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ORIGINAL RESEARCH

Suzuki type unique common fixed point theorem in partial metric spaces using (C)-condition

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Abstract In this paper, we obtain a Suzuki type unique common fixed point theorem using C-condition in partial metric spaces. In addition, we give an example which supports our main theorem.

Keywords Partial metric · Weakly compatible maps · Suzuki-type contraction · C-condition

Mathematics Subject Classification 54H25 · 47H10 · 54E50

Introduction

The notion of a partial metric space was introduced by Matthews [12] as a part of the study of denotational semantics of data flow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation and domain theory in computer science (see [6]).

Matthews [12] and Romaguera [16] and Altun et al. [2] proved some fixed point theorems in partial metric spaces for a single map. For more works on fixed, common fixed point theorems in partial metric spaces, we refer [1, 3–5, 7–11, 13–15, 17–19]).

The aim of this paper is to prove a Suzuki type unique common fixed point theorem for four maps using (C)-condition in partial metric spaces.

First, we give the following theorem of Suzuki [18].

Theorem 1.1 (See [18]) *Let* (X, d) *be a complete metric space and let* T *be a mapping on* X*. Define a non-increasing function* $\theta : [0,1) \to (\frac{1}{2}, 1]$ *by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le \frac{(\sqrt{5}-1)}{2}, \\ (1-r)r^{-2} & \text{if } \frac{(\sqrt{5}-1)}{2} \le r \le 2^{-\frac{1}{2}}, \\ (1+r)^{-1} & \text{if } 2^{-\frac{1}{2}} \le r < 1. \end{cases}$$

Assume that there exists $r \in [0,1)$*, such that*

$$\theta(r)d(x,Tx) \le d(x,y) \Rightarrow d(Tx,Ty) \le rd(x,y)$$

for all $x, y \in X$*. Then, there exists a unique fixed point* z *of* T*. Moreover,* $\lim_n T^n x = z$ *for all* $x \in X$*.*

Definition 1.2 (See [11]) A mapping T on a metric space (X, d) is called a non-expensive mapping if

$$d(Tx,Ty) \le d(x,y), \quad \forall x,y \in X.$$

Definition 1.3 (See [11]) A mapping T on a metric space (X, d) satisfies the C-condition if

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$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in X.$$

First, we recall some basic definitions and lemmas which play crucial role in the theory of partial metric spaces.

Definition 1.4 (See [12]) A partial metric on a nonempty set X is a function $p : X \times X \to R^+$, such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(p_2) $p(x, x) \leq p(x, y), p(y, y) \leq p(x, y)$,
(p_3) $p(x, y) = p(y, x)$,
(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) is called a partial metric space (PMS).

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y), \qquad (1)$$

is a metric on X.

Example 1.5 (See [1, 9, 12]) Consider $X = [0, \infty)$ with $p(x, y) = \max\{x, y\}$. Then, (X, p) is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case, $p^s(x, y) = |x - y|$.

Example 1.6 (See [7]) Let $X = \{[a, b] : a, b, \in \mathbb{R},\ a \leq b\}$ and define $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then, (X, p) is a partial metric space.

We now state some basic topological notions (such as convergence, completeness, and continuity) on partial metric spaces (see [1, 2, 9, 10, 12].)

Definition 1.7

(i) A sequence $\{x_n\}$ in the PMS (X, p) converges to the limit x if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.
(ii) A sequence $\{x_n\}$ in the PMS (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.
(iii) A PMS (X, p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x \in X$, such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

The following lemma is one of the basic results in PMS ([1, 2, 9, 10, 12]).

Lemma 1.8

(i) *A sequence $\{x_n\}$ is a Cauchy sequence in the PMS (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).*
(ii) *A PMS (X, p) is complete if and only if the metric space (X, p^s) is complete. Moreover*

$$\lim_{n\to\infty} p^s(x, x_n) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Next, we give two simple lemmas which will be used in the proof of our main result. For the proofs, we refer to [1].

Lemma 1.9 *Assume $x_n \to z$ as $n \to \infty$ in a PMS (X, p), such that $p(z, z) = 0$. Then, $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

Lemma 1.10 *Let (X, p) be a PMS. Then*

(A) *If $p(x, y) = 0$, then $x = y$.*
(B) *If $x \neq y$, then $p(x, y) > 0$.*

Remark 1.11 If $x = y$, $p(x, y)$ may not be 0.

