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ORIGINAL RESEARCH

Suzuki-type fixed point results in *b*-metric spaces

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Abstract An ingenious approach to generalize Banach contraction principle was adopted by Suzuki in his seminal papers (Proc Am Math Soc 136:1861–1869, 2008, Nonlinear Anal Theory Methods Appl 71:5313–5317, 2009). In this paper we prove certain common fixed point results for generalized Suzuki contractions in the set-up of b-metric spaces, where the b-metric function is not necessarily continuous. Finally, some examples are presented to verify the effectiveness and applicability of our main results.

Keywords Fixed point · b-metric space · Suzuki contarction

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Introduction

There are a lot of generalizations of Banach fixed point principle in the literature. In 2008 Suzuki introduced an interesting generalization of Banach fixed point principle. This interesting fixed-point result is as follows.

Theorem 1 [26] *Let (X, d) be a complete metric space, and let T be a mapping on X. Define a non-increasing function θ from [0, 1) into (1/2, 1] by*

$$\theta(r)=\begin{cases}1, & 0\le r\le \frac{\sqrt{5}-1}{2}\\ \frac{1-r}{r^2}, & \frac{\sqrt{5}-1}{2}\le r\le \frac{1}{\sqrt{2}}\\ \frac{1}{1+r}, & \frac{1}{\sqrt{2}}\le r<1.\end{cases}$$

Assume that there exists $r\in[0,1)$, such that

$$\theta(r)\mathrm{d}(x,Tx)\le \mathrm{d}(x,y)\Longrightarrow \mathrm{d}(Tx,Ty)\le r\mathrm{d}(x,y),$$

for all $x,y\in X$, then there exists a unique fixed-point z of T. Moreover, $\lim_{n\to\infty}T^n x=z$ for all $x\in X$.

Suzuki proved also the following version of Nemytckii fixed point theorem.

Theorem 2 *Let (X, d) be a compact metric space. Let $T:X\to X$ be a selfmap, satisfying for all $x,y\in Xx\neq y$ the condition*

$$\frac{1}{2}\mathrm{d}(x,Tx)\le \mathrm{d}(x,y)\Longrightarrow \mathrm{d}(Tx,Ty)<\mathrm{d}(x,y)$$

Then T has a unique fixed point in X.

This theorem was also generalized in [6].

In addition to the above results, Kikkawa and Suzuki [11] provided a Kannan-type version of the theorems mentioned above. In [21], a Chatterjea-type version is provided, whereas Popescu [20] obtained a Cirić-type version. Recently, Kikkawa and Suzuki also provided multivalued versions in [12, 13].

Very recently, Hussain et al. in [8] have extended Suzuki's Theorems 1 and 2, as well as Popescu's results from [20] to the case of metric-type spaces and cone metric-type spaces.

Czerwik in [5] introduced the concept of b-metric space. Since then, several papers deal with fixed point theory for single-valued and multivalued operators in b-metric spaces (see also [1–5, 7–10, 14–17, 19, 24, 25]). Pacurar [19] proved results on sequences of almost contractions and fixed points in b-metric spaces. Recently, Hussain and Shah [9] obtained results on KKM mappings in cone b-metric spaces. Khamsi ([14, 15]) also showed that each cone metric space has a b-metric structure.

The aim of this paper is to present some common fixed point results for two mappings under generalized contractive condition in b-metric space, where the b-metric function is not necessarily continuous. Because many of the authors in their works have used the b-metric spaces in which the b-metric function is assumed to be continuous. From this point of view the results obtained in this paper generalize and extend several ones obtained earlier concerning b-metric space.

Consistent with [5] and [25, p. 264], the following definition and results will be needed in the sequel.

Definition 1 [5] Let X be a (nonempty) set and $b \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric spaces iff, for all $x, y, z \in X$, the following condition are satisfied:

(b1) $\mathrm{d}(x, y) = 0 \quad \text{iff } x = y$,
(b2) $\mathrm{d}(x, y) = \mathrm{d}(y, x)$,
(b3) $\mathrm{d}(x, z) \leq b[\mathrm{d}(x, y) + \mathrm{d}(y, z)]$.

The pair (X, d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric only if $b = 1$.

We present an example which shows that a b-metric on X need not be a metric on X. (see also [25, p. 264]):

Example 1 [22] Let (X, d) be a metric space, and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $b = 2^{p-1}$.

However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space.

For example, if $X = \mathbb{R}$ is the set of real numbers and $d(x, y) = |x - y|$ is the usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a b-metric on $\mathbb{R}$ with $b = 2$, but is not a metric on $\mathbb{R}$.

