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ORIGINAL RESEARCH

Coupled and tripled coincidence point results under (F, g)-invariant sets in G_b-metric spaces and G-α-admissible mappings

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Abstract In this paper, we prove that coupled and tripled coincidence point theorems under (F, g)-invariant sets for weakly contractive mappings defined on a G-metric space are immediate consequences of corresponding results via rectangular G-α-admissible mappings. This idea can also be applied to obtain coupled and tripled fixed point theorems in other spaces under various contractive conditions which reduces the proof considerably.

Keywords G_b-metric space · Coupled coincidence point · Common coupled fixed point · Tripled coincidence point · Common tripled fixed point · F-invariant set · Admissible mapping

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Introduction and mathematical preliminaries

The concept of generalized metric space, or a G-metric space, was introduced by Mustafa and Sims.

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Definition 1.1 (*G-Metric Space*, [14]) Let X be a nonempty set and $G: X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ iff $x = y = z$;
(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, (symmetry in all three variables);
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Recently, Aghajani et al. [1] motivated by the concept of b-metric [27] introduced the concept of generalized b-metric spaces (G_b-metric spaces) and then they presented some basic properties of G_b-metric spaces.

The following is their definition of G_b-metric spaces.

Definition 1.2 [1] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G: X \times X \times X \to \mathbb{R}^+$ satisfies:

(G_b1) $G(x, y, z) = 0$ if $x = y = z$,
(G_b2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G_b3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(G_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),
(G_b5) $G(x, y, z) \leq s[G(x, a, a) + G(a, y, z)]$ for all $x, y, z, a \in X$ (rectangle inequality).

Then, G is called a generalized b-metric and the pair (X, G) is called a generalized b-metric space or a G_b-metric space.

Each G-metric space is a G_b-metric space with $s = 1$.

Example 1.3 [1] Let (X,G) be a G-metric space and $G_*(x,y,z)=G(x,y,z)^p$, where $p>1$ is a real number. Then, G_* is a G_b-metric with $s=2^{p-1}$.

Example 1.4 [13] Let $X=\mathbb{R}$ and $d(x,y)=|x-y|^2$. We know that (X,d) is a b-metric space with $s=2$. Let $G(x,y,z)=d(x,y)+d(y,z)+d(z,x)$, it is easy to see that (X,G) is not a G_b-metric space. Indeed, (G_b3) is not true for $x=0$, $y=2$ and $z=1$. However, $G(x,y,z)=\max\{d(x,y),d(y,z),d(z,x)\}$ is a G_b-metric on $\mathbb{R}$ with $s=2$.

Proposition 1.5 [1] *Let X be a G_b-metric space. Then, for each $x,y,z,a\in X$, it follows that:*

1. *if $G(x,y,z)=0$ then $x=y=z$,*
2. $G(x,y,z)\le s(G(x,x,y)+G(x,x,z))$,
3. $G(x,y,y)\le 2sG(y,x,x)$,
4. $G(x,y,z)\le s(G(x,a,z)+G(a,y,z))$.

Definition 1.6 [1] Let X be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be:

1. G_b-Cauchy if, for each $\varepsilon>0$ there exists a positive integer n_0 such that for all $m,n,l\ge n_0$, $G(x_n,x_m,x_l)<\varepsilon$;
2. G_b-convergent to a point $x\in X$ if, for each $\varepsilon>0$ there exists a positive integer n_0 such that for all $m,n\ge n_0$, $G(x_n,x_m,x)<\varepsilon$.

Definition 1.7 [1] A G_b-metric space X is called G_b-complete, if every G_b-Cauchy sequence is G_b -convergent in X.

Proposition 1.8 *Let (X,G) and (X',G') be two G_b-metric spaces. Then, a function $f\colon X\to X'$ is G_b-continuous at a point $x\in X$ if and only if it is G_b-sequentially continuous at x, that is, whenever $\{x_n\}$ is G_b-convergent to x, $\{f(x_n)\}$ is G'_b-convergent to $f(x)$.*

Proposition 1.9 *Let (X,G) be a G_b-metric space. A mapping $F\colon X\times X\to X$ is said to be continuous if for any two G_b-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y, respectively, $\{F(x_n,y_n)\}$ is G_b-convergent to $F(x,y)$.*

Existence of fixed points, coupled fixed points and tripled fixed points for contractive type mappings in partially ordered metric spaces has been considered recently by several authors (see, [2, 3, 5–8, 11, 17, 18, 20, 22, 24–26, 40–44]).

Lakshmikantham and Ćirić [11] introduced the notions of mixed g-monotone mapping and coupled coincidence point and proved some coupled coincidence point and common coupled fixed point theorems in partially ordered complete metric spaces.

Definition 1.10 [11] Let $(X,\preceq)$ be a partially ordered set and let $F\colon X\times X\to X$ and $g\colon X\to X$ be two mappings. F has the mixed g-monotone property, if F is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument, that is, for all $x_1,x_2\in X$, $gx_1\preceq gx_2$ implies $F(x_1,y)\preceq F(x_2,y)$ for any $y\in X$ and for all $y_1,y_2\in X$, $gy_1\preceq gy_2$ implies $F(x,y_1)\succeq F(x,y_2)$ for any $x\in X$.

Definition 1.11 [3, 11] An element $(x,y)\in X\times X$ is called

1. a coupled fixed point of mapping $F\colon X\times X\to X$ if $x=F(x,y)$ and $y=F(y,x)$.
2. a coupled coincidence point of mappings $F\colon X\times X\to X$ and $g\colon X\to X$ if $g(x)=F(x,y)$ and $g(y)=F(y,x)$.
3. a common coupled fixed point of mappings $F\colon X\times X\to X$ and $g\colon X\to X$ if $x=g(x)=F(x,y)$ and $y=g(y)=F(y,x)$.

Choudhury and maity [6] have established some coupled fixed point results for mappings with mixed monotone property in partially ordered G-metric spaces. They obtained the following results.

Theorem 1.12 ([6], Theorem 3.1) *Let $(X,\preceq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a complete G-metric space. Let $F\colon X\times X\to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $k\in[0,1)$ such that*

$$G(F(x,y),F(u,v),F(w,z))\le\frac{k}{2}[G(x,u,w)+G(y,v,z)],\tag{1.1}$$

for all $x\preceq u\preceq w$ and $y\succeq v\succeq z$, where either $u\ne w$ or $v\ne z$.

If there exist $x_0,y_0\in X$ such that $x_0\preceq F(x_0,y_0)$ and $y_0\succeq F(y_0,x_0)$, then, F has a coupled fixed point in X, that is, there exist $x,y\in X$ such that $x=F(x,y)$ and $y=F(y,x)$.

Theorem 1.13 ([6], Theorem 3.2) *If in the above theorem, in place of the continuity of F, we assume the following conditions, namely,*

(i) *if a nondecreasing sequence $\{x_n\}\to x$, then $x_n\preceq x$ for all n, and*
(ii) *if a nonincreasing sequence $\{y_n\}\to y$, then $y_n\succeq y$ for all n, then, F has a coupled fixed point.*

Definition 1.14 [4] Let $(X,\preceq)$ be a partially ordered set and G be a G-metric on X. We say that $(X,G,\preceq)$ is regular if the following conditions hold:

(i) If $\{x_n\}$ is a nondecreasing sequence with $x_n\to x$, then $x_n\preceq x$ for all $n\in\mathbb{N}$.

(ii) If $\{x_n\}$ is a nonincreasing sequence with $x_n \to x$, then $x_n \succeq x$ for all $n \in \mathbb{N}$.

Definition 1.15 [6] Let (X, G) be a generalized b-metric space. Mappings $f: X^2 \to X$ and $g: X \to X$ are called compatible if

$$\lim_{n\to\infty} G(gf(x_n, y_n), f(gx_n, gy_n), f(gx_n, gy_n)) = 0$$

and

$$\lim_{n\to\infty} G(gf(y_n, x_n), f(gy_n, gx_n), f(gy_n, gx_n)) = 0$$

hold whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} gx_n$ and $\lim_{n\to\infty} f(y_n, x_n) = \lim_{n\to\infty} gy_n$.

On the other hand, Berinde and Borcut [24] introduced the concept of tripled fixed point and obtained some tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. For a survey of tripled fixed point theorems and related topics, we refer the reader to [24–26].