Definition 1.12 A pair (T, g) is called weakly compatible pair if they commute at coincidence points.

Now, we prove our main result.

Main result

Theorem 2.1 *Let (X, p) be a partial metric space and let $S, T, f, g : X \to X$ be mappings satisfying*

(2.1.1) $\frac{1}{2}\min\{p(fx, Sx), p(gy, Ty)\} \leq p(fx, gy)$ *implies that* $\psi(p(Sx, Ty)) \leq \alpha(M(x, y)) - \beta(M(x, y))$, *for all* x, y in X, *where $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ are such that ψ is an altering distance function, α is continuous, and β is lower semi continuous, $\alpha(0) = \beta(0) = 0$ and $\psi(t) - \alpha(t) + \beta(t) > 0$, for all $t > 0$ and*

$$M(x, y) = \max\left\{ \begin{array}{l} p(fx, gy), p(fx, Sx), p(gy, Ty), \\ \frac{1}{2}[p(fx, Ty) + p(gy, Sx)] \end{array} \right\},$$

(2.1.2) $S(X) \subseteq g(X), T(X) \subseteq f(X)$,
(2.1.3) *either $f(X)$ or $g(X)$ is a complete subspace of X,*
(2.1.4) *the pairs (f, S) and (g, T) are weakly compatible.*

Then, S, T, f and g have a unique common fixed point in X.

Proof Let $x_0 \in X$ be arbitrary point in X. From (2.1.2), there exist sequences of $\{x_n\}$ and $\{y_n\}$ in X, such that
$Sx_{2n} = gx_{2n+1} = y_{2n}$,
$Tx_{2n+1} = fx_{2n+2} = y_{2n+1}, \quad n = 0, 1, 2, \ldots$.

Case (i): Assume that $y_n \neq y_{n+1}$ for all n.
Denote $p_n = p(y_n, y_{n+1})$.
We show that $p_n \leq p_{n-1}, \quad n = 1, 2, 3, \ldots$
Now

$$\frac{1}{2}\min\{p(fx_{2n},Sx_{2n}),p(gx_{2n+1},Tx_{2n+1})\}\le p(fx_{2n},Sx_{2n}) = p(fx_{2n},gx_{2n+1}).$$

From (2.1.1), we get

$$\psi(p(Sx_{2n},Tx_{2n+1}))\le\alpha(M(x_{2n},x_{2n+1}))-\beta(M(x_{2n},x_{2n+1})).$$

$$M(x_{2n},x_{2n+1})=\max\left\{\begin{array}{c}p(y_{2n-1},y_{2n}),p(y_{2n-1},y_{2n}),p(y_{2n},y_{2n+1}),\\ \frac{1}{2}[p(y_{2n-1},y_{2n+1})+p(y_{2n},y_{2n})]\end{array}\right\} =\max\{p_{2n-1},p_{2n}\},\text{ from }(p_4).$$

Hence,
$\psi(p_{2n})\le\alpha(\max\{p_{2n-1},p_{2n}\})-\beta(\max\{p_{2n-1},p_{2n}\})$.

If p_{2n} is maximum, then we have $\psi(p_{2n})\le\alpha(p_{2n})-\beta(p_{2n})$, thus $\psi(p_{2n})-\alpha(p_{2n})+\beta(p_{2n})\le 0$, which is a contradiction.

Hence p_{2n-1} is maximum. Thus

$$\psi(p_{2n})\le\alpha(p_{2n-1})-\beta(p_{2n-1}) \tag{2}$$

$<\psi(p_{2n-1})$.

Since ψ is increasing, we have $p_{2n}\le p_{2n-1}$.

Similarly, we can show that $p_{2n-1}\le p_{2n-2}$.

Thus, $p_n\le p_{n-1},\quad n=1,2,3,\ldots$

Thus, $\{p_n\}$ is a non-increasing sequence of non-negative real numbers and must converge to a real number, say, $l\ge 0$. Suppose $l>0$.

Letting $n\to\infty$ in (2), we get $\psi(l)\le\alpha(l)-\beta(l)$.

Thus, $\psi(l)-\alpha(l)+\beta(l)\le 0$, which is a contradiction. Hence, $l=0$.

