Math Sci (2018) 12:1–6
https://doi.org/10.1007/s40096-017-0238-9

ORIGINAL RESEARCH

The annihilating graph of a ring

Z. Shafiei[1] · M. Maghasedi[1] · F. Heydari[1] · S. Khojasteh[2]

Received: 26 June 2017 / Accepted: 23 September 2017 / Published online: 5 October 2017

Abstract Let A be a commutative ring with unity. The annihilating graph of A, denoted by $\mathbb{G}(A)$, is a graph whose vertices are all non-trivial ideals of A and two distinct vertices I and J are adjacent if and only if $\mathrm{Ann}(I)\mathrm{Ann}(J) = 0$. For every commutative ring A, we study the diameter and the girth of $\mathbb{G}(A)$. Also, we prove that if $\mathbb{G}(A)$ is a triangle-free graph, then $\mathbb{G}(A)$ is a bipartite graph. Among other results, we show that if $\mathbb{G}(A)$ is a tree, then $\mathbb{G}(A)$ is a star or a double star graph. Moreover, we prove that the annihilating graph of a commutative ring cannot be a cycle. Let n be a positive integer number. We classify all integer numbers n for which $\mathbb{G}(\mathbb{Z}_n)$ is a complete or a planar graph. Finally, we compute the domination number of $\mathbb{G}(\mathbb{Z}_n)$.

Keywords Annihilating graph · Diameter · Girth · Planarity

Mathematics Subject Classification 05C10 · 05C25 · 05C40 · 13A99

✉ M. Maghasedi
maghasedi@kiau.ac.ir

Z. Shafiei
zahra.shafiei@kiau.ac.ir

F. Heydari
f-heydari@kiau.ac.ir

S. Khojasteh
s_khojasteh@liau.ac.ir

1 Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran

2 Department of Mathematics, Lahijan Branch, Islamic Azad University, Lahijan, Iran

Introduction

There are many papers on assigning a graph to algebraic structures, for instance see [2–6, 8, 9]. Throughout this paper, all graphs are simple with no loops and multiple edges and A is a commutative ring with non-zero identity. We denote by $\mathbb{I}(A)^*$ and $\mathrm{Max}(A)$, the set of all non-trivial ideals of A and the set of all maximal ideals of A, respectively. A ring having just one maximal ideal is called a local ring and a ring having only finitely many maximal ideals is said to be a semilocal ring. For every ideal I of A, we denote by $\mathrm{Ann}(I)$, the set of elements $a \in A$ such that $aI = 0$.

Let G be a graph with vertex set $V(G)$. If u is adjacent to v, then we write $u - v$. For $u, v \in V(G)$, we recall that a path between u and v is a sequence $u = x_0 - \cdots - x_n = v$ of vertices of G such that for every i with $1 \leq i \leq n$, the vertices x_{i-1} and x_i are adjacent and $x_i \neq x_j$, where $i \neq j$. For every positive integer n, we denote the path of order n, by P_n. For $u, v \in V(G)$ with $u \neq v$, $d(u, v)$ denotes the length of a shortest path between u and v. If there is no such path, then we define $d(u, v) = \infty$. The diameter of G is defined $\mathrm{diam}(G) = \sup\{d(u, v) | u \text{ and } v \text{ are vertices of } G\}$. For any $u \in V(G)$, the degree of u, $\deg(u)$, denotes the number of edges incident with u. The neighborhood of a vertex u is denoted by $N_G(u)$ or simply $N(u)$. A graph G is k-regular if $d(v) = k$ for all $v \in V(G)$; a regular graph is one that is k-regular for some k. We denote the complete graph on n vertices by K_n. A bipartite graph is one whose vertex set can be partitioned into two subsets V_1 and V_2 so that each edge has one end in V_1 and one end in V_2. A complete bipartite graph is a bipartite graph with two partitions V_1 and V_2 in which every vertex in V_1 is joined to every vertex in V_2. The complete bipartite graph with two partitions of size m and n is denoted by $K_{m,n}$. A star graph with center v and n vertices is the complete bipartite graph with part sizes 1 and n such that $\deg(v) = n$. A double-star graph is a union of two star graphs with centers u and v such that u is adjacent to v. We use C_n for the cycle of order n, where $n \geq 3$. If a graph G has a cycle, then the girth of G (notated $\mathrm{gr}(G)$) is defined as the length of a shortest cycle of G; otherwise $\mathrm{gr}(G) = \infty$. A triangle-free graph is a graph which contains no triangle. A clique of a graph is a complete subgraph and the number of vertices in a largest clique of graph G, denoted by $\omega(G)$, is called the clique number of G. Recall that a graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. Also, a dominating set is a subset S of $V(G)$ such that every vertex of $V(G)\backslash S$ is adjacent to at least one vertex in S. The number of vertices in a smallest dominating set denoted by $\gamma(G)$, is called the domination number of G.

Let A be a commutative ring with non-zero identity. The annihilating graph of A, denoted by $\mathbb{G}(A)$, is a graph with the vertex set $\mathbb{I}(A)^*$, and two distinct vertices $I, J \in \mathbb{Z}(A)^*$ are adjacent if and only if $\mathrm{Ann}(I)\mathrm{Ann}(J) = 0$. In this paper, we prove that if A is a ring, then $\mathbb{G}(A)$ is a connected graph, $\mathrm{diam}(\mathbb{G}(A)) \leq 3$ and $\mathrm{gr}(\mathbb{G}(A)) \in \{3, 4, \infty\}$. Also, we prove that for every ring A, if $\mathbb{G}(A)$ is a triangle-free graph, then $\mathbb{G}(A)$ is a bipartite graph. Among other results, we show that if A is a ring and $\mathbb{G}(A)$ is a tree, then $\mathbb{G}(A)$ is a star or a double star graph. Moreover, we prove that the annihilating graph of a ring cannot be a cycle. Also, we obtained some results about $\mathbb{G}(\mathbb{Z}_n)$. We show that $\mathbb{G}(\mathbb{Z}_n)$ is a complete graph if and only if $n \in \{p_1^2, p_1^3, p_1p_2\}$. We also prove that $\mathbb{G}(\mathbb{Z}_n)$ is a planar graph if and only if $n \in \{p_1, p_1^2, \ldots, p_1^8, p_1p_2, p_1^2p_2, p_1^3p_2, p_1^3p_2^2, p_1^4p_2, p_1^2p_2^2, p_1p_2p_3, p_1^2 p_2p_3\}$. Finally, we determine the domination number of $\mathbb{G}(\mathbb{Z}_n)$.

The annihilating graph of A

In this section, we study the diameter and the girth of the annihilating graph of a ring. Also, we classify all rings whose annihilating graphs are complete graph, tree or cycle.

We start with the following lemma.

Lemma 1 *If A is a commutative ring, then* $\gamma(\mathbb{G}(A)) \leq |\mathrm{Max}(A)| \leq \omega(\mathbb{G}(A))$.

