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ORIGINAL RESEARCH

Extension of some theorems to find solution of nonlinear integral equation and homotopy perturbation method to solve it

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Abstract In this paper, the concept of contraction via the measure of non-compactness on a Banach space is investigated by generalizing some results which have been previously discussed in literatures. Furthermore, to validity of the theorems and homotopy perturbation method (HPM), as a technical solution, they are applied on some nonlinear singular integral equations.

Keywords Measure of non-compactness · Integral equation · Homotopy perturbation

Introduction and auxiliary facts

Integral equation is an essential branch of sciences that it has applications in engineering sciences, physical sciences, etc. Measures of non-compactness used for existence of solution fractional integral equations [5], singular Volterra integral equations discussed in [2] and also in [3, 6] Darbo fixed point theorem was created by measures of non-compactness. But we consider solvability of the nonlinear problem with fractional order in the following form:

$$u(s) = g(s) + h(s, u(s)) \int_0^s \frac{(s^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} k(f(s,\xi)) u(\xi)\, d\xi, \quad s \in [0,1], \quad 0 < \alpha \le 1, \quad m > 0, \tag{1.1}$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ and $h(s, u)$ is generated by the superposition operator of H such that $(Hu)(s) = h(s, u(s))$, where $u = u(s)$ defined on [0, 1] in [4]. We prove the existence of some non-decreasing solutions for Eq. (1.1) in $C[0, 1]$ (set of all continuous functions on [0, 1]). In the following for ability and validity of the proposed method, we solve an example of Eq. (1.1) by homotopy perturbation method.

In this section, we suppose $A \neq \emptyset$ and $A \subseteq E$, where $(E, \|\cdot\|)$ is a real Banach space. Also $\mathfrak{M}_E \neq \emptyset$ is a family of bounded subsets of E and $\mathfrak{N}_E$ a subfamily consisting of all relatively compact sets.

Definition 1 [6] A mapping $\eta : \mathfrak{M}_E \to \mathbb{R}^+$ is a measure of non-compactness in E if it satisfies the following conditions:

(1^0) The family $\ker\eta = \{A \in \mathfrak{M}_E : \eta(A) = 0\}$ is nonempty and $\ker\eta \subset \mathfrak{N}_E$,
(2^0) $A \subset B \Rightarrow \eta(A) \le \eta(B)$,
(3^0) $\eta(\bar{A}) = \eta(A)$,
(4^0) $\eta(\mathrm{Conv}A) = \eta(A)$,
(5^0) $\eta(\lambda A + (1-\lambda)B) \le \lambda\eta(A) + (1-\lambda)\eta(B)$ for $\lambda \in [0, 1]$,
(6^0) If $\{A_n\}$ be a sequence of closed sets from m_E such that $A_{n+1} \subset A_n$ for $n \in \mathbb{N}$ and if $\lim_{n\to\infty} \eta(A_n) = 0$, then the set $A_\infty = \bigcap_{n=1}^\infty A_n$ is nonempty.

Extension of Darbo fixed point theorem

For obtaining the generalization of Darbo fixed point theorem (see [1]), we present a new kind of contraction. So throughout this paper we assume that functions of $G, \Theta, \phi : [0, +\infty) \to [0, +\infty)$ satisfy in these conditions:

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(a) $G \in C[0, +\infty)$ and $G(0) = 0 < G(s), \forall s > 0$;
(b) $\phi(s) < \Theta(s), \forall s > 0$ and $\phi(0) = \Theta(0) = 0$;
(c) $\phi(s), \Theta(s) \in C[0, +\infty)$;
(d) Θ is increasing.

Also, let $\mathbb{G} = \{G : G \text{ satisfy (a)}\}$ and $\Psi = \{(\Theta, \phi) : \Theta$ and ϕ satisfy (b), (c) and (d)$\}$.

Now, we illustrate the generalized (Θ, G, ϕ)-contractive mappings via the measure of non-compactness by the following definition and theorem.

Definition 2 Let $v \neq \emptyset$, subset of a Banach space E and $\tau : v \longrightarrow v$ be a mapping. We say that τ is a generalized (Θ, G, ϕ)-contractive mapping if for any $0 < a < b < \infty$ there exist $0 < \rho(a, b) < 1$, $G \in \mathbb{G}$ and $(\Theta, \phi) \in \Psi$ which for all $A \subseteq v$ and η (arbitrary measure of non-compactness), then

$$a \leq G(\eta(A)) \leq b \Longrightarrow \Theta(G(\eta(\tau A))) \leq \rho(a, b)\phi(G(\eta(A))). \quad (2.1)$$

Theorem 1 *Let $v \neq \emptyset$, bounded, closed, convex and subset of a Banach space E and $\tau : v \longrightarrow v$ be a generalized (Θ, G, ϕ)-contractive continuous mapping. Then τ has at least one fixed point in v.*

Proof Let $v_0 = v$, we construct a sequence $\{v_n\}$ where $v_{n+1} = \mathrm{Conv}(\tau v_n)$, for $n \geq 0$. $\tau v_0 = \tau v \subseteq v = v_0, v_1 = \mathrm{Conv}(\tau v_0) \subseteq v = v_0$, therefore by continuing this process, we have

$$v_0 \supseteq v_1 \supseteq \cdots \supseteq v_n \supseteq v_{n+1} \supseteq \cdots$$

