Math Sci (2017) 11:173–179
DOI 10.1007/s40096-017-0210-8

ORIGINAL RESEARCH

Module character inner amenability of Banach algebras

H. Sadeghi[1] · M. Lashkarizadeh Bami[2]

Received: 19 March 2016 / Accepted: 26 January 2017 / Published online: 17 February 2017

Abstract In the present paper, we introduce the notion of module (ϕ, φ)-inner amenability and module character inner amenability for a Banach algebra A which is a Banach module over another Banach algebra $\mathfrak{A}$ with compatible actions. We characterize module (ϕ, φ)-inner amenability and prove some hereditary properties.

Keywords Module (ϕ, φ)-inner amenability · Module character inner amenability · Banach $\mathfrak{A}$-bimodule

Mathematics Subject Classification 46H25

Introduction and preliminaries

Lau [10] introduced a wide class of Banach algebras, called F-algebras, and studied the notion of left amenability for these algebras. In [12], Nasr-Isfahani introduced the concept of inner amenability for Lau algebras. A Lau algebra A was said to be inner amenable if there exists a topological inner invariant mean on the W^*-algebra A^*, that is, a positive linear functional m of norm 1 on A^*, such that $m(f.a) = m(a.f)$ for all $f \in A^*$ and all $a \in P_1(A) = \{a \in A : \|a\| = 1\}$ (or equivalently, for all $a \in A$). Commutative Lau algebras, such as the Fourier algebra $A(G)$ of a locally compact group G, are examples of inner amenable algebras. In addition, the group algebra $L^1(G)$ of any locally compact group G is inner amenable.

Recently, Jabbari et al. [8] have introduced the notion of φ-inner amenability for a Banach algebra A, where $\varphi \in \Delta(A)$, the character space of A. A Banach algebra A was said to be φ-inner amenable if there exists a $m \in A^{**}$ satisfying $m(\varphi) = 1$ and $m(f.a) = m(a.f)(a \in A, f \in A^*)$. A is said to be character inner amenable if and only if A is φ-inner amenable for every $\varphi \in \Delta(A)$.

In [6], Ebrahimi Vishki and Khoddami have investigated the character inner amenability for certain products of Banach algebras consist of projective tensor product $A\widehat{\otimes}B$, Lau product $A \times_\theta B$, where $\theta \in \Delta(B)$ and the module extension $A \oplus X$. For instance, they showed that the projective tensor product $A\widehat{\otimes}B$ is character inner amenable if and only if both A and B are character inner amenable.

Let $\mathfrak{A}$ and A be Banach algebras, such that A be a Banach $\mathfrak{A}$-bimodule with compatible actions

$$\alpha.(ab) = (\alpha.a)b, (ab).\alpha = a(b.\alpha), \alpha.(\beta.a) = (\alpha\beta).a, (a.\beta).\alpha = a.(\beta\alpha),$$

for all $a, b \in A$ and $\alpha \in \mathfrak{A}$.

Let X be a Banach A-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible left actions defined by

$$\alpha.(a.x) = (\alpha.a).x, a.(\alpha.x) = (a.\alpha).x, (\alpha.x).a = \alpha.(x.a)(a \in A, \alpha \in \mathfrak{A}, \mathfrak{x} \in \mathfrak{X}),$$

and similar for the right or two-sided actions. Then, we say that X is a Banach A-$\mathfrak{A}$-module.

✉ H. Sadeghi
h.sadeghi@iaufr.ac.ir

M. Lashkarizadeh Bami
lashkari@sci.ui.ac.ir

1 Department of Mathematics, Fereydan Branch, Islamic Azad University, Isfahan, Iran

2 Department of Mathematics, Faculty of Science, University of Isfahan, P.O. Box 81745-163, Isfahan, Iran

Let $A\widehat{\otimes}A$ be the projective tensor product of A and A which is a Banach A-bimodule and a Banach $\mathfrak{A}$-bimodule by the following actions:

$$\alpha.(a\otimes b)=(\alpha.a)\otimes b, c.(a\otimes b)=(ca)\otimes b(\alpha\in\mathfrak{A}, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}\in\mathfrak{A}),$$

similarly for the right actions. Let $I_{A\widehat{\otimes}A}$ be the closed ideal of $A\widehat{\otimes}A$ generated by elements of the form:

$$\{a.\alpha\otimes b-a\otimes\alpha.b|\alpha\in\mathfrak{A}, \mathfrak{a}, \mathfrak{b}\in\mathfrak{A}\}. \tag{1}$$

Consider the map $\omega_A\in\mathcal{L}(A\widehat{\otimes}A,A)$ defined by $\omega_A(a\otimes b)=ab$ and extended by linearity and continuity. Let J_A be the closed ideal of A generated by

$$\omega(I_{A\widehat{\otimes}A})=\{(a.\alpha)b-a(\alpha.b)\quad|a,b\in A,\alpha\in\mathfrak{A}\}. \tag{2}$$

Then, the module projective tensor product $A\widehat{\otimes}_{\mathfrak{A}}A$, which is $(A\widehat{\otimes}A)/I_{A\widehat{\otimes}A}$ by [16], and the quotient Banach algebra A/J_A are both Banach A-bimodules and Banach $\mathfrak{A}$-bimodules. In addition, A/J_A is A-$\mathfrak{A}$-module with compatible actions when A acts on A/J_A canonically.

Let A and $\mathfrak{A}$ be Banach algebras, such that A is a Banach $\mathfrak{A}$-bimodule with compatible actions. Let $\varphi\in\Delta(\mathfrak{A})\cup\{\}$ and consider the set $\Omega_{A,\varphi}$ of linear continuous maps $\phi:A\to\mathfrak{A}$, such that

$$\phi(ab)=\phi(a)\phi(b),\phi(\alpha.a)=\phi(a.\alpha)=\varphi(\alpha)\phi(a)\quad(a,b\in A,\alpha\in\mathfrak{A}). \tag{3}$$

The concept of module (ϕ,φ)-amenability and module character amenability for Banach algebra A, where $\varphi\in\Delta(\mathfrak{A})$ and $\phi\in\Omega_A$ were introduced by Bodaghi and Amini in [4].

Our aim in this paper is to introduce and study module (ϕ,φ)-inner amenability and module character inner amenability of Banach algebras. We characterize (ϕ,φ)-inner amenability and prove some hereditary properties. Moreover, we investigate that module (ϕ,φ)-inner amenability for certain class of Banach algebras consists of projective tensor product $A\widehat{\otimes}B$, $A\oplus_\infty B$, and $A\oplus_p B$, the l^p-direct sum of A and B, where $1\leq p<\infty$.