Proof Suppose that $\mathfrak{m}_1, \mathfrak{m}_2$ are two distinct maximal ideals of A. Then we have $\mathrm{Ann}(\mathfrak{m}_1)\mathrm{Ann}(\mathfrak{m}_2) \subseteq \mathrm{Ann}(\mathfrak{m}_1) \cap \mathrm{Ann}(\mathfrak{m}_2) \subseteq \mathrm{Ann}(\mathfrak{m}_1 + \mathfrak{m}_2)$. Since $\mathfrak{m}_1 + \mathfrak{m}_2 = A$, we conclude that $\mathrm{Ann}(\mathfrak{m}_1 + \mathfrak{m}_2) = 0$ and so $\mathfrak{m}_1$ is adjacent to $\mathfrak{m}_2$. This implies that $\mathrm{Max}(A)$ is a clique in $\mathbb{G}(A)$. Now, suppose that $I \in \mathbb{Z}(A)^*\backslash\mathrm{Max}(A)$. Let $\mathfrak{m}$ be a maximal ideal containing $\mathrm{Ann}(I)$. Since $\mathrm{Ann}(I)\mathrm{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}\mathrm{Ann}(\mathfrak{m}) = 0$, we deduce that I is adjacent to $\mathfrak{m}$. Hence $\mathrm{Max}(A)$ is a dominating set of $\mathbb{G}(A)$. □

By the previous lemma, if the clique number of $\mathbb{G}(A)$ is finite, then A is a semilocal ring. Also, we have the following result.

Corollary 1 *Let A be a ring. If every maximal ideal of A has finite degree, then $\mathbb{G}(A)$ is a finite graph.*

Proof Since $\mathrm{Max}(A)$ is a clique in $\mathbb{G}(A)$, so $\mathrm{Max}(A)$ is finite. Now, since $\mathrm{Max}(A)$ is a dominating set of $\mathbb{G}(A)$, the result holds. □

Next, we study the diameter and the girth of $\mathbb{G}(A)$.

Theorem 1 *Let A be a ring. Then* $\mathrm{diam}(\mathbb{G}(A)) \leq 3$. *Moreover, if A is a local ring, then* $\mathrm{diam}(\mathbb{G}(A)) \leq 2$.

Proof Assume that I and J are two non-trivial ideals of A. Suppose that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are maximal ideals such that $\mathrm{Ann}(I) \subseteq \mathfrak{m}_1$ and $\mathrm{Ann}(J) \subseteq \mathfrak{m}_2$. Since $\mathrm{Ann}(I)\mathrm{Ann}(\mathfrak{m}_1) \subseteq \mathfrak{m}_1\mathrm{Ann}(\mathfrak{m}_1) = 0$, we conclude that $I = \mathfrak{m}_1$ or I is adjacent to $\mathfrak{m}_1$. Similarly, $J = \mathfrak{m}_2$ or J is adjacent to $\mathfrak{m}_2$. Now, if $\mathfrak{m}_1 = \mathfrak{m}_2$, then $d(I, J) \leq 2$. Otherwise, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are adjacent and so $d(I, J) \leq 3$. Thus $\mathrm{diam}(\mathbb{G}(A)) \leq 3$. (Note that if A has a non-trivial ideal I with $\mathrm{Ann}(I) = 0$, then I is adjacent to all other vertices and hence $\mathrm{diam}(\mathbb{G}(A)) \leq 2$.) Finally, assume that $(A, \mathfrak{m})$ is a local ring. By the proof of Lemma 1, $\mathfrak{m}$ is adjacent to all other vertices, so $\mathrm{diam}(\mathbb{G}(A)) \leq 2$. □

Theorem 2 *Let A be a ring. Then* $\mathrm{gr}(\mathbb{G}(A)) \in \{3, 4, \infty\}$. *Moreover, if A is a local ring and $\mathbb{G}(A)$ contains a cycle, then* $\mathrm{gr}(\mathbb{G}(A)) = 3$.

Proof Clearly, if A has at least three maximal ideals, then $\mathrm{gr}(\mathbb{G}(A)) = 3$. So assume that A has exactly two maximal ideals and $\mathbb{G}(A)$ contains a cycle C. If C is a cycle of length at most 4, then we are done. Otherwise, C contains two adjacent vertices I and J which are not maximal ideals. Suppose that $I \subseteq \mathfrak{m}_1$ and $J \subseteq \mathfrak{m}_2$, where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are maximal ideals of A. Since $\mathrm{Ann}(I)\mathrm{Ann}(\mathfrak{m}_2) \subseteq \mathrm{Ann}(I)Ann(J) = 0$, we deduce that I and $\mathfrak{m}_2$ are adjacent. Similarly, J and $\mathfrak{m}_1$ are adjacent. If $\mathfrak{m}_1 = \mathfrak{m}_2$, then $\mathrm{gr}(\mathbb{G}(A)) = 3$. Otherwise, $\mathrm{gr}(\mathbb{G}(A)) \leq 4$. The last part follows from the proof of Lemma 1. □

The following theorem shows that triangle-free annihilating graphs are bipartite.

Theorem 3 *Let A be a ring. If $\mathbb{G}(A)$ is a triangle-free graph, then $\mathbb{G}(A)$ is a bipartite graph.*

Proof Let $\mathbb{G}(A)$ be a triangle-free graph. Clearly A has at most two maximal ideals. If A is a local ring, then $\mathbb{G}(A)$ is a star and so $\mathbb{G}(A)$ is bipartite. Suppose that A contains exactly two distinct maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$. One can easily see that $\mathbb{G}(A)$ is a bipartite graph with parts $N(\mathfrak{m}_1)$ and $N(\mathfrak{m}_2)$. □

Theorem 4 *Let A be a ring. If $\mathbb{G}(A)$ is a tree, then $\mathbb{G}(A)$ is a star or a double star graph.*

Proof Assume that $\mathbb{G}(A)$ is a tree. It is enough to show that if A has exactly two distinct maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$, then $\mathbb{G}(A)$ is a double star graph. By the proof of Lemma 1, $\mathfrak{m}_1$ is adjacent to $\mathfrak{m}_2$ and every other vertex is adjacent to one of the $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Now, since $\mathbb{G}(A)$ contains no cycles, $\mathbb{G}(A)$ is a double star graph. □

By the previous theorem, we have the following immediate corollary.