Before stating and proving our results, we present some definition and proposition in b-metric space. We recall first the notions of convergence and completeness in a b-metric space.

Definition 2 [3] Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:

(a) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.
(b) Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Proposition 1 (see remark 2.1 in [3]) *In a b-metric space* (X, d) *the following assertions hold*:

(i) *a convergent sequence has a unique limit,*
(ii) *each convergent sequence is Cauchy,*

Definition 3 [3] The b-metric space (X, d) is complete if every Cauchy sequence in X converges.

It should be noted that, in general a b-metric function $d(x, y)$ for $b > 1$ is not jointly continuous in all two of its variables. Now we present an example of a b-metric which is not continuous.

Example 2 (see Example 3 in [8]) Let $X = \mathbb{N} \cup \{\infty\}$ and let $D : X \times X \to \mathbb{R}^+$ be defined by,

$$D(m,n) = \begin{cases} 0 & \text{if } m = n, \\ \left|\frac{1}{m} - \frac{1}{n}\right|, & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m,\ n \text{ is odd and other is odd (and } m \neq n) \text{ or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Then it is easy to see that for all $m, n, p \in X$, we have

$$D(m, p) \leq \frac{5}{2}(D(m, n) + D(n, p)).$$

Thus, (X, D) is a b-metric space with $b = \frac{5}{2}$. In [8], it is proved that $D(x, y)$ is not a continuous function.

Since in general a b-metric is not continuous, we need the following simple lemma about the b-convergent sequences.

Lemma 1 [22] *Let* (X, d) *be a b-metric space with* $b \geq 1$, *and suppose that* $\{x_n\}$ *and* $\{y_n\}$*b-converge tox, y, respectively. Then, we have*

$$\frac{1}{b^2}\mathrm{d}(x, y) \leq \liminf_{n\to\infty} \mathrm{d}(x_n, y_n) \leq \limsup_{n\to\infty} \mathrm{d}(x_n, y_n) \leq b^2\mathrm{d}(x, y).$$

In particular, if $x = y$, *then* $\lim_{n\to\infty} \mathrm{d}(x_n, y_n) = 0$. *Moreover, for each* $z \in X$ *we have*

$$\frac{1}{b}\mathrm{d}(x, z) \leq \liminf_{n\to\infty} \mathrm{d}(x_n, z) \leq \limsup_{n\to\infty} \mathrm{d}(x_n, z) \leq b\mathrm{d}(x, z).$$

Main result

We start our work by proving the following crucial Theorem.

Theorem 3 *Let (X, d) be a complete b-metric space. Let $T, S: X \longrightarrow X$ be two self-maps and $\theta: [0,1) \longrightarrow (\frac{1}{2}, 1]$ be defined by*

$$\theta(r) = \begin{cases} 1, & 0 \le r \le \frac{\sqrt{5}-1}{2} \\ \frac{1-r}{r^2}, & \frac{\sqrt{5}-1}{2} \le r \le \frac{1}{\sqrt{2}} \\ \frac{1}{1+r}, & \frac{1}{\sqrt{2}} \le r < 1. \end{cases} \tag{1}$$

Suppose there exists $r \in [0,1)$ such that for each $x, y \in X$, the following condition is satisfied

$$\frac{1}{b}\theta(r)\min\{d(x,Tx), d(x,Sx)\} \le d(x,y) \Longrightarrow \max\left\{ \begin{array}{l} d(Sx,Sy), d(Tx,Ty), \\ d(Sx,Ty), d(Sy,Tx) \end{array} \right\} \le \frac{r}{b^2} d(x,y). \tag{2}$$

Then T, S have a unique common fixed point $z \in X$.

Proof At first we show that if z is a fixed point of S or T, then z is a common fixed point of T and S. Let z be a fixed point of T that is $Tz = z$ then we show that $Sz = z$. From

$$0 = \frac{1}{b}\theta(r)\min\{d(z,Tz), d(z,Sz)\} \le d(z,Tz),$$

it follows

$$d(Sz,z) \le \max\left\{ \begin{array}{l} d(Sz,STz), d(Tz,T^2z), \\ d(Sz,T^2z), d(STz,Tz) \end{array} \right\} \le \frac{r}{b^2} d(z,Tz) = 0,$$

thus $Sz = z$. Therefore it is enough to show that T have a fixed point. Putting $y = Sx$ in (2)

$$\frac{1}{b}\theta(r)\min\{d(x,Tx), d(x,Sx)\} \le d(x,Sx),$$

it follows

$$\max\left\{ \begin{array}{l} d(Sx,S^2x), d(Tx,TSx), \\ d(Sx,TSx), d(S^2x,Tx) \end{array} \right\} \le \frac{r}{b^2} d(x,Sx), \tag{3}$$

