Math Sci (2015) 9:1–9
DOI 10.1007/s40096-015-0142-0

ORIGINAL RESEARCH

Ranking p-norm generalised fuzzy numbers with different left height and right height using integral values

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Received: 19 May 2014 / Accepted: 13 February 2015 / Published online: 4 March 2015

Abstract This paper considers ranking of generalised fuzzy numbers with different left height and right height using integral values. With the advances in new type of fuzzy number (generalised fuzzy number with different left height and right height) methods should be developed to compare them. Keeping this in view a new modified method has been proposed.

Keywords Generalised fuzzy number · Ranking · p-Norm · Left height · Right height

Introduction

Decision making in engineering, medical and any other real-life problems may be interpreted in terms of fuzzy. This demands ranking or ordering of fuzzy quantities to make a transparent decision. With the advances in fuzzy set theory, different ranking methods are developed. This concept was first proposed by Jain [7]. Some of the literatures that describe different approach of ranking fuzzy quantities are [1, 3, 4, 10, 13–15]. Recently, ranking of trapezoidal fuzzy numbers based on the shadow length has been discussed by Pour et al. [12]. Also, ranking triangular fuzzy numbers by Pareto approach based on two dominance stages is discussed by Bahri et al. [2].

The literatures that are available on ranking fuzzy quantities based on the integral values are [6–9]. These type of methods of ranking fuzzy numbers are based on the convex combination of right and left integral values through an index of optimism found in Liou and Wang [11] and Kim and Park [8]. This concept was further generalised to rank non-normal p-norm trapezoidal fuzzy numbers [6]. However, this method was found insufficient to rank non-normal p-norm fuzzy numbers with different height; keeping this in mind, Kumar et al. [9] developed an approach to overcome those shortcoming's. With the advances of generalised fuzzy numbers (GFNs) with different left height and right height [6], Kumar's approach fails to rank them. Hence, Kumar's approach is only sufficient for ranking fuzzy numbers or non-normal p-norm fuzzy numbers with different height, but the method is insufficient for ranking GFNs with different left height and right height.

Keeping this in view, Kumar's approach has been modified in this paper to rank p-norm GFNs with different left height and right height. This modified method thus handle both normal and non-normal trapezoidal fuzzy number with different height. The modified method can also rank non-normal p-norm trapezoidal fuzzy numbers with different height.

The structure of the paper is as follows. In Sect. 2, some general concept of the GFN is put forwarded. Membership function of GFN is defined. Also the membership function of p-norm GFN with different left height and right height is defined. Section 3 starts with definitions of different integral values of p-norm GFN with different left height and right height, And finally, some properties related to them are discussed in this section. Section 4 describes the

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proposed modified method along with some numerical examples. Finally, in Sect. 4, conclusions are made.

Definitions and notations

In this section, brief review of some concepts of generalised fuzzy number with different left height and right height are put forward.

Generalised fuzzy number

Let $\tilde{A}$ be represented by $(a, b, c, d; h_{\mathrm{L}}, h_{\mathrm{R}})$ on the real line $\mathbb{R}$ such that $-\infty < a \leq b \leq c \leq d < \infty$ is called a GFN with different left height and right height which is bounded and convex. The values a, b, c and d are real, h_{L} is called the left height of the GFN $\tilde{A}$, h_{R} is called the right height of the GFN, $h_{\mathrm{L}} \in [0,1]$ and $h_{\mathrm{R}} \in [0,1]$ [5]. For now, let $\mathbb{F}(\mathbb{R})$ be the set of all GFNs with different left height and right height. If $h_{\mathrm{L}} = h_{\mathrm{R}} = 1$ then the GFN reduces to a standard trapezoidal fuzzy number.

