Math Sci (2015) 9:205–213
DOI 10.1007/s40096-015-0169-2

ORIGINAL RESEARCH

Approximations to the distribution of sum of independent non-identically gamma random variables

H. Murakami[1]

Received: 31 May 2015 / Accepted: 22 October 2015 / Published online: 10 November 2015

Abstract Calculating the sum of independent non-identically distributed random variables is necessary in the scientific field. Computing the probability of the corresponding significance point is important in cases that have a finite sum of random variables. However, it is difficult to evaluate this probability when the number of random variables increases. Under these circumstances, consideration of a more accurate approximation of the distribution function is extremely important. A saddlepoint approximation is performed using upper probabilities from the distribution of the sum of independent non-identically gamma random variables under finite sample sizes. In this study, we compared the results from a saddlepoint approximation to those from normal and moment-based approximations to identify the most appropriate method to use for the distribution function.

Keywords Independent and non-identically distributed · Saddlepoint approximation · Sum of gamma random variables

Introduction

The distribution of the sum of independent identically distributed gamma random variables is well known. However, within the scientific field, it is necessary to know the distribution of the sum of independent non-identically distributed (i.n.i.d.) gamma random variables. For example, it would be necessary to know this distribution for calculating total waiting times where component times are assumed to be independent exponential or gamma random variables. In addition, engineers calculate total excess water flow into a dam as the sum of i.n.i.d. gamma random variables. To calculate the exact probability distribution of the sum of i.n.i.d. gamma random variables, the probability of all possible elements consistent with the sum must be computed. Mathai [12] derived the distribution of the sum of i.n.i.d. gamma random variables by converting the moment-generating function. Additionally, Moschopoulos [13] calculated the distribution of the sum of i.n.i.d. gamma random variables using a simple recursive relation approach. For the detail of the gamma distribution family, we refer the reader to Khodabin and Ahmadabadi [9]. However, Mathai [12] and Moschopoulos [13] derived the density of the sum of i.n.i.d. gamma random variables with infinite summation. This method of computation is intractable in practice, especially in cases in which there is an increase in the number of random variables. An exact calculation is feasible by applying the standard inversion formula to the characteristic function in computer algebra systems, such as Mathematica. However, in these calculations, the probability is estimated with an approximation method. Approximation methods are widely used and have been studied extensively. From a practical view, approximations are typically precise and straightforward to implement in various statistical software programs. Hence, obtaining a more accurate approximation for evaluating the density or the distribution function of i.n.i.d. random variables remains an important area of debate in statistics. In this study, we describe the use of approximation methods to calculate the distribution of the sum of i.n.i.d. gamma random variables in Sect. "A Saddlepoint approximation to the distribution of sum of i.n.i.d. gamma random variables". Furthermore, we

✉ H. Murakami
murakami@gug.math.chuo-u.ac.jp

1 Department of Mathematical Information Science, Tokyo University of Science, Tokyo, Japan

discuss the derivation of the order of errors of suggested approximation for the given distribution. For the approximation presented in this paper, we used the saddlepoint formula employed previously by Daniels [2, 3] and developed by Lugannani [11]. The saddlepoint approximation can be obtained for any statistic or random variable that contains a cumulant generating function. Additionally, the saddlepoint generates accurate probabilities in the tail of distribution. Saddlepoint approximations have been used with great success by several researchers. Excellent discussions of their applications to a range of distributional problems are found in the following studies: Jensen [8], Huzurbazar [7], Kolassa [10], and Butler [1]. Recently, Eisinga et al. [4] discussed the use of the saddlepoint approximation for the sum of i.n.i.d. binomial random variables. Additionally, Murakami [14] and Nadarajah [15] considered the use of the saddlepoint approximation for the sum of i.n.i.d. uniform and beta random variables, respectively. In Sect. "Numerical results", we discuss the results obtained from using the saddlepoint approximation. In Sect. "Concluding remarks", we summarize our conclusions.

A Saddlepoint approximation to the distribution of sum of i.n.i.d. gamma random variables

In this section, we discuss the use of the saddlepoint approximation of the sum of independent non-identically gamma random variables. We assumed that $X_1, \ldots, X_n$ are independent random variables, with shape, $\alpha_i > 0$, and scale parameters, $\beta_i > 0$, for $i = 1, \ldots, n$. Next, we let $S_n = X_1 + X_2 + \cdots + X_n$. The moment-generating function of S_n is

$$M_n(s) = \prod_{i=1}^{n} (1 - \beta_i s)^{-\alpha_i}.$$

It is important to note that Mathai [12] derived the density function of the sum of i.n.i.d. gamma random variables by converting its moment-generating function as follows:

$$\begin{aligned} f_{S_n}(x) = & \left\{ \prod_{i=1}^{n} \beta_i^{\alpha_i} \Gamma(\rho) \right\}^{-1} x^{\rho-1} \exp\left(-\frac{x}{\beta_1}\right) \\ & \times \sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} \left\{ (\alpha_2)_{r_2} \cdots (\alpha_n)_{r_n} \right\} \\ & \times \left[\left(\frac{1}{\beta_1} - \frac{1}{\beta_2} \right) x \right]^{r_2} \cdots \left[\left(\frac{1}{\beta_1} - \frac{1}{\beta_n} \right) x \right]^{r_n} \frac{1}{r_2! \cdots r_n! (\rho)_\rho}, \\ & x > 0, \end{aligned}$$

where $\rho = \alpha_1 + \cdots + \alpha_n$ and $(y)_z$ denote the Pochhammer symbol. In addition, Moschopoulos [13] obtained the density function of the sum of i.n.i.d. gamma random variables using the following simple recursive relation approach:

$$f_{S_n}(x) = \frac{C}{\Gamma(\rho + k) \beta_1^{\rho+k}} \sum_{k=0}^{\infty} \delta_k x^{\rho+k-1} \exp\left(-\frac{x}{\beta_1}\right), \quad x > 0,$$

where

$$C = \prod_{i=1}^{n} \left(\frac{\beta_1}{\beta_i} \right)^{\alpha_i},$$

$$\delta_{k+1} = \frac{1}{k+1} \sum_{i=1}^{k+1} i \omega_i \delta_{k+1-i}, \quad k = 0, 1, 2, \ldots,$$

$$\omega_k = \frac{1}{k} \sum_{i=1}^{n} \alpha_i (1 - \frac{\beta_1}{\beta_i})^k, \quad k = 1, 2, \ldots,$$

with $\delta_0 = 1$. It is difficult to evaluate the exact density of S_n with increasing n.

Herein, we consider an approximation to the distribution of S_n. The cumulant generating function of S_n is

$$\kappa_n(s) = -\sum_{i=1}^{n} \alpha_i \log(1 - \beta_i s).$$

Using the cumulant generating function, the mean, μ, and variance, σ^2, of S_n are given below:

$$\mu = \sum_{i=1}^{n} \alpha_i \beta_i \quad \text{and} \quad \sigma^2 = \sum_{i=1}^{n} \alpha_i \beta_i^2.$$

According to Daniels [3], the saddlepoint approximation of the density function of S_n is as follows:

$$f_s(v) = f_*(v)\{1 + O(n^{-1})\},$$

where

$$f_*(v) = \left\{ 2\pi \kappa_n''(\hat{s}) \right\}^{-\frac{1}{2}} \exp\{\kappa_n(\hat{s}) - \hat{s}v\},$$

$$\kappa_n'(s) = \sum_{i=1}^{n} \frac{\alpha_i \beta_i}{1 - \beta_i s}, \quad \kappa_n''(s) = \sum_{i=1}^{n} \frac{\alpha_i \beta_i^2}{(1 - \beta_i s)^2},$$

and $\hat{s}$ is the root of $\kappa_n'(s) = v$ which is readily solved numerically by the Newton–Raphson algorithm.

Several approaches have been used to further minimize the error of the saddlepoint approximation [5]. For example, one method uses a higher order approximation by including adjustments for the third and fourth cumulants [3]. A higher order saddlepoint approximation uses the following correction term:

$$f_s(v) = f_*(v) \left\{ 1 + \frac{1}{8} \frac{\kappa_n^{(4)}(\hat{s})}{\kappa_n''(\hat{s})^2} - \frac{5}{24} \frac{\kappa_n^{(3)}(\hat{s})^2}{\kappa_n''(\hat{s})^3} + O(n^{-2}) \right\}, \tag{1}$$

where

$$\kappa_n^{(3)}(s) = \sum_{i=1}^{n} \frac{2\alpha_i \beta_i^3}{(1 - \beta_i s)^3}, \quad \kappa_n^{(4)}(s) = \sum_{i=1}^{n} \frac{6\alpha_i \beta_i^4}{(1 - \beta_i s)^4}.$$

The approximate tail probabilities of S_n are determined by numerically integrating Eq. (1).

An alternative approach is to use the Lugannani and Rice [11] for the continuous tail probability approximation as follows:

$$\Pr(S_n < v) \approx \Phi(\hat{w}) - \phi(\hat{w})\left(\frac{1}{\hat{u}} - \frac{1}{\hat{w}}\right),$$

where $\phi(\cdot)$ is the standard normal density function, $\Phi(\cdot)$ is the corresponding cumulative distribution function, and

$$\hat{w} = \sqrt{2(\hat{s}v - \kappa_n(\hat{s}))}\mathrm{sgn}(\hat{s}), \quad \hat{u} = \hat{s}\sqrt{\kappa_n''(\hat{s})},$$

where $\mathrm{sgn}(\hat{s}) = \pm 1$, 0 if $\hat{s}$ is positive, negative, or zero.