for every $x \in X$. Hence,

$$d(Sx,TSx) \le \frac{r}{b^2} d(x,Sx). \tag{4}$$

Now, putting $y = Tx$ in (2)

$$\frac{1}{b}\theta(r)\min\{d(x,Tx), d(x,Sx)\} \le d(x,Tx),$$

it follows

$$\max\left\{ \begin{array}{l} d(Sx,STx), d(Tx,T^2x), \\ d(Sx,T^2x), d(STx,Tx) \end{array} \right\} \le \frac{r}{b^2} d(x,Tx), \tag{5}$$

for every $x \in X$. Hence,

$$d(Tx,T^2x) \le \frac{r}{b^2} d(x,Tx), \tag{6}$$

and

$$d(STx,Tx) \le \frac{r}{b^2} d(x,Tx). \tag{7}$$

Let $x_0 \in X$ be arbitrary and form the sequence $\{x_n\}$ by, $x_{2n+1} = Sx_{2n}$ and $Tx_{2n+1} = x_{2n+2}$ for $n \in \mathbb{N} \cup \{0\}$. We show that $\{x_n\}$ is a Cauchy sequence.

By (4), we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, TSx_{2n}) \\ &\le \frac{r}{b^2} d(x_{2n}, Sx_{2n}) = \frac{r}{b^2} d(x_{2n}, x_{2n+1}). \end{aligned} \tag{8}$$

By (7), we have

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(STx_{2n-1}, Tx_{2n-1}) \\ &\le \frac{r}{b^2} d(x_{2n-1}, Tx_{2n-1}) = \frac{r}{b^2} d(x_{2n-1}, x_{2n}). \end{aligned}$$

Therefore,

$$\begin{aligned} d(x_n, x_{n+1}) &\le \frac{r}{b^2} d(x_{n-1}, x_n) \\ &\le \frac{r^2}{b^4} d(x_{n-2}, x_{n-1}) \\ &\vdots \\ &\le \frac{r^n}{b^{2n}} d(x_0, x_1). \end{aligned}$$

Also, by definition of b-metric spaces for all $m \ge n$, we have

$$\begin{aligned} d(x_n,x_m) &\le b d(x_n,x_{n+1}) + b^2 d(x_{n+1},x_{n+2}) + \cdots + b^{m-n-1} d(x_{m-1},x_m) \\ &\le b\frac{r^n}{b^{2n}} d(x_0,x_1) + b^2 \frac{r^{n+1}}{b^{2n+2}} d(x_0,x_1) + \cdots + b^{m-n-1}\frac{r^{m-1}}{b^{2m-2}} d(x_0,x_1) \\ &= \frac{r^n}{b^{2n-1}} d(x_0,x_1) + \frac{r^{n+1}}{b^{2n}} d(x_0,x_1) + \cdots + \frac{r^{m-1}}{b^{m+n-1}} d(x_0,x_1) \\ &= \frac{r^n}{b^{2n-1}}\left(1 + \frac{r}{b} + \cdots + \frac{r^{m-n-1}}{b^{m-n}}\right) d(x_0,x_1) \\ &\le \frac{r^n}{b^{2n-1}}\left(1 + \frac{r}{b} + \cdots + (\frac{r}{b})^{m-n-1}\right) d(x_0,x_1) \\ &\le \frac{r^n}{b^{2n-1}} d(x_0,x_1)\left(\frac{1}{1-\frac{r}{b}}\right) \longrightarrow 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

So, we have

$$\lim_{n,m\to\infty} d(x_n,x_m) = 0.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, we conclude $\{x_n\}$ converges to z for some $z \in X$. That is

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} x_{2n+1} = z,$$

and

$$\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} x_{2n+2} = z.$$

Let us prove now that

$$d(z, Tx) \le r d(z, x),$$

holds for each $x \ne z$. Since $d(x_{2n}, Sx_{2n}) \longrightarrow 0$, and by Lemma 1

$$\frac{1}{b} d(z, x) \le \limsup_{n\longrightarrow\infty} d(x_{2n}, x),$$

thus $\limsup_{n\longrightarrow\infty} d(x_{2n}, x) > 0$, it follows that there exists a $x_{2n_k} \in X$ such that

$$\frac{1}{b}\theta(r)\min\{d(x_{2n_k}, Sx_{2n_k}), d(x_{2n_k}, Tx_{2n_k})\} \le d(x_{2n_k}, x).$$