Characterization and hereditary properties

We commence this section with the following definition.

Definition 2.1 Let A be a Banach $\mathfrak{A}$-bimodule and let $\varphi\in\Delta(\mathfrak{A})$ and $\phi\in\Omega_A$. Then, A is called module (ϕ,φ)-inner amenable if there exists $m\in A^{**}$, such that $m(\varphi\circ\phi)=1$, $m(f.a)=m(a.f)$ and $m(\alpha.f)=m(f.\alpha)$ for all $a\in A, f\in A^*$ and $\alpha\in\mathfrak{A}$. A Banach $\mathfrak{A}$-bimodule A is called module character inner amenable if it is module (ϕ,φ)-inner amenable for each $\varphi\in\Delta(\mathfrak{A})$ and $\phi\in\Omega_A$.

We note that if $\mathfrak{A}=\mathbb{C}$ and φ is the identity map, then the module (ϕ,φ)-inner amenability and module character inner amenability coincide with ϕ-inner amenability and character inner amenability (see [8] and [6]).

The next theorem characterizes module (ϕ,φ)-inner amenability of Banach algebras that is analogue of Proposition 2.1 of [5] on module (ϕ,φ)-amenable Banach algebras.

Theorem 2.2 *Let A be a Banach $\mathfrak{A}$-bimodule and let $\varphi\in\Delta(\mathfrak{A})$ and $\phi\in\Omega_A$. Then, the following statements are equivalent:*

(i) *A is module (ϕ,φ)-inner amenable;*
(ii) *There exists a bounded net $(a_i)_i$ in A such that $\|aa_i-a_ia\|\longrightarrow0$, $\|\alpha.a_i-a_i.\alpha\|\longrightarrow0(a\in A,\alpha\in\mathfrak{A})$ and $\varphi\circ\phi(a_i)=1$ for all i;*
(iii) *There exists a bounded net $(a_i)_i$ in A such that $\|aa_i-a_ia\|\longrightarrow0$, $\|\alpha.a_i-a_i.\alpha\|\longrightarrow0(a\in A,\alpha\in\mathfrak{A})$ and $\varphi\circ\phi(a_i)\longrightarrow1$.*

Proof (iii) $\Rightarrow$(i) Assume that a net $(a_i)_i$ exists. Let m be a w^*-cluster point of the net $(a_i)_i$ in A^{**}. Then, $\langle m,\varphi\circ\phi\rangle=\lim_i\langle\varphi\circ\phi,a_i\rangle=1$. For every $a\in A$ and $f\in A^*$, we have

$$\begin{aligned}\langle m,f.a\rangle&=\lim_i\langle f.a,a_i\rangle=\lim_i\langle f,aa_i\rangle\\&=\lim_i\langle f,aa_i-a_ia\rangle+\lim_i\langle f,a_ia\rangle\\&=\lim_i\langle a.f,a_i\rangle=\langle m,a.f\rangle,\end{aligned}$$

and similarly, we have $\langle m,f.\alpha\rangle=\langle m,\alpha.f\rangle(\alpha\in\mathfrak{A})$. Therefore, A is module (ϕ,φ)-inner amenable.

(i)$\Rightarrow$ (ii) Suppose that A is module (ϕ,φ)-inner amenable. Then, there exists $m\in A^{**}$ such that $m(\varphi\circ\phi)=1$, $m(f.a)=m(a.f)$, and $m(\alpha.f)=m(f.\alpha)$ for all $a\in A, f\in A^*$ and $\alpha\in\mathfrak{A}$. Choose a net $(u_\beta)_\beta$ in A with $u_\beta\longrightarrow m$ in the w^*-topology of A^{**} and $\|u_\beta\|\leq\|m\|$ for all β. Since $\langle\varphi\circ\phi,u_\beta\rangle\longrightarrow\langle\varphi\circ\phi,m\rangle=1$, passing to a subnet and replacing u_β by $\left(1/\varphi\circ\phi(u_\beta)\right)u_\beta$, we may assume that $\varphi\circ\phi(u_\beta)=1$ and $\|u_\beta\|\leq\|m\|+1$ for all β. Consider the product space A^A endowed with the product of norm topological. Define a linear map $T:A\to A^A$ by $T(b)=\left(ab-ba+\alpha.b-b.\alpha\right)_{a\in A}$, for all $b\in A$ and $\alpha\in\mathfrak{A}$. Let

$$B=\{b\in A:\|b\|\leq\|m\|+1\text{ and }\varphi\circ\phi(\mathrm{b})=1\}\subseteq\mathrm{A}.$$

Clearly, B is convex and so $T(B)$ is a convex subset of A^A. For every $f\in A^*$, we have

$$\begin{aligned}&\langle f, au_\beta - u_\beta a + \alpha.u_\beta - u_\beta.\alpha\rangle\\ &= \langle f, au_\beta\rangle - \langle f, u_\beta a\rangle + \langle f, \alpha.u_\beta\rangle + \langle f, u_\beta.\alpha\rangle\\ &= \langle f.a, u_\beta\rangle - \langle a.f, u_\beta\rangle + \langle f.\alpha, u_\beta\rangle + \langle \alpha.f, u_\beta\rangle\\ &\to \langle m, f.a\rangle - \langle m, a.f\rangle + \langle m, f.\alpha\rangle - \langle m, \alpha.f\rangle\\ &= 0.\end{aligned}$$

This product of weak topologies coincide with topology on A^A (see Theorem 4.3 of [17]). By Mazur's theorem, $0 \in \overline{T(B)}^w = \overline{T(B)}^{\|\cdot\|}$. Therefore, there exists a bounded net $(a_i)_i$ in A, such that $\varphi \circ \phi(a_i) = 1$ and

$$\|aa_i - a_i a\| \longrightarrow 0, \|\alpha.a_i - a_i.\alpha\| \longrightarrow 0 (a \in A, \alpha \in \mathfrak{A}).$$

(ii)⇒ (iii) It is clear. □

Definition 2.3 We say that the Banach algebra $\mathfrak{A}$ acts trivially on A from the left (right) if there is a multiplicative linear functional f on $\mathfrak{A}$, such that $\alpha.a = f(\alpha)a$ (resp. $a.\alpha = f(\alpha)a$) for all $\alpha \in \mathfrak{A}$ and $a \in A$.