If $\exists N \in \mathbb{N}$; $G(\eta(v_N)) = 0$, i.e., $\eta(v_N) = 0$, then v_N is relatively compact. On the other hand, since $\tau(v_N) \subseteq \mathrm{Conv}(\tau v_N) = v_{N+1} \subseteq v_N$ so, τ is compact. Thus from Shauder Theorem (see [1]) we conclude that τ has a fixed point. Otherwise we suppose,

$$G(\eta(v_n)) > 0, \quad \forall n \geq 1. \quad (2.2)$$

If

$$G(\eta(v_{n_0})) < G(\eta(v_{n_0+1})), \quad (2.3)$$

for some $n_0 \in \mathbb{N}$, according to (2.2) and (2.3), we can get

$$0 < a := G(\eta(v_{n_0})) \leq G(\eta(v_{n_0})) < G(\eta(v_{n_0+1})) := b.$$

By considering τ and Definition 2, there exists $0 < \rho(a, b) < 1$ such that

$$\begin{aligned}\Theta(G(\eta(v_{n_0+1}))) = \Theta(G(\eta(\mathrm{conv}(\tau v_{n_0}))) &= \Theta(G(\eta(\tau v_{n_0}))) \\ &\leq \rho(a, b)\phi(G(\eta(v_{n_0}))) \\ &< \rho(a, b)\Theta(G(\eta(v_{n_0}))) \\ &< \rho(a, b)\Theta(G(\eta(v_{n_0+1}))),\end{aligned}$$

which implies that $\rho(a, b) > 1$, and this is a contradiction. So, we can write,

$$G(\eta(v_{n+1})) \leq G(\eta(v_n)),$$

for all $n \in \mathbb{N}$, that is, the sequence $\{G(\eta(v_n))\}$ is non-increasing and nonnegative, we infer that

$$\lim_{n \to \infty} G(\eta(v_n)) = \delta. \quad (2.4)$$

Now, if $\delta > 0$, then

$$0 < a := \delta \leq G(\eta(v_n)) \leq G(\eta(v_0)) =: b, \quad \text{for all } n \geq 0.$$

By considering τ and Definition 2, there exists $0 < \rho(a, b) < 1$ such that

$$\begin{aligned}\Theta(G(\eta(v_{n+1}))) = \Theta(G(\eta(\mathrm{conv}(\tau v_n)))) &= \Theta(G(\eta(\tau v_n))) \\ &\leq \rho(a, b)\phi(G(\eta(v_n))) \\ &< \rho(a, b)\Theta(G(\eta(v_n))),\end{aligned} \quad (2.5)$$

from (2.4) and continuity of the Θ and ϕ in (2.5), we get

$$\Theta(\delta) = \lim_{n \to \infty} \Theta(G(\eta(v_{n+1}))) = \lim_{n \to \infty} \phi(G(\eta(v_n))) = \phi(\delta),$$

from (b) it is concluded that $\delta = 0$ and this is a contradiction. So in the above process $\delta = 0$ and

$$\lim_{n \to \infty} G(\eta(v_n)) = 0.$$

It follows that

$$\lim_{n \to \infty} \eta(v_n) = 0.$$

From $v_n \supseteq v_{n+1}$ and $\tau v_n \subseteq v_n$ for $n \in \mathbb{N}$, as a result of (6^0), we can write

$$v_\infty = \bigcap_{n=1}^{\infty} v_n \neq \emptyset,$$

is a convex closed set, invariant under τ and belongs to $Ker\eta$. The proof is completed by Shauder Theorem (see [1]). □

We consider in the following a result of Theorem 1.

Theorem 2 *Let $v \neq \emptyset$, bounded, closed, convex and subset of a Banach space E and $\tau : v \longrightarrow v$ be continuous function and η be a measure of non-compactness, also $\exists \lambda, 0 < \lambda < 1, G \in \mathbb{G}$ and $(\Theta, \phi) \in \Psi$ such that*

$$\forall A \subseteq v, \quad \Theta(G(\eta(\tau A))) \leq \lambda\phi(G(\eta(A))),$$

then τ has at least one fixed point in v.

Corollary 1 *Let v, τ and η be as mentioned in Theorem 2 and also $\exists \lambda, 0 < \lambda < 1$ such that $\forall A \subseteq v$, $G(\eta(\tau A)) \leq \lambda G(\eta(A))$, then τ has at least one fixed point in v.*

Proof Put in $\Theta(s) = s$ and $\phi(s) = \lambda s$ for each $s \in [0, +\infty)$ and apply Theorem 2. □

Remark 1 Taking $G = I$ in Corollary 1, we obtain the Darbo fixed point theorem.

Corollary 2 *Let v, τ and η be as mentioned in Theorem 2 and also $\exists \lambda, 0 < \lambda < 1$ and $G \in \mathbb{G}$ such that $\forall A \subseteq v$,*

$$\int_0^{G(\eta(\tau A))} f(\xi)\mathrm{d}\xi \leq \lambda \int_0^{G(\eta(A))} f(\xi)\mathrm{d}\xi,$$

where $f : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable, summable and nonnegative function also for each $\epsilon > 0$, $\int_0^{\epsilon} f(\xi)\mathrm{d}\xi > 0$. Then τ has at least one fixed point in v.

Proof Let $\Theta(s) = \int_0^s f(\xi)\mathrm{d}\xi$ and $\phi(s) = \lambda\Theta(s)$ for each $s \in [0, +\infty)$ and apply Theorem 2. □

Corollary 3 *Let v, τ and η be as mentioned in Theorem 2 and we suppose that for any $0 < a < b < \infty$, there exists $0 < \rho(a, b) < 1$ and $(\Theta, \phi) \in \Psi$ such that for all $A \subseteq v$,*

$$a \leq \eta(A) + \phi(\eta(A)) \leq b \Longrightarrow \Theta(\eta(\tau A) + \phi(\eta(\tau A))) \leq \rho(a, b)\phi[\eta(A) + \phi(\eta(A))],$$

where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous function with $\phi(0) = 0$ and $\phi(s) > 0$ for all $s > 0$. Then τ has at least one fixed point in v.