Numerical results

In this section, we investigated the upper probability using the saddlepoint approximations to S_n. In this study, we focused on the Lugannani–Rice formula. Note that Mathai [12] obtained a normal approximation with $n \to \infty$. Moschopoulos [13] derived the density of S_n with infinite summation as follows:

$$f_{S_n}(x) = \frac{C}{\Gamma(\rho+k)\beta_1^{\rho+k}} \sum_{k=0}^{\infty} \delta_k x^{\rho+k-1} \exp\left(-\frac{x}{\beta_1}\right), \quad x > 0.$$

We used a finite number and truncated the infinite series to meet an acceptable precision as published by Moschopoulos [13]. This equation is listed as follows:

$$\begin{aligned} f_{S_n}(x) &= \frac{C}{\Gamma(\rho+k)\beta_1^{\rho+k}} \sum_{k=0}^{\infty} \delta_k x^{\rho+k-1} \exp\left(-\frac{x}{\beta_1}\right) \\ &= \frac{C}{\Gamma(\rho)\beta_1^{\rho}} x^{\rho-1} \exp\left(-\frac{x}{\beta_1}\right) \sum_{k=0}^{\infty} \left(\frac{\delta_k}{(\rho)_k}\right)\left(\frac{x}{\beta_1}\right)^k \\ &\leq \frac{C}{\Gamma(\rho)\beta_1^{\rho}} x^{\rho-1} \exp\left(-\frac{x}{\beta_1}\right) \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{bx}{\beta_1}\right)^k \\ &= \frac{C}{\Gamma(\rho)\beta_1^{\rho}} x^{\rho-1} \exp\left(-\frac{x(1-b)}{\beta_1}\right), \end{aligned}$$

where $(\rho)_k = \rho(\rho+1)\cdots(\rho+k-1)$, $(\rho)_0 = 1$ and $b = \max_{2\leq i\leq n}(1-\beta_1/\beta_i)$. To bound the truncation error with the sum of the first $\ell + 1$, we used the following equation:

$$\begin{aligned} E(w) = &\int_0^w \frac{C}{\Gamma(\rho)\beta_1^{\rho}} x^{\rho-1} \exp\left(-\frac{x(1-b)}{\beta_1}\right) dx \\ &- \int_0^w \frac{C}{\Gamma(\rho+k)\beta_1^{\rho+k}} \sum_{k=0}^{\ell} \delta_k x^{\rho+k-1} \exp\left(-\frac{x}{\beta_1}\right) dx. \end{aligned}$$

In addition, we used another approximation method for the distribution of S_n, a moment-based approximation proposed by Ha and Provost [6]. The distribution of S_n is approximated by the polynomial adjusted $\tilde{f}_k(v)$ such that

$$\tilde{f}_k(v) = \psi(v) \sum_{\ell=0}^{k} \xi_\ell v^\ell,$$

where

$$\begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_k \end{pmatrix} = \begin{pmatrix} m(0) & m(1) & \cdots & m(k-1) & m(k) \\ m(1) & m(2) & \cdots & m(k) & m(k+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m(k) & m(k+1) & \cdots & m(2k-1) & m(2k) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \mathrm{E(M)} \\ \vdots \\ \mathrm{E(M^k)} \end{pmatrix}$$

and $m(k)$ and $\mathrm{E(M^k)}$ denote the kth moment of the adjusted distribution $\psi(v)$ and the kth moment of S_n, respectively.

Herein, we consider the approximation adjusted with the skew-normal distribution as follows:

$$\psi_*(v) = \frac{2}{\sigma}\phi\left(\frac{v-\ell}{g}\right)\Phi\left(\frac{\lambda(v-\ell)}{g}\right),$$

where

$$\begin{aligned} &\ell = \mu - \delta g\sqrt{\frac{2}{\pi}}, \quad g^2 = \sigma^2\left(1 - \frac{2\delta^2}{\pi}\right)^{-1}, \\ &\quad \zeta = \frac{\mathrm{E(M_p^3)} - 3\mu\sigma^2 - \mu^3}{\sigma^3} \\ &\lambda = \frac{\delta}{\sqrt{1-\delta^2}}, \quad |\delta| = \mathrm{Sign}(\zeta)\sqrt{\frac{\pi}{2}\frac{|\xi|^{2/3}}{|\xi|^{2/3} + \left(\frac{4-\pi}{2}\right)^{2/3}}}, \\ &\quad \xi = \min(0.99, |\zeta|). \end{aligned}$$

Then,

$$m(k) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left\{2\exp\left(\ell t + \frac{g^2t^2}{2}\right)\Phi\left(\frac{\lambda g t}{\sqrt{1+\lambda^2}}\right)\right\}\Bigg|_{t=0}.$$

Note that we obtained $\xi_0 = 1, \xi_1 = \xi_2 = 0$ for $k = 2$. Afterwards, the moment-based approximation with skew-normal polynomial was as follows:

$$\tilde{f}_2(v) = \psi_*(v)\sum_{\ell=0}^{2} \xi_\ell v^\ell = \psi_*(v)(1+0+0) = \psi_*(v).$$

An important step for the proposed method is to determine the optimal degrees for the polynomials. We followed the selection rule, which is based on the integrated squared differences between density approximations as previously published by Ha and Provost [6].