Corollary 2 *Let A be a ring. If $\mathbb{G}(A) \cong P_n$, then $n \leq 4$.*

Theorem 5 *The annihilating graph of a ring cannot be a cycle.*

Proof By contrary suppose that $\mathbb{G}(A) \cong C_n$, for some $n \geq 3$. By Theorem 2, we conclude that $n \leq 4$. First assume that $\mathbb{G}(A) \cong C_4$. So A has exactly four non-trivial ideals. By Theorem 2, we deduce that A is not a local ring. Hence by [6, Theorem 8.7], $A \cong F \times S$, where F is a field and S is a ring with exactly one non-trivial ideal. Let $\mathfrak{m}$ be the non-trivial ideal of S. Thus $\mathbb{I}(A)^* = \{0 \times \mathfrak{m}, 0 \times S, F \times 0, F \times \mathfrak{m}\}$. We have $\mathrm{Ann}(0 \times \mathfrak{m}) = F \times \mathfrak{m}$, $\mathrm{Ann}(F \times \mathfrak{m}) = 0 \times \mathfrak{m}$, $\mathrm{Ann}(0 \times S) = F \times 0$ and $\mathrm{Ann}(F \times 0) = 0 \times S$. Therefore, $\mathbb{G}(A)$ is the path $0 \times \mathfrak{m} - F \times \mathfrak{m} - 0 \times S - F \times S$, a contradiction. Next assume that $\mathbb{G}(A) \cong C_3$. Since A has exactly three non-trivial ideals, by [6, Theorem 8.7], A is an Artinian local ring. Let $\mathbb{I}(A)^* = \{I, J, \mathfrak{m}\}$, where $\mathfrak{m}$ is the maximal ideal of A. Suppose that k is the smallest positive integer such that $\mathfrak{m}^k = 0$. So $\mathrm{Ann}(\mathfrak{m}) \neq 0$. With no loss of generality, we consider two cases. Note that the annihilating-ideal graph $\mathbb{AG}(A)$ of A is a graph whose vertex set is the set of all non-zero ideals of A with non-zero annihilator and two distinct vertices I and J are adjacent if and only if $IJ = 0$, see [1].

Case 1 $\mathrm{Ann}(\mathfrak{m}) = \mathfrak{m}$. So $\mathfrak{m}^2 = 0$ and hence $IJ = I\mathfrak{m} = J\mathfrak{m} = 0$. This implies that $\mathbb{AG}(A) \cong \mathbb{G}(A) \cong C_3$. By [1, Corollary 9], $\mathbb{AG}(A)$ cannot be a cycle, a contradiction.

Case 2 $\mathrm{Ann}(\mathfrak{m}) = I$. Thus $I\mathfrak{m} = 0$. So $IJ = 0$ and $\mathfrak{m} = \mathrm{Ann}(I)$. If $\mathfrak{m}J = 0$, then $\mathbb{AG}(A) \cong \mathbb{G}(A) \cong C_3$, a contradiction. Therefore, $\mathfrak{m}J \neq 0$ and hence $\mathbb{AG}(A) \cong P_3$. Now, by [1, Theorem 11], we have $k = 4$ and so $I = \mathfrak{m}^3$ and $J = \mathfrak{m}^2$. This implies that $\mathrm{Ann}(I) = \mathfrak{m}$ and $\mathrm{Ann}(J) = \mathfrak{m}^2$. Thus $\mathbb{G}(A) \cong P_3$, a contradiction. □

Theorem 6 *If $\mathbb{G}(A)$ is a regular graph of finite degree, then $\mathbb{G}(A)$ is a complete graph.*

Proof By Corollary 1, A has finitely many ideals. So A is an Artinian ring. First suppose that $(A, \mathfrak{m})$ is an Artinian local ring. Since $\mathfrak{m}$ is a vertex of $\mathbb{G}(A)$ which is adjacent to all other vertices, we deduce that $\mathbb{G}(A)$ is a complete graph. Now, by [6, Theorem 8.7], we may assume that $A \cong A_1 \times \cdots \times A_n$, where $n \geq 2$ and $(A_i, \mathfrak{m}_i)$ is an Artinian local ring for $i = 1, \ldots, n$. We have $\mathrm{Ann}(0 \times A_2 \times \cdots \times A_n) = A_1 \times 0 \times \cdots \times 0$, $\mathrm{Ann}(\mathfrak{m} \times \mathfrak{A} \times \cdots \times \mathfrak{A}_n) = \mathrm{Ann}(\mathfrak{m}) \times \ \times \cdots \times$, and $\mathrm{Ann}(A_1 \times 0 \times \cdots \times 0) = 0 \times A_2 \times \ \cdots \times A_n$. Let $v_1 = 0 \times A_2 \times \cdots \times A_n$, $v_2 = \mathfrak{m} \times \mathfrak{A} \times \cdots \times \mathfrak{A}_n$ and $v_3 = A_1 \times 0 \times \cdots \times 0$. One can easily see that

$$N(v_1) = \{A_1 \times I_2 \times \cdots \times I_n \mid I_i \text{ is an ideal of } A_i \text{ for } i = 2, \ldots, n\} \backslash \{A\},$$

and

$$N(v_2) = \{I_1 \times I_2 \times \cdots \times I_n \mid I_i \text{ is an ideal of } A_i \text{ for } i = 1, \ldots, n \text{ and } I_1 \neq 0\} \backslash \{A\}.$$

Note that every non-trivial ideal of an Artinian ring A has a non-zero annihilator. Since $\deg(v_1) = \deg(v_2)$, we conclude that A_1 has no proper ideal other than $0, \mathfrak{m}$. Thus

$$N(v_3) = \{0 \times A_2 \times \cdots \times A_n, \mathfrak{m} \times A_2 \times \cdots \times A_n\}.$$

Hence $\deg(v_3) \leq 2$. If $\mathbb{G}(A)$ is a 2-regular graph, then $\mathbb{G}(A)$ is a cycle, a contradiction. Note that by Theorem 1, $\mathbb{G}(A)$ is a connected graph. Therefore, $\mathbb{G}(A)$ is a 1-regular graph. So $\mathbb{G}(A) \cong K_2$ is a complete graph. In this case, $A \cong F_1 \times F_2$, where F_1, F_2 are fields.

Remark 1 Let A be a commutative ring and $\mathfrak{m}$ be a maximal ideal of A with non-zero annihilator. Since $\mathfrak{m}\mathrm{Ann}(\mathfrak{m}) = 0$, we conclude that $\mathfrak{m} \subseteq \mathrm{Ann}(Ann(\mathfrak{m}))$. Now, $\mathrm{Ann}(\mathfrak{m}) \neq 0$ implies that $\mathrm{Ann}(\mathrm{Ann}(\mathfrak{m})) = \mathfrak{m}$.