Proof Let $G(s) = s + \phi(s)$ for each $s \in [0, +\infty)$, and apply Definition 2 and Theorem 1. □

Remark 2 Theorem 3.1 of [5] is special case of Corollary 3.

An immediate consequence of Corollary 3 is the following form.

Corollary 4 *Let v, τ and η be as mentioned in Theorem 2 and also $\exists \lambda, 0 < \lambda < 1$ such that for any nonempty $A \subseteq v$,*

$$\eta(\tau A) + \phi(\eta(\tau A)) \leq \lambda[\eta(A) + \phi(\eta(A))],$$

where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous function with $\phi(0) = 0$ and $\phi(s) > 0$ for all $s > 0$. Then τ has at least one fixed point in v.

Remark 3

(i) Theorem 3.2 of [5] is a special case of Corollary 4.
(ii) Darbo fixed point theorem is concluded from Corollary 4 by taking $\phi \equiv 0$.

Corollary 5 *Let v, τ, G, ϕ, Θ and η be as mentioned in Theorem 2 and also for all $A \subseteq v$,*

$$G(\eta(\tau A)) < \alpha(G(\eta(A)))G(\eta(A)),$$

where $\alpha : \mathbb{R}^+ \longrightarrow [0, 1)$ and $\alpha(s_n) \to 1 \Longrightarrow s_n \to 0$, then τ has at least one fixed point in v.

Proof Let $\Theta(s) = s$ and $\phi(s) = \alpha(s)s$ for each $s \in [0, +\infty)$ and apply Theorem 2. □

Application

We apply Theorem 1 and the above discussion for existence of solution nonlinear integral equations. Consider $C[0, 1]$ as a Banach space with the following norm:

$$||u|| = \max\{|u(s)| : s \geq 0\},$$

and suppose that $A \neq \emptyset$ be a bounded subset of $C[0, 1]$. For $u \in A$ and $\epsilon \geq 0$, we put in,

$$\Omega(u, \epsilon) := \sup\{|u(s) - u(\xi)| : s, \xi \in [0, 1], |s - \xi| \leq \epsilon\},$$
$$\Omega(A, \epsilon) := \sup\{\Omega(u, \epsilon) : u \in A\}, \Omega_0(A) := \lim_{\epsilon \to 0} \Omega(A, \epsilon),$$
$$J(u) := \sup\{|u(\xi) - u(s)| - [u(\xi) - u(s)] : s, \xi \in [0, 1], s \leq \xi\},$$
$$J(A) := \sup\{J(u) : u \in A\}.$$

Thus it is easy that, all of the functions belonging to A are non-decreasing on [0, 1] if and only if $J(A) = 0$. In the following we define η on $\mathfrak{M}_C[0, 1]$ by

$$\eta(A) := \Omega_0(A) + J(A).$$

According to [7], it is straight forward to show that the function of η is a measure of non-compactness on $C[0, 1]$.

Now, we investigate Eq. (1.1) by conditions as follows:

(b_1) $g : [0, 1] \to \mathbb{R}^+$ is a continuous, non-decreasing and nonnegative function on [0, 1];
(b_2) $h : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous function in s and u such that $h([0, 1] \times \mathbb{R}^+) \subseteq \mathbb{R}^+$ and there exists a continuous and non-decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ and for each $s > 0$, $\phi(s) < s$ such that

$$|h(s, u) - h(s, z)| \leq \phi(|u - z|), \quad \forall s \in [0, 1], \quad \forall u, \quad z \in \mathbb{R}, \tag{3.1}$$

also ϕ is superadditive ($\phi(s) + \phi(\xi) \leq \phi(s + \xi)$ for all $s, \xi \in \mathbb{R}^+$);
(b_3) In Eq. (1.1) the operator H satisfies any nonnegative function as u in the condition of $J(Hu) \leq \phi(J(u))$, where ϕ is introduced in (b_2);
(b_4) $f : [0, 1] \times [0, 1] \to \mathbb{R}$ is continuous and also it is non-decreasing in terms of variables s and ξ, separately;
(b_5) $k : \mathrm{Imf} \to \mathbb{R}^+$ is a continuous and non-decreasing function on the compact set Imf;
(b_6) With assumptions $M_1 = \max\{|g(s)| : s \in [0, 1]\}$ and $M_2 = \max\{|h(s, 0)| : s \in [0, 1]\}$, inequality

$$M_1\Gamma(\alpha + 1) + (\phi(r) + M_2)||k||r \leq \Gamma(\alpha + 1)r, \tag{3.2}$$

has a positive solution as r_0, where $\lambda = \dfrac{||k||r_0}{\Gamma(\alpha + 1)} < 1$.

Theorem 3 *Under conditions* (b_1)–(b_6), *Eq.* (1.1) *has at least one non-decreasing solution as* $u = u(\xi) \in C[0,1]$.