For this study, the following notations were utilized: exact probability of S_n, E_{P}, (as proposed by Moschopoulos

Table 1 Numerical results for α % significance level for case 1 ($n = 5$)

v	E_{P}	A_{L}	A_{N}	A_{M}	r.e. A_{L}	r.e. A_{N}	r.e. A_{M}
Case A							
18.4956	0.6000	0.6002	0.5504	0.5997	0.0003	0.0827	0.0005
20.3703	0.7000	0.7002	0.6615	0.6975	0.0002	0.0550	0.0036
22.7281	0.8000	0.8001	0.7826	0.7974	0.0002	0.0218	0.0033
26.3012	0.9000	0.9001	0.9088	0.8998	0.0001	0.0098	0.0003
29.5275	0.9500	0.9500	0.9665	0.9510	0.0000	0.0174	0.0011
32.5345	0.9750	0.9750	0.9892	0.9758	0.0000	0.0145	0.0009
36.2813	0.9900	0.9900	0.9980	0.9901	0.0000	0.0081	0.0001
Case B							
726.059	0.6000	0.6000	0.5824	0.6005	0.0000	0.0293	0.0009
753.042	0.7000	0.7000	0.6872	0.7002	0.0000	0.0183	0.0003
785.493	0.8000	0.8000	0.7952	0.7998	0.0000	0.0060	0.0002
832.073	0.9000	0.9000	0.9045	0.8996	0.0000	0.0050	0.0004
871.945	0.9500	0.9500	0.9574	0.9498	0.0000	0.0078	0.0002
907.574	0.9750	0.9750	0.9817	0.9750	0.0000	0.0069	0.0000
950.242	0.9900	0.9900	0.9944	0.9901	0.0000	0.0044	0.0001
Case C							
10.1143	0.6000	0.6004	0.4883	0.5785	0.0006	0.1861	0.0365
12.2272	0.7000	0.7001	0.6035	0.6937	0.0001	0.1379	0.0532
15.1163	0.8000	0.7999	0.7457	0.8106	0.0002	0.0679	0.0224
19.9089	0.9000	0.8998	0.9070	0.9148	0.0003	0.0077	0.0297
24.5886	0.9500	0.9498	0.9755	0.9537	0.0002	0.0268	0.0107
29.1965	0.9750	0.9749	0.9954	0.9715	0.0001	0.0209	0.0129
35.2135	0.9900	0.9899	0.9997	0.9862	0.0001	0.0098	0.0047
Case D							
61.4894	0.6000	0.6006	0.5605	0.5998	0.0009	0.0659	0.0003
66.2173	0.7000	0.7006	0.6694	0.6993	0.0008	0.0437	0.0010
72.0867	0.8000	0.8005	0.7863	0.7994	0.0006	0.0171	0.0007
80.8566	0.9000	0.9004	0.9073	0.9003	0.0004	0.0081	0.0003
88.6810	0.9500	0.9502	0.9639	0.9504	0.0002	0.0147	0.0003
95.9162	0.9750	0.9751	0.9873	0.9751	0.0001	0.0126	0.0001
104.878	0.9900	0.9901	0.9973	0.9899	0.0001	0.0073	0.0001
Case E							
0.326492	0.6000	0.6088	0.4596	0.5533	0.0146	0.2339	0.0778
0.426459	0.7000	0.7104	0.5691	0.6850	0.0149	0.1870	0.0215
0.570043	0.8000	0.8120	0.7155	0.8228	0.0150	0.1056	0.0285
0.821751	0.9000	0.9107	0.8967	0.9321	0.0119	0.0037	0.0356
1.080503	0.9500	0.9569	0.9759	0.9545	0.0073	0.0273	0.0048
1.345989	0.9750	0.9787	0.9966	0.9638	0.0038	0.0222	0.0115
1.707191	0.9900	0.9913	0.9999	0.9832	0.0013	0.0100	0.0069
Case F							
259.5839	0.6000	0.6000	0.5772	0.6005	0.0001	0.0380	0.0008
272.0152	0.7000	0.7000	0.6832	0.7002	0.0000	0.0240	0.0003
287.0790	0.8000	0.8000	0.7935	0.7998	0.0000	0.0082	0.0002
308.9076	0.9000	0.9000	0.9055	0.8997	0.0000	0.0061	0.0003
327.7724	0.9500	0.9500	0.9592	0.9499	0.0000	0.0097	0.0001
344.7615	0.9750	0.9750	0.9833	0.9751	0.0000	0.0085	0.0001
365.2610	0.9900	0.9900	0.9952	0.9901	0.0000	0.0053	0.0001

