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ORIGINAL RESEARCH

A survey on Christoffel–Darboux type identities of Legendre, Laguerre and Hermite polynomials

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Abstract In this paper, we construct some new Christoffel–Darboux type identities for Legendre, Laguerre and Hermite polynomials. We obtain these types of identities for the derivatives of these polynomials.

Keywords Christoffel–Darboux identity · Cauchy kernel · Legendre polynomials · Laguerre polynomials · Hermite polynomials

Mathematical Subject Classification 33D45 · 33D50 · 45E05

Introduction

In [1], we have simplified the fraction

$$P_n(x,y)=\frac{P_n(y)-P_n(x)}{y-x}, \tag{1}$$

in terms of $P_i(x)$,$P_j(y)$ where $P_n\in\{T_n,U_n,V_n,W_n\}$. Also, for every kind of Chebyshev polynomials, we have obtained the expanded form of the fraction

$$P_{n,s}(x,y)=\frac{P_n^{(s)}(y)-P_n^{(s)}(x)}{y-x},\quad P_n\in\{T_n,U_n,V_n,W_n\}. \tag{2}$$

in terms of $P_i(x)$, $P_j(y)$ where $P_n^{(s)}(x)$ is the sth derivative of $P_n(x)$.

In this paper, we expand the fraction (1) where $P_n(x)$ is Legendre, Laguerre and Hermite polynomials.

Christoffel–Darboux type identities of Legendre, Laguerre and Hermite polynomials

Theorem 2.1 *Let* $\{P_n(x)\}_{n=0}^{\infty}$ *be a sequence of orthogonal polynomials with respect to the weight function* $w(x)$ *on interval* $[a, b]$ *then*

$$\begin{aligned}P_{n+1}(x,y)=\frac{P_{n+1}(y)-P_{n+1}(x)}{y-x}&=\sum_{i=0}^{n}\sum_{j=0}^{n-i}A_{i,j}^{n+1}P_i(x)P_j(y)\\&=\sum_{i=0}^{n}\sum_{j=0}^{i}A_{n-i,j}^{n+1}P_{n-i}(x)P_j(y),\end{aligned} \tag{3}$$

where

$$\begin{aligned}A_{i,j}^{n+1}&=\frac{1}{\gamma_i\gamma_j}\sum_{k=i+j+1}^{n+1}\sum_{v=i}^{k-j-1}C_{n+1,k}\,B_{v,i}\,B_{k-v-1,j}\\&=\frac{1}{\gamma_i\gamma_j}\sum_{k=0}^{n-i-j}\sum_{v=0}^{k}C_{n+1,k+i+j+1}\,B_{v+i,i}\,B_{k+j-v,j}\end{aligned} \tag{4}$$

$$\gamma_i=\int_a^b P_n^2(x)w(x)\,\mathrm{d}x,$$

$$B_{m,n}=\int_a^b x^m P_n(x)w(x)\,\mathrm{d}x,$$

and $C_{n+1,k}$ *is the coefficient of* x^k *in* $P_{n+1}(x)$.

Proof $P_n(x)$ is orthogonal to every polynomial of degree less than n. So, if $i+j>n$ then $A_{i,j}^{n+1}=0$. If $i+j\le n$ then use orthogonality and expanded form of $P_n(x)$ to obtain the result. □

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Corollary 2.1 *If the interval* $[a, b]$ *is symmetric about the origin and* $P_n(-x) = (-1)^n P_n(x)$ *then for even* $n+i+j$, $A^n_{i,j} = 0$.

If the linearization formula of $P_n(x)$ is available then we can compute $A_{i,j}$ coefficients in Eq. (3) by using one sum instead of using double sum in Eq. (4).