Proof We define operators G and τ on $C[0, 1]$ by the formulas

$$(Gu)(s) = \int_0^s \frac{(s^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} k(f(s,\xi))u(\xi)\,\mathrm{d}\xi,$$
$$(\tau u)(s) = g(s) + h(s, u(\xi))(Gu)(s).$$

Firstly, we prove that G is self-map on $C[0, 1]$. Suppose $\epsilon > 0$ is given and let $u \in C[0,1]$ and $s_1, s_2 \in [0,1]$ (without loss of generality) let $s_2 \geq s_1$ and $|s_2 - s_1| \leq \epsilon$. Then,

$$\begin{aligned}
|(Gu)(s_2) - (Gu)(s_1)| &= \Big| \int_0^{s_2} \frac{(s_2^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} k(f(s_2,\xi))u(\xi)\mathrm{d}\xi \\
&- \int_0^{s_1} \frac{(s_1^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} k(f(s_1,\xi))u(\xi)\mathrm{d}\xi \Big| \\
&\leq \Big| \int_0^{s_2} \frac{(s_2^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} k(f(s_2,\xi))u(\xi)\mathrm{d}\xi \\
&- \int_0^{s_2} \frac{(s_2^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} k(f(s_1,\xi))u(\xi)\mathrm{d}\xi \Big| \\
&+ \Big| \int_0^{s_2} \frac{(s_2^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} k(f(s_1,\xi))u(\xi)\mathrm{d}\xi \\
&- \int_0^{s_1} \frac{(s_2^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} k(f(s_1,\xi))u(\xi)\mathrm{d}\xi \Big| \\
&+ | \int_0^{s_1} \frac{(s_2^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} k(f(s_1,\xi))u(\xi)\mathrm{d}\xi \\
&- \int_0^{s_1} \frac{(s_1^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} k(f(s_1,\xi))u(\xi)\mathrm{d}\xi \Big| \\
&\leq \int_0^{s_2} \frac{(s_2^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} |k(f(s_2,\xi)) - k(f(s_1,\xi))||u(\xi)|\mathrm{d}\xi \\
&+ \int_{s_1}^{s_2} \frac{(s_2^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} |k(f(s_1,\xi))||u(\xi)|\mathrm{d}\xi \\
&+ \int_0^{s_1} \frac{|(s_2^m - \xi^m)^{\alpha-1} - (s_1^m - \xi^m)^{\alpha-1}|}{\Gamma(\alpha)} m\xi^{m-1} |k(f(s_1,\xi))||u(\xi)|\mathrm{d}\xi.
\end{aligned}$$

Therefore, if we put

$$\Omega_{\mathrm{kof}}(\epsilon, .) = \sup\{|k(f(s,\xi)) - k(f(s',\xi))| : s, s', \xi \in [0,1] \text{and} |s - s'| \leq \epsilon\},$$

then we have

$$\begin{aligned}
&|(Gu)(s_2) - (Gu)(s_1)| \\
&\leq \frac{||u||\Omega_{kof}(\epsilon, .)}{\Gamma(\alpha)} \int_0^{s_2} (s_2^m - \xi^m)^{\alpha-1} m\xi^{m-1}\mathrm{d}\xi \\
&\quad + \frac{||u||||k||}{\Gamma(\alpha)} \int_{s_1}^{s_2} (s_2^m - \xi^m)^{\alpha-1} m\xi^{m-1}\mathrm{d}\xi \\
&+ \frac{||u||||k||}{\Gamma(\alpha)} \int_0^{s_1} [(s_1^m - \xi^m)^{\alpha-1} - (s_2^m - \xi^m)^{\alpha-1}] m\xi^{m-1}\mathrm{d}\xi \\
&\leq \frac{||u||\Omega_{\mathrm{kof}}(\epsilon, .)}{\Gamma(\alpha)} \frac{s_2^{m\alpha}}{\alpha} + \frac{||u||||k||}{\Gamma(\alpha)} \frac{(s_2^m - s_1^m)^{\alpha}}{\alpha} \\
&\quad + \frac{||u||||k||}{\Gamma(\alpha)} \left[\frac{(s_2^m - s_1^m)^{\alpha}}{\alpha} + \frac{s_1^{m\alpha}}{\alpha} - \frac{s_2^{m\alpha}}{\alpha} \right] \\
&\leq \frac{||u||\Omega_{\mathrm{kof}}(\epsilon, .)}{\Gamma(\alpha+1)} + \frac{2||u||||k||}{\Gamma(\alpha+1)} (s_2^m - s_1^m)^{\alpha}.
\end{aligned}$$

Obviously, from the uniform continuity of the function *kof* on the set $[0,1] \times [0,1]$ we can get $\Omega_{\mathrm{kof}}(\epsilon, .) \to 0$ as $\epsilon \to 0$. Thus $Gu \in C[0,1]$, and consequently, $\tau u \in C[0,1]$. Also, we have

$$\begin{aligned}
|(Gu)(s)| &\leq \int_0^s \frac{(s^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} |k(f(s,\xi))||u(\xi)|\mathrm{d}\xi \\
&\leq \frac{||k||||u||}{\Gamma(\alpha)} \int_0^s (s^m - \xi^m)^{\alpha-1} m\xi^{m-1}\mathrm{d}\xi \leq \frac{||k||||u||}{\Gamma(\alpha+1)}
\end{aligned} \tag{3.3}$$

for all $s \in [0,1]$. Therefore,

$$\begin{aligned}
|(\tau u)(s)| &\leq |g(s)| + |h(s,u)||Gu(\xi)| \\
&\leq M_1 + [|h(s,u) - h(s,0)| + |h(s,0)|] \frac{||k||||u||}{\Gamma(\alpha+1)} \\
&\leq M_1 + (\phi(||u||) + M_2) \frac{||k||||u||}{\Gamma(\alpha+1)}.
\end{aligned}$$

Hence,

$$||\tau u|| \leq M_1 + (\phi(||u||) + M_2) \frac{||k||||u||}{\Gamma(\alpha+1)}.$$

Thus, if $||u|| \leq r_0$ we conclude the following estimation by assumption (b_6)

$$||\tau u|| \leq M_1 + (\phi(r_0) + M_2) \frac{||k||r_0}{\Gamma(\alpha+1)} \leq r_0.$$