The membership function of GFN $\tilde{A}$ with different left height and right height is as given below

$$\mu_{\tilde{A}}(x) = \begin{cases} \mu_1(x), & \text{if } a \leq x \leq b; \\ \mu_2(x), & \text{if } b \leq x \leq c; \\ \mu_3(x), & \text{if } c \leq x \leq d; \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where $\mu_1 : [a,b] \longrightarrow [0, h_{\mathrm{L}}]$, $\mu_2 : [b,c] \longrightarrow [h_{\mathrm{L}}, h_{\mathrm{R}}]$ (or $[h_{\mathrm{R}}, h_{\mathrm{L}}]$) and $\mu_3 : [c,d] \longrightarrow [0, h_{\mathrm{R}}]$ are continuous. The functions $\mu_1(x)$ and $\mu_3(x)$ are strictly increasing and strictly decreasing, respectively. The function $\mu_2(x)$ is strictly increasing when $h_{\mathrm{L}} < h_{\mathrm{R}}$ and strictly decreasing when $h_{\mathrm{L}} > h_{\mathrm{R}}$. Then the inverse of $\mu_{\tilde{A}}(x)$ is

$$\mu_{\tilde{A}}^{-1}(y) = \begin{cases} \mu_1^{-1}(y), & \text{if } 0 \leq y \leq h_{\mathrm{L}}; \\ \mu_2^{-1}(y), & \text{if } h_{\mathrm{L}} \leq y \leq h_{\mathrm{R}}, \text{or } (h_{\mathrm{R}} \leq y \leq h_{\mathrm{L}}); \\ \mu_3^{-1}(y), & \text{if } 0 \leq y \leq h_{\mathrm{R}}; \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where $\mu_1^{-1} : [0, h_{\mathrm{L}}] \longrightarrow [a,b]$, $\mu_2^{-1} : [h_{\mathrm{L}}, h_{\mathrm{R}}]$ (or $[h_{\mathrm{R}}, h_{\mathrm{L}}]$) $\longrightarrow [b,c]$ and $\mu_3^{-1} : [0, h_{\mathrm{R}}] \longrightarrow [c,d]$ are continuous. The function $\mu_1^{-1}(x)$ and $\mu_3^{-1}(x)$ are strictly increasing and strictly decreasing, respectively. The function $\mu_2^{-1}(x)$ is strictly increasing when $h_{\mathrm{L}} < h_{\mathrm{R}}$ and strictly decreasing when $h_{\mathrm{L}} > h_{\mathrm{R}}$.

Let $\tilde{A} = (a, b, c, d; h_{\mathrm{L}}, h_{\mathrm{R}})$ be a trapezoidal GFN with different left height and right height then the membership function is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{h_{\mathrm{L}}(x-a)}{b-a}, & \text{if } a \leq x \leq b; \\ \dfrac{h_{\mathrm{L}}(c-b) + (h_{\mathrm{R}} - h_{\mathrm{L}})(x-b)}{c-b}, & \text{if } b \leq x \leq c; \\ \dfrac{h_{\mathrm{R}}(x-d)}{c-d}, & \text{if } c \leq x \leq d; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Definition 2.1.1 A GFN $\tilde{A}_p = (a, b, c, d; h_{\mathrm{L}}, h_{\mathrm{R}})_p$ is said to be a p-norm GFN with different left height and right height if its membership function is given by

$$\mu_{\tilde{A}_p}(x) = \begin{cases} f_{\tilde{A}_p}^{\mathrm{L}}(x) = h_{\mathrm{L}}\left[1 - \left(\dfrac{x-b}{a-b}\right)^p\right]^{\frac{1}{p}}, & \text{if } a \leq x \leq b; \\ f_{\tilde{A}_p}^{\mathrm{M}}(x) = h_{\mathrm{L}} + (h_{\mathrm{R}} - h_{\mathrm{L}})\left[1 - \left(\dfrac{x-c}{b-c}\right)^p\right]^{\frac{1}{p}}, & \text{if } b \leq x \leq c; \\ f_{\tilde{A}_p}^{\mathrm{R}}(x) = h_{\mathrm{R}}\left[1 - \left(\dfrac{x-c}{d-c}\right)^p\right]^{\frac{1}{p}}, & \text{if } c \leq x \leq d; \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

where p is a positive integer. The functions $f_{\tilde{A}_p}^{\mathrm{L}} : [a,b] \longrightarrow [0, h_{\mathrm{L}}]$ and $f_{\tilde{A}_p}^{\mathrm{R}} : [c,d] \longrightarrow [0, h_{\mathrm{R}}]$ are both continuous as well as strictly increasing and strictly decreasing functions, respectively. The function $f_{\tilde{A}_p}^{\mathrm{M}} : [b,c] \longrightarrow [h_{\mathrm{L}}, h_{\mathrm{R}}]$(or $[h_{\mathrm{R}}, h_{\mathrm{L}}]$) is strictly increasing (decreasing) when $h_{\mathrm{L}} < h_{\mathrm{R}} (h_{\mathrm{R}} < h_{\mathrm{L}})$. When p is one, the p-norm GFN with different left height and right height reduces to trapezoidal GFN with different left height and right height as defined by Eq. (3).

