Similarly, for $\mathbb{T}=h'\mathbb{Z}, h'>0$,, etc., there exists $s_0\geq 0$ such that (4.1) oscillates on $[t_0,\infty)_{\mathbb{T}}$.

Example 4.3 For $\mathbb{T}=\mathbb{R}$, a second-order non-linear differential equation

$$y''(t)+\frac{1}{t}y'(t)+y(t)+\sum_{n=1}^{\infty}\frac{1}{(2n-1)^n}(y(t-h))^{(2n-1)}=0. \tag{4.5}$$

Thus for each $t\in[2,\infty)_{\mathbb{R}}$, we obtain O_2 holds, as follows

$$\int_2^{\infty}\frac{1}{e_{\xi(t)}(t,2)}\mathrm{d}t=\int_2^{\infty}e_{-\frac{1}{t}}(t,2)\,\mathrm{d}t$$
$$=\int_2^{\infty}\exp\left(\int_2^t-\frac{1}{s}\,\mathrm{d}s\right)\,\mathrm{d}t=\int_2^{\infty}\frac{2}{t}\,\mathrm{d}t=\infty.$$

Therefore, from Eq. (4.3) and in view of corollary 3.3, we conclude that (4.5) oscillates on $[2,\infty)$.

Conclusion

The oscillation of a function is a number that quantifies how much a function varies between its extreme values as it approaches infinity or some other point. It is very important qualitative property of a function. There are many criteria for oscillations, out of which two important are Kamenev and Philos type. In this present work, we have established such criteria for Eq. (1.2). First we have proved Lemmas 2.11–2.13 and then used Riccati transformation technique to establish the main result. We have obtained some new conditions for the oscillations. It is also important to note that the results presented in this paper are not valid for $\mathbb{T}=\overline{q^{\mathbb{Z}}}$. The reason is given in the Remark 2.9. We have also constructed few examples for some time scale $\mathbb{T}$, i.e., $\mathbb{T}=\mathbb{R}, \mathbb{T}=\mathbb{Z}$ and $\mathbb{T}=h'\mathbb{Z}$ to illustrate the results.

Acknowledgements The authors would like to thank the reviewers's careful reviewing and helpful comments and suggestions. The research of first author "Shekhar Singh Negi" is supported by the University Grants Commission of India (UGC India) under Sr. no. 2061440966, Ref. no. 22/06/2014(i)EU-V. Yonghui Xia was supported by the National Natural Science Foundation of China under Grant (No. 11671176 and No. 11271333), Natural Science Foundation of Zhejiang Province under Grant (No. LY15A010007), the Scientific Research Funds of Huaqiao University and China Postdoctoral Science Foundation (No.2014M562320).

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