Assumption (2) implies that for such x_{2n_k}

$$\begin{aligned} d(Sx_{2n_k}, Tx) &\le \max\left\{ \begin{matrix} d(Sx_{2n_k}, Sx), d(Tx_{2n_k}, Tx) \\ d(Sx_{2n_k}, Tx), d(Sx, Tx_{2n_k}) \end{matrix} \right\} \\ &\le \frac{r}{b^2} d(x_{2n_k}, x), \end{aligned}$$

hence by Lemma 1

$$\begin{aligned} \frac{1}{b} d(z, Tx) &\le \limsup_{n\to\infty} d(Sx_{2n_k}, Tx) \le \frac{r}{b^2} \limsup_{n\to\infty} d(x_{2n_k}, x) \\ &\le \frac{r}{b} d(z, x), \end{aligned}$$

thus for each $x \ne z$ we get that

$$d(z, Tx) \le rd(z, x). \tag{9}$$

We will prove that

$$d(T^n z, z) \le d(Tz, z), \tag{10}$$

for each $n \in \mathbb{N}$. For $n = 1$ this relation is obvious. Suppose that it holds for some $m \in \mathbb{N}$. If $T^m z = z$ then $T^{m+1}z = Tz$ it follows that the above inequality is true. If $T^m z \ne z$, we can apply (9) and the induction hypothesis, we get that

$$\begin{aligned} d(z, T^{m+1}z) &\le rd(z, T^m z) \\ &\le rd(Tz, z) \le d(Tz, z), \end{aligned}$$

and (10) is proved by induction.

In order to prove that $Tz = z$. We consider two possible cases.

Case I. $0 \le r < \frac{1}{\sqrt{2}}$ (and hence $\theta(r) \le \frac{1-r}{r^2}$). We will prove first that

$$d(T^n z, Tz) \le \frac{r}{b} d(Tz, z) \tag{11}$$

for each $n \in \mathbb{N}$. For $n = 1$ it is obvious. For $n = 2$ it follows from (6). Suppose that (11) holds for some $n > 2$. Since

$$\begin{aligned} d(Tz, z) &\le bd(z, T^n z) + bd(T^n z, Tz) \\ &\le bd(z, T^n z) + rd(z, Tz), \end{aligned}$$

hence $(1 - r)d(z, Tz) \le bd(z, T^n z)$. It follows [using (6) with $x = T^{n-1}z$] that

$$\begin{aligned} \frac{1}{b}\theta(r)\min\{d(ST^n z, T^n z), d(T^n z, T^{n+1}z)\} &\le \frac{1-r}{br^2} d(T^n z, T^{n+1}z) \\ &\le \frac{1-r}{br^n} d(T^n z, T^{n+1}z) \\ &\le \frac{1-r}{br^n}\cdot\frac{r^n}{b^{2n}} d(z, Tz) \\ &= \frac{1-r}{b^{2n+1}} d(z, Tz) \\ &\le \frac{1}{b^{2n}} d(z, T^n z) \\ &\le d(z, T^n z). \end{aligned}$$

Assumptions (2) and (10) imply that

$$\begin{aligned} &\max\{d(ST^n z, Sz), d(ST^n z, Tz), \\ &d(T^{n+1}z, Tz), d(Sz, T^{n+1}z)\} \le \frac{r}{b^2} d(z, T^n z) \\ &\le \frac{r}{b^2} d(z, Tz) \le \frac{r}{b} d(z, Tz). \end{aligned}$$

Thus

$$d(T^{n+1}z, Tz) \le \frac{r}{b} d(Tz, z). \tag{12}$$

So relation (11) is proved by induction.

Now $Tz \ne z$ and (11) imply that $T^n z \ne z$ for each $n \in \mathbb{N}$. Hence, (9) implies that

$$\begin{aligned} d(z, T^{n+1}z) &\le rd(z, T^n z) \\ &\le r^2 d(z, T^{n-1}z) \\ &\vdots \\ &\le r^n d(z, Tz). \end{aligned}$$

Hence $\lim_{n\to\infty} d(z, T^{n+1}z) = 0$. On the other hand using Lemma 1, we have

$$\frac{1}{b} d(z, \liminf_{n\longrightarrow\infty} T^{n+1}z) \le \liminf_{n\longrightarrow\infty} d(z, T^{n+1}z) = 0,$$

so

$$d(z, \liminf_{n\longrightarrow\infty} T^{n+1}z) = 0.$$

Similarly,

$$d(z, \limsup_{n\longrightarrow\infty} T^{n+1}z) = 0,$$

therefore $d(z, \lim_{n\longrightarrow\infty} T^{n+1}z) = 0$.