Christoffel–Darboux type identities of Hermite polynomials

Theorem 3.1 *Let* $H_n(x)$ *be Hermite polynomial of degree n then*

$$H_n(x,y) = \frac{H_n(y) - H_n(x)}{y - x} = \sum_{i=0}^{n-1}\sum_{j=0}^{n-i-1} A^n_{i,j} H_i(x) H_j(y),$$

where

$$\begin{aligned} A^n_{i,j} &= \left(\frac{1-(-1)^{i+j+n}}{2}\right) \\ &\times \left\{ \frac{1}{2^i i!}\sum_{k=0}^{j} (-1)^{\left(\frac{3n-3j+6k+i+1}{2}\right)} \frac{2^{\left(\frac{n-j+i-1}{2}\right)}}{k!} \binom{n}{j-k} \right. \\ &\times \Gamma\left(\frac{n-j+i+2k+1}{2}\right) + \frac{1}{2^j j!}\sum_{k=0}^{i} (-1)^{\left(\frac{3n-3i+6k+j+1}{2}\right)} \\ &\left. \times \frac{2^{\left(\frac{n-i+j-1}{2}\right)}}{k!}\binom{n}{i-k} \times \Gamma\left(\frac{n-i+j+2k+1}{2}\right) \right\}. \end{aligned} \tag{5}$$

Proof First, we prove that

$$\begin{aligned} H_{m,n}(x,y) &= P.V. \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{H_m(x)H_n(y)e^{-x^2}e^{-y^2}}{y-x}\, dy dx \\ &= 2^{\frac{m+n-1}{2}}(-1)^{n+1}\sin\left(\frac{m+n}{2}\pi\right)\Gamma\left(\frac{m+n+1}{2}\right)\pi, \\ & m,n = 0,1,2,\ldots. \end{aligned} \tag{6}$$

From [5], use the Hilbert transform of $H_n(y)e^{-y^2}$ to obtain

$$\begin{aligned} P.V. \int_{-\infty}^{\infty} \frac{H_n(y)e^{-y^2}}{y-x}\, dy &= (2\pi)^{n+1}\sqrt{\pi}(-1)^{n+1} \\ &\times \int_0^{\infty} f^n e^{-\pi^2 f^2} \sin\left(2\pi f x + \frac{n\pi}{2}\right) df \end{aligned}$$

So

$$\begin{aligned} H_{m,n}(x,y) &= P.V. \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{H_m(x)H_n(y)e^{-x^2}e^{-y^2}}{y-x}\, dy dx \\ &= (2\pi)^{n+1}\sqrt{\pi}(-1)^{n+1}\int_0^{\infty}\int_{-\infty}^{\infty} f^n H_m(x) e^{-x^2} e^{-\pi^2 f^2} \\ &\times \sin\left(2\pi f x + \frac{n\pi}{2}\right) dx df. \end{aligned} \tag{7}$$

On the other hand, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-x^2} H_m(x) \sin\left(2\pi f x + \frac{n\pi}{2}\right) dx \\ &= \begin{cases} 0, & m+n \text{ is even}, \\ 2\int_0^{\infty} e^{-x^2} H_m(x)\sin\left(2\pi f x + \frac{n\pi}{2}\right) dx, & m+n \text{ is odd}, \end{cases} \end{aligned} \tag{8}$$

So, if $m+n$ is even then $H_{m,n}(x,y) = 0$. If $m+n$ is odd, then use relation (8) and integration by parts and Rodrigue's formula of Hermite polynomials to obtain

$$\begin{aligned} &\int_0^{\infty} e^{-x^2} H_m(x) \sin\left(2\pi f x + \frac{n\pi}{2}\right) dx \\ &= (2\pi f)^m \sin\left(\frac{m+n}{2}\pi\right) \int_0^{\infty} e^{-x^2}\cos(2\pi f x)\, dx. \end{aligned} \tag{9}$$

From the relations (7), (9) by using change of the variable $\pi f = y$, we obtain

$$\begin{aligned} H_{m,n}(x,y) &= 2^{m+n+2}(-1)^{n+1}\sqrt{\pi}\sin\left(\frac{m+n}{2}\pi\right) \\ &\times \int_0^{\infty} y^{m+n} e^{-y^2}\, dy \int_0^{\infty} e^{-x^2}\cos(2xy)\, dx \\ &= 2^{m+n+2}(-1)^{n+1}\sqrt{\pi}\sin\left(\frac{m+n}{2}\pi\right) \\ &\times \int_0^{\infty} y^{m+n} e^{-y^2}\left(\frac{\sqrt{\pi}}{2}e^{-y^2}\right) dy \\ &= 2^{\frac{m+n-1}{2}}(-1)^{n+1}\sin\left(\frac{m+n}{2}\pi\right)\Gamma\left(\frac{m+n+1}{2}\right)\pi \end{aligned} \tag{10}$$

