Asymptotic equilibrium of integro-differential equations with infinite delay

Le Anh Minh¹ · Dang Dinh Chau²

Received: 25 June 2014 / Accepted: 2 September 2015 / Published online: 21 September 2015
 © The Author(s) 2015. This article is published with open access at Springerlink.com

Abstract The asymptotic equilibrium problems of ordinary differential equations in a Banach space have been considered by several authors. In this paper, we investigate the asymptotic equilibrium of the integro-differential equations with infinite delay in a Hilbert space.

Keywords Asymptotic equilibrium · Integro-differential equations · Infinite delay

Introduction

The asymptotic equilibrium problems of ordinary differential equations in a Banach space have been considered by several authors, Mitchell and Mitchell [3], Bay et al. [1], but the results for the asymptotic equilibrium of integro-differential equations with infinite delay still is not presented. In this paper, we extend the results in [1] to a class of integro-differential equations with infinite delay in a Hilbert space H which has the following form:

$$\begin{cases} \frac{dx(t)}{dt} = A(t) \left(x(t) + \int_{-\infty}^t k(t-\theta)x(\theta)d\theta \right), & t \geq 0, \\ x(t) = \varphi(t), & t \leq 0 \end{cases} \quad (1)$$

where $A(t) : H \rightarrow H$, φ in the phase space $\mathcal{B}$, and x_t is defined as

$$x_t(\theta) = x(t + \theta), \quad -\infty < \theta \leq 0.$$

Preliminaries

We assume that the phase space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into H satisfying the following fundamental axioms (we refer reader to [2])

(A₁) For $a > 0$, if x is a function mapping $(-\infty, a]$ into H , such that $x \in \mathcal{B}$ and x is continuous on $[0, a]$, then for every $t \in [0, a]$ the following conditions hold:

- (i) x_t belongs to $\mathcal{B}$;
- (ii) $\|x(t)\| \leq G\|x_t\|_{\mathcal{B}}$;
- (iii) $\|x_t\|_{\mathcal{B}} \leq K(t) \sup_{s \in [0, t]} \|x(s)\| + M(t)\|x_0\|_{\mathcal{B}}$

where G is a positive constant, $K, M : [0, \infty) \rightarrow [0, \infty)$, K is continuous, M is locally bounded, and they are independent of x .

(A₂) For the function x in (A₁), x_t is a $\mathcal{B}$ -valued continuous function for t in $[0, a]$.

(A₃) The space $\mathcal{B}$ is complete.

✉ Le Anh Minh
 leanhminh@hdu.edu.vn
 Dang Dinh Chau
 chaudida@gmail.com

¹ Department of Mathematical Analysis, Hong Duc University, Thanh Hóa, Vietnam

² Department of Mathematics, Hanoi University of Science, VNU, Hanoi, Vietnam

Example 1

- (i) Let BC be the space of all bounded continuous functions from $(-\infty, 0]$ to H , we define $C^0 := \{\varphi \in BC : \lim_{\theta \rightarrow -\infty} \varphi(\theta) = 0\}$ and $C^\infty := \{\varphi \in BC : \lim_{\theta \rightarrow -\infty} \varphi(\theta) \text{ exists in } H\}$ endowed with the norm

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta \in (-\infty, 0]} \|\varphi(\theta)\|$$

then C^0, C^∞ satisfies (A_1) – (A_3) . However, BC satisfies (A_1) and (A_3) , but (A_2) is not satisfied.

- (ii) For any real constant γ , we define the functional spaces C_γ by

$$C_\gamma = \left\{ \varphi \in C((-\infty, 0], X) : \lim_{\theta \rightarrow -\infty} e^{\gamma\theta} \varphi(\theta) \text{ exists in } H \right\}$$

endowed with the norm

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta \in (-\infty, 0]} e^{\gamma\theta} \|\varphi(\theta)\|.$$

Then conditions (A_1) – (A_3) are satisfied in C_γ .

Remark 1 In this paper, we use the following acceptable hypotheses on $K(t)$, $M(t)$ in (A_1) (iii) which were introduced by Hale and Kato [2] to estimate solutions as $t \rightarrow \infty$,

(γ_1) $K = K(t)$ is a constant for all $t \geq 0$;

(γ_2) $M(t) \leq M$ for all $t \geq 0$ and some M .

Example 2 For the functional space C_γ in Example 1, the hypotheses (γ_1) and (γ_2) are satisfied if $\gamma \geq 0$.