For the proof of the following result, we refer to Lemma 3.13 of [1].

Lemma 2.4 *Let $\mathfrak{A}$ acts on A trivially from the left or right and A/J_A has a right bounded approximate identity, then for each $\alpha \in \mathfrak{A}$ and $a \in A$ we have $f(\alpha)a - a.\alpha \in J_A$.*

Let $\varphi \in \Delta(\mathfrak{A})$ and $\phi \in \Omega_A$. Clearly, $\phi\big((a.\alpha)b - a(\alpha.b)\big) = 0(\alpha \in \mathfrak{A}, \mathfrak{a}, \mathfrak{b} \in \mathfrak{A})$, and hence, $\phi = 0$ on J_A and $\tilde{\phi} : A/J_A \longrightarrow \mathfrak{A}$ given by $\tilde{\phi}(a + J_A) = \phi(a)$ is well defined. Then, $\tilde{\phi} \in \Omega_{A/J_A}$.

Proposition 2.5 *Let A be a Banach $\mathfrak{A}$-bimodule, and let $\mathfrak{A}$ acts on A trivially from the left and A/J_A has a bounded approximate identity. Then A/J_A is $\varphi \circ \tilde{\phi}$-inner amenable for every $\varphi \in \Delta(\mathfrak{A})$ and $\phi \in \Omega_A$.*

Proof Let $(e_\alpha + J_A)_\alpha$ be a bounded approximate identity of A/J_A and let $\varphi \in \Delta(\mathfrak{A})$ and $\phi \in \Omega_A$. Then, $\varphi \circ \tilde{\phi}(e_\alpha + J_A) \longrightarrow 1$. Clearly

$$\|(a + J_A)(e_\alpha + J_A) - (e_\alpha + J_A)(a + J_A)\| \longrightarrow 0$$

for all $a + J_A \in A/J_A$. By Proposition 2.2 of [6], A/J_A is $\varphi \circ \tilde{\phi}$-inner amenable. □

Note that in the above proposition, both left and right actions of $\mathfrak{A}$ on A/J_A are trivial, by Lemma 2.4. Therefore, for every $\alpha \in \mathfrak{A}$, we have

$$\begin{aligned}&\|\alpha.(e_\alpha + J_A) - (e_\alpha + J_A).\alpha\|\\ &= \|\alpha.e_\alpha + J_A - e_\alpha.\alpha + J_A\|\\ &= \|f(\alpha)(e_\alpha + J_A) - f(\alpha)(e_\alpha + J_A)\| = 0.\end{aligned}$$

Then, the net $(e_\alpha + J_A)_\alpha$ satisfies condition (iii) of Theorem 2.2, and hence, A/J_A is module $(\tilde{\phi}, \varphi)$-inner amenable.

Remark 2.6 A inverse semigroup is a discrete semigroup S, such that for each $s \in S$, there is a unique element $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an idempotent if $e^2 = e^* = e$. The set of idempotent elements of S is denoted by E_S. Define the relation $\leq$ on E_S by $e \leq d \Leftrightarrow ed = e(e, d \in E_S)$. Then, E_S is a commutative subsemigroup of S, and $l^1(E_S)$ may be regarded as a subalgebra of $l^1(S)$.

Let s be an inverse semigroup with the set of idempotents E_S. We let $l^1(E_S)$ acts on $l^1(S)$ by multiplication from the right and trivially from the left, that is

$$\delta_e.\delta_s = \delta_s \delta_s.\delta_e = \delta_{se} = \delta_s * \delta_e (e \in E_S, s \in S).$$

By these actions, $l_1(S)$ becomes a Banach $l_1(E_S)$-module. In this case

$$J_{l^1(S)} = \{\delta_{set} - \delta_{st} | e \in E_S, s, t \in S\}.$$

We consider an equivalence relation on S as follows $s \approx t \Leftrightarrow \delta_s - \delta_t \in J_{l^1(S)}(s, t \in S)$. For inverse semigroup S, the quotient semigroup $S/\approx$ is discrete group and so $l^1(S/\approx)$ has an identity (see [2, 13]). Indeed, $S/\approx$ is homomorphic to the maximal group homomorphic image G_S of S (see [11, 14]). It is also shown in Theorem 3.3 of [15] that $l^1(S)/J_{l^1(S)} \cong l^1(S/\approx) = l^1(G_S)$ is a commutative $l^1(E_S)$-bimodule with the following actions:

$$\delta_e.\delta_{[s]} = \delta_{[s]}, \delta_{[s]}.\delta_e = \delta_{[se]}(s \in S, e \in E_S),$$

where $[s]$ denotes the equivalence class of s in G_S.

It is shown in [4] that the maps φ and ϕ satisfying (3) exist for $l^1(S)$.

Example 2.7 Let S be an inverse semigroup with the set of idempotents E_S. Consider $l^1(S)$ as a Banach module over $l^1(E_S)$ with the trivial left action and natural right action. Then, by Proposition 2.5, $l^1(G_S)$ is $\varphi \circ \tilde{\phi}$-inner amenable (module $(\tilde{\phi}, \varphi)$-inner amenable) for all $\varphi \in \Delta(l^1(E_S))$ and $\phi \in \Omega_{l^1(S)}$.

Example 2.8 Let A be a commutative Banach algebra and commutative $\mathfrak{A}$-bimodule (i.e., $\alpha.a = a.\alpha(a \in A, \alpha \in \mathfrak{A})$). Let $\varphi \in \Delta(\mathfrak{A}), \phi \in \Omega_{\mathfrak{A}}$ and let $a \in A$ be such that $\varphi \circ \phi(a) = 1$. put $m = \widehat{a}$. Then, $m(\varphi \circ \phi) = \widehat{a}(\varphi \circ \phi) = \varphi \circ \phi(a) = 1$ and clearly, $m(f.a) = m(a.f)$ and $m(\alpha.f) = m(f.\alpha)$ for all $a \in A, \alpha \in \mathfrak{A}$. Therefore, A is module (ϕ, φ)-inner amenable. In particular, if S is a commutative inverse semigroup, then $l^1(S)$ is commutative and commutative $l^1(E_S)$-bimodule. Therefore, $l^1(S)$ is module (ϕ, φ)-inner amenable for all $\varphi \in \Delta(l^1(E_S))$ and $\phi \in \Omega_{l^1(S)}$.