Definition 1.16 ([24, 25]) Let $(X, \preceq)$ be a partially ordered set, $f: X^3 \to X$ and $g: X \to X$. An element $(x, y, z) \in X^3$ is called

1. a tripled fixed point of f if $f(x, y, z) = x$, $f(y, x, y) = y$, and $f(z, y, x) = z$.
2. a tripled coincidence point of the mappings f and g if $f(x, y, z) = gx$, $f(y, x, y) = gy$ and $f(z, y, x) = gz$.
3. a tripled common fixed point of f and g if $x = g(x) = f(x, y, z)$, $y = g(y) = f(y, x, y)$ and $z = g(z) = f(z, y, x)$.
4. We say that f has the mixed g-monotone property if $f(x, y, z)$ is g-nondecreasing in x, g-nonincreasing in y and g-nondecreasing in z, that is, if for any $x, y, z \in \mathcal{X}$,

$$x_1, x_2 \in X, gx_1 \preceq gx_2 \Rightarrow f(x_1, y, z) \preceq f(x_2, y, z),$$
$$y_1, y_2 \in X, gy_1 \preceq gy_2 \Rightarrow f(x, y_1, z) \succeq f(x, y_2, z)$$

and

$$z_1, z_2 \in X, gz_1 \preceq gz_2 \Rightarrow f(x, y, z_1) \preceq f(x, y, z_2).$$

Definition 1.17 Let (X, G) be a generalized b-metric space. Mappings $f: X^3 \to X$ and $g: X \to X$ are called compatible if

$$\lim_{n\to\infty} G(gf(x_n, y_n, z_n), f(gx_n, gy_n, gz_n), f(gx_n, gy_n, gz_n)) = 0,$$
$$\lim_{n\to\infty} G(gf(y_n, x_n, y_n), f(gy_n, gx_n, gy_n), f(gy_n, gx_n, gy_n)) = 0$$

and

$$\lim_{n\to\infty} G(gf(z_n, y_n, x_n), f(gz_n, gy_n, gx_n), f(gz_n, gy_n, gx_n)) = 0$$

hold whenever $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences in X such that $\lim_{n\to\infty} f(x_n, y_n, z_n) = \lim_{n\to\infty} gx_n$, $\lim_{n\to\infty} f(y_n, x_n, y_n) = \lim_{n\to\infty} gy_n$ and $\lim_{n\to\infty} f(z_n, y_n, x_n) = \lim_{n\to\infty} gz_n$.

let $\psi: [0, +\infty) \to [0, +\infty)$ satisfies:

(i) ψ is continuous and nondecreasing,
(ii) $\psi(t) = 0$ if and only if t = 0.

That is, ψ is an altering distance function.

Batra and Vashistha [36] introduced the concept of an (F, g)-invariant set which is a generalization of the F-invariant set introduced by Samet and Vetro [37].

Definition 1.18 [36] Let (X, d) be a metric space and let $F: X \times X \to X$ and $g: X \to X$ be given mappings. Let M be a nonempty subset of X^4. We say that M is an (F, g)-invariant subset of X^4 if and only if, for all $x, y, z, w \in X$,

(i) $(x, y, z, w) \in M$ iff $(w, z, y, x) \in M$;
(ii) $(g(x), g(y), g(z), g(w)) \in M$ implies that $(F(x, y), F(y, x), F(z, w), F(w, z)) \in M$.

Definition 1.19 [35] Let (X, G) be a G-metric space and let $F: X \times X \to X$ be a given mapping. Let M be a nonempty subset of X^6. We say that M is an F^*-invariant subset of X^6 if and only if, for all $x, y, z, u, v, w \in X$,

1. $(x, y, z, u, v, w) \in M$ iff $(w, v, u, z, y, x) \in M$;
2. $(x, y, z, u, v, w) \in M$ implies that $(F(x, y), F(y, x), F(z, u), F(u, z), F(v, w), F(w, v)) \in M$.

Definition 1.20 [35] Let (X, G) be a G-metric space and let $F: X \times X \to X$ and $g: X \to X$ are given mappings. Let M be a nonempty subset of X^6. We say that M is an (F^*, g)-invariant subset of X^6 if and only if, for all $x, y, z, u, v, w \in X$,

1. $(x, y, z, u, v, w) \in M$ iff $(w, v, u, z, y, x) \in M$;
2. $(gx, gy, gz, gu, gv, gw) \in M$ implies that $(F(x, y), F(y, x), F(z, u), F(u, z), F(v, w), F(w, v)) \in M$.

Definition 1.21 (*corrected from* [35]) Let (X, G) be a G-metric space and let M be a subset of X^6. We say that M satisfies the *transitive property* if and only if, for all $x, y, z, u, v, w, a, b \in X$,

$(x, y, z, u, z, u) \in M$ and $(z, u, v, w, a, b) \in M$ then $(x, y, v, w, a, b) \in M$.

Samet et al. [29] defined the notion of α-admissible mapping as follows.

Definition 1.22 Let T be a self-mapping on X and let $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Definition 1.23 [30] Let (X, G) be a G-metric space, T be a self-mapping on X and $\alpha : X^3 \to [0, +\infty)$ be a function. We say that T is an G-α-admissible mapping if

$$x, y, z \in X, \quad \alpha(x, y, z) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty, Tz) \geq 1.$$

Following the recent work in [31], Hussain et al. [23] presented the following definition in the setting of G-metric spaces.

Definition 1.24 [23] Let (X, G) be a G-metric space, $f, g : X \to X$ and $\alpha : X^3 \to [0, +\infty)$. We say that f is a rectangular G-α-admissible mapping with respect to g if

(R1) $\alpha(gx, gy, gz) \geq 1$ implies $\alpha(fx, fy, fz) \geq 1$, $x, y, z \in X$,

(R2) $\begin{cases} \alpha(gx, gy, gy) \geq 1 \\ \alpha(gy, gz, gw) \geq 1 \end{cases}$ implies $\alpha(gx, gz, gw) \geq 1$, $x, y, z, w \in X$.

Definition 1.25 Let (X, G) be a G-metric space, $F : X \times X \to X$, $g : X \to X$ and $\alpha : (X^2)^3 \to [0, +\infty)$. We say that F is a rectangular G-α-admissible mapping with respect to g if

(R1)

$$\begin{aligned} &\alpha((gx, gy), (gu, gv), (ga, gb)) \geq 1 \quad \text{implies} \\ &\alpha((F(x, y), F(y, x)), (F(u, v), F(v, u)), (F(a, b), F(b, a))) \geq 1, \end{aligned} \tag{1.2}$$

where $x, y, u, v, a, b \in X$,

(R2)

$$\begin{aligned} &\alpha((gx, gy), (gu, gv), (gu, gv)) \geq 1, \quad \text{and} \\ &\alpha((gu, gv), (ga, gb), (gc, gd)) \geq 1 \quad \text{implies that} \\ &\alpha((gx, gy), (ga, gb), (gc, gd)) \geq 1, \end{aligned} \tag{1.3}$$

where $x, y, u, v, a, b, c, d \in X$.

Definition 1.26 Let (X, G) be a G-metric space, $F : X \times X \times X \to X$, $g : X \to X$ and $\alpha : (X^3)^3 \to [0, +\infty)$. We say that F is a rectangular G-α-admissible mapping with respect to g if

(R1)

$$\begin{aligned} &\alpha((gx, gy, gz), (gu, gv, gw), (ga, gb, gc)) \geq 1 \text{ implies} \\ &\alpha((F(x, y, z), F(y, x, y), F(z, y, x)), (F(u, v, w), F(v, u, v), \\ &F(w, v, u)), (F(a, b, c), F(b, a, b), F(c, b, a))) \geq 1, \end{aligned} \tag{1.4}$$

where $x, y, z, u, v, w, a, b, c \in X$,

(R2)

$$\begin{aligned} &\alpha((gx, gy, gz), (gu, gv, gw), (gu, gv, gw)) \geq 1 \text{ and} \\ &\alpha((gu, gv, gw), (ga, gb, gc), (gd, ge, gf)) \geq 1 \text{ implies} \\ &\alpha((gx, gy, gz), (ga, gb, gc), (gd, ge, gf)) \geq 1, \end{aligned} \tag{1.5}$$

where $x, y, z, u, v, w, a, b, c, d, e, f \in X$.

Using the following coincidence point result, Hussain et al. obtained some interesting coupled and tripled coincidence point results which we use them in obtaining the main results here.

Theorem 1.27 [23] *Let (X, G) be a generalized b-metric space and let $f, g : X \to X$ satisfy the following condition:*

$$\alpha(gx, gy, gz)\psi(sG(fx, fy, fz)) \leq \psi(G(gx, gy, gz)) - \varphi(G(gx, gy, gz)) \tag{1.6}$$

for all $x, y, z \in X$, where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are two altering distance mappings, $\alpha : X^3 \to [0, +\infty)$ and f is a rectangular G-α-admissible mapping w.r.t. g. Then, f and g have a coincidence point if,

(i) *$f(X) \subseteq g(X)$;*
(ii) *there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0, fx_0) \geq 1$;*
(iii) *f and g are continuous and compatible and (X, G) is complete, or.*
(iii)′ *one of f(X) or g(X) is complete and whenever $\{x_n\}$ in X be a sequence such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$*

Lemma 1.28 [23] *Let (X, G) be a generalized b-metric space (with the parameter s).*

(a) *If a mapping $\Omega_2^m : X^2 \times X^2 \times X^2 \to \mathbb{R}^+$ is given by*

$$\Omega_2^m(X, U, A) = \max\{G(x, u, a), G(y, v, b)\},$$
$$X = (x, y), \ U = (u, v) \text{ and } A = (a, b) \in X^2,$$

then (X^2, Ω_2^m) is a generalized b-metric space (with the same parameter s). The space (X^2, Ω_2^m) is G_b-complete iff (X, G) is G_b-complete.