So, for odd $m+n$, we have

$$H_{m,n}(x,y) = 2^{\frac{m+n-1}{2}}(-1)^{\frac{m+3n+1}{2}}\Gamma\left(\frac{m+n+1}{2}\right)\pi \tag{11}$$

The famous linearization formula of Hermite polynomials is [2]

$$\begin{aligned} H_m(x)H_n(x) &= 2^m m! \sum_{k=0}^{m} \frac{1}{2^k k!}\binom{n}{m-k} H_{n-m+2k}(x), \\ & m \le n. \end{aligned} \tag{12}$$

By using the relations (11) and (12), we can obtain $A^n_{i,j}$ in relation (5). □

Corollary 3.1 *The* $A^n_{i,j}$ *coefficients in relation* (5) *can be computed as follows*:

$$\begin{aligned} A^n_{i,j} &= \frac{n!}{2^{i+j} i! j! \pi}\sum_{k=i+j+1}^{n}\sum_{v=i}^{k-j-1}(-1)^{\frac{n-k}{2}}\left(\frac{1+(-1)^{n+k}}{2}\right) \\ &\times \left(\frac{1+(-1)^{v+i}}{2}\right)\left(\frac{1+(-1)^{k+j-v-1}}{2}\right) \\ &\times \frac{2^k v!(k-v-1)!}{k!(v-i)!(k-v-j-1)!\left(\frac{n-k}{2}\right)!}\Gamma\left(\frac{v-i+1}{2}\right)\Gamma\left(\frac{k-v-j}{2}\right). \end{aligned}$$

Now, we can obtain Christoffel–Darboux type identities for the derivatives of Hermite polynomials.

Corollary 3.2 *Let*

$$H_n^{(s)}(x,y) = \frac{H_n^{(s)}(y) - H_n^{(s)}(x)}{y-x} = \sum_{i=0}^{n-s-1} \sum_{j=0}^{n-s-i-1} A_{i,j}^{n,s} H_i(x) H_j(y), \quad s = 0\ldots n, \tag{13}$$

where

$$A_{i,j}^{n,s} = \frac{2^s n!}{(n-s)!} \left(\frac{1-(-1)^{n-s+i+j}}{2} \right) \times \left\{ \frac{1}{2^i i!} \sum_{k=0}^{j} (-1)^{\left(\frac{3n-3s-3j+6k+i+1}{2}\right)} \frac{2^{\left(\frac{n-s-j+i-1}{2}\right)}}{k!} \times \binom{n-s}{j-k} \Gamma\left(\frac{n-s-j+i+2k+1}{2} \right) + \frac{1}{2^j j!} \sum_{k=0}^{i} (-1)^{\left(\frac{3n-3s-3i+6k+j+1}{2}\right)} \frac{2^{\left(\frac{n-s-i+j-1}{2}\right)}}{k!} \times \binom{n-s}{i-k} \Gamma\left(\frac{n-s-i+j+2k+1}{2} \right) \right\}. \tag{14}$$

Christoffel–Darboux type identities of Legendre polynomials

Theorem 4.1 *Let* $P_n(x)$ *be Legendre polynomial of degree* n then

$$P_n(x,y) = \frac{P_n(y) - P_n(x)}{y-x} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} A_{i,j}^n P_i(x) P_j(y), \tag{15}$$

where

$$A_{i,j}^n = -\frac{1}{2}(2i+1)(2j+1) \sum_{k=0}^{\min(i,j)} \frac{1+(-1)^{i+j+n-1}}{i+j+n-2k+1} \left\{ \frac{B_{i,j}^k}{i+j-n-2k} - \frac{B_{\min(i,j),n}^k}{\min(i,j) - \max(i,j) + n - 2k} \right\},$$

$$B_{i,j}^k = \frac{B_{i-k} B_k B_{j-k}}{B_{i+j-k}} \left(\frac{2i+2j-4k+1}{2i+2j-2k+1} \right),$$

$$B_k = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{k!}, \quad k = 1, 2, 3, \ldots.$$

$$B_0 = 1. \tag{16}$$

Proof Legendre function of the second kind is defined by

$$Q_n(x) = -\frac{1}{2} P.V. \int_{-1}^{1} \frac{P_n(y)}{y-x} \, dy,$$

and

$$P.V. \int_{-1}^{1} P_m(x) Q_n(x) \, dx = \frac{1+(-1)^{m+n}}{(m-n)(m+n+1)}, \quad m \neq n. \tag{17}$$