Consequently, the operator τ maps the ball $B_{r_0} \subset C[0,1]$ into itself. To prove continuity of τ on B_{r_0}, let $\{u_n\}$ be a sequence in B_{r_0} such that $u_n \to u$. We have to show that $\tau u_n \to \tau u$. In fact, $\forall s \in [0,1]$, we have

$$\begin{aligned}
|(Gu_n)(s) - (Gu)(s)| &= \Big| \int_0^s \frac{(s^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} k(f(s,\xi))u_n(\xi)\mathrm{d}\xi \\
&\quad - \int_0^s \frac{(s^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} k(f(s,\xi))u(\xi)\mathrm{d}\xi \Big| \\
&\leq \int_0^s \frac{(s^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} |k(f(s,\xi))||u_n(\xi) - u(\xi)|\mathrm{d}\xi,
\end{aligned}$$

thus

$$||Gu_n - Gu|| \leq \frac{||k||}{\Gamma(\alpha+1)} ||u_n - u||.$$

As,

$$\begin{aligned}
&|(\tau u_n)(s) - (\tau u)(s)| = |h(s,u_n(s))(Gu_n)(s) - h(s,u(s))(Gu)(s)| \\
&\leq |h(s,u_n(s))(Gu_n)(s) - h(s,u(s))(Gu_n)(s)| \\
&+ |h(s,u(s))(Gu_n)(s) - h(s,u(s))(Gu)(s)| \\
&\leq |h(s,u_n(s)) - h(s,u(s))||(Gu_n)(s)| + |h(s,u(s))||(Gu_n)(s) - (Gu)(s)| \\
&\leq \phi(|u_n(s) - u(s)|) \int_0^s \frac{(s^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} |k(f(s,\xi))||u_n(\xi)|\mathrm{d}\xi \\
&+ (\phi(|u(s)|) + M_2) \int_0^s \frac{(s^m - \xi^m)^{\alpha-1}}{\Gamma(\alpha)} m\xi^{m-1} |k(f(s,\xi))||u_n(\xi) - u(\xi)|\mathrm{d}\xi.
\end{aligned}$$

It follows that

$$||\tau u_n - \tau u|| \leq \phi(||u_n - u||)\frac{||k||}{\Gamma(\alpha+1)}||u_n|| + (\phi(||u||) + M_2)\frac{||k||}{\Gamma(\alpha+1)}||u_n - u||.$$

So τ is continuous on B_{r_0}. we introduce,

$$B_{r_0}^{\sim} = \{u \in B_{r_0} : u(s) \geq 0, \quad \text{for } s \in [0,1]\} \subseteq B_{r_0}.$$

Obviously $B_{r_0}^{\sim} \neq \emptyset$ is bounded, closed and convex. By assumptions $(b_1), (b_2)$ and (b_5), if $u(s) \geq 0$ then $(\tau u)(s) \geq 0$ for all $s \in [0,1]$. Thus τ projects $B_{r_0}^{\sim}$ into itself. Moreover τ is continuous on $B_{r_0}^{\sim}$. Let $A \neq \emptyset$ be a subset of $B_{r_0}^{\sim}$, also $\epsilon > 0$ and

$$s_1,\, s_2 \in [0,1]; |s_2 - s_1| \leq \epsilon.$$

For simplicity, we suppose that $s_2 \geq s_1$. So we get

$$\begin{aligned}
&|(\tau u)(s_2) - (\tau u)(s_1)| \\
&= |g(s_2) + h(s_2, u(s_2))(Gu)(s_2) - g(s_1) - h(s_1, u(s_1))(Gu)(s_1)| \\
&\leq |g(s_2) - g(s_1)| + |h(s_2, u(s_2))(Gu)(s_2) - h(s_1, u(s_2))(Gu)(s_2)| \\
&+ |h(s_1, u(s_2))(Gu)(s_2) - h(s_1, u(s_1))(Gu)(s_2)| \\
&+ |h(s_1, u(s_1))(Gu)(s_2) - h(s_1, u(s_1))(Gu)(s_1)| \\
&\leq |g(s_2) - g(s_1)| + |h(s_2, u(s_2)) - h(s_1, u(s_2))||(Gu)(s_2)| \\
&+ |h(s_1, u(s_2)) - h(s_1, u(s_1))||(Gu)(s_2)| \\
&+ |h(s_1, u(s_1))||(Gu)(s_2) - (Gu)(s_1)| \\
&\leq \Omega(g,\epsilon) + \rho_{r_0}(h,\epsilon)\frac{||u||||k||}{\Gamma(\alpha+1)} + \phi(|u(s_2) - u(s_1)|)\frac{||u||||k||}{\Gamma(\alpha+1)} \\
&+ (\phi(||u||) + M_2)\left[\frac{||u||\Omega_{\text{kof}}(\epsilon,.)}{\Gamma(\alpha+1)} + \frac{2||u||||k||}{\Gamma(\alpha+1)}(s_2^m - s_1^m)^\alpha\right],
\end{aligned}$$

where we denoted

$$\rho_{r_0}(h,\epsilon) = \sup\{|h(s,u) - h(s',u)| : s, s' \in [0,1], u \in [0, r_0], |s - s'| \leq \epsilon\}.$$