Let (X, G) be a generalized b-metric space, $F: X^2 \to X$ and $g: X \to X$. In the rest of this paper unless otherwise stated, for all $x, y, u, v, z, w \in X$, let

$$N_F^m(x,y,u,v,z,w) = \max\{G(F(x,y),F(u,v),F(z,w)), G(F(y,x),F(v,u),F(w,z))\}$$

and

$$N_g^m(x,y,u,v,z,w) = \max\{G(gx,gu,gz),G(gy,gv,gw)\}.$$

Theorem 1.29 [23] *Let (X, G) be a generalized b-metric space with the parameter s and let $F: X^2 \to X$ and $g: X \to X$. Assume that*

$$\alpha((gx,gy),(gu,gv),(gz,gw)) \psi(sN_F^m(x,y,u,v,z,w)) \leq \psi(N_g^m(x,y,u,v,z,w)) - \varphi(N_g^m(x,y,u,v,z,w)), \tag{1.7}$$

for all $x,y,u,v,z,w \in X$, where $\psi, \varphi: [0,\infty) \to [0,\infty)$ are altering distance functions, $\alpha: (X^2)^3 \to [0,\infty)$ and F is a rectangular G-α-admissible mapping with respect to g. Assume also that

1. $F(X^2) \subseteq g(X)$;
2. *there exist $x_0, y_0 \in X$ such that*

$$\alpha((gx_0,gy_0),(F(x_0,y_0),F(y_0,x_0)),(F(x_0,y_0),F(y_0,x_0))) \geq 1$$

and

$$\alpha((gy_0,gx_0),(F(y_0,x_0),F(x_0,y_0)),(F(y_0,x_0),F(x_0,y_0))) \geq 1.$$

Also, suppose that either

(a) *F and g are continuous, the pair (F, g) is compatible and (X, G) is G_b-complete, or*
(b) *$(g(X), G)$ is G_b-complete and assume that whenever $\{x_n\}$ and $\{y_n\}$ in X be sequences such that*

$$\alpha((x_n,y_n),(x_{n+1},y_{n+1}),(x_{n+1},y_{n+1})) \geq 1$$

and

$$\alpha((y_n,x_n),(y_{n+1},x_{n+1}),(y_{n+1},x_{n+1})) \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$, $y_n \to y$ as $n \to +\infty$, we have

$$\alpha((x_n,y_n),(x,y),(x,y)) \geq 1$$

and

$$\alpha((y_n,x_n),(y,x),(y,x)) \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$. Then, F and g have a coupled coincidence point in X.

Theorem 1.30 [23] *In addition to the hypotheses of Theorem 1.29, suppose that for all (x,y) and $(x^*,y^*) \in X^2$, there exists $(u,v) \in X^2$, such that $\alpha((gx,gy),(gu,gv),(gu,gv)) \geq 1$ and $\alpha((gx^*,gy^*),(gu,gv),(gu,gv)) \geq 1$. Then, F and g have a unique common coupled fixed point of the form (a,a).*

Let $\Omega_2^a: X^2 \times X^2 \times X^2 \to \mathbb{R}^+$ is given by

$$\Omega_2^a(X,U,A) = \frac{G(x,u,a)+G(y,v,b)}{2},$$
$$X=(x,y),\ U=(u,v) \quad \text{and} \quad \mathrm{A}=(\mathrm{a},\mathrm{b}) \in \mathrm{X}^2,$$

then (X^2, Ω_2^a) is a generalized b-metric space (with the same parameter s).

Let (X, G) be a generalized b-metric space, $F: X^2 \to X$ and $g: X \to X$. For all $x,y,u,v,z,w \in X$, let

$$N_F^a(x,y,u,v,z,w) = \frac{G(F(x,y),F(u,v),F(z,w))+G(F(y,x),F(v,u),F(w,z))}{2}$$

and

$$N_g^a(x,y,u,v,z,w) = \frac{G(gx,gu,gz)+G(gy,gv,gw)}{2}.$$

Remark 1.31 [23] The result of Theorems 1.29 and 1.30 holds, if we replace Ω_2^m, N_F^m and N_g^m by Ω_2^a, N_F^a and N_g^a, respectively.

Coupled fixed point results under (F^*, g)-invariant sets

Definition 2.1 Let (X, G) be a G_b-metric space and $M \subseteq X^6$. We say that X is M-regular if and only if the following hypothesis holds:

Whenever $\{x_n\}$ and $\{y_n\}$ in X be sequences such that

$$(x_n,y_n,x_{n+1},y_{n+1},x_{n+1},y_{n+1}) \in M$$

and

$$(y_n,x_n,y_{n+1},x_{n+1},y_{n+1},x_{n+1}) \in M$$

for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$, $y_n \to y$ as $n \to +\infty$, we have

$$(x_n,y_n,x,y,x,y) \in M$$

and

$$(y_n,x_n,y,x,y,x) \in M$$

for all $n \in \mathbb{N} \cup \{0\}$.

Theorem 2.2 *Let (X, G_b) be a G_b-metric space with the parameter s, $F: X^2 \to X$, $g: X \to X$ and M be a nonempty subset of X^6. Assume that*

$$\psi(sN_F^m(x,y,u,v,z,w)) \leq \psi(N_g^m(x,y,u,v,z,w)) - \varphi(N_g^m(x,y,u,v,z,w)), \tag{2.1}$$

for all $x, y, u, v, z, w \in X$ *with* $(gx, gu, gz, gy, gv, gw) \in M$, *where* $\psi, \varphi : [0, \infty) \to [0, \infty)$ *are altering distance functions. Assume also that*

1. $F(X^2) \subseteq g(X)$;
2. *M is an* (F^*, g)*-invariant set which satisfies the transitive property;*
3. there exist $x_0, y_0 \in X$ *such that* $(gx_0, gy_0, F(x_0, y_0), F(y_0, x_0), F(x_0, y_0), F(y_0, x_0)) \in M$.

Also, suppose that either

(a) *F and g are continuous, the pair* (F, g) *is compatible and* (X, G) *is* G_b*-complete, or*
(b) (X, G_b) *is M-regular and* $(g(X), G)$ *is* G_b*-complete. Then, F and g have a coupled coincidence point in X.*

Proof Define $\alpha : (X^2)^3 \to [0, +\infty)$ by

$$\alpha((x, y), (u, v), (a, b)) = \begin{cases} 1, & \text{if } (x, y, u, v, a, b) \in M \\ 0, & \text{otherwise.} \end{cases}$$

First, we prove that F is a rectangular G-α-admissible mapping w.r.t. g. Hence, we assume that $\alpha((gx, gy), (gu, gv), (ga, gb)) \geq 1$. Therefore, we have $(gx, gy, gu, gv, ga, gb) \in M$. Since, M is an (F^*, g)-invariant subset of X^6, then

$$(F(x, y), F(y, x), F(u, v), F(v, u), F(a, b), F(b, a)) \in M$$

which implies that

$$\alpha((F(x, y), F(y, x)), (F(u, v), F(v, u)), (F(a, b), F(b, a))) \geq 1.$$

Now, let $\alpha((x, y), (a, b), (a, b)) \geq 1$ and $\alpha((a, b), (u, v), (u, v)) \geq 1$, then $(x, y, a, b, a, b) \in M$ and $(a, b, u, v, u, v) \in M$. Consequently, as M satisfies the transitive property, we deduce that $(x, y, a, b, u, v) \in M$, that is, $\alpha((x, y), (a, b), (u, v)) \geq 1$. Thus, F is a rectangular G-α-admissible mapping w.r.t. g.

From (2.1) and the definition of α,

$$\begin{aligned} &\alpha((gx, gy), (gu, gv), (gz, gw)) \\ &\psi(sN_F^m(x, y, u, v, z, w)) \leq \psi(N_g^m(x, y, u, v, z, w)) \\ &- \varphi(N_g^m(x, y, u, v, z, w)), \end{aligned} \tag{2.2}$$

for all $x, y, u, v, z, w \in X$. Moreover, from (2) there exist $x_0, y_0 \in \mathcal{X}$ such that

$$\alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0)), (F(x_0, y_0), F(y_0, x_0))) \geq 1$$

and

$$\alpha((gy_0, gx_0), (F(y_0, x_0), F(x_0, y_0)), (F(y_0, x_0), F(x_0, y_0))) \geq 1.$$

Hence, all the conditions of Theorem 1.29 are satisfied and so F and g have a coupled coincidence point. □

In the following theorem, we give a sufficient condition for the uniqueness of the common coupled fixed point (see also [24]).

Theorem 2.3 *In addition to the hypotheses of Theorem 2.2, suppose that for all* (x, y) *and* $(x^*, y^*) \in X^2$, *there exists* $(u, v) \in X^2$, *such that* $(gx, gy, gu, gv, gu, gv) \in M$ *and* $(gx^*, gy^*, gu, gv, gu, gv) \in M$. *Then, F and g have a unique common coupled fixed point of the form* (a, a).

Remark 2.4 In Theorem 2.2, we can replace the contractive condition (2.1) by the following:

$$\begin{aligned} \psi(sN_F^a(x, y, u, v, z, w)) \leq{}& \psi(N_g^a(x, y, u, v, z, w)) \\ &- \varphi(N_g^a(x, y, u, v, z, w)). \end{aligned} \tag{2.3}$$

In Theorem 2.2, if we take $\psi(t) = t$ for all $t \in [0, \infty)$, we obtain the following result.