Definition 2.1.2 The inverse function of $\mu_{\tilde{A}_p}(x)$ as given by membership function in (4) is given by

$$\mu_{\tilde{A}_p}^{-1}(y) = \begin{cases} g_{\tilde{A}_p}^{\mathrm{L}}(y) = b + (a-b)\left[1 - \left(\dfrac{y}{h_{\mathrm{L}}}\right)^p\right]^{\frac{1}{p}}, & \text{if } 0 \leq y \leq h_{\mathrm{L}}; \\ g_{\tilde{A}_p}^{\mathrm{M}}(y) = c + (b-c)\left[1 - \left(\dfrac{y - h_{\mathrm{L}}}{h_{\mathrm{R}} - h_{\mathrm{L}}}\right)^p\right]^{\frac{1}{p}}, & \text{if } h_{\mathrm{L}} \leq y \leq h_{\mathrm{R}}, \text{or } (h_{\mathrm{R}} \leq y \leq h_{\mathrm{L}}); \\ g_{\tilde{A}_p}^{\mathrm{R}}(y) = c + (d-c)\left[1 - \left(\dfrac{y}{h_{\mathrm{R}}}\right)^p\right]^{\frac{1}{p}}, & \text{if } 0 \leq y \leq h_{\mathrm{R}}; \end{cases} \tag{5}$$

The functions $g^{\mathrm{L}}_{\tilde{A}_p} : [0, h_{\mathrm{L}}] \longrightarrow [a, b]$ and $g^{\mathrm{R}}_{\tilde{A}_p} : [0, h_{\mathrm{R}}] \longrightarrow [c, d]$ are both continuous as well as strictly increasing and strictly decreasing functions, respectively. The function $g^{\mathrm{M}}_{\tilde{A}_p}:[h_{\mathrm{L}}, h_{\mathrm{R}}](\text{or } [h_{\mathrm{R}}, h_{\mathrm{L}}]) \longrightarrow [b, c]$ is strictly increasing (decreasing) when $h_{\mathrm{L}} < h_{\mathrm{R}} (h_{\mathrm{R}} < h_{\mathrm{L}})$.

Total integral value

Convex combination of right and left integral values through an index of optimism is called the total integral value [8, 11]. The middle integral value is zero for normal and non-normal p-norm trapezoidal fuzzy numbers with different height. However, for a p-norm GFNs with different left height and right height this integral value has to be counted for a transparent decision. Keeping this in view, following definitions are put forwarded.

Definition 3.1 If $\tilde{A}$ is a fuzzy number with different left height and right height as defined by the membership function (1) and the inverse membership function given by (2) then the left integral value of $\tilde{A}$ is defined as

$$I_{\mathrm{L}}(\tilde{A}) = \int_0^{h_{\mathrm{L}}} \mu_1^{-1}(y)\mathrm{d}y \tag{6}$$

Definition 3.2 If $\tilde{A}$ is a fuzzy number with different left height and right height as defined by the membership function (1) and the inverse membership function given by (2) then the right integral value of $\tilde{A}$ is defined as

$$I_{\mathrm{R}}(\tilde{A}) = \int_0^{h_{\mathrm{R}}} \mu_3^{-1}(y)\mathrm{d}y \tag{7}$$

Definition 3.3 If $\tilde{A}$ is a fuzzy number with different left height and right height as defined by the membership function (1) and the inverse membership function given by (2) then the middle integral value of $\tilde{A}$ is defined as

$$I_{\mathrm{M}}(\tilde{A}) = \int_{h_{\mathrm{L}}}^{h_{\mathrm{R}}} \mu_2^{-1}(y)\mathrm{d}y \quad \text{or} \quad I_{\mathrm{M}}(\tilde{A}) = \int_{h_{\mathrm{R}}}^{h_{\mathrm{L}}} \mu_2^{-1}(y)\mathrm{d}y \tag{8}$$