Definition 1 Equation (1) has an asymptotic equilibrium if every solution of it has a finite limit at infinity and, for every $h_0 \in H$, there exists a solution $x(t)$ of it such that $x(t) \rightarrow h_0$ as $t \rightarrow \infty$.

Main results

Now, we consider the asymptotic equilibrium of Eq. (1) which satisfies the following assumptions:

- (M₁) $A(t)$ is a strongly continuous bounded linear operator for each $t \in \mathbb{R}^+$;
 (M₂) $A(t)$ is a self-adjoint operator for each $t \in \mathbb{R}^+$;
 (M₃) k satisfies

$$\int_0^{+\infty} |k(\theta)| d\theta = L < +\infty;$$

and

- (M₄) There exists a constant $T > 0$ such that

$$\sup_{h \in S(0,1)} \int_T^\infty \|A(t)h\| dt < q < \frac{1}{\kappa}, \quad (2)$$

herein $S(0, 1)$ is a unit ball in H , $\kappa = L(K + M) + 1$, where K, M, L are given in (γ_1) , (γ_2) and (M_3) .

Theorem 1 If (M_1) , (M_2) , (M_3) and (M_4) are satisfied, then Eq. (1) has an asymptotic equilibrium.

Proof We shall begin with showing that all solutions of (1) has a finite limit at infinity. Indeed, Eq. (1) may be rewritten as

$$\frac{dx(t)}{dt} = A(t) \left(x(t) + \int_{-\infty}^0 k(-\theta)x_t(\theta) d\theta \right),$$

then for $t \geq s \geq T$ we have

$$x(t) = x(s) + \int_s^t A(\tau) \left(x(\tau) + \int_{-\infty}^0 k(-\theta)x_\tau(\theta) d\theta \right) d\tau$$

and

$$\begin{aligned} \|x(t)\| &= \sup_{h \in S(0,1)} \left| \left\langle x(s) + \int_s^t A(\tau) \left(x(\tau) + \int_{-\infty}^0 k(-\theta)x_\tau(\theta) d\theta \right) d\tau, h \right\rangle \right| \\ &\leq \|x(s)\| + \sup_{h \in S(0,1)} \int_s^t \left| \left\langle x(\tau) + \int_{-\infty}^0 k(-\theta)x_\tau(\theta) d\theta, A(\tau)h \right\rangle \right| d\tau \\ &\leq \|x(s)\| + q \left((LK + 1) \sup_{\xi \in [0,t]} \|x(\xi)\| + LM \|\varphi\|_{\mathcal{B}} \right) \end{aligned} \quad (3)$$

implies

$$\|x(t)\| \leq \|x(s)\| + q((LK + 1)\|x(t)\| + LM\|\varphi\|_{\mathcal{B}})$$

or

$$\|x(t)\| \leq \frac{\|x(s)\| + qLM\|\varphi\|_{\mathcal{B}}}{1 - q(LK + 1)} \quad (4)$$

where

$$\|x(t)\| = \sup_{0 \leq \xi \leq t} \|x(\xi)\|.$$

Now, we conclude that $x(t)$ is bounded since

$$0 < q < \frac{1}{\kappa} = \frac{1}{L(K+M)+1} < \frac{1}{LK+1} \Rightarrow q(LK+1) < 1$$

and by (4).

Putting

$$M^* = \sup_{t \in \mathbb{R}} \|x(t)\|,$$

we have

$$\begin{aligned} \|x(t) - x(s)\| &= \sup_{h \in S(0,1)} |\langle x(t) - x(s), h \rangle| \\ &\leq \sup_{h \in S(0,1)} \left| \int_s^t \left\langle A(\tau) \left(x(\tau) + \int_{-\infty}^0 k(-\theta)x_\tau(\theta)d\theta \right), h \right\rangle d\tau \right|, \\ &\leq [M^*(LK+1) + LM\|\varphi\|_{\mathcal{B}}] \sup_{h \in S(0,1)} \int_s^t \|A(\tau)h\| d\tau \rightarrow 0 \end{aligned}$$

as $t \geq s \rightarrow +\infty$. That means all solutions of (1) have a finite limit at infinity. To complete the proof, it remains to show that for any $h_0 \in H$, there exists a solution $x(t)$ of (1) such that

$$\lim_{t \rightarrow +\infty} x(t) = h_0.$$

Indeed, let h_0 be an arbitrary fixed element of H ; we choose the initial function φ belongs to $\mathcal{B}$ such that $\varphi(0) = h_0$ and $\|\varphi\|_{\mathcal{B}} \leq \|h_0\|$ and consider the functional

$$\begin{aligned} g_1(t, h) &= \langle h_0, h \rangle \\ &\quad - \int_t^\infty \left\langle A(\tau) \left(h_0 + \int_{-\infty}^\tau k(\tau - \theta)x_0(\theta)d\theta \right), h \right\rangle d\tau \end{aligned}$$