Corollary 2.5 *Let* (X, G_b) *be a* G_b*-metric space with the parameter* s, $F : X^2 \to X$, $g : X \to X$ *and M be a nonempty subset of* X^6. *Assume that*

$$\begin{aligned} &\frac{G(F(x, y), F(u, v), F(z, w)) + G(F(y, x), F(v, u), F(w, z))}{2} \\ &\leq \frac{1}{s}\frac{G(gx, gu, gz) + G(gy, gv, gw)}{2} \\ &\quad - \frac{1}{s}\varphi\left(\frac{G(gx, gu, gz) + G(gy, gv, gw)}{2}\right), \end{aligned} \tag{2.4}$$

for all $x, y, u, v, z, w \in X$ *with* $(gx, gu, gz, gy, gv, gw) \in M$, *where* $\varphi : [0, \infty) \to [0, \infty)$ *is an altering distance function. Assume also that*

1. $F(X^2) \subseteq g(X)$;
2. M is an (F^*, g)*-invariant set which satisfies the transitive property*;
3. there exist $x_0, y_0 \in X$ *such that* $(gx_0, gy_0, F(x_0, y_0), F(y_0, x_0), F(x_0, y_0), F(y_0, x_0)) \in M$.

Also, suppose that either

(a) *F and g are continuous, the pair* (F, g) *is compatible and* (X, G) *is* G_b*-complete, or*
(b) (X, G_b) *is M-regular and* $(g(X), G)$ *is* G_b*-complete. Then, F and g have a coupled coincidence point in X.*

Tripled coincidence point results via (F^*, g)-invariant sets

In this section, we prove some tripled coincidence and tripled common fixed point results.

Definition 3.1 Let (X, G) be a G-metric space and let $F: X \times X \times X \to X$ and $g: X \to X$ are given mappings. Let M be a nonempty subset of X^9. We say that M is an (f^*, g)-invariant subset of X^9 if and only if, for all $x, y, z, u, v, w, a, b, c \in X$,

1. $(x, y, z, u, v, w, a, b, c) \in M$ iff $(c, b, a, w, v, u, z, y, x) \in M$;
2. $$(gx, gy, gz, gu, gv, gw, ga, gb, gc) \in M$$ implies that $$(F(x,y,z), F(y,x,y), F(z,y,x), F(u,v,w), F(v,u,v), F(w,v,u), F(a,b,c), F(b,a,b), F(c,b,a)) \in M.$$

Definition 3.2 Let (X, G) be a G-metric space and let M be a subset of X^9. We say that M satisfies the *transitive property* if and only if, for all $x, y, z, u, v, w, a, b, c, d, e, f \in X$,

$$(x, y, z, u, v, w, u, v, w) \in M \text{ and } (u, v, w, a, b, c, d, e, f) \in M \text{ then } (x, y, z, a, b, c, d, e, f) \in M.$$

Definition 3.3 Let (X, G) be a G_b-metric space and $M \subseteq X^9$. We say that X is M-regular if and only if the following hypothesis holds:

Whenever $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X be sequences such that

$$(x_n, y_n, z_n, x_{n+1}, y_{n+1}, z_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}) \in M,$$
$$(y_n, x_n, y_n, y_{n+1}, x_{n+1}, y_{n+1}, y_{n+1}, x_{n+1}, y_{n+1}) \in M$$

and

$$(z_n, y_n, x_n, z_{n+1}, y_{n+1}, x_{n+1}, z_{n+1}, y_{n+1}, x_{n+1}) \in M$$

for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$, $y_n \to y$ and $z_n \to z$ as $n \to +\infty$, we have

$$(x_n, y_n, z_n, x, y, z, x, y, z) \in M,$$
$$(y_n, x_n, y_n, y, x, y, y, x, y) \in M$$

and

$$(z_n, y_n, x_n, z, y, x, z, y, x) \in M$$

for all $n \in \mathbb{N} \cup \{0\}$.

Lemma 3.4 [23] *Let (X, G) be a generalized b-metric space (with the parameter s).*

If a mapping $\Omega_3^m : X^3 \times X^3 \times X^3 \to \mathbb{R}^+$ is given by

$$\Omega_3^m(X, U, A) = \max\{G(x, u, a), G(y, v, b), G(z, w, c)\},$$

for all $X = (x, y, z), U = (u, v, w)$ and $A = (a, b, c) \in X^3$, then (X^3, Ω_3^m) is an generalized b-metric space (with the same parameter s). The space (X^3, Ω_3^m) is G_b-complete iff (X, G) is G_b-complete.

Let (X, G) be a generalized b-metric space, $f: X^3 \to X$ and $g: X \to X$. For all $x, y, z, u, v, w, a, b, c \in X$, let

$$M_F^m(x, y, z, u, v, w, a, b, c) = \max\{G(F(x,y,z), F(u,v,w), F(a,b,c)), G(F(y,x,y), F(v,u,v), F(b,a,b)), G(F(z,y,x), F(w,v,u), F(c,b,a))\}$$

and

$$M_g^m(x, y, z, u, v, w, a, b, c) = \max\{G(gx, gu, ga), G(gy, gv, gb), G(gz, gw, gc)\}.$$

Theorem 3.5 [23] *Let (X, G) be a generalized b -metric space with the parameter s, $F: X^3 \to X$ and $g: X \to X$. Assume that*

$$\alpha((gx, gy, gz), (gu, gv, gw), (ga, gb, gc))\psi(sM_F^m(x, y, z, u, v, w, a, b, c)) \leq \psi(M_g^m(x, y, z, u, v, w, a, b, c)) - \varphi(M_g^m(x, y, z, u, v, w, a, b, c)), \quad (3.1)$$

for all $x, y, z, u, v, w, a, b, c \in X$ where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions and $\alpha : (X^3)^3 \to [0, \infty)$ is a mapping such that F is a rectangular G-α-admissible mapping w.r.t. g. Assume also that

1. *$F(X^3) \subseteq g(X)$;*
2. *there exist $x_0, y_0, z_0 \in X$*

such that

$$\alpha((gx_0, gy_0, gz_0), (F(x_0,y_0,z_0), F(y_0,x_0,y_0), F(z_0,y_0,x_0)), (F(x_0,y_0,z_0), F(y_0,x_0,y_0), F(z_0,y_0,x_0))) \geq 1,$$
$$\alpha((gy_0, gx_0, gy_0), (F(y_0,x_0,y_0), F(x_0,y_0,z_0), F(y_0,x_0,y_0)), (F(y_0,x_0,y_0), F(x_0,y_0,z_0), F(y_0,x_0,y_0))) \geq 1$$

and

$$\alpha((gz_0, gy_0, gx_0), (F(z_0,y_0,x_0), F(y_0,x_0,y_0), F(x_0,y_0,z_0)), (F(z_0,y_0,x_0), F(y_0,x_0,y_0), F(x_0,y_0,z_0))) \geq 1.$$

Also, suppose that either

(a) *F and g are continuous, the pair (F, g) is compatible and (X, G) is G_b-complete, or*
(b) *$(g(X), G)$ is G_b-complete and assume that whenever $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X be sequences such that*

$$\alpha((x_n,y_n,z_n),(x_{n+1},y_{n+1},z_{n+1}),(x_{n+1},y_{n+1},z_{n+1}))\geq 1,$$
$$\alpha((y_n,x_n,y_n),(y_{n+1},x_{n+1},y_{n+1}),(y_{n+1},x_{n+1},y_{n+1}))\geq 1$$

and

$$\alpha((z_n,y_n,x_n),(z_{n+1},y_{n+1},x_{n+1}),(z_{n+1},y_{n+1},x_{n+1}))\geq 1$$

for all $n\in\mathbb{N}\cup\{0\}$ *and* $x_n\to x, y_n\to y$ *and* $z_n\to z$ *as* $n\to+\infty$, *we have*

$$\alpha((x_n,y_n,z_n),(x,y,z),(x,y,z))\geq 1,$$
$$\alpha((y_n,x_n,y_n),(y,x,y),(y,x,y))\geq 1$$

and

$$\alpha((z_n,y_n,x_n),(z,y,x),(z,y,x))\geq 1$$

for all $n\in\mathbb{N}\cup\{0\}$.

Then, F and g have a tripled coincidence point in X.

Theorem 3.6 [23] *In addition to the hypotheses of Theorem 3.5, suppose that for all* (x,y,z) *and* $(x^*,y^*,z^*)\in X^3$, *there exists* $(u,v,w)\in X^3$, *such that*

$$\alpha((gx,gy,gz),(gu,gv,gw),(gu,gv,gw))\geq 1$$

and

$$\alpha((gx^*,gy^*,gz^*),(gu,gv,gw),(gu,gv,gw))\geq 1.$$

Then, F and g have a unique common tripled fixed point of the form (a,a,a).

Let

$$\Omega_3^a(X,U,A)=\frac{G(x,u,a)+G(y,v,b)+G(z,w,c)}{3},$$
$$X=(x,y,z),U=(u,v,w),A=(a,b,c)\in\mathcal{X}^3,$$

$$M_F^a(x,y,z,u,v,w,a,b,c)=\frac{G(F(x,y,z),F(u,v,w),F(a,b,c))+G(F(y,x,y),F(v,u,v),F(b,a,b))+G(F(z,y,x),F(w,v,u),F(c,b,a))}{3}$$

and

$$M_g^a(x,y,z,u,v,w,a,b,c)=\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gw,gc)}{3}.$$

Remark 3.7 [23] In Theorem 3.5, we can replace the contractive condition (3.1) by the following:

$$\begin{aligned}\alpha((gx,gy,gz),(gu,gv,gw),(ga,gb,gc))\psi&\\ (sM_F^a(x,y,z,u,v,w,a,b,c))&\\ \leq\psi(M_g^a(x,y,z,u,v,w,a,b,c))&\\ -\varphi(M_g^a(x,y,z,u,v,w,a,b,c)).&\end{aligned}\tag{3.2}$$

Following tripled fixed point results in G_b-metric spaces can be obtained.