Table 2 Numerical results for α % significance level for case 2 ($n = 10$)

v	E_P	A_L	A_N	A_M	r.e. A_L	r.e. A_N	r.e. A_M
Case A							
26.3644	0.6000	0.6002	0.5579	0.5997	0.0003	0.0702	0.0006
28.5636	0.7000	0.7002	0.6677	0.6992	0.0003	0.0462	0.0012
31.3020	0.8000	0.8002	0.7858	0.7995	0.0002	0.0177	0.0006
35.4041	0.9000	0.9001	0.8909	0.9004	0.0001	0.0090	0.0004
39.0681	0.9500	0.9501	0.9647	0.9503	0.0001	0.0155	0.0004
42.4555	0.9750	0.9750	0.9878	0.9750	0.0000	0.0132	0.0000
46.6462	0.9900	0.9900	0.9975	0.9898	0.0000	0.0075	0.0002
Case B							
1096.671	0.6000	0.6000	0.5855	0.6003	0.0000	0.0242	0.0005
1128.481	0.7000	0.7000	0.6895	0.7002	0.0000	0.0150	0.0003
1166.561	0.8000	0.8000	0.7961	0.7999	0.0000	0.0049	0.0001
1220.920	0.9000	0.9000	0.9038	0.8998	0.0000	0.0042	0.0002
1267.189	0.9500	0.9500	0.9562	0.9499	0.0000	0.0066	0.0001
1308.347	0.9750	0.9750	0.9807	0.9750	0.0000	0.0059	0.0000
1357.419	0.9900	0.9900	0.9938	0.9901	0.0000	0.0038	0.0001
Case C							
3.62148	0.6000	0.6186	0.4116	0.4838	0.0310	0.3140	0.1936
4.71105	0.7000	0.7107	0.5012	0.6281	0.0152	0.2840	0.1027
6.54256	0.8000	0.8050	0.6494	0.8154	0.0062	0.1882	0.0193
10.1893	0.9000	0.9011	0.8732	0.9645	0.0013	0.0298	0.0716
14.2326	0.9500	0.9500	0.9763	0.9552	0.0000	0.0277	0.0055
18.5259	0.9750	0.9747	0.9980	0.9638	0.0003	0.0236	0.0115
24.4566	0.9900	0.9898	1.0000	0.9829	0.0002	0.0101	0.0170
Case D							
65.5973	0.6000	0.6007	0.5653	0.6000	0.0012	0.0578	0.0000
69.7384	0.7000	0.7007	0.6732	0.6995	0.0010	0.0038	0.0007
74.8464	0.8000	0.8007	0.7880	0.7995	0.0008	0.0151	0.0006
82.4243	0.9000	0.9005	0.9065	0.9001	0.0006	0.0073	0.0001
89.1426	0.9500	0.9503	0.9625	0.9503	0.0003	0.0132	0.0003
95.3289	0.9750	0.9752	0.9863	0.9751	0.0002	0.0116	0.0001
102.966	0.9900	0.9901	0.9968	0.9899	0.0001	0.0069	0.0001
Case E							
3.24261	0.6000	0.6023	0.5000	0.5879	0.0038	0.1666	0.0201
3.83360	0.7000	0.7019	0.6131	0.6957	0.0028	0.1241	0.0061
4.63308	0.8000	0.8013	0.7506	0.8058	0.0017	0.0618	0.0072
5.95032	0.9000	0.9007	0.9061	0.9091	0.0008	0.0067	0.0102
7.23386	0.9500	0.9504	0.9739	0.9524	0.0004	0.0251	0.0025
8.49852	0.9750	0.9752	0.9947	0.9723	0.0002	0.0202	0.0028
10.1529	0.9900	0.9901	0.9996	0.9871	0.0001	0.0097	0.0029
Case F							
325.9534	0.6000	0.6000	0.5823	0.6005	0.0000	0.0295	0.0009
338.0179	0.7000	0.7000	0.6871	0.7002	0.0000	0.0184	0.0003
352.5290	0.8000	0.8000	0.7952	0.7998	0.0000	0.0060	0.0003
373.3628	0.9000	0.9000	0.9046	0.8996	0.0000	0.0051	0.0005
391.1991	0.9500	0.9500	0.9575	0.9498	0.0000	0.0078	0.0002
407.1389	0.9750	0.9750	0.9818	0.0000	0.0000	0.0069	0.0000
426.2288	0.9900	0.9900	0.9944	0.9901	0.0000	0.0044	0.0001