Example 2.9 Let $S = (\mathbb{N}, \wedge)$ be the inverse semigroup of positive integers with the minimum operation. Let $A = l^1(S), \mathfrak{A} = \mathfrak{l}(\mathfrak{E}_{\mathfrak{S}})$ and $\mathfrak{A}$ acts on A by the following actions:

$$\delta_e.\delta_s = \delta_s.\delta_e = \delta_{se} \quad (e \in E_S, s \in S).$$

A is module amenable (see page 42 of [3]). By Theorem 2.1 of [4], A is module character amenable. Let $\varphi \in \Delta(\mathfrak{A})$ and $\phi \in \Omega_A$. Then, there exist $m \in A^{**}$, such that $m(f.a) = \varphi \circ \phi(a)m(f), m(f.\alpha) = \varphi(\alpha)m(f)$ and $m(\varphi \circ \phi) = 1$ for every $f \in A^*, a \in A$ and $\alpha \in \mathfrak{A}$. Let $(a_\alpha)_\alpha$ be a net in A converging to m in the w^*-topology of A^{**}. Since A is commutative and commutative $\mathfrak{A}$-bimodule, for every $a \in A, f \in A^*$

$$\begin{aligned} m(f.a) &= \lim_\alpha \langle f.a, a_\alpha\rangle = \lim_\alpha \langle f, aa_\alpha\rangle \\ &= \lim_\alpha \langle f, a_\alpha a\rangle = \lim_\alpha \langle a.f, a_\alpha\rangle \\ &= m(a.f). \end{aligned}$$

Similarly, for every $\alpha \in \mathfrak{A}$ and $f \in A^*$, we have

$$m(f.\alpha) = m(\alpha.f).$$

Thus, A is module (ϕ, φ)-inner amenable. Therefore, A is module character inner amenable.

The proof of the following proposition is adapted from that of Proposition 2.3 of [4].

Proposition 2.10 *Let A and B be Banach $\mathfrak{A}$-bimodules and let h be an $\mathfrak{A}$-module homomorphism with dense range. If $\phi \in \Omega_B, \varphi \in \Delta(\mathfrak{A})$ and A is module $(\phi \circ h, \varphi)$-inner amenable, then B is module (ϕ, φ)-inner amenable.*

Proof Let $m \in A^{**}$ be such that $m\big(\varphi \circ (\phi \circ h)\big) = 1, m(f.a) = m(a.f)$ and $m(f.\alpha) = m(\alpha.f)$ for all $a \in A, f \in A^*$ and $\alpha \in \mathfrak{A}$. Define $m_B \in B^{**}$ by $m_B(g) = m(g \circ h)(g \in B^*)$. We show that $m_B\big(g.b\big) = m_B\big(b.g\big)(b \in B)$. For see this let $b \in B$ be such that $h(a) = b$. One can easily check that $(g.h(a)) \circ h = (g \circ h).a$ and $(h(a).g) \circ h = a.(g \circ h)$. Hence for every $g \in B^*$

$$\begin{aligned} m_B(g.b) &= m_B\big(g.h(a)\big) = m\big((g.h(a)) \circ h\big) \\ &= m\big((g \circ h).a\big) = m\big(a.(g \circ h)\big) \\ &= m\big((h(a).g) \circ h\big) = m_B\big(h(a).g\big) \\ &= m_B(b.g). \end{aligned}$$

By density of the range of h and the continuity of h, we conclude that $m_B\big(g.b\big) = m_B\big(b.g\big)(b \in B)$. In addition, for every $\alpha \in \mathfrak{A}$, we have

$$\begin{aligned} m_B(g.\alpha) &= m\big((g.\alpha) \circ h\big) = m\big((g \circ h).\alpha\big) \\ &= m\big(\alpha.(g \circ h)\big) = m\big((\alpha.g) \circ h)\big) \\ &= m_B(\alpha.g). \end{aligned}$$

Furthermore, $m_B(\varphi \circ \phi) = m\big((\varphi \circ \phi) \circ h\big) = m\big(\varphi \circ (\phi \circ h)\big) = 1$. Therefore, B is module (ϕ, φ)-inner amenable. □

Corollary 2.11 *Let A and B be Banach $\mathfrak{A}$-bimodules and let h be an $\mathfrak{A}$-module homomorphism with dense range. Then the module character inner amenability of A implies the module character inner amenability of B. In particular, if A is module character inner amenable, then so is A/J_A.*

The proof idea of the following result is taken from the proof of Lemma 2.6 of [4].

Proposition 2.12 Let A be a Banach $\mathfrak{A}$-bimodule and I be a closed ideal and $\mathfrak{A}$-submodule of A, and let $\varphi \in \Delta(\mathfrak{A})$ and $\phi \in \Omega_A$ be such that $\phi|_I \neq 0$. If A is module (ϕ, φ)-inner amenable, then I is module $(\phi|_I, \varphi)$-inner amenable.

Proof Let $m \in A^{**}$ satisfy $m(\varphi \circ \phi) = 1$, $m(f.a) = m(a.f)$ and $m(\alpha.f) = m(f.\alpha)$ for all $a \in A, f \in A^*$ and $\alpha \in \mathfrak{A}$. By a similar argument as in the proof of Lemma 3.1 of [9], one can define a bounded linear functional n on I^* by, $n(g) = m(f)$ for all $g \in I^*$, where f is an arbitrary element of A^* extending g. Now, for every $g \in I^*, a \in I$ and $\alpha \in \mathfrak{A}$, we have

$$n(g.a) = m(f.a) = m(a.f) = n(a.g),$$

and

$$n(g.\alpha) = m(f.\alpha) = m(\alpha.f) = n(\alpha.g).$$

In addition, $n(\varphi \circ \phi|_I) = m(\varphi \circ \phi) = 1$. Therefore, I is module $(\phi|_I, \varphi)$-inner amenable. □

We need to recall the following remark from [4] to give the next result:

Remark 2.13 Let $\mathfrak{A}$ be a Banach algebra and $\mathfrak{A}$ be the unitization of $\mathfrak{A}$ which is $\mathfrak{A} = \mathfrak{A} \oplus \mathbb{C}$ is a unital Banach algebra which contains $\mathfrak{A}$ as a closed ideal. Let A be a Banach $\mathfrak{A}$-module. Then, A is a Banach $\mathfrak{A}$-bimodule with the following module actions:

$$(\alpha, \lambda).a = \alpha.a + \lambda a, a.(\alpha, \lambda) = a.\alpha + \lambda a(\lambda \in \mathbb{C}, \alpha \in \mathfrak{A}, \mathfrak{a} \in \mathfrak{A}).$$