Thus

$$\lim_{n\to\infty}p(y_n,y_{n+1})=0. \tag{3}$$

Hence, from (p_2), we get

$$\lim_{n\to\infty}p(y_n,y_n)=0. \tag{4}$$

By definition of p^s, (3), and (4), we get

$$\lim_{n\to\infty}p^s(y_n,y_{n+1})=0. \tag{5}$$

Now, we prove that $\{y_{2n}\}$ is a Cauchy sequence in (X,p^s). On contrary, suppose that $\{y_{2n}\}$ is not Cauchy.

There exist $\epsilon>0$ and monotone increasing sequences of natural numbers $\{2m_k\}$ and $\{2n_k\}$, such that $n_k>m_k$,

$$p^s(y_{2m_k},y_{2n_k})\ge\epsilon \tag{6}$$

and

$$p^s(y_{2m_k},y_{2n_k-2})<\epsilon. \tag{7}$$

From (6) and (7), we obtain

$$\begin{aligned}\epsilon&\le p^s(y_{2m_k},y_{2n_k})\\&\le p^s(y_{2m_k},y_{2n_k-2})+p^s(y_{2n_k-2},y_{2n_k-1})+p^s(y_{2n_k-1},y_{2n_k})\\&\le\epsilon+p^s(y_{2n_k-2},y_{2n_k-1})+p^s(y_{2n_k-1},y_{2n_k}).\end{aligned}$$

Letting $k\to\infty$ and then using (5), we get

$$\lim_{k\to\infty}p^s(y_{2m_k},y_{2n_k})=\epsilon. \tag{8}$$

Hence, from definition of p^s and (4), we have

$$\lim_{k\to\infty}p(y_{2m_k},y_{2n_k})=\frac{\epsilon}{2}. \tag{9}$$

Letting $k\to\infty$ and then using (8) and (5) in $|p^s(y_{2n_k+1},y_{2m_k})-p^s(y_{2n_k},y_{2m_k})|\le p^s(y_{2n_k+1},y_{2n_k})$ we obtain

$$\lim_{k\to\infty}p^s(y_{2n_k+1},y_{2m_k})=\epsilon. \tag{10}$$

Hence, we have

$$\lim_{k\to\infty}p(y_{2n_k+1},y_{2m_k})=\frac{\epsilon}{2}. \tag{11}$$

Letting $k\to\infty$ and then using (8) and (5) in $|p^s(y_{2n_k},y_{2m_k-1})-p^s(y_{2n_k},y_{2m_k})|\le p^s(y_{2m_k-1},y_{2m_k})$, we get

$$\lim_{k\to\infty}p^s(y_{2n_k},y_{2m_k-1})=\epsilon. \tag{12}$$

Hence, we have

$$\lim_{k\to\infty}p(y_{2n_k},y_{2m_k-1})=\frac{\epsilon}{2}. \tag{13}$$

Letting $k\to\infty$ and then using (12) and (5) in $|p^s(y_{2m_k-1},y_{2n_k+1})-p^s(y_{2m_k-1},y_{2n_k})|\le p^s(y_{2n_k+1},y_{2n_k})$ we obtain

$$\lim_{k\to\infty}p^s(y_{2m_k-1},y_{2n_k+1})=\epsilon. \tag{14}$$

Hence, we get

$$\lim_{k\to\infty}p(y_{2m_k-1},y_{2n_k+1})=\frac{\epsilon}{2}. \tag{15}$$

If

$$\frac{1}{2}\min\{p(y_{2m_k-1},y_{2m_k}),p(y_{2n_k},y_{2n_k+1})\}>p(y_{2m_k-1},y_{2n_k}),$$

then letting $k\to\infty$, we get $0\ge\frac{\epsilon}{2}$ from (3) and (13).

It is a contradiction. Hence $\frac{1}{2}\min\{p(y_{2m_k-1},y_{2m_k}),p(y_{2n_k},y_{2n_k+1})\}\le p(y_{2m_k-1},y_{2n_k})=p(fx_{2m_k},gx_{2n_k+1})$.