Theorem 3.8 *Let* (X,G) *be a* G_b *-metric space with the parameter* s, $F:X^3\to X$, $g:X\to X$ *and M be a nonempty subset of* X^9. *Assume that*

$$\psi(sM_F^m(x,y,z,u,v,w,a,b,c))\leq\psi(M_g^m(x,y,z,u,v,w,a,b,c))-\varphi(M_g^m(x,y,z,u,v,w,a,b,c)),\tag{3.3}$$

for all $x,y,z,u,v,w,a,b,c\in X$ *with* $(gx,gu,ga,gy,gv,gb,gz,gw,gc)\in M$, *where* $\psi,\varphi:[0,\infty)\to[0,\infty)$ *are altering distance functions. Assume also that*

1. $F(X^3)\subseteq g(X)$;
2. M *is an* (F^*,g)*-invariant set which satisfies the transitive property*;
3. *there exist* $x_0,y_0,z_0\in X$ *such that*

$$(gx_0,gy_0,,gz_0,F(x_0,y_0,z_0),F(y_0,x_0,y_0),F(z_0,y_0,x_0),F(x_0,y_0,z_0),F(y_0,x_0,y_0),F(z_0,y_0,x_0))\in M.$$

Also, suppose that either

(a) F *and* g *are continuous, the pair* (F,g) *is compatible and* (X,G) *is* G_b*-complete, or*

(b) (X,G) *is M-regular and* $(g(X),G)$ *is* G_b*-complete.*

Then, F and g have a tripled coincidence point in X.

Theorem 3.9 *In addition to the hypotheses of Theorem 3.5, suppose that for all* (x,y,z) *and* $(x^*,y^*,z^*)\in X^3$, *there exists* $(u,v,w)\in X^3$, *such that*

$$(gx,gy,gz,gu,gv,gw,gu,gv,gw)\in M$$

and

$(gx^*, gy^*, gz^*, gu, gv, gw, gu, gv, gw) \in M.$

Then, F and g have a unique common tripled fixed point of the form (a, a, a).

Remark 3.10 In Theorem 3.8, we can replace the contractive condition (3.3) by the following:

$$\psi(sM_F^a(x,y,z,u,v,w,a,b,c)) \le \psi(M_g^a(x,y,z,u,v,w,a,b,c)) - \varphi(M_g^a(x,y,z,u,v,w,a,b,c)). \quad (3.4)$$

Coupled and tripled fixed point results for ψ-contractions in G-metric spaces

Let Ψ denotes the set of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying $\psi^{-1}(0) = 0$, $\psi(t) < t$ for all $t > 0$ and $\lim_{r\to t^+} \psi(r) < t$ for all $t > 0$.

Using the following coincidence point result, we obtain some coupled and tripled coincidence point results under (F^*, g)-invariant sets.

Lemma 4.1 [23] *Let f be a rectangular G-α-admissible mapping w.r.t. g such that* $f(X) \subseteq g(X)$*. Assume that there exists* $x_0 \in X$ *such that* $\alpha(fx_0, fx_0, gx_0) \ge 1$*. Define sequence* $\{y_n\}$ *by* $y_n = gx_n = fx_{n-1}$*. Then,*

$\alpha(y_n, y_m, y_m) \ge 1$ for all $n, m \in \mathbb{N}$ with $n < m$.

Theorem 4.2 *Let* (X, G) *be a G-metric space and let* $f, g : X \to X$ *satisfy the following condition:*

$$\alpha(gx, gy, gz)G(fx, fy, fz) \le \psi(G(gx, gy, gz)) \quad (4.1)$$

for all $x, y, z \in X$*, where* $\psi \in \Psi$, $\alpha : X^3 \to [0, +\infty)$ *and f is a rectangular G-α-admissible mapping w.r.t. g.*

Then, f and g have a coincidence point if,

- (i) $f(X) \subseteq g(X)$;
- (ii) *there exists* $x_0 \in X$ *such that* $\alpha(fx_0, fx_0, gx_0) \ge 1$;
- (iii) *f and g are continuous, g commutes with f and* (X, G) *is complete, or.*
- (iii)′ $(g(X), G)$ *is G-complete and assume that whenever* $\{x_n\}$ *in X be a sequence such that* $\alpha(x_n, x_{n+1}, x_{n+1}) \ge 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to +\infty$*, we have* $\alpha(x_n, x, x) \ge 1$ *for all* $n \in \mathbb{N} \cup \{0\}$

Proof Let $x_0 \in X$ be such that $\alpha(fx_0, fx_0, gx_0) \ge 1$. According to (i) one can define the sequence $\{y_n\}$ as $y_{n+1} = gx_{n+1} = fx_n$ for all $n = 0, 1, 2, \ldots$.

As $\alpha(gx_1, gx_1, gx_0) = \alpha(fx_0, fx_0, gx_0) \ge 1$ and since f is a G-α-admissible mapping with respect to g, then $\alpha(y_2, y_2, y_1) = \alpha(fx_1, fx_1, fx_0) \ge 1$. Continuing this process, we get $\alpha(y_{n+1}, y_{n+1}, y_n) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$.

If $y_n = y_{n+1}$, then x_n is a coincidence point of f and g. Now, assume that $y_n \ne y_{n+1}$ for all n, that is,

$$G(y_n, y_{n+1}, y_{n+2}) > 0, \quad (4.2)$$

for all n. Let $G_n = G(y_n, y_{n+1}, y_{n+2})$. Then, from (4.1) we obtain that

$$\begin{aligned} G(y_{n+1}, y_{n+2}, y_{n+3}) &\le \alpha(y_n, y_{n+1}, y_{n+2})G(y_{n+1}, y_{n+2}, y_{n+3}) \\ &= \alpha(gx_n, gx_{n+1}, gx_{n+2})G(fx_n, fx_{n+1}, fx_{n+2}) \\ &\le \psi(G(y_n, y_{n+1}, y_{n+2})) < G(y_n, y_{n+1}, y_{n+2}). \end{aligned} \quad (4.3)$$

So, we have proved that $G_{n+1} \le G_n$ for each $n \in \mathbb{N}$, and so there exists $r \ge 0$ such that $\lim_{n\to\infty} G_n = r \ge 0$.

Suppose that $r > 0$. Then, from (4.3), by taking the limit as $n \to \infty$, since $\psi \in \Psi$ we have

$$r \le \lim_{n\to\infty} \psi(G_n) = \lim_{G_n \to r^+} \psi(G_n) < r,$$

a contradiction. Hence,

$$\lim_{n\to\infty} G_n = \lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+2}) = 0. \quad (4.4)$$

Since $y_{n+1} \ne y_{n+2}$ for every n, so by property $(G3)$ we obtain

$$G(y_n, y_{n+1}, y_{n+1}) \le 2G(y_n, y_n, y_{n+1}) \le 2G(y_n, y_{n+1}, y_{n+2}).$$

Hence,

$$\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1}) = \lim_{n\to\infty} G(y_n, y_n, y_{n+1}) = 0. \quad (4.5)$$

Now, we prove that $\{y_{2n}\}$ is a G-Cauchy sequence. Assume on contrary that $\{y_{2n}\}$ is not a G-Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{y_{2m_k}\}$ and $\{y_{2n_k}\}$ of $\{y_{2n}\}$ such that m_k is the smallest index for which $m_k > n_k > k$ and

$$G(y_{2n_k}, y_{2m_k}, y_{2m_k}) \ge \varepsilon. \quad (4.6)$$

This means that

$$G(y_{2n_k}, y_{2m_k-1}, y_{2m_k-1}) < \varepsilon. \quad (4.7)$$

Since f is a rectangular G-α-admissible mapping with respect to g, then from Lemma 4.1 $\alpha(y_{2n_k}, y_{2m_k-1}, y_{2m_k-1}) \ge 1$. Now, from (4.1) we have

$$\begin{aligned} G(y_{2n_k+1}, y_{2m_k}, y_{2m_k}) &\le \alpha(y_{2n_k}, y_{2m_k-1}, y_{2m_k-1})G(y_{2n_k+1}, y_{2m_k}, y_{2m_k}) \\ &= \alpha(gx_{2n_k}, gx_{2m_k-1}, gx_{2m_k-1})G(fx_{2n_k}, fx_{2m_k-1}, fx_{2m_k-1}) \\ &\le \psi(G(y_{2n_k}, y_{2m_k-1}, y_{2m_k-1})) \\ &< G(y_{2n_k}, y_{2m_k-1}, y_{2m_k-1}), \end{aligned} \quad (4.8)$$

as $2n_k \ne 2m_k - 1$.