Let $A^\sharp = (A \oplus \mathfrak{A}, \cdot)$, where the multiplication $\cdot$ is defined through

$$(a, u) \cdot (b, v) = (ab + a.v + u.b, uv)(a, b \in A, u, v \in \mathfrak{A}).$$

Then, with the actions defined by

$$u.(a, v) = (u.a, uv), (a, v).u = (a.u, vu)(a \in A, u, v \in \mathfrak{A}),$$

$A^\sharp$ is a unital $\mathfrak{A}$-module Banach algebra with the identity $e_{A^\sharp} = (0, e_{\mathfrak{A}})$, where $e_{\mathfrak{A}} = (0, 1)$. Now, suppose that $\phi \in \Omega_A$ and $\varphi^\#$ is the extension of φ on $\mathfrak{A}$ defined by $\varphi^\#(\alpha, \lambda) = \varphi(\alpha) + \lambda(a \in A, \alpha \in \mathfrak{A}, \lambda \in \mathbb{C}$. If $u = (\alpha, \lambda) \in \mathfrak{A}$, it is easy to see that

$$\phi(a.u) = \phi(u.a) = \varphi^{\#}(u)\phi(a)(a \in A). \quad (4)$$

Define $\phi^{\sharp} : A^{\sharp} \longrightarrow \mathfrak{A}$ by

$$\phi^{\sharp}(a,u) = (\phi(a), \varphi^{\#}(u))(a \in A, u \in \mathfrak{A}). \quad (5)$$

Using (4), one can show that $\phi^{\sharp}$ is multiplicative and

$$\phi^{\sharp}(u.(a,v)) = \phi^{\sharp}((a,v).u) = \varphi^{\#}(u)\phi^{\sharp}(a,v)(a \in A, u, v \in \mathfrak{A}).$$

Therefore, $\phi^{\sharp}$ is an extension of ϕ, such that $\phi^{\sharp}(0,u) = \varphi^{\#}(u)$ is the extension $h_0 = \tilde{0}$ of the zero function given by (5).

The proof of the following theorem is inspired by the proof of Proposition 2.7 of [4].

Proposition 2.14 *Let A be a Banach $\mathfrak{A}$-bimodule, and let $\varphi \in \Delta(\mathfrak{A})$ and $\phi \in \Omega_A$. Then, A is module (ϕ, φ)-inner amenable if and only if $A^{\sharp}$ is module $(\phi^{\sharp}, \varphi^{\#})$-inner amenable.*

Proof Let $A^{\sharp}$ be module $(\phi^{\sharp}, \varphi^{\#})$-inner amenable. Since the image of $\phi^{\sharp}|_A$ is included in $\mathfrak{A}$, by 2.12, we conclude that A is module (ϕ, φ)-inner amenable.

Conversely, suppose that A is module (ϕ, φ)-inner amenable. Then, there exists a $m \in A^{**}$, such that $m(\varphi \circ \phi) = 1$, $m(f.a) = m(a.f)$, and $m(\alpha.f) = m(f.\alpha)$ for all $a \in A, f \in A^*$ and $\alpha \in \mathfrak{A}$. By Remark 2.13, we may identify the dual space $(A^{\sharp})^*$ with $A^* \oplus \mathbb{C}h_0$, where $h_0|_A = 0$ and $h_0(e_{A^{\sharp}}) = 1$. Define $n \in (A^{\sharp})^{**}$ by $n(f) = m(f)(f \in A^*)$ and $n(h_0) = 0$. Since A is an ideal and $\mathfrak{A}$-submodule of $A^{\sharp}$, it follows that $h_0.a = 0$ and $h_0.\alpha = 0$ for all $a \in A$ and $\alpha \in \mathfrak{A}$. A simple computation shows that

$$n\big((f + \lambda h_0).(a + \lambda' e_{A^{\sharp}})\big) = n\big((a + \lambda' e_{A^{\sharp}}).(f + \lambda h_0)\big),$$

and

$$n\big((f + \lambda h_0).u\big) = n\big(u.(f + \lambda h_0)\big),$$

for all $f \in A^*, a \in A, u \in \mathfrak{A}$ and $\lambda, \lambda' \in \mathbb{C}$. For $f \in A^*$, consider the map $\bar{f} : A^{\sharp} \longrightarrow \mathbb{C}$ defined by $\bar{f}(a,u) = f(a) + \tilde{\varphi}(u)(a \in A, u \in \mathfrak{A})$. Thus, $\varphi^{\#} \circ \phi^{\sharp} = \overline{\varphi \circ \phi}$, and hence

$$n(\varphi^{\#} \circ \phi^{\sharp}) = n(\overline{\varphi \circ \phi}) = n(\varphi \circ \phi + \lambda h_0) = m(\varphi \circ \phi) = 1.$$

Therefore, $A^{\sharp}$ is module $(\phi^{\sharp}, \varphi^{\#})$-inner amenable.

Module inner amenability of certain Banach algebras

Let $A \widehat{\otimes} B$ be the projective tensor product of two Banach algebras A and B. For every $f \in A^*$ and $g \in B^*$, let $f \otimes g$ denote the element of $(A \widehat{\otimes} B)^*$ satisfying, $f \otimes g(a \otimes b) = f(a)g(b)(a \in A, b \in B)$. In addition, note that $A \widehat{\otimes} B$ is a Banach $\mathfrak{A} \widehat{\otimes} \mathfrak{A}$-bimodule with the following actions:

$$(\alpha \otimes \beta).(a \otimes b) = (\alpha.a) \otimes (\beta.b)(a \in A, b \in B, \alpha, \beta \in \mathfrak{A}),$$

and similarly for right action. For $\varphi_1, \varphi_2 \in \Delta(\mathfrak{A}), \psi \in \Omega_{\mathfrak{A}}(= \Omega_{\mathfrak{A},\varphi})$ and $\phi \in \Omega_A(= \Omega_{A,\varphi_2})$, define $(\phi \otimes \psi) : A \widehat{\otimes} B \to \mathfrak{A} \widehat{\otimes} \mathfrak{A}$ by $(\phi \otimes \psi)(a \otimes b) = \phi(a) \otimes \psi(b)(a \in A, b \in B)$. Clearly, $\phi \otimes \psi \in \Omega_{A \widehat{\otimes} B}(= \Omega_{A \widehat{\otimes} B, \varphi_1 \otimes \varphi_2})$ and $\varphi_1 \otimes \varphi_2 \in \Delta(\mathfrak{A} \widehat{\otimes} \mathfrak{A})$. In addition, if $\overline{\varphi} \in \Delta(\mathfrak{A} \widehat{\otimes} \mathfrak{A})$, then $\overline{\varphi} = \varphi_1 \otimes \varphi_2$, where $\varphi_1, \varphi_2 \in \Delta(\mathfrak{A})$ (see [4]).