We have

$$\begin{aligned} |g_1(t, h)| &\leq \|h_0\|(\|h\| + \int_t^{+\infty} \|x_0(\tau)\| \\ &\quad + \int_{-\infty}^\tau k(\tau - \theta)x_0(\theta)d\theta\| \|A(\tau)h\| d\tau). \end{aligned}$$

Since $x_0(\tau) \equiv h_0$, then

$$|g_1(t, h)| \leq \|h_0\|(\|h\| + q\kappa).$$

It follows from Riesz representation theorem that there exists an element $x_1(t)$ in H , such that

$$g_1(t, h) = \langle x_1(t), h \rangle$$

and

$$\|x_1(t)\| \leq \|h_0\|(1 + q\kappa).$$

Now, we consider the functional

$$\begin{aligned} g_2(t, h) &= \langle h_0, h \rangle \\ &\quad - \int_t^{+\infty} \left\langle A(\tau) \left(x_1(t) + \int_{-\infty}^\tau k(\tau - \theta)x_1(\theta)d\theta \right), h \right\rangle d\tau. \end{aligned}$$

By an argument analogous to the previous one, we get

$$|g_2(t, h)| \leq \|h_0\|(\|h\| + q\kappa + (q\kappa)^2)$$

and there exists an element $x_2(t)$ in H , such that

$$g_2(t, h) = \langle x_2(t), h \rangle$$

with

$$\|x_2(t)\| \leq \|h_0\|(1 + q\kappa + (q\kappa)^2).$$

Continuing this process, we obtain the linear continuous functional

$$\begin{aligned} g_n(t, h) &= \langle h_0, h \rangle \\ &\quad - \int_t^{+\infty} \left\langle A(\tau) \left(x_{n-1}(t) + \int_{-\infty}^\tau k(\tau - \theta)x_{n-1}(\theta)d\theta \right), h \right\rangle d\tau \end{aligned} \quad (5)$$

and $x_n(t) \in H$ such that

$$g_n(t, h) = \langle x_n(t), h \rangle$$

satisfies the following estimate

$$\|x_n(t)\| \leq (1 + q\kappa + (q\kappa)^2 + \cdots + (q\kappa)^n)\|h_0\| \leq \frac{\|h_0\|}{1 - q\kappa}.$$

Futhermore,

$$\|x_n(t) - x_{n-1}(t)\| \leq \|h_0\|(q\kappa)^n.$$

This inequality shows that $\{x_n(t)\}$ is uniformly convergent on $[T, +\infty)$ since $q\kappa < 1$. Put

$$x(t) = \lim_{n \rightarrow +\infty} x_n(t).$$

In (5), let $n \rightarrow +\infty$, we have

$$\begin{aligned} \langle x(t), h \rangle &= \langle h_0, h \rangle \\ &\quad - \int_t^{+\infty} \left\langle A(\tau) \left(x(t) + \int_{-\infty}^\tau k(\tau - \theta)x(\theta)d\theta \right), h \right\rangle d\tau \end{aligned} \quad (6)$$

and since

$$|\langle x_n(t), h_0 \rangle| < \int_T^{+\infty} \|x_{n-1}(\tau)\| \\ + \int_{-\infty}^{\tau} k(\tau - \theta) x_{n-1}(\theta) d\theta \|A(\tau)h\| d\tau$$

or

$$|\langle x_n(t), h_0 \rangle| \leq \frac{\|h_0\|q}{1 - q\kappa},$$

we have $x_n(t) \rightarrow h_0$ as $q \rightarrow 0$, which means that there exists a solution of (1) converging to h_0 . The theorem is proved.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License ([http://](http://creativecommons.org/licenses/by/4.0/)

creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Bay, N.S., Hoan, N.T., Man, N.M.: On the asymptotic equilibrium and asymptotic equivalence of differential equations in Banach spaces. *Ukr. Math. J.* **60**(5), 716–729 (2008)
2. Hale, J.K., Kato, J.: Phase space for retarded equations with infinite delay. *Funkcialaj Ekvacioj* **21**, 11–41 (1978)
3. Mitchell, A.R., Mitchell, R.W.: Asymptotic equilibrium of ordinary differential systems in a Banach space. *Theory Comput. Syst.* **9**(3), 308–314 (1975)