Lemma 2 *Let A be a local ring with non-zero maximal ideal $\mathfrak{m}$. If $I \in \mathbb{Z}(A)^*$ and $\mathrm{Ann}(I) = \mathrm{Ann}(\mathfrak{m})$, then I is adjacent to all other vertices of $\mathbb{G}(A)$.*

Proof Suppose that $\mathrm{Ann}(I) = \mathrm{Ann}(\mathfrak{m})$. Let J be a non-trivial ideal of A and $J \neq I$. Since $\mathrm{Ann}(J) \subseteq \mathfrak{m}$, we deduce that $\mathrm{Ann}(J)\mathrm{Ann}(I) \subseteq \mathfrak{m}\mathrm{Ann}(\mathfrak{m}) = 0$. Hence I and J are adjacent. The proof is complete. □

Theorem 7 *Let A be a local ring with non-zero maximal ideal $\mathfrak{m}$ such that $\mathrm{Ann}(\mathfrak{m}) \neq 0$. Then $\mathbb{G}(A)$ is a complete graph if and only if $\mathrm{Ann}(I) = \mathrm{Ann}(\mathfrak{m})$, for every ideal $I \in \mathbb{Z}(A)^* \backslash \{\mathrm{Ann}(\mathfrak{m})\}$.*

Proof Suppose that $\mathbb{G}(A)$ is a complete graph and let $I \in \mathbb{Z}(A)^* \backslash \{\mathrm{Ann}(\mathfrak{m})\}$. Since $\mathrm{Ann}(\mathfrak{m}) \neq 0, A$, we conclude that $\mathrm{Ann}(\mathfrak{m})$ is a vertex of $\mathbb{G}(A)$ and hence is adjacent to

I. Thus $\mathrm{Ann}(I)\mathrm{Ann}(Ann(\mathfrak{m})) = 0$. By Remark 1, $\mathrm{Ann}(\mathrm{Ann}(\mathfrak{m})) = \mathfrak{m}$. So $\mathrm{Ann}(I)\mathfrak{m} = 0$ which implies that $\mathrm{Ann}(I) \subseteq \mathrm{Ann}(\mathfrak{m})$. In other hand, since $I \subseteq \mathfrak{m}$, we deduce that $\mathrm{Ann}(\mathfrak{m}) \subseteq \mathrm{Ann}(I)$. Therefore, $\mathrm{Ann}(I) = \mathrm{Ann}(\mathfrak{m})$. Conversely, suppose that $\mathrm{Ann}(I) = \mathrm{Ann}(\mathfrak{m})$, for every ideal $I \in \mathbb{Z}(A)^*\backslash\{\mathrm{Ann}(\mathfrak{m})\}$. Assume that $I, J \in \mathbb{Z}(A)^*\backslash\{\mathrm{Ann}(\mathfrak{m})\}$ and $I \neq J$. Since $\mathrm{Ann}(I) = \mathrm{Ann}(J) = Ann(\mathfrak{m})$, we conclude that $\mathrm{Ann}(I)\mathrm{Ann}(J) = Ann(\mathfrak{m})\mathrm{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}\mathrm{Ann}(\mathfrak{m}) = 0$. Hence I and J are adjacent. Now, since $\mathrm{Ann}(\mathrm{Ann}(\mathfrak{m}))\mathrm{Ann}(I) = \mathfrak{m}\mathrm{Ann}(I) = \mathfrak{m}\mathrm{Ann}(\mathfrak{m}) = 0$, then $\mathrm{Ann}(\mathfrak{m})$ is adjacent to all other vertices. Thus $\mathbb{G}(A)$ is a complete graph. □

Theorem 8 *Let A be an Artinian local ring with non-zero maximal ideal $\mathfrak{m}$. Then $\mathbb{G}(A)$ is a complete graph if and only if either $\mathfrak{m}^2 = 0$ or $\mathfrak{m}^3 = 0$ and $IJ = \mathfrak{m}^2$, for every ideal $I, J \in \mathbb{Z}(A)^*\backslash\{\mathfrak{m}^2\}$.*

Proof First assume that $\mathfrak{m}^2 = 0$. Thus $\mathfrak{m} \subseteq \mathrm{Ann}(\mathfrak{m})$ and hence $\mathrm{Ann}(\mathfrak{m}) = \mathfrak{m}$. Let $I \in \mathbb{Z}(A)^*$. Since $I \subseteq \mathfrak{m}$, we deduce that $\mathfrak{m} = \mathrm{Ann}(\mathfrak{m}) \subseteq \mathrm{Ann}(I)$. So $\mathrm{Ann}(I) = \mathfrak{m}$. Now, Theorem 9 implies that $\mathbb{G}(A)$ is a complete graph. Next assume that $\mathfrak{m}^3 = 0$ and $IJ = \mathfrak{m}^2$, for every ideal $I, J \in \mathbb{Z}(A)^*\backslash\{\mathfrak{m}^2\}$. Note that $\mathfrak{m}^2 \neq 0$. Hence $\mathrm{Ann}(\mathfrak{m}) \neq \mathfrak{m}$ and $\mathrm{Ann}(\mathfrak{m}^2) = \mathfrak{m}$. Since $\mathrm{Ann}(\mathfrak{m})\mathrm{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}\mathrm{Ann}(\mathfrak{m}) = 0$, we conclude that $\mathrm{Ann}(\mathfrak{m}) = \mathfrak{m}^2$. Let $I \in \mathbb{Z}(A)^*\backslash\{\mathfrak{m}^2\}$. Since $I\mathrm{Ann}(I) = 0 \neq \mathfrak{m}^2$, we deduce that $\mathrm{Ann}(I) = \mathfrak{m}^2 = \mathrm{Ann}(\mathfrak{m})$. Thus by Theorem 9, $\mathbb{G}(A)$ is complete. Conversely, suppose that $\mathbb{G}(A)$ is a complete graph. Let k be the smallest positive integer such that $\mathfrak{m}^k = 0$. If $k = 2$, we are done. Assume that $k \geq 3$. So $\mathrm{Ann}(\mathfrak{m}) \neq \mathfrak{m}$. Since $\mathfrak{m} \subseteq \mathrm{Ann}(\mathfrak{m}^{k-1})$, we conclude that $\mathrm{Ann}(\mathfrak{m}^{k-1}) = \mathfrak{m}$. Now, by Theorem 9, $\mathrm{Ann}(\mathfrak{m}) = \mathfrak{m}^{k-1}$. In other hand, since $\mathfrak{m}^{k-2} \subseteq \mathrm{Ann}(\mathfrak{m}^2)$, then $\mathrm{Ann}(\mathfrak{m}^2) \neq \mathfrak{m}^{k-1} = \mathrm{Ann}(\mathfrak{m})$. This implies that $\mathfrak{m}^2 = \mathrm{Ann}(\mathfrak{m}) = \mathfrak{m}^{k-1}$. Therefore, $k = 3$ and so we have $\mathfrak{m}^3 = 0$, $\mathrm{Ann}(\mathfrak{m}) = \mathfrak{m}^2$, and $\mathrm{Ann}(\mathfrak{m}^2) = \mathfrak{m}$. Finally, suppose that $I, J \in \mathbb{Z}(A)^*\backslash\{\mathfrak{m}^2\}$. Since $\mathfrak{m}IJ \subseteq \mathfrak{m}^3 = 0$, we deduce that $IJ = 0$ or $\mathrm{Ann}(IJ) = \mathfrak{m}$. If $IJ = 0$, then $I \subseteq \mathrm{Ann}(J) = \mathfrak{m}^2$ and hence $\mathfrak{m} = \mathrm{Ann}(\mathfrak{m}^2) \subseteq \mathrm{Ann}(I) = \mathfrak{m}^2$, a contradiction. Thus $\mathrm{Ann}(IJ) = \mathfrak{m}$ and so Theorem 7 implies that $IJ = \mathfrak{m}^2$. The proof is complete. □

We close this section by the following theorem which is a classification of rings whose annihilating graphs are complete.