According to mean value theorem ($|s_2^m - s_1^m|^\alpha \leq m^\alpha |s_2 - s_1|^\alpha$) in the last inequality, we conclude that,

$$\begin{aligned}
&|(\tau u)(s_2) - (\tau u)(s_1)| \\
&\leq \Omega(g,\epsilon) + \rho_{r_0}(h,\epsilon)\frac{||u||||k||}{\Gamma(\alpha+1)} + \phi(|u(s_2) - u(s_1)|)\frac{||u||||k||}{\Gamma(\alpha+1)} \\
&+ (\phi(||u||) + M_2)\left[\frac{||u||\Omega_{\text{kof}}(\epsilon,.)}{\Gamma(\alpha+1)} + \frac{2||u||||k||}{\Gamma(\alpha+1)}(m\epsilon)^\alpha\right].
\end{aligned}$$

Hence,

$$\Omega(\tau u,\epsilon) \leq \Omega(g,\epsilon) + \rho_{r_0}(h,\epsilon)\frac{r_0||k||}{\Gamma(\alpha+1)} + \phi(\Omega(u,\epsilon))\frac{r_0||k||}{\Gamma(\alpha+1)} + (\phi(r_0) + M_2)\left[\frac{r_0\Omega_{\text{kof}}(\epsilon,.)}{\Gamma(\alpha+1)} + \frac{2r_0||k||}{\Gamma(\alpha+1)}(m\epsilon)^\alpha\right].$$

By computing supremum on A, we can write

$$\Omega(\tau A,\epsilon) \leq \Omega(g,\epsilon) + \rho_{r_0}(h,\epsilon)\frac{r_0||k||}{\Gamma(\alpha+1)} + \phi(\Omega(A,\epsilon))\frac{r_0||k||}{\Gamma(\alpha+1)} + (\phi(r_0) + M_2)\left[\frac{r_0\Omega_{\text{kof}}(\epsilon,.)}{\Gamma(\alpha+1)} + \frac{2r_0||k||}{\Gamma(\alpha+1)}(m\epsilon)^\alpha\right].$$

Since g is continuous on [0, 1] and also, h and kof are uniform continuous on $[0,1] \times [0, r_0]$ and $[0,1] \times [0,1]$, respectively, so when $\epsilon \to 0$ then $\Omega(g,\epsilon) \to 0$, $\rho_{r_0}(h,\epsilon) \to 0$, $\Omega_{\text{kof}}(\epsilon,.) \to 0$ and also in the following, we have

$$\Omega_0(\tau A) \leq \frac{r_0||k||}{\Gamma(\alpha+1)}\phi(\Omega_0(A)). \tag{3.4}$$

Suppose $u \in A$ and $s_1, s_2 \in [0,1]$ such that $s_1 < s_2$, thus

$$\begin{aligned}
&|(\tau u)(s_2) - (\tau u)(s_1)| - [(\tau u)(s_2) - (\tau u)(s_1)] \\
&= |g(s_2) + h(s_2,u(s_2))(Gu)(s_2) - g(s_1) - h(s_1,u(s_1))(Gu)(s_1)| \\
&- [g(s_2) + h(s_2,u(s_2))(Gu)(s_2) - g(s_1) - h(s_1,u(s_1))(Gu)(s_1)] \\
&\leq \{|g(s_2) - g(s_1)| - [g(s_2) - g(s_1)]\} + |h(s_2,u(s_2))(Gu)(s_2) \\
&- h(s_1,u(s_1))(Gu)(s_2)| + |h(s_1,u(s_1))(Gu)(s_2) - h(s_1,u(s_1))(Gu)(s_1)| \\
&- \{[h(s_2,u(s_2))(Gu)(s_2) - h(s_1,u(s_1))(Gu)(s_2)] \\
&+ [h(s_1,u(s_1))(Gu)(s_2) - h(s_1,u(s_1))(Gu)(s_1)] \\
&\leq \{|h(s_2,u(s_2)) - h(s_1,u(s_1))| - [h(s_2,u(s_2)) - h(s_1,u(s_1))]\}(Gu)(s_2) \\
&+ h(s_1,u(s_1))\{|(Gu)(s_2) - (Gu)(s_1)| - [(Gu)(s_2) - (Gu)(s_1)]\} \\
&\leq J(Hu)\frac{r_0||k||}{\Gamma(\alpha+1)}.
\end{aligned}$$

Also we conclude that,

$$J(\tau u) \leq \phi(J(u))\frac{r_0||k||}{\Gamma(\alpha+1)},$$

and consequently,

$$J(\tau A) \leq \frac{r_0||k||}{\Gamma(\alpha+1)}\phi(J(A)). \tag{3.5}$$

From (3.4) and (3.5) and the definition of η, we get

$$\begin{aligned}
\eta(\tau A) &= \Omega_0(\tau A) + J(\tau A) \leq \frac{r_0||k||}{\Gamma(\alpha+1)}\phi(\Omega_0(A)) + \frac{r_0||k||}{\Gamma(\alpha+1)}\phi(J(A)) \\
&\leq \frac{r_0||k||}{\Gamma(\alpha+1)}(\phi(\Omega_0(A)) + \phi(J(A))) \leq \frac{r_0||k||}{\Gamma(\alpha+1)}(\phi(\Omega_0(A) + J(A))) \\
&\leq \lambda\phi(\eta(A)).
\end{aligned}$$

By the above inequality and because $\frac{r_0||k||}{\Gamma(\alpha+1)} < 1$, with applying Theorem 1 for in the case of $G(s) = \Theta(s) = 1$, we complete the proof. Also, such a solution is non-decreasing in Remark 3 and the definition of μ, was given in Sect. 2. □

Corollary 6 *Let the conditions of Theorem 3 be satisfied, then some of the integral equations with fractional order have at least one solution in* $C[0,1]$*, such as in the case of (i, ii, iii):*

(i) *for* $m = 1$,

$$u(s) = g(s) + h(s,u(s)) \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} k(f(s,\xi))u(\xi)\,\mathrm{d}\xi,$$

(ii) *for* $m = 1$ *and* $h(s,u(s)) = 1$,

$$u(s) = g(s) + \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} k(f(s,\xi))u(\xi)\,\mathrm{d}\xi,$$

(iii) *for* $m = 1, k = I, h(s,u(s)) = 1$ *and* $g(s) = 0$,

$$u(s) = \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(s,\xi)u(\xi)\,\mathrm{d}\xi.$$

Now, we consider an example by applying Theorem 3.