Thus $T^{n+1}z \longrightarrow z$ and, using Lemma 1 in (12), we have

$$\frac{1}{b} d(z, Tz) \le \limsup_{n\longrightarrow\infty} d(T^{n+1}z, Tz) \le \frac{r}{b} d(z, Tz),$$

which implies that $\mathrm{d}(z, Tz) = 0$, a contradiction.

Case II. $\frac{1}{\sqrt{2}} \le r < 1$ (and so $\theta(r) = \frac{1}{1+r}$). We will prove that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\frac{1}{b(1+r)}\min\{\mathrm{d}(x_{n_k}, Sx_{n_k}), \mathrm{d}(x_{n_k}, Tx_{n_k}) \le \mathrm{d}(x_{n_k}, z) \quad (13)$$

holds for each $k \in \mathbb{N}$. Suppose the contrary

$$\frac{1}{b(1+r)}\mathrm{d}(x_n, Tx_n) \ge \frac{1}{b(1+r)}\min\{\mathrm{d}(x_n, Sx_n), \mathrm{d}(x_n, Tx_n)\} > \mathrm{d}(x_n, z),$$

and

$$\frac{1}{b(1+r)}\mathrm{d}(x_n, Sx_n) \ge \frac{1}{b(1+r)}\min\{\mathrm{d}(x_n, Sx_n), \mathrm{d}(x_n, Tx_n)\} > \mathrm{d}(x_n, z),$$

holds for each $n \in \mathbb{N}$. Now if n is odd then

$$\frac{1}{b(1+r)}\mathrm{d}(x_{2n+1}, Tx_{2n+1}) \ge \frac{1}{b(1+r)}\min\{\mathrm{d}(x_{2n+1}, Sx_{2n+1}), \mathrm{d}(x_{2n+1}, Tx_{2n+1}) > \mathrm{d}(x_{2n+1}, z),$$

if n is even then

$$\frac{1}{b(1+r)}\mathrm{d}(x_{2n}, Sx_{2n}) \ge \frac{1}{b(1+r)}\min\{\mathrm{d}(x_{2n}, Sx_{2n}), \mathrm{d}(x_{2n}, Tx_{2n}) > \mathrm{d}(x_{2n}, z),$$

holds for each $n \in \mathbb{N}$. Then from (8) we have

$$\begin{aligned}
\mathrm{d}(x_{2n}, x_{2n+1}) &\le b\mathrm{d}(x_{2n}, z) + b\mathrm{d}(x_{2n+1}, z) \\
&< \frac{b}{b(1+r)}\mathrm{d}(x_{2n}, Sx_{2n}) + \frac{b}{b(1+r)}\mathrm{d}(x_{2n+1}, Tx_{2n+1}) \\
&= \frac{1}{1+r}\mathrm{d}(x_{2n}, x_{2n+1}) + \frac{1}{1+r}\mathrm{d}(x_{2n+1}, x_{2n+2}) \\
&\le \frac{1}{1+r}\mathrm{d}(x_{2n}, x_{2n+1}) + \frac{r}{b^2(1+r)}\mathrm{d}(x_{2n}, x_{2n+1}) \\
&\le \frac{1}{1+r}\mathrm{d}(x_{2n}, x_{2n+1}) + \frac{r}{1+r}\mathrm{d}(x_{2n}, x_{2n+1}) \\
&= \mathrm{d}(x_{2n}, x_{2n+1}),
\end{aligned}$$

which is impossible. Hence one of the following inequalities is satisfied for each $n \in \mathbb{N}$:

$$\frac{1}{b}\theta(r)\min\{\mathrm{d}(x_{2n}, Sx_{2n}), \mathrm{d}(x_{2n}, Tx_{2n}) \le \mathrm{d}(x_{2n}, z).$$

or

$$\frac{1}{b}\theta(r)\min\{\mathrm{d}(x_{2n+1}, Sx_{2n+1}), \mathrm{d}(x_{2n+1}, Tx_{2n+1}) \le \mathrm{d}(x_{2n+1}, z).$$