Theorem 9 *Let A be a commutative ring. If $\mathbb{G}(A) \cong K_n$, then one of the following holds:*

(i) *$(A, \mathfrak{m})$ is an Artinian local ring with $\mathfrak{m}^2 = 0$.*
(ii) *$(A, \mathfrak{m})$ is an Artinian local ring with $\mathfrak{m}^3 = 0$ and $IJ = \mathfrak{m}^2$, for every ideal $I, J \in \mathbb{Z}(A)^*\backslash\{\mathfrak{m}^2\}$.*
(iii) *$A \cong F_1 \times F_2$, where F_1, F_2 are fields.*

Proof Suppose that $\mathbb{G}(A) \cong K_n$, for some positive integer n. So A is an Artinian ring. By Theorem 8, if A is a local ring, then the cases (ii) or (iii) occur. Otherwise, by the proof of Theorem 6, $A \cong F_1 \times F_2$, where F_1, F_2 are fields. □

The annihilating graph of $\mathbb{Z}_n$

In this section, we study the case that $A = \mathbb{Z}_n$. Throughout this section, without loss of generality, we assume that $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, where p_i's are distinct primes and α_i's are positive integers. It is easy to see that $\mathbb{I}(\mathbb{Z}_n) = \{d\mathbb{Z}_n : d$ divides $n\}$ and $|\mathbb{I}(\mathbb{Z}_n)^*| = \prod_{i=1}^{s}(\alpha_i + 1) - 2$. We denote the least common multiple and the greatest common divisor of integers a and b by $[a, b]$ and (a, b), respectively. Also, we write $a|b$ ($a \not| b$) if a divides b (a does not divide b). We begin with the following lemma.

Lemma 3 *If $p_1^{\beta_1} \cdots p_s^{\beta_s}\mathbb{Z}_n \in \mathbb{Z}(\mathbb{Z}_n)^*$, then* $\mathrm{Ann}(p_1^{\beta_1} \cdots p_s^{\beta_s}\mathbb{Z}_n) = p_1^{\alpha_1-\beta_1} \cdots p_s^{\alpha_s-\beta_s}\mathbb{Z}_n$.

Proof Let $d = p_1^{\beta_1} \cdots p_s^{\beta_s}$ and $d' = p_1^{\alpha_1-\beta_1} \cdots p_s^{\alpha_s-\beta_s}$. Clearly, $d\mathbb{Z}_n d'\mathbb{Z}_n = 0$ and so $d'\mathbb{Z}_n \subseteq \mathrm{Ann}(d\mathbb{Z}_n)$. Let $r \in \mathrm{Ann}(d\mathbb{Z}_n)$. Then n divides rd. Since $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ and $d = p_1^{\beta_1} \cdots p_s^{\beta_s}$, so $p_1^{\alpha_1-\beta_1} \cdots p_s^{\alpha_s-\beta_s}$ divides r. This implies that $r \in d'\mathbb{Z}_n$ and $\mathrm{Ann}(d\mathbb{Z}) \subseteq d'\mathbb{Z}_n$. The proof is complete. □

Remark 2 Let $d_1\mathbb{Z}_n, d_2\mathbb{Z}_n \in \mathbb{Z}(\mathbb{Z}_n)^*$ and let $d_1 = p_1^{\beta_1} \cdots p_s^{\beta_s}$, $d_2 = p_1^{\gamma_1} \cdots p_s^{\gamma_s}$. Then $d_1\mathbb{Z}_n$ and $d_2\mathbb{Z}_n$ are adjacent if and only if $p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ divides $p_1^{2\alpha_1-(\beta_1+\gamma_1)} \cdots p_s^{2\alpha_s-(\beta_s+\gamma_s)}$ which implies that $\alpha_i \geq \beta_i + \gamma_i$, for $i = 1, \ldots, s$. Also, if $(d_1, d_2) = 1$ then $d_1\mathbb{Z}_n$ and $d_2\mathbb{Z}_n$ are adjacent.

Lemma 4 *If $d = p_1^{\beta_1} \cdots p_s^{\beta_s}$, then $\prod_{i=1}^{s}(\alpha_i - \beta_i + 1) - 2 \leq \deg(d\mathbb{Z}_n) \leq \prod_{i=1}^{s}(\alpha_i - \beta_i + 1) - 1$.*

Proof If $p_1^{\gamma_1} \cdots p_s^{\gamma_s}\mathbb{Z}_n$ and $d\mathbb{Z}_n$ are adjacent, then by Remark 2, $0 \leq \gamma_i \leq \alpha_i - \beta_i$. On the other hand, $p_1^{\gamma_1} \cdots p_s^{\gamma_s} \notin \{1, d\}$ which implies that $\deg(d\mathbb{Z}_n) \in \{\prod_{i=1}^{s}(\alpha_i - \beta_i + 1) - 2, \prod_{i=1}^{s}(\alpha_i - \beta_i + 1) - 1\}$. □

Next, we study the girth of $\mathbb{G}(\mathbb{Z}_n)$.

Theorem 10 *Let n be a positive integer number. Then $\mathrm{gr}(\mathbb{G}(\mathbb{Z}_n)) \in \{3, \infty\}$. Moreover, $\mathbb{G}(\mathbb{Z}_n)$ is a tree if and only if $n \in \{p_1^2, p_1^3, p_1p_2, p_1^2p_2\}$.*

Proof If $s \geqslant 3$, then $p_1\mathbb{Z}_n - p_2\mathbb{Z}_n - p_3\mathbb{Z}_n - p_1\mathbb{Z}_n$ is a 3-cycle in $\mathbb{G}(\mathbb{Z}_n)$. Therefore $\mathrm{gr}(\mathbb{G}(\mathbb{Z}_n)) = 3$. Now, consider two following cases:

Case 1 $s = 1$. If $\alpha_1 \geqslant 4$, then it is easy to see that $p_1\mathbb{Z}_n - p_1^2\mathbb{Z}_n - p_1^3\mathbb{Z}_n - p_1\mathbb{Z}_n$ is a triangle in $\mathbb{G}(\mathbb{Z}_n)$ and so $\mathrm{gr}(\mathbb{G}(\mathbb{Z}_n)) = 3$. Also, it is clear that if $n = p_1^2$ or $n = p_1^3$, then $\mathrm{gr}(\mathbb{G}(\mathbb{Z}_n)) = \infty$.

Case 2 $s = 2$. If $\alpha_1 \geqslant 3$, then $p_1\mathbb{Z}_n - p_2\mathbb{Z}_n - p_1^2\mathbb{Z}_n - p_1\mathbb{Z}_n$ is a 3-cycle in $\mathbb{G}(\mathbb{Z}_n)$. This yields that $\mathrm{gr}(\mathbb{G}(\mathbb{Z}_n)) = 3$. Now, suppose that $\alpha_1, \alpha_2 \in \{1, 2\}$. Whit out lose of generality we may assume the following three subcases:

Subcase 1 $n = p_1p_2$. Then $\mathbb{G}(\mathbb{Z}_n) \cong K_2$ and $\mathrm{gr}(\mathbb{G}(\mathbb{Z}_n)) = \infty$.