Definition 3.4 If $\tilde{A}$ is a fuzzy number with different left height and right height as defined by the membership function (1), then the total integral value with index of optimism α is defined as

$$I^{\alpha}_{\mathrm{T}}(\tilde{A}) = \alpha(I_{\mathrm{R}}(\tilde{A}) + I_{\mathrm{M}}(\tilde{A})) + (1 - \alpha)(I_{\mathrm{L}}(\tilde{A}) + I_{\mathrm{M}}(\tilde{A}))$$

Proposition 3.1 *Let $\tilde{A}_p = (a, b, c, d; h_{\mathrm{L}}, h_{\mathrm{R}})_p$ be a p-norm GFN with different left height and right height with membership function (4), where p is a positive integer. Then:*

1. *The left membership function $f^{\mathrm{L}}_{\tilde{A}_p}(x)$ is continuous and strictly increasing function and its left integral value is*

$$I_{\mathrm{L}}(\tilde{A}_p) = bh_{\mathrm{L}} + \frac{a-b}{p} h_{\mathrm{L}} \frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{2}{p}+1\right)} \tag{9}$$

where $\Gamma(x)$ is Euler's gamma function, defined by $\int_0^{\infty} y^{x-1}\mathrm{e}^{-y}\mathrm{d}y$.

2. *The right membership function $f^{\mathrm{R}}_{\tilde{A}_p}(x)$ is continuous and strictly decreasing function and its right integral value is*

$$I_{\mathrm{R}}(\tilde{A}_p) = ch_{\mathrm{R}} + \frac{d-c}{p} h_{\mathrm{R}} \frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{2}{p}+1\right)} \tag{10}$$

3. *The middle membership function $f^{\mathrm{M}}_{\tilde{A}_p}(x)$ is continuous and strictly increasing and strictly decreasing when $h_{\mathrm{L}} < h_{\mathrm{R}}$ and $h_{\mathrm{R}} < h_{\mathrm{L}}$, respectively. The middle integral value is given by*

$$I_{\mathrm{M}}(\tilde{A}_p) = (h_{\mathrm{R}} - h_{\mathrm{L}})c + \frac{b-c}{p}(h_{\mathrm{R}} - h_{\mathrm{L}}) \times \frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{2}{p}+1\right)} \tag{11}$$

4. The total integral value with optimism α *is*

$$\begin{aligned} I^{\alpha}_{\mathrm{T}}(\tilde{A}_p) &= \alpha ch_{\mathrm{R}} + (1-\alpha)bh_{\mathrm{L}} + (h_{\mathrm{R}} - h_{\mathrm{L}})c \\ &\quad + \{\alpha h_{\mathrm{R}}(d-c) + (1-\alpha)h_{\mathrm{L}}(a-b) \\ &\quad + (h_{\mathrm{R}} - h_{\mathrm{L}})(b-c)\} \frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{p \times \Gamma\left(\frac{2}{p}+1\right)} \end{aligned} \tag{12}$$

Proof Continuity of the left membership function $f^{\mathrm{L}}_{\tilde{A}_p}(x)$ is trivial. Also, this function is strictly increasing and its integral values are inherited from [6]. Similarly, for the right membership function $f^{\mathrm{L}}_{\tilde{A}_p}(x)$.

Trivially, the function $f^{\mathrm{M}}_{\tilde{A}_p}(x)$ is continuous. Now,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} f^{\mathrm{M}}_{\tilde{A}_p}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{h_{\mathrm{L}} + (h_{\mathrm{R}} - h_{\mathrm{L}})\left[1 - \left(\frac{x-c}{b-c}\right)^p\right]^{\frac{1}{p}}\right\} \\ &= (h_{\mathrm{R}} - h_{\mathrm{L}})\frac{1}{p}\left[1 - \left(\frac{x-c}{b-c}\right)^p\right]^{\frac{1}{p}-1} \\ &\quad \times \left[-p\left(\frac{x-c}{b-c}\right)^{p-1}\left(\frac{1}{b-c}\right)\right] \end{aligned}$$