The technique of proof of the following theorem (one side) is similar to that of Theorem 2.8 of [4].

Theorem 3.1 *Let A and B be Banach $\mathfrak{A}$-bimodules, and let $\varphi_1, \varphi_2 \in \Delta(\mathfrak{A})$ and $\phi \in \Omega_A, \psi \in \Omega_B$. Then $A \widehat{\otimes} B$ is module $(\phi \otimes \psi, \varphi_1 \otimes \varphi_2)$-inner amenable (as $\mathfrak{A} \widehat{\otimes} \mathfrak{A}$ module) if and only if A is module (ϕ, φ_1)-inner amenable and B is module (ψ, φ_2)-inner amenable.*

Proof Suppose that $A \widehat{\otimes} B$ is module $(\phi \otimes \psi, \varphi_1 \otimes \varphi_2)$-inner amenable. Then, there exists $m \in (A \widehat{\otimes} B)^{**}$, such that $m\big((\varphi_1 \otimes \varphi_2) \circ (\phi \otimes \psi)\big) = 1$ and

$$m\big(f \otimes g.(a \otimes b)\big) = m\big((a \otimes b).f \otimes g\big), m\big(f \otimes g.(\alpha \otimes \beta)\big) = m\big((\alpha \otimes \beta).f \otimes g\big),$$

for all $a \otimes b \in A \widehat{\otimes} B, f \in A^*, g \in B^*$ and $\alpha \otimes \beta \in \mathfrak{A} \widehat{\otimes} \mathfrak{A}$. Define $m_A : A^* \to \mathbb{C}$ by $m_A(f) = m\big(f \otimes (\varphi_2 \circ \psi)\big)(f \in A^*)$. Therefore, $m_A(\varphi_1 \circ \phi) = m\big((\varphi_1 \circ \phi) \otimes (\varphi_2 \circ \psi)\big) = m\big((\varphi_1 \otimes \varphi_2) \circ (\phi \otimes \psi)\big) = 1$. Choose $b_0 \in A$, such that $\varphi_2 \circ \psi(b_0) = 1$. Therefore, for every $a \in A$ and $f \in A^*$, we have

$$\begin{aligned} m_A(f.a) &= m\big((f.a) \otimes (\varphi_2 \circ \psi)\big) \\ &= m\big((f.a) \otimes (\varphi_2 \circ \psi).b_0\big) \\ &= m\big(f \otimes (\varphi_2 \circ \psi).(a \otimes b_0)\big) \\ &= m\big((a \otimes b_0).f \otimes (\varphi_2 \circ \psi)\big) \\ &= m\big((a.f) \otimes b_0.(\varphi_2 \circ \psi)\big) \\ &= m_A(a.f). \end{aligned}$$

Similarly, for every $\alpha \in \mathfrak{A}$, if we take $\beta \in \mathfrak{A}$, such that $\varphi_2(\beta) = 1$, then

$$\begin{aligned} m_A(f.\alpha) &= m\big((f.\alpha) \otimes (\varphi_2 \circ \psi)\big) \\ &= m\big((f.\alpha) \otimes (\varphi_2 \circ \psi).\beta\big) \\ &= m\big(f \otimes (\varphi_2 \circ \psi).(\alpha \otimes \beta)\big) \\ &= m\big((\alpha \otimes \beta).f \otimes (\varphi_2 \circ \psi)\big) \\ &= m\big((\alpha.f) \otimes \beta.(\varphi_2 \circ \psi)\big) \\ &= m\big((\alpha.f) \otimes (\varphi_2 \circ \psi)\big) \\ &= m_A(\alpha.f), \end{aligned}$$

for all $f \in A^*$. Therefore, A is module (φ, ϕ_1)-inner amenable. Similarly, one can prove that B is module (ψ, ϕ_2)-inner amenable.

For the converse, let A is module (ϕ, φ_1)-inner amenable and B is module (ψ, φ_2)-inner amenable. Then, by Theorem 2.2, there exist bounded nets $(a_i)_i$ in A and $(b_j)_j$ in B with bounds M_1 and M_2, respectively, such that

$$\varphi_1 \circ \phi(a_i) = 1, \|aa_i - a_i a\| \longrightarrow 0, \|\alpha.a_i - a_i.\alpha\| \longrightarrow 0 (a \in A, \alpha \in \mathfrak{A}),$$

and

$$\varphi_2 \circ \psi(b_j) = 1, \|bb_j - b_j b\| \longrightarrow 0, \|\alpha.b_j - b_j.\alpha\| \longrightarrow 0 (b \in B, \alpha \in \mathfrak{A}).$$

Consider the bounded net $(a_i \otimes b_j)_{(i,j)}$ in $A \widehat{\otimes} B$. Therefore, $\big((\varphi_1 \otimes \varphi_2) \circ (\phi \otimes \psi)\big)(a_i \otimes b_j) = \varphi_1 \circ \phi(a_i)\varphi_2 \circ \quad \psi(b_j) = 1$. Let $\mathfrak{F} = \sum_{l=}^{\mathfrak{N}} \alpha_l \otimes \beta_l \in \mathfrak{A} \widehat{\otimes} \mathfrak{A}$, then

$$\begin{aligned} &\|\mathfrak{F}.(a_i \otimes b_j) - (a_i \otimes b_j).\mathfrak{F}\| \\ &= \| \sum_{l=1}^{N} \left[(\alpha_l.a_i - a_i.\alpha_l) \otimes \beta_l.b_j + a_i.\alpha_l \otimes (\beta_l.b_j - b_j.\beta_l) \right] \| \\ &\le \sum_{l=1}^{N} M_2 \|\beta_l\| \|\alpha_l.a_i - a_i.\alpha_l\| + \sum_{l=1}^{N} M_1 \|\alpha_l\| \|\beta_l.b_j - b_j.\beta_l\| \longrightarrow 0. \end{aligned}$$