From (2.1.1), we have

$$\begin{aligned}&\psi(p(y_{2m_k},y_{2n_k+1}))\\&=\psi(p(Sx_{2m_k},Tx_{2n_k+1}))\\&\leq\alpha\left(\max\left\{\begin{array}{c}p(y_{2m_k-1},y_{2n_k}),p(y_{2m_k-1},y_{2m_k}),p(y_{2n_k},y_{2n_k+1}),\\ \frac{1}{2}[p(y_{2m_k-1},y_{2n_k+1})+p(y_{2n_k},y_{2m_k})]\end{array}\right\}\right)\\&-\beta\left(\max\left\{\begin{array}{c}p(y_{2m_k-1},y_{2n_k}),p(y_{2m_k-1},y_{2m_k}),p(y_{2n_k},y_{2n_k+1}),\\ \frac{1}{2}[p(y_{2m_k-1},y_{2n_k+1})+p(y_{2n_k},y_{2m_k})]\end{array}\right\}\right).\end{aligned}$$

Letting $k\to\infty$ and then using (11), (13), (3), (15), and (9), we have

$$\begin{aligned}\psi\left(\frac{\epsilon}{2}\right)&\leq\alpha\left(\max\left\{\frac{\epsilon}{2},0,0,\frac{1}{2}\left[\frac{\epsilon}{2}+\frac{\epsilon}{2}\right]\right\}\right)\\&\quad-\beta\left(\max\left\{\frac{\epsilon}{2},0,0,\frac{1}{2}\left[\frac{\epsilon}{2}+\frac{\epsilon}{2}\right]\right\}\right)\\&=\alpha\left(\frac{\epsilon}{2}\right)-\beta\left(\frac{\epsilon}{2}\right)\\&<\psi\left(\frac{\epsilon}{2}\right),\end{aligned}$$

which is a contradiction. Hence, $\{y_{2n}\}$ is Cauchy.

In addition, $|p^s(y_{2n+1},y_{2m+1})-p^s(y_{2n},y_{2m})|\leq p^s(y_{2n+1},y_{2n})+p^s(y_{2m},y_{2m+1})$.

Letting $n,m\to\infty$, we have

$$\lim_{n,m\to\infty}p^s(y_{2n+1},y_{2m+1})=0.$$

Hence, $\{y_{2n+1}\}$ is Cauchy. Thus $\{y_n\}$ is a Cauchy sequence in (X,p^s).

Hence, we have $\lim_{n,m\to\infty}p^s(y_n,y_m)=0$.

Now, from the definition of p^s and from (4), we obtain

$$\lim_{n,m\to\infty}p(y_n,y_m)=0. \tag{16}$$

Therefore, $\{y_n\}$ is Cauchy sequence in X.

Suppose $g(X)$ is complete.

Since $y_{2n}=Sx_{2n}=gx_{2n+1}$, it follows $\{y_{2n}\}\subseteq g(X)$ is a Cauchy sequence in the complete metric space $(g(X),p^s)$, it follows that $\{y_{2n}\}$ converges in $(g(X),p^s)$.

Thus, $\lim_{n\to\infty}p^s(y_{2n},u)=0$ for some $u\in g(X)$.

That is, $y_{2n}\to u=gt\in g(X)$ for some $t\in X$.

Since $\{y_n\}$ is Cauchy in X and $\{y_{2n}\}\to u$, it follows that $\{y_{2n+1}\}\to u$.

From Lemma (1.2.5), we get

$$p(u,u)=\lim_{n\to\infty}p(y_{2n+1},u)=\lim_{n\to\infty}p(y_{2n},u)=\lim_{n,m\to\infty}p(y_n,y_m). \tag{17}$$

From (16) and (17), we obtain

$$p(u,u)=\lim_{n\to\infty}p(y_{2n+1},u)=\lim_{n\to\infty}p(y_{2n},u)=0. \tag{18}$$

Now, we claim that, for each $n\geq 1$, at least, one of the following assertions holds:

$$\frac{1}{2}p(y_{2n-1},y_{2n})\leq p(y_{2n-1},u)\ \text{or}\ \frac{1}{2}p(y_{2n},y_{2n+1})\leq p(y_{2n},u).$$

On the contrary, suppose that

$$\frac{1}{2}p(y_{2n-1},y_{2n})>p(y_{2n-1},u)\ \text{and}\ \frac{1}{2}p(y_{2n},y_{2n+1})>p(y_{2n},u)$$

for some $n\geq 1$.

Then we have

$$\begin{aligned}p_{2n-1}=p(y_{2n-1},y_{2n})&\leq p(y_{2n-1},u)+p(u,y_{2n})-p(u,u)\\&<\frac{1}{2}[p(y_{2n-1},y_{2n})+p(y_{2n},y_{2n+1})]\\&\leq\frac{1}{2}[p_{2n-1}+p_{2n}]\\&\leq p_{2n-1},\end{aligned}$$

which is a contradiction, and so, the claim holds.