Since $0 \leq \left(\frac{x-c}{b-c}\right)^p \leq 1$, it trivially follows that $\frac{\mathrm{d}}{\mathrm{d}x} f^{\mathrm{M}}_{\tilde{A}_p}(x) \geq 0$ if $h_{\mathrm{R}} - h_{\mathrm{L}} \geq 0$ and $\frac{\mathrm{d}}{\mathrm{d}x} f^{\mathrm{M}}_{\tilde{A}_p}(x) \leq 0$ if $h_{\mathrm{R}} - h_{\mathrm{L}} \leq 0$. Hence the function $f^{\mathrm{M}}_{\tilde{A}_p}(x)$ is strictly increasing when $h_{\mathrm{R}} \geq h_{\mathrm{L}}$ and strictly decreasing when $h_{\mathrm{R}} \leq h_{\mathrm{L}}$. Also,

$$I_{\mathrm{M}}(\tilde{A}_p) = \int_{h_{\mathrm{L}}(\text{or } h_{\mathrm{R}})}^{h_{\mathrm{R}}(\text{or } h_{\mathrm{L}})} \left[c + (b-c)\left[1 - \left(\frac{y-h_{\mathrm{L}}}{h_{\mathrm{R}}-h_{\mathrm{L}}}\right)^p\right]^{\frac{1}{p}}\right] \mathrm{d}x$$
$$= (h_{\mathrm{R}} - h_{\mathrm{L}})c + \frac{b-c}{p}(h_{\mathrm{R}} - h_{\mathrm{L}}) \frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{2}{p}+1\right)}.$$

Now, the total integral value with optimism α is

$$\begin{aligned} I_{\mathrm{T}}^{\alpha}(\tilde{A}_p) &= \alpha\{I_{\mathrm{R}}(\tilde{A}_p) + I_{\mathrm{M}}(\tilde{A}_p)\} + (1-\alpha)\{I_{\mathrm{L}}(\tilde{A}_p) + I_{\mathrm{M}}(\tilde{A}_p)\} \\ &= \alpha c h_{\mathrm{R}} + (1-\alpha) b h_{\mathrm{L}} + (h_{\mathrm{R}} - h_{\mathrm{L}})c \\ &\quad + \{\alpha h_{\mathrm{R}}(d-c) + (1-\alpha)h_{\mathrm{L}}(a-b) \\ &\quad + (h_{\mathrm{R}} - h_{\mathrm{L}})(b-c)\} \frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{p \times \Gamma\left(\frac{2}{p}+1\right)} \end{aligned}$$

□

Remark 3.1 For a pessimistic decision maker $\alpha = 0$, for an optimistic decision maker $\alpha = 1$ and for a moderate decision maker $\alpha = 0.5$.

Corollary 3.1 *For an pessimistic decision maker* $(\alpha = 0)$ *the total integral* $I_{\mathrm{T}}^{\alpha}(\tilde{A}) = I_{\mathrm{L}}(\tilde{A}) + I_{\mathrm{M}}(\tilde{A})$, *for optimistic decision maker* $(\alpha = 1)$ *the total integral value* $I_{\mathrm{T}}^{\alpha}(\tilde{A}) = I_{\mathrm{R}}(\tilde{A}) + I_{\mathrm{M}}(\tilde{A})$ *and for moderate decision maker* $(\alpha = 0.5)$ *the total integral value* $I_{\mathrm{T}}^{\alpha}(\tilde{A}) = I_{\mathrm{M}}(\tilde{A}) + \frac{1}{2}\{I_{\mathrm{R}}(\tilde{A}) + I_{\mathrm{L}}(\tilde{A})\}$.

Note 1 The above total integral values for pessimistic decision maker, optimistic decision maker and moderate decision maker reduce to $I_{\mathrm{T}}^{\alpha}(\tilde{A}) = I_{\mathrm{L}}(\tilde{A})$, $I_{\mathrm{T}}^{\alpha}(\tilde{A}) = I_{\mathrm{R}}(\tilde{A})$ and $I_{\mathrm{T}}^{\alpha}(\tilde{A}) = \frac{1}{2}\{I_{\mathrm{R}}(\tilde{A}) + I_{\mathrm{L}}(\tilde{A})\}$, respectively, for a non-normal p-norm trapezoidal fuzzy number. This is because $I_{\mathrm{M}}(\tilde{A}) = 0$ when $h_{\mathrm{L}} = h_{\mathrm{R}}$.