Now, let $\mathfrak{G} \in \mathfrak{A} \widehat{\otimes} \mathfrak{A}$, so there exist sequences $(\alpha_l)_l \subseteq \mathfrak{A}$ and $(\beta_l)_l \subseteq \mathfrak{A}$, such that $\mathfrak{G} = \sum_{l=}^{\infty} \alpha_l \otimes \beta_l$ with $\sum_{l=1}^{\infty} \|\alpha_l\| \|\beta_l\| < \infty$. By the same argument as in the proof of the Theorem 3.1 of [6], for every $\mathfrak{G} \in \mathfrak{A} \widehat{\otimes} \mathfrak{A}$, one can show that $\|\mathfrak{G}.(a_i \otimes b_j) - (a_i \otimes b_j).\mathfrak{G}\| \longrightarrow 0$. Similarly, we may show that $\|G(a_i \otimes b_j) - (a_i \otimes b_j)G\| \longrightarrow 0$ for all $G \in A \widehat{\otimes} B$. Therefore, Proposition 2.2 implies that $A \widehat{\otimes} B$ is module $(\phi \otimes \psi, \varphi_1 \otimes \varphi_2)$-inner amenable.

Let A and B be Banach algebras, it is well known that $A \oplus_\infty B$ and $A \oplus_p B$, the l^p-direct sum of A and B, are Banach algebras with respect to the canonical multiplication defined by

$$(a, b)(c, d) := (ac, bd) \quad (a, c \in A, b, d \in B),$$

and norms $\|(a, b)\| = \max\{\|a\|, \|b\|\}$ and $\|(a, b)\| = (\|a\|^p + \|b\|^p)^{\frac{1}{p}} (a \in A, b \in B)$. Furthermore, if A and B are two Banach $\mathfrak{A}$-bimodules, then $A \oplus_\infty B$ and $A \oplus_p B$ are Banach $\mathfrak{A}$-bimodules under the module actions

$$\alpha.(a, b) = (\alpha.a, \alpha.b), \quad (a, b).\alpha = (a.\alpha, b.\alpha) \quad (a \in A, b \in B, \alpha \in \mathfrak{A}).$$

Before stating the next theorem, we note that if for every $\varphi \in \Delta(\mathfrak{A}), \phi \in \Omega_{\mathfrak{A}}$ and $\psi \in \Omega_B$, we define $(0, \psi) : A \oplus_p B \to \mathfrak{A}$ and $(\phi, 0) : A \oplus_p B \to \mathfrak{A}$ by

$$(0, \psi)(a, b) = \psi(b), (\phi, 0)(a, b) = \phi(a)(a \in A, b \in B),$$

where $1 \le p \le \infty$, then $(0, \psi)$ and $(\phi, 0) \in \Omega_{A \oplus_p B} (= \Omega_{A \oplus_p B, \varphi})$.

Theorem 3.2 *Let A and B be two $\mathfrak{A}$-bimodule Banach algebras, $\varphi \in \Delta(\mathfrak{A}), \phi \in \Omega_A, \psi \in \Omega_B$ and $1 \le p \le \infty$. Then the following statements are valid:*

(i) *$A \oplus_p B$ is module $\big((\phi, 0), \varphi\big)$-inner amenable if and only if A is module (ϕ, φ)-inner amenable.*
(ii) *$A \oplus_p B$ is module $\big((0, \psi), \varphi\big)$-inner amenable if and only if B is module (ψ, φ)-inner amenable.*

Proof (i) Assume that $A \oplus_p B$ is module $\big((\phi, 0), \varphi\big)$-inner amenable. By Theorem 2.2, there exists a net $(a_i, b_i)_i$ in $A \oplus_p B$, such that $\varphi \circ (\phi, 0)(a_i, b_i) \longrightarrow 1$ and

$$\|(a, b).(a_i, b_i) - (a_i, b_i).(a, b)\| \longrightarrow 0, \|\alpha.(a_i, b_i) - (a_i, b_i).\alpha\| \longrightarrow 0,$$

for all $(a, b) \in A \oplus_p B$ and $\alpha \in \mathfrak{A}$. Consider the bounded net $(a_i)_i$ in A. One can easily show that $\|aa_i - a_i a\| \longrightarrow 0$ and $\|\alpha.a_i - a_i.\alpha\| \longrightarrow 0$ for all $a \in A, \alpha \in \mathfrak{A}$. In addition, it is clear that $\varphi \circ \phi(a_i) \longrightarrow 1$. Therefore, Theorem 2.2 implies that A is module (ϕ, φ)-inner amenable.

Conversely, suppose that A is module (ϕ, φ)-inner amenable. Then, there exists a bounded net $(a_i)_i$ in A, such that $\varphi \circ \phi(a_i) \longrightarrow 1$, $\|aa_i - a_i a\| \longrightarrow 0$ and $\|\alpha.a_i - a_i.\alpha\| \longrightarrow 0$ for all $a \in A, \alpha \in \mathfrak{A}$. Clearly, the bounded net $(a_i, 0)_i \subset A \oplus_p B$ satisfies in the condition (iii) of Theorem 2.2. Therefore, $A \oplus_p B$ is module $\big((\phi, 0), \varphi\big)$-inner amenable.

Similarly, we can prove (ii). □

Corollary 3.3 *Let A and B be two $\mathfrak{A}$-bimodule Banach algebras and $1 \le p \le \infty$. Then $A \oplus_p B$ is module $\big((\phi, 0), \varphi\big)$-inner amenable and module $\big((0, \psi), \varphi\big)$-inner amenable for every $\varphi \in \Delta(\mathfrak{A}), \phi \in \Omega_A$ and $\psi \in \Omega_B$ if and only if both A and B are module character inner amenable.*

We note that for two Banach algebras A and B, a direct verification shows that

$$\Delta(A \oplus_p B) = (\Delta(A) \times \{0\}) \cup (\{0\} \times \Delta(B)), 1 \le p \le \infty.$$

Now, if we take $\mathfrak{A} = \mathbb{C}$ and φ is the identity map in the above corollary, then we obtain that $A \oplus_p B$ is character inner amenable if and only if both A and B are character inner amenable. Therefore, the above corollary generalizes Proposition 4.2 of [6].

Let A be a Banach algebra and X be a Banach A-bimodule. The l^1-direct sum of A and X, denoted by $A \oplus_1 X$, with the product defined by

$$(a,x)(a',x') = (aa', a.x' + x.a')(a,a' \in A, x,x' \in X),$$

is a Banach algebra that is called the module extension Banach algebra of A and X.