Sub case(a): Suppose $\frac{1}{2}p(y_{2n-1},y_{2n})\leq p(y_{2n-1},u)$.

Suppose $Tt\neq u$.

We have

$$\begin{aligned}\frac{1}{2}\min\{p(fx_{2n},Sx_{2n}),p(gt,Tt)\}&\leq\frac{1}{2}p(fx_{2n},Sx_{2n})\\&=\frac{1}{2}p(y_{2n-1},y_{2n})\\&\leq p(y_{2n-1},u)\\&=p(fx_{2n},gt).\end{aligned}$$

From (2.1.1), we get

$$\begin{aligned}&\psi(p(Sx_{2n},Tt))\leq\alpha(M(x_{2n},t))-\beta(M(x_{2n},t))\\&\leq\alpha\left(\max\left\{\begin{array}{c}p(fx_{2n},gt),p(fx_{2n},Sx_{2n}),p(gt,Tt),\\ \frac{1}{2}[p(fx_{2n},Tt)+p(gt,Sx_{2n})]\end{array}\right\}\right)\\&-\beta\left(\max\left\{\begin{array}{c}p(fx_{2n},gt),p(fx_{2n},Sx_{2n}),p(gt,Tt),\\ \frac{1}{2}[p(fx_{2n},Tt)+p(gt,Sx_{2n})]\end{array}\right\}\right).\end{aligned}$$

Letting $n\to\infty$ and using (17), (18), we get

$$\begin{aligned}\psi(p(u,Tt))&\leq\alpha\left(\max\left\{\begin{array}{c}p(u,gt),p(u,u),p(gt,Tt),\\ \frac{1}{2}[p(u,Tt)+p(gt,u)]\end{array}\right\}\right)\\&\quad-\beta\left(\max\left\{\begin{array}{c}p(u,gt),p(u,u),p(gt,Tt),\\ \frac{1}{2}[p(u,Tt)+p(gt,u)]\end{array}\right\}\right)\\&=\alpha\left(\max\left\{\begin{array}{c}p(u,u),p(u,u),p(u,Tt),\\ \frac{1}{2}[p(u,Tt)+p(u,u)]\end{array}\right\}\right)\\&\quad-\beta\left(\max\left\{\begin{array}{c}p(u,u),p(u,u),p(u,Tt),\\ \frac{1}{2}[p(u,Tt)+p(u,u)]\end{array}\right\}\right)\\&\leq\alpha(p(u,Tt))-\beta(p(u,Tt))<\psi(p(u,Tt)).\end{aligned}$$

It is a contradiction. Hence, $Tt = u = gt$.

Since the pair (g, T) is weakly compatible, we have $gu = Tu$.

Suppose $Tu \neq u$.

Since $\frac{1}{2}\min\{p(fx_{2n}, Sx_{2n}), p(gu, Tu)\} \leq p(fx_{2n}, gu)$, from (2.1.1), we get

$$\psi(p(Sx_{2n}, Tu)) \leq \alpha\left(\max\left\{\begin{matrix} p(fx_{2n}, gu), p(fx_{2n}, Sx_{2n}), p(gu, Tu), \\ \frac{1}{2}[p(fx_{2n}, Tu) + p(gu, Sx_{2n})] \end{matrix}\right\}\right) - \beta\left(\max\left\{\begin{matrix} p(fx_{2n}, gu), p(fx_{2n}, Sx_{2n}), p(gu, Tu), \\ \frac{1}{2}[p(fx_{2n}, Tu) + p(gu, Sx_{2n})] \end{matrix}\right\}\right).$$

Letting $n \to \infty$, we have

$$\begin{aligned}\psi(p(u, Tu)) &\leq \alpha\left(\max\left\{\begin{matrix} p(u, gu), p(u, u), p(gu, Tu), \\ \frac{1}{2}[p(u, Tu) + p(gu, u)] \end{matrix}\right\}\right) \\ &\quad - \beta\left(\max\left\{\begin{matrix} p(u, gu), p(u, u), p(gu, Tu), \\ \frac{1}{2}[p(u, Tu) + p(gu, u)] \end{matrix}\right\}\right) \\ &\leq \alpha(p(u, Tu)) - \beta(p(u, Tu)) \\ &< \psi(p(u, Tu)),\end{aligned}$$

which is a contradiction.

Hence, $Tu = u$.

Therefore, $u = Tu = gu$.

Since $T(X) \subseteq f(X)$, then there exists $v \in X$, such that $Tu = fv = u$.