Arithmetic operations

The arithmetic of p-norm GFNs with different left height and right height are reviewed from Chen [5]. Let $\tilde{A}_p = (a, b, c, d; h_{\mathrm{L}}, h_{\mathrm{R}})_p$ and $\tilde{B}_p = (q, r, s, t; h'_{\mathrm{L}}, h'_{\mathrm{R}})_p$ be p-norm GFNs with different left height and right height. Then

1. $\tilde{A}_p \oplus \tilde{B}_p = (a+q, b+r, c+s, d+t; \min(h_{\mathrm{L}}, h'_L), \min(h_{\mathrm{R}}, h'_R))_p$
2. $\tilde{A}_p \ominus \tilde{B}_p = (a-t, b-s, c-r, d-q; \min(h_{\mathrm{L}}, h'_L), \min(h_{\mathrm{R}}, h'_R))_p$
3. $$\lambda\tilde{A} = \begin{cases} (\lambda a, \lambda b, \lambda c, \lambda d; h_{\mathrm{L}}, h_{\mathrm{R}})_p & \lambda > 0 \\ (\lambda d, \lambda c, \lambda b, \lambda a; h_{\mathrm{L}}, h_{\mathrm{R}})_p & \lambda < 0 \end{cases}$$

The proposed method

In this section, a method of ranking p-norm GFNs with different left height and right height is presented. The method calculates total integral value on the basis of left integral value, right integral value and middle integral value. Ranking is done on the basis of these evaluated total integral values. Let $\tilde{A}_p = (a, b, c, d; h_{\mathrm{L}}, h_{\mathrm{R}})_p$ and $\tilde{B}_p = (q, r, s, t; h'_{\mathrm{L}}, h'_{\mathrm{R}})_p$ be p-norm GFNs with different left height and right height. Then

1. $\tilde{A}_p \succ \tilde{B}_p$ if $I_{\mathrm{T}}^{\alpha}(\tilde{A}_p) > I_{\mathrm{T}}^{\alpha}(\tilde{B}_p)$,
2. $\tilde{A}_p \prec \tilde{B}_p$ if $I_{\mathrm{T}}^{\alpha}(\tilde{A}_p) < I_{\mathrm{T}}^{\alpha}(\tilde{B}_p)$,
3. $\tilde{A}_p \sim \tilde{B}_p$ if $I_{\mathrm{T}}^{\alpha}(\tilde{A}_p) = I_{\mathrm{T}}^{\alpha}(\tilde{B}_p)$,

The following are the steps involved in this ranking method:

Step 1. Find $h_1 = \min(h_{\mathrm{L}}, h'_L)$ and $h_2 = \min(h_{\mathrm{R}}, h'_R)$.

Step 2. Find $I_{\mathrm{L}}(\tilde{A}_p)$, $I_{\mathrm{R}}(\tilde{A}_p)$, $I_{\mathrm{M}}(\tilde{A}_p)$ and $I_{\mathrm{L}}(\tilde{B}_p)$, $I_{\mathrm{R}}(\tilde{B}_p)$, $I_{\mathrm{M}}(\tilde{B}_p)$, such that

$$I_{\mathrm{L}}(\tilde{A}_p) = \int_0^{h_1} g_{\tilde{A}_p}^{\mathrm{L}}(x)\mathrm{d}x, \quad \text{where}$$
$$g_{\tilde{A}_p}^{\mathrm{L}}(x) = b + (a-b)\left[1 - \left(\frac{x}{h_1}\right)^p\right]^{\frac{1}{p}}, \tag{13}$$
$$= bh_1 + \frac{a-b}{p} h_1 \frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{2}{p}+1\right)}$$

$$I_{\mathrm{R}}(\tilde{A}_p) = \int_0^{h_2} g_{\tilde{A}_p}^{\mathrm{R}}(x)\mathrm{d}x, \quad \text{where}$$
$$g_{\tilde{A}_p}^{\mathrm{R}}(x) = c + (d-c)\left[1 - \left(\frac{x}{h_2}\right)^p\right]^{\frac{1}{p}}, \tag{14}$$
$$= ch_2 + \frac{d-c}{p} h_2 \frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{2}{p}+1\right)}$$