In other words, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that (13) holds for each $k \in \mathbb{N}$. Hence assumption (2) implies that

$$\begin{aligned}
\mathrm{d}(Sx_{2n}, Tz) &\le \max\left\{\begin{array}{l}\mathrm{d}(Sx_{2n}, Sz), \mathrm{d}(Tx_{2n}, Tz), \\ \mathrm{d}(Tz, Sx_{2n}), \mathrm{d}(Sz, Tx_{2n})\end{array}\right\} \\
&\le \frac{r}{b^2}\mathrm{d}(x_{2n}, z).
\end{aligned}$$

or

$$\begin{aligned}
\mathrm{d}(Tx_{2n+1}, Tz) &\le \max\left\{\begin{array}{l}\mathrm{d}(Sx_{2n+1}, Sz), \mathrm{d}(Tx_{2n+1}, Tz), \\ \mathrm{d}(Tz, Sx_{2n+1}), \mathrm{d}(Sz, Tx_{2n+1})\end{array}\right\} \\
&\le \frac{r}{b^2}\mathrm{d}(x_{2n+1}, z).
\end{aligned}$$

By Lemma 1, we get

$$\begin{aligned}
\frac{1}{b}\mathrm{d}(z, Tz) &\le \limsup_{n\longrightarrow\infty} \mathrm{d}(\mathrm{S}x_{2n}, \mathrm{T}z) \le \frac{r}{b^2}\limsup_{n\longrightarrow\infty} \mathrm{d}(x_{2n}, z) \\
&\le \frac{r}{b}\mathrm{d}(z, z) = 0,
\end{aligned}$$

or

$$\begin{aligned}
\frac{1}{b}\mathrm{d}(z, Tz) &\le \limsup_{n\longrightarrow\infty} \mathrm{d}(T x_{2n+1}, \mathrm{T}z) \le \frac{r}{b^2}\limsup_{n\longrightarrow\infty} \mathrm{d}(x_{2n+1}, z) \\
&\le \frac{r}{b}\mathrm{d}(z, z) = 0,
\end{aligned}$$

hence $\mathrm{d}(z, Tz) \le 0$, which is possible only if $Tz = z$.

Thus, we have proved that z is a fixed point of T. The uniqueness of the common fixed point follows easily from (2). Indeed, if z, z' are two common fixed points of T,

$$\frac{1}{b}\theta(r)\min\{\mathrm{d}(z, Tz), \mathrm{d}(z, Sz)\} \le \mathrm{d}(z, z'),$$

then (2) implies that

$$\begin{aligned}
\mathrm{d}(z, z') &= \max\{\mathrm{d}(Sz, Sz'), \mathrm{d}(Tz, Tz'), \mathrm{d}(Sz, Tz'), \mathrm{d}(Sz', Tz)\} \\
&\le \frac{r}{b^2}\mathrm{d}(z, z'),
\end{aligned}$$

which is possible only if $z = z'$. This proves that z is a unique common fixed point of T and S. □

According to Theorem 3 we get the following result.

Corollary 1 *Let (X, d) be a complete b-metric space, and let T be a mapping on X. Define a non-increasing function θ from [0, 1) into (1/2, 1] by (1).*

Suppose there exists $r \in [0, 1)$ such that for each $x, y \in X$, the following condition is satisfied

$$\frac{1}{b}\theta(r)\mathrm{d}(x, Tx) \le \mathrm{d}(x, y) \Longrightarrow \mathrm{d}(Tx, Ty) \le \frac{r}{b^2}\mathrm{d}(x, y),$$

then there exists a unique fixed-point z of T. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

Proof It is enough set $S = T$ in the Theorem 3 then the desired result is obtained. □

Remark 1 Note that for $b = 1$, Corollary 1 reduces to Theorem.

Corollary 2 *Let* (X, d) *be a complete b-metric space, and* $f, S, T : X \longrightarrow X$ *be three self-maps and* $\theta : [0,1) \longrightarrow (\frac{1}{2}, 1]$ *be defined by* (1).

Suppose there exists $r \in [0,1)$ *such that for each* $x, y \in X$, *the following condition is satisfied*

$$\frac{1}{b}\theta(r)\min\{d(x,fTx), d(x,fSx)\} \le d(x,y) \Longrightarrow \max\left\{\begin{array}{l} d(fSx,fSy), d(fTx,fTy), \\ d(fSx,fTy), d(fSy,fTx) \end{array}\right\} \le \frac{r}{b^2}d(x,y).$$

Also, if f *is one to one,* $fS = Sf$ *and* $fT = Tf$, *then we have* f, T and S *have a unique common fixed point* $z \in X$.