Therefore

$$A_{i,j}^n = \frac{1}{4}(2i+1)(2j+1) \int_{-1}^{1} \int_{-1}^{1} \frac{P_n(y) - P_n(x)}{y-x} P_i(x) P_j(y) \, dy \, dx = -\frac{1}{2}(2i+1)(2j+1) \int_{-1}^{1} P_i(x) \Big(P_j(x) Q_n(x) - P_n(x) Q_j(x) \Big) \, dx. \tag{18}$$

The following famous linearization formula of Legendre polynomials is Neumann-Adams formula [2]:

$$P_m(x) P_n(x) = \sum_{k=0}^{m} \frac{B_{m-k} B_k B_{n-k}}{B_{m+n-k}} \left(\frac{2m+2n-4k+1}{2m+2n-2k+1} \right) P_{m+n-2k}(x), \quad m \leq n,$$

$$B_k = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{k!}, \quad k = 1, 2, 3, \ldots.$$

$$B_0 = 1. \tag{19}$$

Now, use the relations (17), (18) and (19) to obtain the result. □

Corollary 4.1 *The* $A_{i,j}^n$ *coefficients in relation* (15) *can be computed as follows*:

$$A_{i,j}^n = \frac{(2i+1)(2j+1)}{2^{n+i+j+2}} \sum_{k=i+j+1}^{n} \sum_{v=i}^{k-j-1} (-1)^{\frac{n-k}{2}} \left(\frac{1+(-1)^{n+k}}{2} \right) \times \left(\frac{1+(-1)^{v+i}}{2} \right) \left(\frac{1+(-1)^{k+j-v-1}}{2} \right) \times \frac{(n+k)! v! (k-v-1)!}{k!(v-i)!(k-v-j-1)! \left(\frac{n-k}{2}\right)! \left(\frac{n+k}{2}\right)!} \frac{\Gamma\left(\frac{v-i+1}{2}\right) \Gamma\left(\frac{k-v-j}{2}\right)}{\Gamma\left(\frac{v+i+3}{2}\right) \Gamma\left(\frac{k-v+j+2}{2}\right)}$$

Now, we can obtain Christoffel–Darboux type identities for the derivatives of Legendre polynomials.

From [4], for the case $\gamma = 0$, we can derive

$$P_n^{(s)}(x) = \sum_{k=0}^{n-s} a_k^s P_k(x), \quad s = 0\ldots n, \tag{20}$$

where

$$a_k^s = \left(\frac{1+(-1)^{n+k+s}}{2} \right) \frac{2k+1}{2^{s-2}(s-1)!} \frac{(n+k+s-1)!}{(n+k+s+2)!} \times \frac{\left(\frac{n-k+s}{2} - 1\right)!}{\left(\frac{n-k-s}{2}\right)!} \frac{\left(\frac{n+k-s}{2} + 1\right)!}{\left(\frac{n+k+s}{2} - 1\right)!}. \tag{21}$$

Corollary 4.2

$$\frac{P_n^{(s)}(y)-P_n^{(s)}(x)}{y-x}=\sum_{i=0}^{n-s-1}\sum_{j=0}^{n-s-i-1}A_{i,j}^{n,s}P_i(x)P_j(y),\quad s=0\ldots n, \tag{22}$$

where

$$\begin{aligned}A_{i,j}^{n,s}&=-\frac{1}{2}(2i+1)(2j+1)\sum_{k=i+j+1}^{n-s}\sum_{v=0}^{\min(i,j)}\left(\frac{1+(-1)^{n+k+s}}{2}\right)\\&\times\frac{2k+1}{2^{s-2}(s-1)!}\frac{(n+k+s-1)!}{(n+k+s+2)!}\\&\times\frac{\left(\frac{n-k+s}{2}-1\right)!}{\left(\frac{n-k-s}{2}\right)!}\frac{\left(\frac{n+k-s}{2}+1\right)!}{\left(\frac{n+k+s}{2}-1\right)!}\frac{1+(-1)^{i+j+k-1}}{i+j+k-2v+1}\\&\times\left\{\frac{B_{i,j}^{v}}{i+j-k-2v}-\frac{B_{\min(i,j),k}^{v}}{\min(i,j)-\max(i,j)+k-2v}\right\},\\B_{i,j}^{v}&=\frac{B_{i-v}B_vB_{j-v}}{B_{i+j-v}}\left(\frac{2i+2j-4v+1}{2i+2j-2v+1}\right),\\B_v&=\frac{1\cdot3\cdot5\cdots(2v-1)}{v!},\quad v=1,2,3,\ldots.\\B_0&=1.\end{aligned}\tag{23}$$