Example 4 Suppose, integral equation with singular kernel and fractional order is given in the following form,

$$u(s) = \frac{1}{5}s^3 + \frac{2su(s)}{5(1+s)} \int_0^s \frac{2\xi}{\Gamma(\frac{1}{2})\sqrt{s^2-\xi^2}} \left[\frac{1}{8}(s+\xi) + \frac{1}{4}\right] u(\xi)\,\mathrm{d}\xi, \tag{3.6}$$

where $s \in [0,1]$. In this example, we have $g(s) = \frac{1}{5}s^3$ and this function satisfies assumption $(b1)$ and $M_1 = \frac{1}{5}$. Here $f(s,\xi) = \frac{1}{4}\sqrt{s+\xi}$ and this function satisfies assumption (b_4). Let $k : [0, \frac{\sqrt{2}}{4}] \to \mathbb{R}^+$ be given by $k(z) = 2z^2 + \frac{1}{4}$, then k satisfying assumption (b_5) with $||k|| = \frac{1}{2}$. Moreover, the function $h(s,u) = \dfrac{2su}{5(1+s)}$ satisfies hypothesis (b_2) with assumption $\phi(s) = \dfrac{1}{5}s$,

$$|h(s,u) - h(s,z)| \le \frac{1}{5}|u - z| = \phi(|u - z|), \quad \forall u,z \in \mathbb{R},\ s \in [0,1],$$

also h satisfies in (b_3). In fact, by choosing an arbitrary nonnegative function $u \in C[0,1]$ and $s_1, s_2 \in [0,1]$ ($s_1 \le s_2$), we can write

$$\begin{aligned}
&|(Hu)(s_2) - (Hu)(s_1)| - [(Hu)(s_2) - (Hu)(s_1)] \\
&= |h(s_2,u(s_2)) - h(s_1,u(s_1))| - [h(s_2,u(s_2)) - h(s_1,u(s_1))] \\
&= \left|\frac{2s_2u(s_2)}{5(1+s_2)} - \frac{2s_1u(s_1)}{5(1+s_1)}\right| - \left[\frac{2s_2u(s_2)}{5(1+s_2)} - \frac{2s_1u(s_1)}{5(1+s_1)}\right] \\
&\le \left|\frac{2s_2u(s_2)}{5(1+s_2)} - \frac{2s_2u(s_1)}{5(1+s_2)}\right| + \left|\frac{2s_2u(s_1)}{5(1+s_2)} - \frac{2s_1u(s_1)}{5(1+s_1)}\right| \\
&\quad - \left[\frac{2s_2u(s_2)}{5(1+s_2)} - \frac{2s_2u(s_1)}{5(1+s_2)} + \frac{2s_2u(s_1)}{5(1+s_2)} - \frac{2s_1u(s_1)}{5(1+s_1)}\right] \\
&\le \frac{2s_2}{5(1+s_2)}|u(s_2) - u(s_1)| + \left|\frac{2s_2}{5(1+s_2)} - \frac{2s_1}{5(1+s_1)}\right| u(s_1) \\
&\quad - \frac{2s_2}{5(1+s_2)}[u(s_2) - u(s_1)] - \left[\frac{2s_2}{5(1+s_2)} - \frac{2s_1}{5(1+s_1)}\right] u(s_1) \\
&\le \frac{2s_2}{5(1+s_2)}\{|u(s_2) - u(s_1)| - [u(s_2) - u(s_1)]\} \\
&\le \frac{2s_2}{5(1+s_2)}J(u) \le \frac{1}{5}J(u) = \phi(J(u)).
\end{aligned}$$

According to the example, (3.2) converts to this form,

$$\frac{1}{5}\Gamma\left(\frac{3}{2}\right) + \frac{1}{10}r^2 \le \Gamma\left(\frac{3}{2}\right)r,$$

and $r_0 = 1$ is as a positive solution of it. Also,

$$\lambda = \frac{||k||r_0}{\Gamma\left(\frac{3}{2}\right)} = \frac{1}{\sqrt{\pi}} < 1.$$

Thus, Theorem 3 guarantees that Eq. (3.6) has a non-decreasing solution.

Homotopy perturbation method (HPM) for solving functional I.E.

In this section, we solve functional integral Eq. (3.6) by using (HPM). In [10], perturbation method which depends on a small parameter can be led to imprecise solution by choosing unsuitable small parameter. But homotopy perturbation method introduced in [9], by an important concept of topology it can convert a nonlinear problem to a finite number of linear problems without dependence to the small parameter, this independence is very important. For introducing homotopy perturbation method according to the above-mentioned references, we consider the nonlinear problem:

$$\begin{aligned} &M(u) - g(s) = 0, \quad s \in D \\ &\Lambda\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad n \in \Upsilon, \end{aligned} \tag{4.1}$$

where M and Λ are differential and boundary operators, respectively, also $g(s)$ is a known analytic function and Υ is the boundary of the domain D. we assume operator M is divided into linear and nonlinear operators such as ℓ and $\aleph$. So, we can write Eq. (4.1) to this form,