Proof By Theorem 3, fT, fS have a unique common fixed point $z \in X$. That is $fSz = fTz = z$, since f is one to one it follows that $Sz = Tz$. From

$$0 = \frac{1}{b}\theta(r)\min\{d(z,fTz), d(z,fSz)\} \le d(z,Tz),$$

it follows that

$$\begin{aligned} d(z,Tz) &\le \max\left\{\begin{array}{l} d(fSz,fSTz), d(fTz,fT^2z), \\ d(fSz,fT^2z), d(fSTz,fTz) \end{array}\right\} \\ &= \max\left\{\begin{array}{l} d(fSz,SfTz), d(fTz,TfTz), \\ d(fSz,TfTz), d(SfTz,fTz) \end{array}\right\} \\ &= \max\left\{\begin{array}{l} d(z,Sz), d(z,Tz), \\ d(z,Tz), d(Sz,z) \end{array}\right\} \\ &\le \frac{r}{b^2}d(z,Tz), \end{aligned}$$

it follows that $Tz = Sz = z$, hence $fz = fTz = z$. □

Corollary 3 Let (X, d) be a complete metric space, and $f, S, T : X \longrightarrow X$ be three self-maps and $\theta : [0,1) \longrightarrow (\frac{1}{2}, 1]$ be defined by (1).

Suppose there exists $r \in [0,1)$ such that for each $x, y \in X$, the following condition is satisfied

$$\theta(r)\min\{d(x,fTx), d(x,fSx)\} \le d(x,y) \Longrightarrow \max\left\{\begin{array}{l} d(fSx,fSy), d(fTx,fTy), \\ d(fSx,fTy), d(fSy,fTx) \end{array}\right\} \le r d(x,y).$$

Also, if f is one to one, $fS = Sf$ and $fT = Tf$, then we have f, T and S have a unique common fixed point $z \in X$.

Proof It is enough to set $b = 1$ in the Corollary 2 then the desired result is obtained. □

Now, in order to support the useability of our results, let us introduce the following examples.

Let $X = [0, \infty)$. Define $d : X \times X \to \mathbb{R}^+$ by

$$d(x,y) = \begin{cases} 0, & x = y \\ (x+y)^2, & x \ne y. \end{cases}$$

for all $x, y \in X$. Then (X, d) is a complete b-metric space for $b = 2$. Define two maps $T, S : X \to X$ by

$$\begin{aligned} T(x) &= \ln(1 + \frac{1}{4\sqrt{2}}x), \\ S(x) &= \ln(1 + \frac{1}{8\sqrt{2}}x) \end{aligned}$$

for $x \in X$. Then for each $x, y \in X$ we have

$$\begin{aligned} \frac{1}{2}\theta(r)\min\{d(x,Tx), d(x,Sx)\} &= \frac{1}{4}\min\left\{\begin{array}{l} (x + \ln(1 + \frac{1}{4\sqrt{2}}x))^2, \\ (x + \ln(1 + \frac{1}{8\sqrt{2}}x))^2 \end{array}\right\} \\ &= \frac{1}{4}(x + \ln(1 + \frac{1}{8\sqrt{2}}x))^2 \\ &\le \frac{1}{4}(x + \frac{1}{8\sqrt{2}}x)^2 = \frac{1}{4}(1 + \frac{1}{8\sqrt{2}})^2 x^2 \\ &\le x^2 \le (x+y)^2 = d(x,y). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \max\left\{\begin{array}{l} d(Sx,Sy), d(Tx,Ty), \\ d(Sx,Ty), d(Sy,Tx) \end{array}\right\} &= \max\left\{\begin{array}{l} (\ln(1+\frac{1}{8\sqrt{2}}x) + \ln(1+\frac{1}{8\sqrt{2}}y))^2, \\ (\ln(1+\frac{1}{4\sqrt{2}}x) + \ln(1+\frac{1}{4\sqrt{2}}y))^2, \\ (\ln(1+\frac{1}{8\sqrt{2}}x) + \ln(1+\frac{1}{4\sqrt{2}}y))^2, \\ (\ln(1+\frac{1}{4\sqrt{2}}x) + \ln(1+\frac{1}{8\sqrt{2}}y))^2 \end{array}\right\} \\ &\le \max\left\{\begin{array}{l} (\frac{1}{8\sqrt{2}}x + \frac{1}{8\sqrt{2}}y)^2, \\ (\frac{1}{4\sqrt{2}}x + \frac{1}{4\sqrt{2}}y)^2, \\ (\frac{1}{8\sqrt{2}}x + \frac{1}{4\sqrt{2}}y)^2, \\ (\frac{1}{4\sqrt{2}}x + \frac{1}{8\sqrt{2}}y)^2 \end{array}\right\} \\ &\le (\frac{1}{4\sqrt{2}}x + \frac{1}{4\sqrt{2}}y)^2 = [\frac{1}{4\sqrt{2}}(x+y)]^2 \\ &\le \frac{1}{4}.\frac{1}{2\sqrt{2}}(x+y)^2 = \frac{r}{b^2}d(x,y). \end{aligned}$$

Thus T and S satisfy all the hypotheses of Theorem 3 and hence T and S have a unique common fixed point. Indeed, $r = \frac{1}{2\sqrt{2}} < \frac{\sqrt{5}-1}{2}$, $\theta(r) = \frac{1}{2}$ and 0 is the unique common fixed point of T and S.