Christoffel–Darboux type identities of Laguerre polynomials

The famous linearization formula of associated laguerre polynomials is Feldheim formula [6]

$$L_m^{\alpha}L_n^{\beta}(x)=\sum_{k=0}^{m+n}\sum_{v=0}^{k}(-1)^{m+n+k}\binom{k}{v}\binom{m+\alpha}{n-k+v}\binom{n+\beta}{m-v}\times L_k^{\alpha+\beta}(x).$$

In spite of Hermite and Legendre polynomials, the linearization formula of Laguerre polynomials is presented by double summation. The coefficients $A_{i,j}^n$ of Hermite and Legendre polynomials are obtained from (5) and (16) by one summation, and in the following relations, the $A_{i,j}^n$ coefficients of Laguerre polynomials are given by double summation.

Corollary 5.1 *Let $L_n^m(x)$ be associated laguerre polynomials of degree n then*

$$L_{n+1}^m(x,y)=\frac{L_{n+1}^m(y)-L_{n+1}^m(x)}{y-x}=\sum_{i=0}^{n}\sum_{j=0}^{n-i}A_{i,j}^{n+1}L_i^m(x)L_j^m(y), \tag{24}$$

where

$$A_{i,j}^{n+1}=\sum_{k=i+j+1}^{n+1}\sum_{v=i}^{k-j-1}(-1)^{i+j+k}\frac{i!j!}{k!}\times\frac{(m+v)!(m+k-v-1)!}{(m+i)!(m+j)!}\binom{m+n+1}{n-k+1}\binom{v}{i}\binom{k-v-1}{j} \tag{25}$$

The related formula for Laguerre polynomials of degree n is

$$L_{n+1}(x,y)=\frac{L_{n+1}(y)-L_{n+1}(x)}{y-x}=\sum_{i=0}^{n}\sum_{j=0}^{n-i}A_{i,j}^{n+1}L_i(x)L_j(y), \tag{26}$$

where

$$A_{i,j}^{n+1}=\sum_{k=i+j+1}^{n+1}\sum_{v=i}^{k-j-1}(-1)^{i+j+k}\frac{\binom{n+1}{k}\binom{v}{i}\binom{k-v-1}{j}}{(v+1)\binom{k}{v+1}} \tag{27}$$

From [3], we have

$$\frac{d^s}{dx^s}L_n^m(x)=(-1)^s\sum_{k=0}^{n-s}\binom{n-k-1}{s-1}L_k^m(x),\quad s=0\ldots n.$$

Let

$$L_n^{m,s}(x)=\frac{d^s}{dx^s}L_n^m(x),$$

then

$$\frac{L_n^{m,s}(y)-L_n^{m,s}(x)}{y-x}=\sum_{i=0}^{n-s-1}\sum_{j=0}^{n-s-i-1}A_{i,j}^{n,s}L_i^m(x)L_j^m(y),\quad s=0\ldots n,$$

where

$$A_{i,j}^{n,s}=\sum_{k=i+j+1}^{n-s}\sum_{k'=i+j+1}^{k}\sum_{v=i}^{k'-j-1}(-1)^{i+j+s+k'}\frac{i!j!}{k'!}\frac{(m+v)!(m+k'-v-1)!}{(m+i)!(m+j)!}\times\binom{n-k-1}{s-1}\binom{m+k}{k-k'}\binom{v}{i}\binom{k'-v-1}{j}.$$

Conclusion

In this paper, we obtained some new Christoffel–Darboux type identities for Legendre, Laguerre and Hermite polynomials. We also obtained these types of identities for the derivatives of these polynomials. These formulas are good theoretically and the correctness of the obtained formulas are checked by Maple 17, and Some of these formulas are not efficient numerically.

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