Subcase 2 $n = p_1^2p_2$. Then $\mathbb{G}(\mathbb{Z}_n) \cong P_4$ and so $\mathrm{gr}(\mathbb{G}(\mathbb{Z}_n)) = \infty$. Note that, $p_1p_2\mathbb{Z}_n - p_1\mathbb{Z}_n - p_2\mathbb{Z}_n - p_1^2\mathbb{Z}_n$.

Subcase 3 $n = p_1^2p_2^2$. Then $p_1\mathbb{Z}_n - p_1p_2\mathbb{Z}_n - p_2\mathbb{Z}_n - p_1\mathbb{Z}_n$ is a triangle in $\mathbb{G}(\mathbb{Z}_n)$. Hence $\mathrm{gr}(\mathbb{G}(\mathbb{Z}_n)) = 3$. □

Now, we compute some numerical invariants of $\mathbb{G}(\mathbb{Z}_n)$, namely domination number and clique number.

Theorem 11 *If n is a positive integer number, then $\gamma(\mathbb{G}(\mathbb{Z}_n)) = s$.*

Proof We note that $\mathrm{Max}(\mathbb{Z}_n) = \{p_1\mathbb{Z}_n, \ldots, p_s\mathbb{Z}_n\}$. Hence by Theorem 1, we find that $\gamma(\mathbb{G}(\mathbb{Z}_n)) \leq s$. Next, we prove that $\gamma(\mathbb{G}(\mathbb{Z}_n)) \geq s$. Let D be a smallest dominating set for $\mathbb{G}(\mathbb{Z}_n)$ and let $I_j = p_j^{\alpha_j - 1}\prod_{i \neq j} p_i^{\alpha_i}\mathbb{Z}_n$, for $j = 1, \ldots, s$. We have $N(I_j) = \{p_j\mathbb{Z}_n\}$. This implies that $\{I_j, p_j\mathbb{Z}_n\} \cap D \neq \varnothing$, for every j, $1 \leq j \leq s$. Therefore $|D| \geq s$ and so $\gamma(\mathbb{G}(\mathbb{Z}_n)) = s$. □

Theorem 12 *If $n = p^\alpha$, then* $\omega(\mathbb{G}(\mathbb{Z}_n)) = \begin{cases} \dfrac{\alpha}{2}, & \text{if } \alpha \text{ is even;} \\ \dfrac{\alpha+1}{2}, & \text{otherwise.} \end{cases}$

Proof First suppose that α is even. By Remark 2, $p^r\mathbb{Z}_n$ and $p^{r'}\mathbb{Z}_n$ are adjacent, where $1 \leq r, r' \leq \alpha/2$. This yields that $A = \{p^r\mathbb{Z}_n : r = 1, \ldots, \alpha/2\}$ is a clique in $\mathbb{G}(\mathbb{Z}_n)$. We claim that A is a maximum clique in $\mathbb{G}(\mathbb{Z}_n)$. By contradiction, suppose that $\{p^{r_1}\mathbb{Z}_n, \ldots, p^{r_{\alpha/2+1}}\mathbb{Z}_n\}$ is a clique in $\mathbb{G}(\mathbb{Z}_n)$. Clearly, $1 \leq r_i \leq \alpha$, for $i = 1, \ldots, \alpha/2 + 1$. With no loss of generality, we may assume that $r_1 \geq \alpha/2 + 1$. By Remark 2, we conclude that $\deg(p^{r_1}\mathbb{Z}_n) \leq \alpha/2$, a contradiction. Therefore $\{p^r\mathbb{Z}_n : r = 1, \ldots, \alpha/2\}$ is a maximum clique in $\mathbb{G}(\mathbb{Z}_n)$ and $\omega(\mathbb{G}(\mathbb{Z}_n)) = \alpha/2$. Similarly, $\{p^r\mathbb{Z}_n : r = 1, \ldots, (\alpha+1)/2\}$ is a maximum clique in $\mathbb{G}(\mathbb{Z}_n)$, where α is odd. This completes the proof. □

Theorem 13 *$\mathbb{G}(\mathbb{Z}_n)$ is a complete graph if and only if $n \in \{p_1^2, p_1^3, p_1p_2\}$.*

Proof One side is obvious. For the other side assume that $\mathbb{G}(\mathbb{Z}_n)$ is a complete graph. By Theorem 9, we find that $s = 1, 2$. For the case $s = 1$, we have $\mathrm{Max}(\mathbb{Z}_n) = \{p_1\mathbb{Z}_n\}$. Hence by Theorem 9, $\alpha_1 = 2, 3$. Also, if $s = 2$, then Theorem 9 implies that $\alpha_1 = \alpha_2 = 1$. Therefore $n = p_1p_2$. □

If $n = p_1^3p_2^2$ and $v_1 = p_1p_2\mathbb{Z}_n, v_2 = p_1p_2^2\mathbb{Z}_n, v_3 = p_1\mathbb{Z}_n, v_4 = p_1^2p_2\mathbb{Z}_n, v_5 = p_1^2\mathbb{Z}_n, v_6 = p_2^2\mathbb{Z}_n, v_7 = p_1^2p_2^2\mathbb{Z}_n, v_8 = p_1^2p_2^2\mathbb{Z}_n, v_9 = p_2\mathbb{Z}_n, v_{10} = p_1^3p_2\mathbb{Z}_n$, then we have the following graph (Fig. 1):

Also, if $n = p_1^2p_2p_3$ and $v_1 = p_1\mathbb{Z}_n, v_2 = p_1p_2\mathbb{Z}_n, v_3 = p_1p_3\mathbb{Z}_n, v_4 = p_2\mathbb{Z}_n, v_5 = p_1^2\mathbb{Z}_n, v_6 = p_2p_3\mathbb{Z}_n, v_7 = p_3\mathbb{Z}_n, v_8 = p_1p_2p_3\mathbb{Z}_n, v_9 = p_1^2p_2\mathbb{Z}_n, v_{10} = p_1^2p_3\mathbb{Z}_n$, then we have the following graph (Fig. 2):

Now, we investigate the planarity of $\mathbb{G}(\mathbb{Z}_n)$. We will frequently need a celebrated theorem due to Kuratowski.