Table 3 Numerical results for α % significance level for case 3 ($n = 15$)

v	E_P	A_L	A_N	A_M	r.e. A_L	r.e. A_N	r.e. A_M
Case A							
38.7943	0.6000	0.6002	0.5679	0.6001	0.0003	0.0535	0.0002
41.1704	0.7000	0.7002	0.6757	0.6996	0.0002	0.0347	0.0006
44.0891	0.8000	0.8001	0.7897	0.7995	0.0002	0.0128	0.0007
48.3914	0.9000	0.9001	0.9068	0.9000	0.0001	0.0075	0.0000
52.1756	0.9500	0.9501	0.9620	0.9503	0.0001	0.0127	0.0003
55.6331	0.9750	0.9750	0.9857	0.9752	0.0000	0.0110	0.0002
59.8647	0.9900	0.9900	0.9965	0.9900	0.0000	0.0066	0.0000
Case B							
1588.181	0.6000	0.6000	0.5880	0.6002	0.0000	0.0200	0.0003
1625.798	0.7000	0.7000	0.6914	0.7002	0.0000	0.0123	0.0002
1670.661	0.8000	0.8000	0.7990	0.8000	0.0000	0.0039	0.0000
1734.395	0.9000	0.9000	0.9032	0.8999	0.0000	0.0036	0.0001
1788.379	0.9500	0.9500	0.9553	0.9499	0.0000	0.0055	0.0001
1836.205	0.9750	0.9750	0.9798	0.9750	0.0000	0.0050	0.0000
1893.005	0.9900	0.9900	0.9932	0.9900	0.0000	0.0033	0.0000
Case C							
368.9250	0.6000	0.6004	0.5773	0.6005	0.0007	0.0379	0.0008
385.2496	0.7000	0.7006	0.6977	0.7001	0.0008	0.0242	0.0002
405.0313	0.8000	0.8008	0.7931	0.7997	0.0010	0.0086	0.0004
433.7166	0.9000	0.9012	0.9052	0.8996	0.0013	0.0057	0.0004
458.5522	0.9500	0.9516	0.9590	0.9499	0.0017	0.0095	0.0001
480.9795	0.9750	0.9772	0.9833	0.9751	0.0023	0.0085	0.0001
508.1583	0.9900	0.9925	0.9952	0.9901	0.0025	0.0053	0.0001
Case D							
236.3144	0.6000	0.6001	0.5777	0.6005	0.0002	0.0372	0.0008
245.9966	0.7000	0.7001	0.6834	0.7002	0.0002	0.0237	0.0003
257.7243	0.8000	0.8001	0.7933	0.7998	0.0001	0.0083	0.0002
274.7178	0.9000	0.9001	0.9052	0.8997	0.0001	0.0058	0.0003
289.4113	0.9500	0.9500	0.9589	0.9499	0.0000	0.0094	0.0001
302.6551	0.9750	0.9750	0.9831	0.9751	0.0000	0.0084	0.0001
318.6544	0.9900	0.9900	0.9952	0.9901	0.0000	0.0052	0.0001
Case E							
1.78598	0.6000	0.6156	0.4898	0.5948	0.0260	0.1836	0.0253
2.10679	0.7000	0.7173	0.5966	0.6977	0.0247	0.1478	0.0032
2.55103	0.8000	0.8174	0.7318	0.8139	0.0218	0.0852	0.0174
3.31351	0.9000	0.9131	0.8962	0.9176	0.0146	0.0043	0.0196
4.09487	0.9500	0.9576	0.9724	0.9525	0.0080	0.0236	0.0027
4.90124	0.9750	0.9787	0.9953	0.9675	0.0038	0.0208	0.0077
6.00860	0.9900	0.9912	0.9998	0.9846	0.0012	0.0099	0.0055
Case F							
624.837	0.6000	0.6000	0.5866	0.6003	0.0000	0.0223	0.0004
642.419	0.7000	0.7000	0.6903	0.7002	0.0000	0.0138	0.0003
663.431	0.8000	0.8000	0.7965	0.0000	0.0000	0.0044	0.0000
693.358	0.9000	0.9000	0.9036	0.8998	0.0000	0.0040	0.0002
718.771	0.9500	0.9500	0.9558	0.9499	0.0000	0.0061	0.0001
741.331	0.9750	0.9750	0.9803	0.9750	0.0000	0.0055	0.0000
768.173	0.9900	0.9900	0.9935	0.9900	0.0000	0.0036	0.0000

[13]); normal approximation, A_{N}; saddlepoint approximation with Lugannani–Rice formula, A_{L}; moment-based approximation with skew-normal polynomial, A_{M}; and the relative error of approximations, r.e. (Tables 1, 2, 3). We used different values for $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$. These values were grouped into cases 1–3. Herein, we assumed that α_i and β_i for $n = 5, 10$ and 15 as follows:

Case 1: α_i and β_i were simulated from Case A: Uniform distribution with interval [0, 3] independently as

$$\alpha_i = (1.04022, 1.52149, 2.96165, 0.77156, 1.93264)$$

$$\beta_i = (2.93353, 2.60821, 2.49735, 1.57684, 1.05720)$$

Case B: Poisson distribution with parameter $\lambda = 10$ independently as

$$\alpha_i = (9, 10, 18, 8, 11)$$
$$\beta_i = (17, 14, 13, 10, 9)$$

Case C: Lognormal distribution with parameters location $\mu = 0$ and scale $\sigma = 2$ independently as

$$\alpha_i = (0.05459, 0.87723, 0.98562, 1.37783, 6.40726)$$

$$\beta_i = (1.68872, 0.39881, 0.25645, 6.14009, 0.18285)$$

Case D: Gamma distribution with parameters shape $\gamma = 2.0$ and scale $\xi = 1.5$ independently as

$$\alpha_i = (1.96560, 1.89408, 3.00261, 4.28812, 3.01364)$$

$$\beta_i = (6.99957, 5.68468, 3.15081, 3.49359, 3.32123)$$

Case E: Exponential distribution with parameter $\lambda = 2.0$ independently as

$$\alpha_i = (0.52959, 0.33946, 0.00643, 0.67897, 0.21986)$$

$$\beta_i = (0.01120, 0.06997, 0.09169, 0.32160, 0.52149)$$

Case F: Binomial distribution with parameter $N = 20$, $p = 0.3$ independently as

$$\alpha_i = (5, 6, 11, 5, 7)$$
$$\beta_i = (10, 8, 8, 6, 5)$$

Case 2: α_i and β_i were simulated from Case A: Uniform [0, 3] independently as

$$\alpha_i = (0.22417, 1.14752, 0.50906, 1.98942, 2.72316, 2.50722, 1.28708, 0.52985, 2.61593, 2.06543)$$

$$\beta_i = (2.85308, 0.91297, 2.36745, 0.57299, 2.95146, 2.01277, 2.77988, 0.36263, 0.10206, 1.98484)$$

Case B: Poisson distribution with parameter $\lambda = 10$ independently as

$$\alpha_i = (6, 9, 7, 11, 14, 13, 9, 7, 14, 11)$$
$$\beta_i = (15, 8, 12, 7, 17, 11, 15, 6, 5, 11)$$

Case C: Lognormal distribution with parameters location $\mu = 0$ and scale $\sigma = 2$ independently as

$$\alpha_i = (0.11560, 10.5703, 0.18732, 0.05774, 9.51800, 0.32369, 0.01573, 0.38308, 0.15837, 0.50578)$$

$$\beta_i = (0.57765, 0.00716, 0.04465, 0.01232, 0.13829, 0.69592, 0.62148, 7.70751, 0.11445, 0.04548)$$

Case D: Gamma distribution with parameters shape $\gamma = 2.0$ and scale $\xi = 1.5$ independently as

$$\alpha_i = (2.17916, 1.10074, 1.40375, 0.55393, 3.18918, 3.27868, 5.79357, 0.74198, 2.54858, 1.52722)$$

$$\beta_i = (4.51414, 3.89484, 2.66041, 3.88386, 1.83366, 2.45540, 2.29499, 2.67110, 1.79620, 6.17920)$$

Case E: Exponential distribution with parameter $\lambda = 2.0$ independently as

$$\alpha_i = (0.63848, 0.49992, 0.68366, 0.20362, 0.39683, 0.03837, 1.13507, 0.31168, 0.24129, 0.29315)$$

$$\beta_i = (1.54349, 0.15572, 1.81201, 0.21383, 0.25284, 0.31853, 0.27480, 0.62515, 0.89060, 0.21336)$$

Case F: Binomial distribution with parameter $N = 20$, $p = 0.3$ independently as

$$\alpha_i = (5, 5, 5, 7, 6, 9, 3, 6, 7, 6)$$
$$\beta_i = (3, 7, 2, 7, 6, 6, 6, 5, 4, 7)$$

Case 3: α_i and β_i were simulated from Case A: Uniform distribution with interval [0, 3] independently as

$$\alpha_i = (0.58008, 2.22637, 2.51611, 1.12297, 2.29383, 1.18906, 1.57483, 0.31849, 2.53235, 2.27937, 2.91811, 2.57865, 0.79358, 1.86520, 2.46924)$$

$$\beta_i = (0.28821, 1.91593, 2.49916, 0.86430, 0.84279, 0.11141, 1.67239, 2.36912, 2.71671, 0.77938, 1.87762, 0.17339, 0.96113, 0.79465, 1.37481)$$

Case B: Poisson distribution with parameter $\lambda = 10$ independently as

$$\alpha_i = (14, 8, 7, 6, 11, 8, 9, 10, 14, 9, 10, 7, 13, 12, 15)$$

$$\beta_i = (11, 8, 12, 10, 16, 11, 4, 7, 5, 10, 8, 10, 18, 6, 14)$$

Case C: Lognormal distribution with parameter location $\mu = 0$ and scale $\sigma = 2$ independently as

$$\alpha_i = (0.65787, 2.10546, 27.7337, 5.30692, 1.32748, 6.60756, 0.06394, 0.44348, 0.62371, 2.24283, 2.82459, 9.36521, 0.82459, 0.27752, 0.78127)$$

$$\beta_i = (0.35243, 0.03793, 9.39041, 0.64510, 5.61762, 1.26038, 33.0086, 2.74834, 0.82037, 9.86410, 9.98734, 0.85253, 18.6350, 0.06063, 0.15637)$$