Suppose $Sv \neq fv$.

Since $\frac{1}{2}\min\{p(fv, Sv), p(gu, Tu)\} \leq p(fv, gu)$, from (2.1.1), we have

$$\begin{aligned}\psi(p(Sv, fv)) &= \psi(p(Sv, Tu)) \\ &\leq \alpha(M(v, u)) - \beta(M(v, u)) \\ &\leq \alpha\left(\max\left\{\begin{matrix} p(fv, gu), p(fv, Sv), p(gu, Tu), \\ \frac{1}{2}[p(fv, Tu) + p(gu, Sv)] \end{matrix}\right\}\right) \\ &\quad - \beta\left(\max\left\{\begin{matrix} p(fv, gu), p(fv, Sv), p(gu, Tu), \\ \frac{1}{2}[p(fv, Tu) + p(gu, Sv)] \end{matrix}\right\}\right) \\ &\leq \alpha(p(Sv, Tu)) - \beta(p(Sv, Tu)) \\ &< \psi(p(Sv, Tu)) \\ &= \psi(p(Sv, fv)).\end{aligned}$$

Hence, $Sv = fv = u$.

Since the pair (f, S) is weakly compatible, we have $fu = Su$.

Suppose $Su \neq u$.

Since $\frac{1}{2}\min\{p(fu, Su), p(gt, Tt)\} \leq p(fu, gt)$, from (2.1.1), we have

$$\begin{aligned}\psi(p(Su, u)) &= \psi(p(Su, Tt)) \\ &\leq \alpha\left(\max\left\{\begin{matrix} p(fu, gt), p(fu, Su), p(gt, Tt), \\ \frac{1}{2}[p(fu, Tt) + p(gt, Su)] \end{matrix}\right\}\right) \\ &\quad - \beta\left(\max\left\{\begin{matrix} p(fu, gt), p(fu, Su), p(gt, Tt), \\ \frac{1}{2}[p(fu, Tt) + p(gt, Su)] \end{matrix}\right\}\right) \\ &\leq \alpha(p(Su, Tt)) - \beta(p(Su, Tt)) < \psi(p(Su, u)).\end{aligned}$$

Is a contradiction. Hence, $u = Su = fu$.

Thus, $Tu = gu = Su = fu = u$.

Hence, u is a common fixed point of S, T, f and g.

Let w be another common fixed point of S, T, f and g.

Since $\frac{1}{2}\min\{p(fu, Su), p(gw, Tw)\} \leq p(fu, gw)$, from (2.1.1), we obtain

$$\begin{aligned}\psi(p(u, w)) &= \psi(p(Su, Tw)) \\ &\leq \alpha\left(\max\left\{\begin{matrix} p(fu, gw), p(fu, Su), p(gw, Tw), \\ \frac{1}{2}[p(fu, Tw) + p(gw, Su)] \end{matrix}\right\}\right) \\ &\quad - \beta\left(\max\left\{\begin{matrix} p(fu, gw), p(fu, Su), p(gw, Tw), \\ \frac{1}{2}[p(fu, Tw) + p(gw, Su)] \end{matrix}\right\}\right) \\ &\leq \alpha\left(\max\left\{\begin{matrix} p(u, w), p(u, u), p(w, w), \\ \frac{1}{2}[p(u, w) + p(w, u)] \end{matrix}\right\}\right) \\ &\quad - \beta\left(\max\left\{\begin{matrix} p(u, w), p(u, u), p(w, w), \\ \frac{1}{2}[p(u, w) + p(w, u)] \end{matrix}\right\}\right) \\ &\leq \alpha(p(u, w)) - \beta(p(u, w)) \\ &< \psi(p(u, w)),\end{aligned}$$

which is a contradiction. Hence, $u = w$.

Thus, u is the unique common fixed point of S, T, f and g.

Sub case(b) : Suppose $\frac{1}{2}p(y_{2n}, y_{2n+1}) \leq p(y_{2n}, u)$.

In this case, also, we can prove that u is the unique common fixed point of S, T, f and g by proceeding as in Subcase(a).

Case(ii): Suppose $y_{2m} = y_{2m+1}$ for some m.