Proposition 1 [7, Theorem 10.30] *A graph is planar if and only if it contains no subdivision of either K_5 or $K_{3,3}$.*

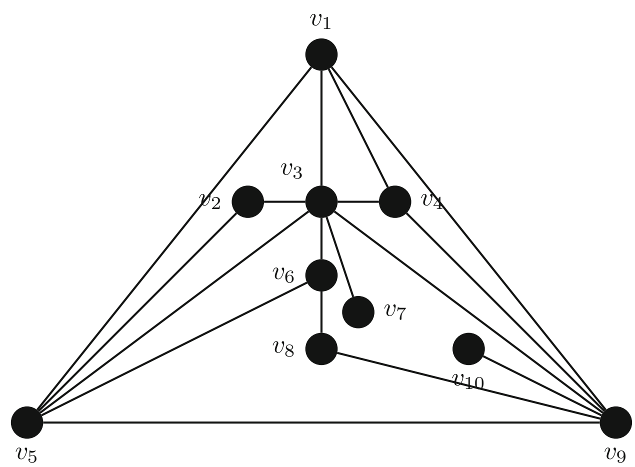

Fig. 1 $\mathbb{G}(\mathbb{Z}_{p_1^3p_2^2})$

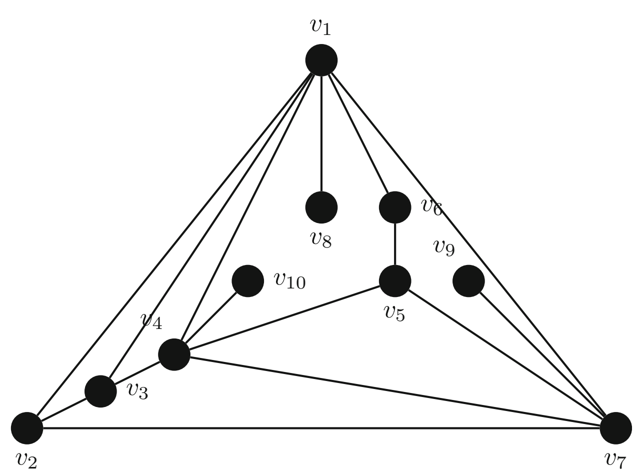

Fig. 2 $\mathbb{G}(\mathbb{Z}_{p_1^2p_2p_3})$

Theorem 14 *$\mathbb{G}(\mathbb{Z}_n)$ is a planar graph if and only if $n \in \{p_1, p_1^2, \ldots, p_1^8, p_1p_2, p_1^2p_2, p_1^3p_2, p_1^3p_2^2, p_1^4p_2, p_1^2p_2^2, p_1p_2p_3, p_1^2p_2p_3\}$.*

Proof One side is obvious. For the other side assume that $\mathbb{G}(\mathbb{Z}_n)$ is a planar graph. If $s \geq 5$, then $\{p_1\mathbb{Z}_n, \ldots, p_5\mathbb{Z}_n\}$ is a clique, a contradiction. Therefore $s \leq 4$. Consider two following cases:

Case 1 $s = 1$. If $\alpha_1 \geq 9$, then $\{p_1\mathbb{Z}_n, p_1^2\mathbb{Z}_n, \ldots, p_1^5\mathbb{Z}_n\}$ is a clique, a contradiction. Hence $\alpha_1 \leq 8$. It is clear that if $\alpha_1 \leq 5$, then $|V(\mathbb{G}(\mathbb{Z}_n))| \leq 4$ and so $\mathbb{G}(\mathbb{Z}_n)$ is a planar graph. If $\alpha_1 = 6$, then $|V(\mathbb{G}(\mathbb{Z}_n))| = 5$. On the other hand $p_1^4\mathbb{Z}_n$ and $p_1^5\mathbb{Z}_n$ are two non adjacent vertices. Now, by Theorem 1, we find that $\mathbb{G}(\mathbb{Z}_n)$ is a planar graph. If $\alpha_1 = 7$, then $|V(\mathbb{G}(\mathbb{Z}_n))| = 6$. Also, $N(p_1^6\mathbb{Z}_n) = \{p_1\mathbb{Z}_n\}$. Therefore by Theorem 1, $\mathbb{G}(\mathbb{Z}_n)$ is a planar graph. If $\alpha_1 = 8$, then $|V(\mathbb{G}(\mathbb{Z}_n))| = 7$. It is easy to see that $N(p_1^7\mathbb{Z}_n) = \{p_1\mathbb{Z}_n\}$, $N(p_1^6\mathbb{Z}_n) = \{p_1\mathbb{Z}_n, p_1^2\mathbb{Z}_n\}$ and $N(p_1^5\mathbb{Z}_n) = \{p_1\mathbb{Z}_n, p_1^2\mathbb{Z}_n, p_1^3\mathbb{Z}_n\}$. Hence $\mathbb{G}(\mathbb{Z}_n)$ contains no subdivision of either K_5 or $K_{3,3}$. Therefore by Theorem 1, $\mathbb{G}(\mathbb{Z}_n)$ is a planar graph.

Case 2 $2 \leq s \leq 4$. If $\alpha_1, \alpha_2 \geq 3$, then vertices of the set $\{p_1\mathbb{Z}_n, p_1^2\mathbb{Z}_n, p_1^3\mathbb{Z}_n\}$ are adjacent to the vertices of the set $\{p_2\mathbb{Z}_n, p_2^2\mathbb{Z}_n, p_2^3\mathbb{Z}_n\}$, and so $K_{3,3}$ is a subgraph of $\mathbb{G}(\mathbb{Z}_n)$, a contradiction. Hence we may assume that $\alpha_2, \ldots, \alpha_s \leq 2$. If $\alpha_1 \geq 5$, then two sets $\{p_1\mathbb{Z}_n, p_1^2\mathbb{Z}_n, p_1^3\mathbb{Z}_n\}$ and $\{p_2\mathbb{Z}_n, p_1p_2\mathbb{Z}_n, p_1^2p_2\mathbb{Z}_n\}$ imply that $\mathbb{G}(\mathbb{Z}_n)$ contains $K_{3,3}$, a contradiction. Therefore $\alpha_1 \leq 4$. There are three following subcases:

Subcase 1 $s = 2$. Since $\alpha_1 \leq 4$ and $\alpha_2 \leq 2$, $n \in \{p_1p_2, p_1^2p_2, p_1^3p_2, p_1^4p_2, p_1p_2^2, p_1^2p_2^2, p_1^3p_2^2, p_1^4p_2^2\}$. With no loss of generality we may assume that $n \in \{p_1p_2, p_1^2p_2, p_1^3p_2, p_1^4p_2, p_1^2p_2^2, p_1^3p_2^2, p_1^4p_2^2\}$. It is clear that if $n \in \{p_1p_2, p_1^2p_2\}$, then $|V(\mathbb{G}(\mathbb{Z}_n))| \leq 4$ and so $\mathbb{G}(\mathbb{Z}_n)$ is a planar graph. If $n = p_1^3p_2$, then $|V(\mathbb{G}(\mathbb{Z}_n))| = 6$. Clearly, $N(p_1^3\mathbb{Z}_n) = \{p_2\mathbb{Z}_n\}$ and $N(p_1^2p_2\mathbb{Z}_n) = \{p_1\mathbb{Z}_n\}$. This implies that $\mathbb{G}(\mathbb{Z}_n)$ contains no subdivision of either K_5 or $K_{3,3}$. Therefore by Theorem 1, $\mathbb{G}(\mathbb{Z}_n)$ is a planar graph. If $n = p_1^4p_2$, then $|V(\mathbb{G}(\mathbb{Z}_n))| = 8$. Clearly, $N(p_1^4\mathbb{Z}_n) = \{p_2\mathbb{Z}_n\}$, $N(p_1^3p_2\mathbb{Z}_n) = \{p_1\mathbb{Z}_n\}$ and $N(p_1^2p_2\mathbb{Z}_n) = \{p_1\mathbb{Z}_n, p_1^2\mathbb{Z}_n\}$. Hence $\mathbb{G}(\mathbb{Z}_n)$ contains no subdivision of either K_5 or $K_{3,3}$. Therefore by Theorem 1, $\mathbb{G}(\mathbb{Z}_n)$ is a planar graph. If $n = p_1^2p_2^2$, then $|V(\mathbb{G}(\mathbb{Z}_n))| = 7$. Clearly, $N(p_1^2p_2\mathbb{Z}_n) = \{p_2\mathbb{Z}_n\}$, $N(p_1p_2^2\mathbb{Z}_n) = \{p_1\mathbb{Z}_n\}$ and $N(p_1p_2\mathbb{Z}_n) = \{p_1\mathbb{Z}_n, p_2\mathbb{Z}_n\}$. Hence $\mathbb{G}(\mathbb{Z}_n)$ contains no subdivision of either K_5 or $K_{3,3}$. Therefore by Theorem 1, $\mathbb{G}(\mathbb{Z}_n)$ is a planar graph. If $n = n = p_1^3p_2^2$, then by Fig.1, we find that $\mathbb{G}(\mathbb{Z}_n)$ is planar. If $n = p_1^4p_2^2$, then two sets $\{p_1\mathbb{Z}_n, p_1^2\mathbb{Z}_n, p_1^3\mathbb{Z}_n\}$ and $\{p_2\mathbb{Z}_n, p_2^2\mathbb{Z}_n, p_1p_2\mathbb{Z}_n\}$ imply that $K_{3,3}$ is a subgraph of $\mathbb{G}(\mathbb{Z}_n)$, a contradiction.

Subcase 2 $s = 3$. If $\alpha_1 \geq 3$, then two sets $\{p_1\mathbb{Z}_n, p_1^2\mathbb{Z}_n, p_1^3\mathbb{Z}_n\}$ and $\{p_2\mathbb{Z}_n, p_3\mathbb{Z}_n, p_2p_3\mathbb{Z}_n\}$ imply that $K_{3,3}$ is a subgraph of $\mathbb{G}(\mathbb{Z}_n)$, a contradiction. Hence $\alpha_1 \leq 2$ and $n \in \{p_1p_2p_3, p_1^2p_2p_3, p_1p_2^2p_3, p_1p_2p_3^2, p_1^2p_2^2p_3, p_1^2p_2p_3^2, p_1p_2^2p_3^2, p_1^2p_2^2p_3^2\}$. With no loss of generality we may assume that $n \in \{p_1p_2p_3, p_1^2p_2p_3, p_1^2p_2^2p_3, p_1^2p_2^2p_3^2\}$. If $n = p_1p_2p_3$, then $\deg(p_1p_2\mathbb{Z}_n) = \deg(p_1p_3\mathbb{Z}_n) = \deg(p_2p_3\mathbb{Z}_n) = 1$ and $\deg(p_1\mathbb{Z}_n) = \deg(p_2\mathbb{Z}_n) = \deg(p_3\mathbb{Z}_n) = 2$. This yields that $\mathbb{G}(\mathbb{Z}_n)$ is a planar graph. If $n = p_1^2p_2p_3$, then by Fig.2, we conclude that $\mathbb{G}(\mathbb{Z}_n)$ is planar. If $n \in \{p_1^2p_2^2p_3, p_1^2p_2^2p_3^2\}$, then two sets $\{p_1\mathbb{Z}_n, p_2\mathbb{Z}_n, p_1p_2\mathbb{Z}_n\}$ and $\{p_3\mathbb{Z}_n, p_2p_3\mathbb{Z}_n, p_1p_2p_3\mathbb{Z}_n\}$ imply that $K_{3,3}$ is a subgraph of $\mathbb{G}(\mathbb{Z}_n)$, a contradiction.

Subcase 3 $s = 4$. If $\alpha_2, \alpha_3 \geq 2$, then $\{p_1\mathbb{Z}_n, p_1p_2\mathbb{Z}_n, p_1p_3\mathbb{Z}_n, p_1p_4\mathbb{Z}_n, p_1p_5\mathbb{Z}_n\}$ is a clique, a contradiction. Similarly, we conclude that at most one of the element of the set $\{\alpha_2, \alpha_3, \alpha_4\}$ can be more than 2. Therefore with no loss of generality we may assume that $\alpha_3 = \alpha_4 = 1$. If $\alpha_1 \geq 3$ and $\alpha_2 = 1$, then $\{p_1\mathbb{Z}_n, p_2\mathbb{Z}_n, p_3\mathbb{Z}_n, p_4\mathbb{Z}_n, p_1^2\mathbb{Z}_n\}$ is a clique, a contradiction. Otherwise, two sets $\{p_1\mathbb{Z}_n, p_2\mathbb{Z}_n, p_1p_2\mathbb{Z}_n\}$ and $\{p_3\mathbb{Z}_n, p_4\mathbb{Z}_n, p_3p_4\mathbb{Z}_n\}$ imply that $K_{3,3}$ is a subgraph of $\mathbb{G}(\mathbb{Z}_n)$, a contradiction. □

References

1. Aalipour, G., Akbari, S., Nikandish, R., Nikmehr, M.J., Shaveisi, F.: The classification of the annihilating-ideal graph of a commutative ring. Algebra Colloq. **21**(02), 249–256 (2014)
2. Akbari, S., Heydari, F.: The regular graph of a noncommutative ring. Bull. Aust. Math. Soc. **89**, 132–140 (2014)
3. Akbari, S., Heydari, F., Maghasedi, M.: The intersection graph of a group. J. Algebra Appl. **14**, 1550065 (2015)
4. Akbari, S., Khojasteh, S., Yousefzadehfard, A.: The proof of a conjecture in Jacobson graph of a commutative ring. J. Algebra Appl. **14**(10), 1550107 (2015)
5. Akbari, S., Khojasteh, S.: Commutative rings whose cozero-divisor graphs are unicyclic or of bounded degree. Commun. Algebra **42**, 1594–1605 (2014)
6. Atiyah, M.F., Macdonald, I.G.: Introduction to Commutative Algebra. Addison-Wesley, Reading (1969)
7. Bondy, J.A., Murty, U.S.R.: Graph Theory, Graduate Texts in Mathematics, vol. 244. Springer, New York (2008)
8. Jafari Rad, N., Jafari, S.H.: A note on the intersection graphs of subspaces of a vector space. Ars Comb. **125**, 401–407 (2016)
9. Jafari Rad, N., Jafari, S.H., Mojdeh, D.A.: On domination in zero-divisor graphs. Can. Math. Bull. **56**, 407–411 (2013)