Using $(G5)$, we obtain that

$$G(y_{2n_k}, y_{2m_k}, y_{2m_k}) \leq G(y_{2n_k}, y_{2m_k-1}, y_{2m_k-1}) + G(y_{2m_k-1}, y_{2m_k}, y_{2m_k}).$$

Taking the limit as $k \to \infty$ and using (4.5) and (4.7), we obtain that

$$\lim_{k\to\infty} G(y_{2n_k}, y_{2m_k-1}, y_{2m_k-1}) \geq \varepsilon. \tag{4.9}$$

Hence, by (4.7), we have

$$\lim_{k\to\infty} G(y_{2n_k}, y_{2m_k-1}, y_{2m_k-1}) = \varepsilon. \tag{4.10}$$

Using $(G5)$, we obtain that

$$G(y_{2n_k}, y_{2m_k}, y_{2m_k}) \leq G(y_{2n_k}, y_{2n_k+1}, y_{2n_k+1}) + G(y_{2n_k+1}, y_{2m_k}, y_{2m_k}).$$

Taking the upper limit as $k \to \infty$ and using (4.5) and (4.7), we obtain that

$$\lim_{k\to\infty} G(y_{2n_k+1}, y_{2m_k}, y_{2m_k}) \geq \varepsilon. \tag{4.11}$$

Using $(G5)$, we obtain that

$$\begin{aligned} G(y_{2n_k+1}, y_{2m_k}, y_{2m_k}) &\leq G(y_{2n_k+1}, y_{2n_k}, y_{2n_k}) + G(y_{2n_k}, y_{2m_k}, y_{2m_k}) \\ &\leq G(y_{2n_k+1}, y_{2n_k}, y_{2n_k}) + G(y_{2n_k}, y_{2m_k-1}, y_{2m_k-1}) + G(y_{2m_k-1}, y_{2m_k}, y_{2m_k}). \end{aligned}$$

Taking the upper limit as $k \to \infty$ and using (4.5) and (4.7), we obtain that

$$\lim_{k\to\infty} \sup G(y_{2n_k+1}, y_{2m_k}, y_{2m_k}) \leq \varepsilon. \tag{4.12}$$

Consequently, from (4.11),

$$\lim_{k\to\infty} \sup G(y_{2n_k+1}, y_{2m_k}, y_{2m_k}) = \varepsilon. \tag{4.13}$$

Taking the upper limit as $k \to \infty$ in (4.8) and using (4.7) and (4.9), we obtain that

$$\epsilon \leq \lim_{k\to\infty} G(y_{2n_k+1}, y_{2m_k}, y_{2m_k}) \leq \lim_{k\to\infty} \psi(G(y_{2n_k}, y_{2m_k-1}, y_{2m_k-1})) < \epsilon,$$

a contradiction. It follows that $\{y_n\}$ is a G-Cauchy sequence in X. Suppose first that (iii) holds. Then, there exists

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \in X.$$

Further, since f and g are continuous and g commutes with f, we get

$$fz = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = gz.$$

It means that f and g have a coincidence point.

In the case (iii′), if we assume that $g(X)$ is G-complete, then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu = z$$

for some $u \in X$. Also, from (iii′) we have $\alpha(gx_n, gu, gu) \geq 1$. Applying (4.1) with $x = x_n$ and $y = z = u$, we have:

$$\begin{aligned} G(fx_n, fu, fu) &\leq \alpha(gx_n, gu, gu) G(fx_n, fu, fu)) \\ &\leq \psi(G(gx_n, gu, gu)) < G(gx_n, gu, gu). \end{aligned} \tag{4.14}$$

It follows that $G(fx_n, fu, fu) \to 0$ when $n \to \infty$, that is, $fx_n \to fu$. Uniqueness of the limit yields that $fu = z = gu$. Hence, f and g have a coincidence point $u \in X$.

Theorem 4.3 *Let (X, G) be a G-metric space and let $F : X^2 \to X$ and $g : X \to X$. Assume that*

$$\alpha((gx, gy), (gu, gv), (gz, gw)) N_F^m(x, y, u, v, z, w) \leq \psi(N_g^m(x, y, u, v, z, w)), \tag{4.15}$$

for all $x, y, u, v, z, w \in X$, where $\psi \in \Psi$, $\alpha : (X^2)^3 \to [0, \infty)$ and F is a rectangular G-α-admissible mapping with respect to g. Assume also that

1. *$F(X^2) \subseteq g(X)$;*
2. *there exist $x_0, y_0 \in X$ such that*

$$\alpha((F(x_0, y_0), F(y_0, x_0)), (F(x_0, y_0), F(y_0, x_0)), (gx_0, gy_0),) \geq 1$$

and

$$\alpha((F(y_0, x_0), F(x_0, y_0)), (F(y_0, x_0), F(x_0, y_0)), (gy_0, gx_0),) \geq 1.$$

Also, suppose that either

(a) *F and g are continuous, g commutes with F and (X, G) is complete, or*

(b) *$(g(X), G)$ is complete and assume that whenever $\{x_n\}$ and $\{y_n\}$ in X be sequences such that*

$$\alpha((x_n, y_n), (x_{n+1}, y_{n+1}), (x_{n+1}, y_{n+1})) \geq 1$$

and

$$\alpha((y_n, x_n), (y_{n+1}, x_{n+1}), (y_{n+1}, x_{n+1})) \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$, $y_n \to y$ as $n \to +\infty$, we have

$$\alpha((x_n, y_n), (x, y), (x, y)) \geq 1$$

and

$$\alpha((y_n, x_n), (y, x), (y, x)) \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$. *Then, F and g have a coupled coincidence point in X.*

Theorem 4.4 *In addition to the hypotheses of Theorem 4.3, suppose that for all* (x, y) *and* $(x^*, y^*) \in X^2$, *there exists* $(u, v) \in X^2$, *such that* $\alpha((gx, gy), (gu, gv), (gu, gv)) \geq 1$ *and* $\alpha((gx^*, gy^*), (gu, gv), (gu, gv)) \geq 1$. *Then, F and g have a unique common coupled fixed point of the form* (a, a).

Remark 4.5 The result of Theorems 4.3 and 4.4 holds, if we replace N_F^m and N_g^m by N_F^a and N_g^a, respectively.

Theorem 4.6 *Let* (X, G) *be a G-metric space and let* $F : X^2 \to X$ *and* $g : X \to X$ *and M be a nonempty subset of* X^6. *Assume that*

$$N_F^m(x, y, u, v, z, w) \leq \psi(N_g^m(x, y, u, v, z, w)), \tag{4.16}$$

for all $x, y, u, v, z, w \in X$ *with* $(gx, gu, gz, gy, gv, gw) \in M$, *where* $\psi \in \Psi$. *Assume also that*

1. $F(X^2) \subseteq g(X)$;
2. *M is an* (F^*, g)*-invariant set which satisfies the transitive property;*
3. *there exist* $x_0, y_0 \in X$ *such that* $(F(x_0, y_0), F(y_0, x_0), F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$.

Also, suppose that either

(a) *F and g are continuous, g commutes with F and* (X, G) *is complete, or*
(b) (X, G) *is M-regular and* $(g(X), G)$ *is G-complete. Then, F and g have a coupled coincidence point in X.*

Theorem 4.7 *In addition to the hypotheses of Theorem 4.6, suppose that for all* (x, y) *and* $(x^*, y^*) \in X^2$, *there exists* $(u, v) \in X^2$, *such that* $(gx, gy, gu, gv, gu, gv) \in M$ *and* $(gx^*, gy^*, gu, gv, gu, gv) \in M$. *Then, F and g have a unique common coupled fixed point of the form* (a, a).

We have the following corollary which is Theorems 3.1 and 3.2 of [35], but more general in contractive condition.

Corollary 4.8 *Let* (X, G) *be a G-metric space and let* $F : X^2 \to X$ *and* $g : X \to X$ *and M be a nonempty subset of* X^6. *Assume that*

$$\frac{G(gx, gu, gz) + G(gy, gv, gw)}{2} \leq \psi\left(\frac{G(F(x, y), F(u, v), F(z, w)) + G(F(y, x), F(v, u), F(w, z))}{2}\right), \tag{4.17}$$

for all $x, y, u, v, z, w \in X$ *with* $(gx, gu, gz, gy, gv, gw) \in M$, *where* $\psi \in \Psi$. *Assume also that*

1. $F(X^2) \subseteq g(X)$;
2. *M is an* (F^*, g)*-invariant set which satisfies the transitive property;*
3. there exist $x_0, y_0 \in X$ *such that* $(F(x_0, y_0), F(y_0, x_0), F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$.