Case D: Gamma distribution with parameter shape $\gamma = 2.0$ and scale $\xi = 1.5$ independently as

$$\alpha_i = (6.89214, 8.05464, 1.88477, 7.43300, 3.20878, 3.90603, 0.939177, 1.5469, 12.2503, 5.63549, 1.49472, 3.16031, 1.32145, 1.69085, 0.815398)$$

$$\beta_i = (2.62198, 1.1429, 4.84865, 1.37041, 7.08359, 5.63902, 3.86697, 4.96188, 3.69634, 7.9203, 6.06043, 4.08442, 0.501449, 3.76872, 9.79386)$$

Case E: Exponential distribution with parameter $\lambda = 2.0$ independently as

$$\alpha_i = (0.32981, 0.46440, 0.08766, 0.29351, 0.16135, 1.34496, 0.17082, 0.05857, 0.34622, 0.11345, 0.31936, 0.23438, 0.21816, 0.22887, 0.06095)$$

$$\beta_i = (0.05570, 0.88621, 0.78224, 0.42277, 1.07697, 0.23754, 0.43072, 0.20499, 0.24145, 0.38523, 0.13308, 0.19507, 1.63801, 0.15344, 0.11302)$$

Case F: Binomial distribution with parameter $N = 20$, $p = 0.3$ independently as

$$\alpha_i = (6, 5, 8, 6, 7, 3, 7, 9, 6, 8, 6, 7, 7, 7, 8)$$
$$\beta_i = (9, 4, 4, 6, 4, 7, 6, 7, 7, 6, 7, 7, 3, 7, 8)$$

The results listed in Tables 1, 2 and 3 indicate that the A_M approximation was more suitable than the normal approximation for the distribution of S_n. In support of this, we observed that the A_L approximation was more accurate than the A_M approximation in all cases tested. Therefore, we suggest estimating the probability using the A_L approximation in cases with large n.

Concluding remarks

In this paper, we considered both the saddlepoint and moment-based approximations on the distribution of the sum of i.n.i.d. gamma random variables. Use of the saddlepoint approximation was an accurate method for calculating distribution. From our results, we determined that the precision of the saddlepoint approximation was superior to both the normal and moment-based approximations.

Acknowledgments The author would like to thank the editor and the referee for their valuable comments and suggestions. The author appreciates that this research is supported by the Grant-in-Aid for Young Scientists B of JSPS, KAKENHI Number 26730025.

References

1. Butler, R.W.: Saddlepoint Approximations with Applications. Cambridge University Press, Cambridge (2007)
2. Daniels, H.E.: Saddlepoint approximations in statistics. Ann. Math. Stat. **25**, 631–650 (1954)
3. Daniels, H.E.: Tail probability approximations. Int. Stat. Rev. **55**, 37–48 (1987)
4. Eisinga, R., Grotenhuis, M.T., Pelzer, B.: Saddlepoint approximation for the sum of independent non-identically distributed binomial random variables. Stat. Neerl. **67**, 190–201 (2013)
5. Gillespie, C.S., Renshaw, E.: An improved saddlepoint approximation. Math. Biosci. **208**, 359–374 (2007)
6. Ha, H.-T., Provost, S.B.: A viable alternative to resorting to statistical tables. Commun. Stat. Simul. Comput. **36**, 1135–1151 (2007)
7. Huzurbazar, S.: Practical saddlepoint approximations. Am. Stat. **53**, 225–232 (1999)
8. Jensen, J.L.: Saddlepoint Approximations. Oxford University Press, Oxford (1995)
9. Khodabin, M., Ahmadabadi, A.: Some properties of generalized gamma distribution. Math. Sci. **4**, 9–28 (2010)
10. Kolassa, J.E.: Series Approximation Methods in Statistics. Springer-Verlag, New York (2006)
11. Lugannani, R., Rice, S.O.: Saddlepoint approximation for the distribution of the sum of independent random variables. Adv. Appl. Probab. **12**, 475–490 (1980)
12. Mathai, A.M.: Storage capacity of a dam with gamma type inputs. Ann. Inst. Stat. Math. **34**, 591–597 (1982)
13. Moschopoulos, P.G.: The distribution of the sum of independent gamma random variables. Ann. Inst. Stat. Math. **37**, 541–544 (1985)
14. Murakami, H.: A saddlepoint approximation to the distribution of the sum of independent non-identically uniform random variables. Stat. Neerl. **68**, 267–275 (2014)
15. Nadarajah, S., Jiang, X., Chu, J.: A saddlepoint approximation to the distribution of the sum of independent non-identically beta random variables. Stat. Neerl. **69**, 102–114 (2015)