If A is $\mathfrak{A}$-bimodule and X is a Banach A-$\mathfrak{A}$-module, then $A \oplus_1 X$ is Banach $\mathfrak{A}$-bimodules under the module actions:

$$\alpha.(a,x) = (\alpha.a, \alpha.x), (a,x).\alpha = (a.\alpha, x.\alpha) \quad (a \in A, x \in X, \alpha \in \mathfrak{A}). \tag{6}$$

Let A and B be Banach algebras and let X be a Banach A, B-module; that is, a left A-module and a right B-module satisfying $\|axb\| \leq \|a\|\|x\|\|b\|, (a \in A, b \in B, x \in X)$. The corresponding triangular Banach algebra

$$\tau = \left\{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} : a \in A, x \in X, b \in B \right\},$$

is equipped with the usual 2×2-matrix operations and the norm

$$\| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \| = \|a\| + \|x\| + \|b\|.$$

This Banach algebra were introduced by Forrest and Marcoux in [7]. Note that τ can be identified with the module extension $(A \oplus_1 B) \oplus_1 X$, in which X is considered as a $A \oplus_1 B$-module under the operations:

$$(a,b).x = ax, x.(a,b) = xb(a \in A, b \in B, x \in X).$$

Furthermore, if A and B are two Banach $\mathfrak{A}$-bimodules and X is a Banach $A \oplus_1 B$-$\mathfrak{A}$-module, then τ is Banach $\mathfrak{A}$-bimodules under the module actions defined as (6).

Let $\phi \in \Omega_A$ and define $\tilde{\phi} : A \oplus_1 X \longrightarrow \mathfrak{A}$ by $\tilde{\phi}(a,x) = \phi(a)(a \in A, x \in X)$. Then, $\tilde{\phi} \in \Omega_{A \oplus_1 X}$.

Using Theorem 2.2, we can routinely prove the following proposition and so we omit its proof.

Proposition 3.4 *Let A be $\mathfrak{A}$-bimodule and X be a Banach A-$\mathfrak{A}$-module and let $\varphi \in \Delta(\mathfrak{A})$ and $\phi \in \Omega_A$. Then $A \oplus_1 X$ is module $(\tilde{\phi}, \varphi)$-inner amenable if and only if there exists a bounded net $(a_i, x_i)_i$ in $A \oplus_1 X$* satisfying

(i) $\varphi \circ \phi(a_i) \longrightarrow 1$ *and* $\|aa_i - a_i a\| \longrightarrow 0, \|\alpha.a_i - a_i.\alpha\| \longrightarrow 0$ and $\|\alpha.x_i - x_i.\alpha\| \longrightarrow 0$ for all $a \in A$ and $\alpha \in \mathfrak{A}$,

(ii) $\|x.a_i - a_i.x\| \longrightarrow 0$ *for all* $x \in X$, *and*

(iii) $\|a.x_i - x_i.a\| \longrightarrow 0$ *for all* $a \in A$.

Corollary 3.5 *Let A be $\mathfrak{A}$-bimodule and X be a Banach A-$\mathfrak{A}$-module and let $\varphi \in \Delta(\mathfrak{A})$ and $\phi \in \Omega_A$. If $A \oplus_1 X$ is module $(\tilde{\phi}, \varphi)$-inner amenable, then A is module (ϕ, φ)-inner amenable.*

Corollary 3.6 *Let Aand B be $\mathfrak{A}$-bimodules and X be a Banach $A \oplus_1 B$-$\mathfrak{A}$-module. If τ is module character inner amenable, then so are A and B.*

Proof Suppose that τ is module character inner amenable. By Corollary 3.5, $A \oplus_1 B$ is module character inner amenable. Therefore, corollary 3.3 implies that A and B are module character inner amenable. □

Acknowledgements We are thankful to the referees for their valuable suggestions and comments.

References

1. Amini, M., Bodaghi, A., Babaee, R.: Module derivations into iterated duals of Banach algebras , Proc. Rom. Aca., Series A, 12, 277–284 (2011)
2. Amini, M., Bodaghi, A., Ebrahimi Bagha, D.: Module amenability of the second dual and module topological center of semigroup algebras. Semigroup Forum **80**, 302–312 (2010)
3. Bami, M. L., Valaei, M., Amini, A.: Module amenability and weak module amenability of Banach algebras, U.P.B. Sci. Bull., Series A, 76, 35–44 (2014)
4. Bodaghi, A., Amini, M.: Module character amenability of Banach algebras. Arch. Math. **99**, 353–365 (2012)
5. Bodaghi, A., Ebrahimi, H., Lashkarizadeh Bami, M., Nemati, M.: Module mean for Banach algebras, U.P.B. Sci. Bull., Series A, 78, 21–30 (2016)
6. Ebrahimi Vishki, H.R., Khoddami, A.R.: Character inner amenability of certain Banach algebras. Colloq. Math. **122**, 225–232 (2011)
7. Forrest, B.E., Marcoux, L.W.: Derivations on triangular Banach algebras. Indiana Univ. Math. J. **45**, 441–462 (1996)
8. Jabbari, A., Mehdi Abad, T., Zaman Abadi, M.: On φ-inner amenable Banach algebras. Colloq. Math. **122**, 1–10 (2011)
9. Kaniuth, E., Lau, A., Pym, J.: On φ-amenability of Banach algebras. Math. Proc. Camp. Phil. Soc. **144**, 85–96 (2008)
10. Lau, A.T.: Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups. Fund. Math. **118**, 161–175 (1983)
11. Mun, W.D.: A class of irreducible matrix representation of an arbitrary inverse semigroup. Proc. Glasgow Math. Assoc **5**, 41–48 (1961)
12. Nasr-Isfahani, R.: Inner amenability of Lau algebras. Arch. Math. (Brno) **37**, 45–55 (2001)
13. Pourmahmood Aghababa, H.: (Super) module amenability, module topological centre and semigroup algebras. Semigroup Forum **81**, 344–356 (2010)
14. Pourmahmood Aghababa, H.: H.: A note on two equivalence relations on inverse semigroups. Semigroup Forum **48**, 200–202 (2012)
15. Rezavand, R., Amini, M., Sattari, M.H., Ebrahim Bagha, D.: Module Arens regularity for semigroup algebras. Semigroup Forum **77**, 300–305 (2008)
16. Rieffel, M.A.: Induced Banach representations of Banach algebras and locally compact groups. J. Funct. Anal. **1**, 443–491 (1967)
17. Schaefer, H.H.: Topological Vector Space, Springer-Verlag (1979)