Also, suppose that either

(a) *F and g are continuous, g commutes with F and* (X, G) *is complete, or*
(b) (X, G) *is M-regular and* $(g(X), G)$ *is G-complete. Then, F and g have a coupled coincidence point in X.*

Theorem 4.9 *Let* (X, G) *be a G-metric space and let* $F : X^3 \to X$ *and* $g : X \to X$. *Assume that*

$$\alpha((gx, gy, gz), (gu, gv, gw), (ga, gb, gc)) M_F^m(x, y, z, u, v, w, a, b, c) \leq \psi(M_g^m(x, y, z, u, v, w, a, b, c)), \tag{4.18}$$

for all $x, y, z, u, v, w, a, b, c \in X$ *where* $\psi \in \Psi$ *and* $\alpha : (X^3)^3 \to [0, \infty)$ *is a mapping such that F is a rectangular G-α-admissible mapping w.r.t. g. Assume also that*

1. $F(X^3) \subseteq g(X)$;
2. *there exist* $x_0, y_0, z_0 \in X$

such that

$$\alpha((F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)), (F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)), (gx_0, gy_0, gz_0)) \geq 1,$$
$$\alpha((F(y_0, x_0, y_0), F(x_0, y_0, z_0), F(y_0, x_0, y_0)), (F(y_0, x_0, y_0), F(x_0, y_0, z_0), F(y_0, x_0, y_0)), (gy_0, gx_0, gy_0)) \geq 1$$

and

$$\alpha((F(z_0, y_0, x_0), F(y_0, x_0, y_0), F(x_0, y_0, z_0)), (F(z_0, y_0, x_0), F(y_0, x_0, y_0), F(x_0, y_0, z_0)), (gz_0, gy_0, gx_0)) \geq 1.$$

Also, suppose that either

(a) *F and g are continuous, g commutes with F and* (X, G) *is G-complete, or*
(b) $(g(X), G)$ *is G-complete and assume that whenever* $\{x_n\}, \{y_n\}, \{z_n\}$ *in X be sequences such that*

$$\alpha((x_n, y_n, z_n), (x_{n+1}, y_{n+1}, z_{n+1}), (x_{n+1}, y_{n+1}, z_{n+1})) \geq 1,$$
$$\alpha((y_n, x_n, y_n), (y_{n+1}, x_{n+1}, y_{n+1}), (y_{n+1}, x_{n+1}, y_{n+1})) \geq 1$$

and

$$\alpha((z_n, y_n, x_n), (z_{n+1}, y_{n+1}, x_{n+1}), (z_{n+1}, y_{n+1}, x_{n+1})) \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$, $y_n \to y$ *and* $z_n \to z$ *as* $n \to +\infty$, *we have*

$$\alpha((x_n, y_n, z_n), (x, y, z), (x, y, z)) \geq 1,$$
$$\alpha((y_n, x_n, y_n), (y, x, y), (y, x, y)) \geq 1$$

and

$$\alpha((z_n, y_n, x_n), (z, y, x), (z, y, x)) \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$. *Then, F and g have a tripled coincidence point in X.*

Theorem 4.10 *In addition to the hypotheses of Theorem 4.9, suppose that for all* (x, y, z) *and* $(x^*, y^*, z^*) \in X^3$, *there exists* $(u, v, w) \in X^3$, *such that*

$$\alpha((gx, gy, gz), (gu, gv, gw), (gu, gv, gw)) \geq 1$$

and

$$\alpha((gx^*, gy^*, gz^*), (gu, gv, gw), (gu, gv, gw)) \geq 1.$$

Then, F and g have a unique common tripled fixed point of the form (a, a, a).

Remark 4.11 [23] In Theorem 4.9, we can replace the contractive condition (4.18) by the following:

$$\alpha((gx, gy, gz), (gu, gv, gw), (ga, gb, gc))M_F^a(x, y, z, u, v, w, a, b, c) \leq \psi(M_g^a(x, y, z, u, v, w, a, b, c)). \tag{4.19}$$

Following tripled fixed point results in G-metric spaces can be obtained.

Theorem 4.12 *Let* (X, G) *be a G-metric space,* $F : X^3 \to X$, $g : X \to X$ *and M be a nonempty subset of* X^9. *Assume that*

$$M_F^m(x, y, z, u, v, w, a, b, c) \leq \psi(M_g^m(x, y, z, u, v, w, a, b, c)), \tag{4.20}$$

for all $x, y, z, u, v, w, a, b, c \in \mathcal{X}$ *with* $(gx, gu, ga, gy, gv, gb, gz, gw, gc) \in M$, *where* $\psi \in \Psi$. *Assume also that*

1. $F(X^3) \subseteq g(X)$;
2. *M is an* (F^*, g)*-invariant set which satisfies the transitive property;*
3. *there exist* $x_0, y_0, z_0 \in X$ *such that*

$$(F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0), F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0), gx_0, gy_0, , gz_0) \in M.$$

Also, suppose that either

(a) *F and g are continuous, g commutes with F and* (X, G) *is G-complete, or*
(b) (X, G) *is M-regular and* $(g(X), G)$ *is G-complete.*

Then, f and g have a tripled coincidence point in X.

Theorem 4.13 *In addition to the hypotheses of Theorem 4.12, suppose that for all* (x, y, z) *and* $(x^*, y^*, z^*) \in X^3$, *there exists* $(u, v, w) \in \mathcal{X}^3$, *such that*

$$(gx, gy, gz, gu, gv, gw, gu, gv, gw) \in M$$

and

$$(gx^*, gy^*, gz^*, gu, gv, gw, gu, gv, gw) \in M.$$

Then, F and g have a unique common tripled fixed point of the form (a, a, a).

Remark 4.14 In Theorem 3.8, we can replace the contractive condition (4.20) by the following:

$$M_F^a(x, y, z, u, v, w, a, b, c) \leq \psi(M_g^a(x, y, z, u, v, w, a, b, c)). \tag{4.21}$$

Application to integral equations

As an application of the (coupled) fixed point theorems established in Sect. 4, we study the existence and uniqueness of a solution for a Fredholm nonlinear integral equation.

To compare our results to the ones in [12, 39], we shall consider the same integral equation, that is,

$$x(t) = \int_a^b (K_1(t, s) + K_2(t, s))(f(s, x(s)) + g(s, x(s)))ds + h(t), \tag{5.1}$$

where $t \in I = [a, b]$.

Consider the space $X = C([0, T], \mathbb{R})$ of continuous functions defined on $I = [a, b]$.

Assume that the functions K_1, K_2, f and g fulfill the following conditions:

Assumption 5.1 (i) $K_1(t, s) \geq 0$ and $K_2(t, s) \leq 0$, for all $t, s \in I$;

(iii)

$$\sup_{t \in I}\left[\left(\int_a^b K_1(t, s)ds\right)^p + \left(\int_a^b -K_2(t, s)ds\right)^p\right] \leq \frac{r}{2^{3p-2}}, \tag{5.2}$$

for an $0 \leq r < 1$.

(ii) There exists $M \subseteq X^6$ which satisfies the transitive property and

1. $(x, y, z, u, v, w) \in M$ iff $(w, v, u, z, y, x) \in M$;

2. $(x, y, z, u, v, w) \in M$ implies that

$$\Bigg(\int_a^b K_1(t,s)[f(s,x(s)) + g(s,y(s))]\mathrm{d}s + \int_a^b K_2(t,s)[f(s,y(s)) + g(s,x(s))]\mathrm{d}s,$$

$$\int_a^b K_1(t,s)[f(s,y(s)) + g(s,x(s))]\mathrm{d}s + \int_a^b K_2(t,s)[f(s,x(s)) + g(s,y(s))]\mathrm{d}s,$$

$$\int_a^b K_1(t,s)[f(s,z(s)) + g(s,u(s))]\mathrm{d}s + \int_a^b K_2(t,s)[f(s,u(s)) + g(s,z(s))]\mathrm{d}s,$$

$$\int_a^b K_1(t,s)[f(s,u(s)) + g(s,z(s))]\mathrm{d}s + \int_a^b K_2(t,s)[f(s,z(s)) + g(s,u(s))]\mathrm{d}s,$$

$$\int_a^b K_1(t,s)[f(s,v(s)) + g(s,w(s))]\mathrm{d}s + \int_a^b K_2(t,s)[f(s,w(s)) + g(s,v(s))]\mathrm{d}s,$$

$$\int_a^b K_1(t,s)[f(s,w(s)) + g(s,v(s))]\mathrm{d}s + \int_a^b K_2(t,s)[f(s,v(s)) + g(s,w(s))]\mathrm{d}s\Bigg) \in M$$

for all $x, y, z, u, v, w \in X$.

(*iii*) For all $x, y \in X$, the following Lipschitzian type conditions hold:

$$0 \le f(t,x(t)) - f(t,y(t)) \le |x(t) - y(t)| \tag{5.3}$$

and

$$-|x(t) - y(t)| \le g(t,x(t)) - g(t,y(t)) \le 0; \tag{5.4}$$

(*iv*) For all (x, y) and $(x^*, y^*) \in X^2$, there exists $(u, v) \in X^2$, such that $(x, y, u, v, u, v) \in M$ and $(x^*, y^*, u, v, u, v) \in M$.

Motivated by [12], we present the following definition.

Definition 5.2 A pair $(\alpha, \beta) \in X^2$ is called an M-coupled solution of Eq. (5.1) if, for all $s \in I$,

$$\Bigg(\alpha, \beta, \int_a^b K_1(\cdot,s)[f(s,\alpha(s)) + g(s,\beta(s))]\mathrm{d}s + \int_a^b K_2(\cdot,s)[f(s,\beta(s)) + g(s,\alpha(s))]\mathrm{d}s + h(\cdot),$$

$$\int_a^b K_1(\cdot,s)[f(s,\beta(s)) + g(s,\alpha(s))]\mathrm{d}s + \int_a^b K_2(\cdot,s)[f(s,\alpha(s)) + g(s,\beta(s))]\mathrm{d}s + h(\cdot),$$

$$\int_a^b K_1(\cdot,s)[f(s,\alpha(s)) + g(s,\beta(s))]\mathrm{d}s + \int_a^b K_2(\cdot,s)[f(s,\beta(s)) + g(s,\alpha(s))]\mathrm{d}s + h(\cdot),$$

$$\int_a^b K_1(\cdot,s)[f(s,\beta(s)) + g(s,\alpha(s))]\mathrm{d}s + \int_a^b K_2(\cdot,s)[f(s,\alpha(s)) + g(s,\beta(s))]\mathrm{d}s + h(\cdot)\Bigg) \in M.$$

Theorem 5.3 *Consider the integral equation* (5.1) *with* $K_1, K_2 \in C(I \times I, \mathbb{R})$ *and* $h \in C(I, \mathbb{R})$.