Assume that $y_{2m+1} \neq y_{2m+2}$.

$$M(x_{2m+2}, x_{2m+1}) = \max\left\{\begin{array}{c} p(y_{2m+1}, y_{2m}), p(y_{2m+1}, y_{2m+2}), p(y_{2m}, y_{2m+1}), \\ \frac{1}{2}[p(y_{2m+1}, y_{2m+1}) + p(y_{2m}, y_{2m+2})] \end{array}\right\}.$$

However, $p(y_{2m+1}, y_{2m}) = p(y_{2m+1}, y_{2m+1}) \leq p(y_{2m+1}, y_{2m+2})$, from (p_2) and

$$\begin{aligned}\frac{1}{2}&[p(y_{2m+1}, y_{2m+1}) + p(y_{2m}, y_{2m+2})] \\ &\leq \frac{1}{2}[p(y_{2m}, y_{2m+1}) + p(y_{2m+1}, y_{2m+2})], \text{ from } (p_4) \\ &\leq \frac{1}{2}[p(y_{2m+1}, y_{2m+2}) + p(y_{2m+1}, y_{2m+2})] \\ &= p(y_{2m+1}, y_{2m+2}).\end{aligned}$$

Hence, $M(x_{2m+2}, x_{2m+1}) = p(y_{2m+1}, y_{2m+2})$.

$$\begin{aligned}\text{Since } &\frac{1}{2}\min\{p(fx_{2m+2}, Sx_{2m+2}), p(gx_{2m+1}, Tx_{2m+1})\} \\ &\leq p(gx_{2m+1}, Tx_{2m+1}) \\ &= p(fx_{2m+2}, gx_{2m+1}),\end{aligned}$$

from (2.1.1), we get

$$\begin{aligned}\psi(p(y_{2m+2}, y_{2m+1})) &= \psi(p(Sx_{2m+2}, Tx_{2m+1})) \\ &\leq \alpha(M(x_{2m+2}, x_{2m+1})) - \beta(M(x_{2m+2}, x_{2m+1})) \\ &= \alpha(p(y_{2m+2}, y_{2m+1})) - \beta(p(y_{2m+2}, y_{2m+1})) \\ &< \psi(p(y_{2m+2}, y_{2m+1})).\end{aligned}$$

It is a contradiction. Hence, $y_{2m+2} = y_{2m+1}$.

Continuing in this way, we can conclude that $y_n = y_{n+k}$ for all $k > 0$.

Thus, $\{y_n\}$ is a Cauchy sequence.

The rest of the proof follows as in Case(i). □

The following example illustrates our Theorem 2.1

Example 2.2 Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Let $f, g, S, T : X \to X, f(x) = \frac{x}{2}, g(x) = \frac{x}{3}, S(x) = \frac{x}{4}$ and $T(x) = \frac{x}{6}$, Let $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ be defined by $\psi(t) = 4t$, $\alpha(t) = 7t$ and $\beta(t) = \frac{7t}{2}$. Clearly, ψ is an altering distance function and α is continuous and β is lower semi continuous, $\alpha(0) = \beta(0) = 0$ and $\psi(t) - \alpha(t) + \beta(t) = \frac{t}{2} > 0$, for all $t > 0$.

Now

$$\begin{aligned}\frac{1}{2}\min\{p(fx, Sx), p(gy, Ty)\} &= \frac{1}{2}\min\{\max\{fx, Sx\}, \max\{gy, Ty\}\} \\ &= \frac{1}{2}\min\left\{\max\left\{\frac{x}{2}, \frac{x}{4}\right\}, \max\left\{\frac{y}{3}, \frac{y}{6}\right\}\right\} \\ &= \frac{1}{2}\min\left\{\frac{x}{2}, \frac{y}{3}\right\} \\ &\leq \frac{1}{2}\max\left\{\frac{x}{2}, \frac{y}{3}\right\} \\ &\leq p(fx, gy).\end{aligned}$$

$$\begin{aligned}\psi(p(Sx, Ty)) &= 4p(Sx, Ty) \\ &= 4\max\left\{\frac{x}{4}, \frac{y}{6}\right\} \\ &= 4 \times \frac{1}{2}\max\left\{\frac{x}{2}, \frac{y}{3}\right\} \\ &= 2p(fx, gy) \\ &\leq 2M(x, y) \\ &\leq 7M(x, y) - \frac{7}{2}M(x, y).\end{aligned}$$

So

$$\psi(p(Sx, Ty)) \leq \alpha(M(x, y)) - \beta(M(x, y)).$$

Therefore, all of the conditions of Theorem 2.1 are satisfied and 0 is the unique common fixed point of S, T, f and g.

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