Inspired by [8, Example 4] and [26, Example 1], we present the following example:

Example 3 Let $X = \{(0,0), (10,12), (12,10), (40,42), (42,40)\} \subset \mathbb{R}^2$. Define $d : X \times X \to \mathbb{R}^+$ by

$$d((x_1,y_1),(x_2,y_2)) = (x_1 - x_2)^2 + (y_1 - y_2)^2,$$

for all $x = (x_1, y_1), y = (x_2, y_2) \in X$. Then (X, d) is a complete b-metric space for $b = 2$. Define two maps $T, S : X \to X$ by

$$\begin{cases} T(0,0) = T(10,12) = T(12,10) = (0,0), \\ T(40,42) = (10,12), \\ T(42,40) = (12,10). \end{cases}$$

$$\begin{cases} S(0,0) = S(10,12) = S(12,10) = (0,0), \\ \qquad S(40,42) = (12,10), \\ \qquad S(42,40) = (10,12). \end{cases}$$

Then for each $x, y \in X$, if

$$\frac{1}{2}\frac{\sqrt{2}}{\sqrt{2}+1}\min\{d(x,Tx), d(x,Sx)\} \leq d(x,y),$$

this implies that

$$\max\left\{ \begin{array}{c} \mathrm{d}(Sx,Sy), \mathrm{d}(Tx,Ty), \\ \mathrm{d}(Sx,Ty), \mathrm{d}(Sy,Tx) \end{array} \right\} \leq \frac{r}{4}\mathrm{d}(x,y).$$

Because,

i) $\frac{1}{2}\frac{\sqrt{2}}{\sqrt{2}+1}\min\{\mathrm{d}((0,0),T(0,0)), \mathrm{d}((0,0),S(0,0))\} \leq \mathrm{d}(0,0),y), \quad \forall y \in X.$

ii) $\frac{1}{2}\frac{\sqrt{2}}{\sqrt{2}+1}\min\{\mathrm{d}((10,12),T(10,12)), \mathrm{d}((10,12),S(10,12))\} \leq \mathrm{d}(10,12),y), \quad \forall y = (0,0),(40,42),(42,40).$

iii) $\frac{1}{2}\frac{\sqrt{2}}{\sqrt{2}+1}\min\{\mathrm{d}((12,10),T(12,10)), \mathrm{d}((12,10),S(12,10))\} \leq \mathrm{d}(12,10),y), \quad \forall y = (0,0),(40,42),(42,40).$

iv) $\frac{1}{2}\frac{\sqrt{2}}{\sqrt{2}+1}\min\{\mathrm{d}((40,42),T(40,42)), \mathrm{d}((40,42),S(40,42))\} \leq \mathrm{d}(40,42),y), \quad \forall y = (0,0),(10,12),(12,10).$

v) $\frac{1}{2}\frac{\sqrt{2}}{\sqrt{2}+1}\min\{\mathrm{d}((42,40),T(42,40)), \mathrm{d}((42,40),S(42,40))\} \leq \mathrm{d}(42,40),y), \quad \forall y = (0,0),(10,12),(12,10).$

On the other hand, in all of the cases we have

$$\max\left\{ \begin{array}{c} \mathrm{d}(Sx,Sy), \mathrm{d}(Tx,Ty), \\ \mathrm{d}(Sx,Ty), \mathrm{d}(Sy,Tx) \end{array} \right\} \leq \frac{1}{4\sqrt{2}}\mathrm{d}(x,y).$$

Thus T satisfy all the hypotheses of Theorem 3 and hence T has a unique fixed point. Indeed, $r = \frac{1}{\sqrt{2}}$, $\theta(r) = \frac{\sqrt{2}}{\sqrt{2}+1}$ and (0, 0) is the unique common fixed point of T and S.

But,

$$\mathrm{d}(T(40,42), T(42,40)) \leq \frac{r}{4}\mathrm{d}((40,42),(42,40)),$$

that is $2^2 + 2^2 \leq \frac{r}{4}(2^2 + 2^2)$ this implies that $r \geq 4$. It is contradiction. This proves that Theorem 1 is not applicable to T.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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