Suppose that there exists an M*-coupled solution* (α, β) *of* (5.1) *and that Assumption* 5.1 *is satisfied. Then, the integral equation* (5.1) *has a unique solution in* $C(I, \mathbb{R})$.

Proof It is well known that X is a complete b-metric space with respect to the sup metric

$$d(x, y) = \sup_{t \in I} |x(t) - y(t)|^p, \; x, y \in C(I, \mathbb{R}).$$

Define

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}.$$

It is easy to see that (X, G) is a complete G_b-metric space with $s = 2^{p-1}$(see, Example 1.3).

Define now the mapping $F : X \times X \to X$ by

$$F(x,y)(t) = \int_a^b K_1(t,s)[f(s,x(s)) + g(s,y(s))]ds + \int_a^b K_2(t,s)[f(s,y(s)) + g(s,x(s))]ds + h(t), \quad \text{for all } \; t \in I.$$

According to the computations done by Berinde in [39],

$$\begin{aligned}
&F(x,y)(t) - F(u,v)(t) \\
&= \int_a^b K_1(t,s)[f(s,x(s)) + g(s,y(s))]\mathrm{d}s \\
&\quad + \int_a^b K_2(t,s)[f(s,y(s)) + g(s,x(s))]\mathrm{d}s \\
&\quad - \int_a^b K_1(t,s)[f(s,u(s)) + g(s,v(s))]\mathrm{d}s \\
&\quad - \int_a^b K_2(t,s)[f(s,v(s)) + g(s,u(s))]\mathrm{d}s \\
&= \int_a^b K_1(t,s)[f(s,x(s)) - f(s,u(s)) + g(s,y(s)) \\
&\quad - g(s,v(s))]\mathrm{d}s + \int_a^b K_2(t,s)[f(s,y(s)) - f(s,v(s)) \\
&\quad + g(s,x(s)) - g(s,u(s))]\mathrm{d}s \\
&= \int_a^b K_1(t,s)[(f(s,x(s)) - f(s,u(s))) \\
&\quad - (g(s,v(s)) - g(s,y(s)))]\mathrm{d}s \\
&\quad - \int_a^b K_2(t,s)[(f(s,y(s)) - f(s,v(s))) \\
&\quad - (g(s,u(s)) - g(s,x(s)))]\mathrm{d}s \\
&\le \int_a^b K_1(t,s)[|x(s) - u(s)| + |v(s) - y(s)|]\mathrm{d}s \\
&\quad - \int_a^b K_2(t,s)[|y(s) - v(s)| + |u(s) - x(s)|]\mathrm{d}s.
\end{aligned} \tag{5.5}$$

Hence, by (5.5), in view of the fact that $K_2(t,s) \le 0$, we obtain that

$$\begin{aligned}
|F(x,y)(t) - F(u,v)(t)| \le& \int_a^b K_1(t,s)[|x(s) - u(s)| + |v(s) - y(s)|]\mathrm{d}s \\
&- \int_a^b K_2(t,s)[|y(s) - v(s)| + |u(s) - x(s)|]\mathrm{d}s,
\end{aligned} \tag{5.6}$$

as all quantities in the right-hand side of (5.5) are non-negative.

Now, from (5.5) we have

$$\begin{aligned}
&|F(x,y)(t) - F(u,v)(t)|^p \\
&\le \begin{pmatrix} \int_a^b K_1(t,s)[|x(s) - u(s)| + |v(s) - y(s)|]\mathrm{d}s \\ - \int_a^b K_2(t,s)[|y(s) - v(s)| + |u(s) - x(s)|]\mathrm{d}s \end{pmatrix}^p \\
&\le 2^{p-1} \begin{pmatrix} \left(\int_a^b K_1(t,s)\mathrm{d}s\right)^p (|x(s) - u(s)| + |v(s) - y(s)|)^p \\ + \left(-\int_a^b K_2(t,s)\mathrm{d}s\right)^p (|y(s) - v(s)| + |u(s) - x(s)|)^p \end{pmatrix} \\
&\le 2^{p-1} \begin{pmatrix} \left(\int_a^b K_1(t,s)\mathrm{d}s\right)^p 2^{p-1}(|x(s) - u(s)|^p + |v(s) - y(s)|^p) \\ + \left(-\int_a^b K_2(t,s)\mathrm{d}s\right)^p 2^{p-1}(|y(s) - v(s)|^p + |u(s) - x(s)|^p) \end{pmatrix} \\
&\le 2^{2p-2} \begin{bmatrix} \left(\left(\int_a^b K_1(t,s)ds\right)^p + \left(-\int_a^b K_2(t,s)\mathrm{d}s\right)^p\right) d(x,u) \\ + \left(\left(\int_a^b K_1(t,s)\mathrm{d}s\right)^p + \left(-\int_a^b K_2(t,s)\mathrm{d}s\right)^p\right) d(v,y) \end{bmatrix}.
\end{aligned}$$

So, we have

$$\begin{aligned}
|F(x,y)(t) - F(u,v)(t)|^p \le& 2^{2p-2}\left(\left(\int_a^b K_1(t,s)\mathrm{d}s\right)^p \right. \\
&\left. + \left(-\int_a^b K_2(t,s)\mathrm{d}s\right)^p\right)[d(x,u) + d(y,v)].
\end{aligned} \tag{5.7}$$

Similarly, one can obtain that

$$\begin{aligned}
|F(u,v)(t) - F(z,t)(t)|^p \le& 2^{2p-2}\left(\left(\int_a^b K_1(t,s)\mathrm{d}s\right)^p \right. \\
&\left. + \left(-\int_a^b K_2(t,s)\mathrm{d}s\right)^p\right)[d(u,z) + d(v,t)]
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
|F(z,w)(t) - F(x,y)(t)|^p \le& 2^{2p-2}\left(\left(\int_a^b K_1(t,s)\mathrm{d}s\right)^p \right. \\
&\left. + \left(-\int_a^b K_2(t,s)\mathrm{d}s\right)^p\right)[d(x,z) + d(y,w)].
\end{aligned} \tag{5.9}$$

Taking the supremum with respect to t and using (5.2) we get,

$$\begin{aligned}
&G(F(x,y), F(u,v), F(z,w)) \\
&= \max\{\sup_{t\in I}|F(x,y)(t) - F(u,v)(t)|^p, \sup_{t\in I}|F(u,v)(t) \\
&\quad - F(z,w)(t)|^p, \sup_{t\in I}|F(z,w)(t) - F(x,y)(t)|^p\} \\
&\le 2^{2p-2}\sup_{t\in I}\left[\left(\int_a^b K_1(t,s)\mathrm{d}s\right)^p + \left(\int_a^b -K_2(t,s)\mathrm{d}s\right)^p\right] \\
&\quad \max\{d(x,u) + d(y,v), d(u,z) + d(v,w), d(x,z) + d(y,w)\} \\
&\le 2^{2p-2}\sup_{t\in I}\left[\left(\int_a^b K_1(t,s)\mathrm{d}s\right)^p + \left(\int_a^b -K_2(t,s)\mathrm{d}s\right)^p\right] \\
&\quad \times [G(x,u,z) + G(y,v,w)] \\
&\le \frac{2^{2p-2}\cdot r}{2^{3p-2}}[G(x,u,z) + G(y,v,w)] \\
&= \frac{r}{2^p}[G(x,u,z) + G(y,v,w)] \\
&\le \frac{r}{2^{p-1}}[\max\{G(x,u,z), G(y,v,w)\}]
\end{aligned}$$

and analogously,

$$G(F(y,x), F(v,u), F(w,z)) \le \frac{r}{2^{p-1}}[\max\{G(x,u,z), G(y,v,w)\}].$$

Combination of the above inequalities is just the contractive condition (2.1) in Theorem 2.2 with $\psi(t) = t$ and $\varphi(t) = (1-r)t$, for all $t > 0$.

Now, let $(\alpha, \beta) \in X^2$ be an M-coupled solution of (5.1). Thus, all hypotheses of Theorem 2.2 are satisfied.

This proves that F has a coupled fixed point (x_*, y_*) in X^2. From (iv) and by Theorem 2.3 it follows that $x_* = y_*$, that is, $x_* = F(x_*, x_*)$, and therefore $x_* \in C(I, \mathbb{R})$ is the solution of the integral equation (5.1). □

Conclusion

After that Samet and Vetro [37] introduced the concept of F-invariant set and used it to obtain coupled fixed point results in usual metric spaces, several authors have obtained different coincidence point results in various classes of generalized metric spaces (see, e.g., [32–38]). As we saw in the present paper, (also, see [22, 23]) we showed that

1. coupled and tripled fixed point results can be deduced from corresponding fixed point theorems,
2. coupled and tripled fixed point results via invariant subsets can be deduced from coupled and tripled fixed point results via the concept of α-admissible mappings.

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