$$\ell(u) + \aleph(u) - g(s) = 0. \tag{4.2}$$

Homotopy perturbation $H(v,p)$ can be written as follows [9]:

$$\begin{aligned} &H : D \times [0,1] \longrightarrow \Re, \\ &H(v,p) = (1-p)[\ell(v) - \ell(v_0)] + p[M(v) - g(s)] = 0, \end{aligned} \tag{4.3}$$

where p is an embedding parameter, v is an approximation of u and v_0 is an initial approximation of u. Of course some kinds of modifications of homotopy perturbation method can be seen in [8, 11]. We solve nonlinear integral Eq. (3.6) by Eq. (4.3). Let us consider Eq. (3.6) to the following form,

$$u(s)=\frac{1}{5}s^3+u(s)\int_0^s\frac{4s\xi\left(\frac{1}{8}(s+\xi)+\frac{1}{4}\right)}{5\sqrt{\pi}(1+s)\sqrt{s^2-\xi^2}}u(\xi)\,\mathrm{d}\xi, \tag{4.4}$$

the general form of Eq. (4.4) is as follows:

$$u(s)-u(s)\int_0^s k(s,\xi)u(\xi)\,\mathrm{d}\xi=\frac{1}{5}s^3, \tag{4.5}$$

according to the nonlinear Eq. (4.1), we can write,

$$M(u(s))=g(s);\;\; g(s)=\frac{1}{5}s^3. \tag{4.6}$$

In the homotopy perturbation (4.3) we approximate solution of Eq. (4.5) in terms of power series of p,

$$v=v_0+pv_1+p^2v_2+\cdots=\sum_{i=0}^{\infty}p^iv_i. \tag{4.7}$$

Also in Eq. (4.3), we choose linear and nonlinear operators to these forms,

$$\ell(v)=v, M(v)=v(s)-v(s)\int_0^s k(s,\xi)v(\xi)\,\mathrm{d}\xi, g(s)=\frac{1}{5}s^3.$$

So, we can write,

$$\begin{aligned}H(v,p)&=(1-p)(v-v_0)\\&+p\left[v(s)-v(s)\int_0^s k(s,\xi)v(\xi)\mathrm{d}\xi-\frac{1}{5}s^3\right]=0,\end{aligned} \tag{4.8}$$

by substituting Eq. (4.7) in the homotopy formula Eq. (4.8), we have

$$\begin{aligned}&pv_1+p^2v_2+\cdots+pv_0-pv_0(s)\int_0^s k(s,\xi)v_0(\xi)\,\mathrm{d}\xi\\&-p^2v_0(s)\int_0^s k(s,\xi)v_1(\xi)\,\mathrm{d}\xi\\&-p^2v_1(s)\int_0^s k(s,\xi)v_0(\xi)\,\mathrm{d}\xi+\cdots-p\frac{1}{5}s^3=0,\end{aligned} \tag{4.9}$$

with ordering the above relations in terms of p powers, we have

$$\begin{aligned}&p^1:(v_1+v_0-v_0(s)\int_0^s k\left(s,\xi)v_0(\xi)\,\mathrm{d}\xi-\frac{1}{5}s^3\right),\\&p^2:(v_2-v_0(s)\int_0^s k(s,\xi)v_1(\xi)\,\mathrm{d}\xi-v_1(s)\int_0^s k(s,\xi)v_0(\xi)\,\mathrm{d}\xi),\\&p^3:\cdots\end{aligned}$$

By considering to Eq. (4.9), we put in the coefficients of p powers equal to zero and by suitable choosing initial guess $v_0(s)$, we obtain

$$\begin{aligned}&v_0(s)=\frac{1}{5}s^3,\\&v_1(s)=v_0(s)\int_0^s k(s,\xi)v_0(\xi)\,\mathrm{d}\xi,\\&v_2(s)=v_0(s)\int_0^s k(s,\xi)v_1(\xi)\,\mathrm{d}\xi-v_1(s)\int_0^s k(s,\xi)v_0(\xi)\,\mathrm{d}\xi,\end{aligned} \tag{4.10}$$

where $k(s,\xi)$ is given by Eq. (4.4), therefore,

$$v_1(s)=\frac{4s^4}{125\sqrt{\pi}(1+s)}\int_0^s\frac{\frac{1}{8}(s+\xi)+\frac{1}{4}}{\sqrt{s^2-\xi^2}}\xi^4\,\mathrm{d}\xi=\frac{s^8(128s+45\pi(2+s))}{(240)(250)\sqrt{\pi}(1+s)}.$$

By taking two terms of Eq. (4.7) into account, we can approximate the solution of Eq. (4.4) as follows:

$$v(s)=\frac{1}{5}s^3+\frac{s^8(128s+45\pi(2+s))}{240\times 250\sqrt{\pi}(1+s)}. \tag{4.11}$$

By substituting (4.11) in Eq. (4.4) and comparing both sides of it, we reach absolute errors in points (see Table 1).

Table 1 Absolute errors for Eq. (4.4) by HPM

t	Absolute errors
0.0	0.0
0.1	5.9×10^{-18}
0.2	4.8×10^{-14}
0.3	9.3×10^{-12}
0.4	3.9×10^{-10}
0.5	7.0×10^{-9}
0.6	7.5×10^{-8}
0.7	5.5×10^{-7}
0.8	3.1×10^{-6}
0.9	1.4×10^{-5}
1.0	5.7×10^{-5}

Conclusion

In this paper, we try to introduce a mixed plan of pure and applied mathematics, where measure of non-compactness on a Banach space is used for the generalization of Darbo fixed point theorem for existence of solution singular integral equations with fractional order. Also, by homotopy perturbation method we obtain an approximation of a solution with high accuracy.

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