$$I_{\mathrm{M}}(\tilde{A}_p) = \int_{h_1(\text{or } h_2)}^{h_2(\text{or } h_1)} g_{\tilde{A}_p}^{\mathrm{M}}(x)\mathrm{d}x,$$
$$\text{where} \quad g_{\tilde{A}_p}^{\mathrm{M}}(x) = c + (b-c)\left[1 - \left(\frac{x-h_1}{h_2-h_1}\right)^p\right]^{\frac{1}{p}}, \tag{15}$$
$$= c(h_2 - h_1) + \frac{b-c}{p}(h_2 - h_1) \frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{2}{p}+1\right)}$$

$$I_{\mathrm{L}}(\tilde{B}_p) = \int_0^{h_1} g_{\tilde{B}_p}^{\mathrm{L}}(x)\mathrm{d}x,$$
$$\text{where} \quad g_{\tilde{B}_p}^{\mathrm{L}}(x) = r + (q-r)\left[1 - \left(\frac{x}{h_1}\right)^p\right]^{\frac{1}{p}}, \tag{16}$$
$$= rh_1 + \frac{q-r}{p} h_1 \frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{2}{p}+1\right)}$$

$$I_{\mathrm{R}}(\tilde{B}_p)=\int_0^{h_2} g^{\mathrm{R}}_{\tilde{B}_p}(x)\mathrm{d}x, \quad \text{where}$$

$$\begin{aligned} g^{\mathrm{R}}_{\tilde{B}_p}(x) &= s+(t-s)\left[1-\left(\frac{x}{h_2}\right)^p\right]^{\frac{1}{p}}, \\ &= sh_2+\frac{t-s}{p}h_2\frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{2}{p}+1\right)} \end{aligned} \tag{17}$$

$$I_{\mathrm{M}}(\tilde{B}_p)=\int_{h_1(\text{or } h_2}^{h_2(\text{or } h_1)} g^{\mathrm{M}}_{\tilde{B}_p}(x)\mathrm{d}x, \quad \text{where}$$

$$\begin{aligned} g^{\mathrm{M}}_{\tilde{B}_p}(x) &= s+(r-s)\left[1-\left(\frac{x-h_1}{h_2-h_1}\right)^p\right]^{\frac{1}{p}}, \\ &= s(h_2-h_1)+\frac{r-s}{p}(h_2-h_1)\frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{2}{p}+1\right)} \end{aligned} \tag{18}$$

Step 3. Find $I^{\alpha}_{\mathrm{T}}(\tilde{A}_p)$ and $I^{\alpha}_{\mathrm{T}}(\tilde{B}_p)$, which are given by

$$\begin{aligned} I^{\alpha}_{\mathrm{T}}(\tilde{A}_p) &= \alpha ch_2+(1-\alpha)bh_1+(h_2-h_1)c \\ &\quad +\{\alpha h_2(d-c)+(1-\alpha)h_1(a-b) \\ &\quad +(h_2-h_1)(b-c)\}\frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{p\times\Gamma\left(\frac{2}{p}+1\right)}, \end{aligned} \tag{19}$$

$$\begin{aligned} I^{\alpha}_{\mathrm{T}}(\tilde{B}_p) &= \alpha sh_2+(1-\alpha)rh_1+(h_2-h_1)s \\ &\quad +\{\alpha h_2(t-s)+(1-\alpha)h_1(q-r) \\ &\quad +(h_2-h_1)(r-s)\}\frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{p\times\Gamma\left(\frac{2}{p}+1\right)}. \end{aligned} \tag{20}$$

Step 4. Check $I^{\alpha}_{\mathrm{T}}(\tilde{A}_p)>I^{\alpha}_{\mathrm{T}}(\tilde{B}_p)$ or $I^{\alpha}_{\mathrm{T}}(\tilde{A}_p)<I^{\alpha}_{\mathrm{T}}(\tilde{B}_p)$ or $I^{\alpha}_{\mathrm{T}}(\tilde{A}_p)=I^{\alpha}_{\mathrm{T}}(\tilde{B}_p)$.

Case (i) If $I^{\alpha}_{\mathrm{T}}(\tilde{A}_p)>I^{\alpha}_{\mathrm{T}}(\tilde{B}_p)$ then $\tilde{A}_p\succ\tilde{B}_p$.
Case (ii) If $I^{\alpha}_{\mathrm{T}}(\tilde{A}_p)<I^{\alpha}_{\mathrm{T}}(\tilde{B}_p)$ then $\tilde{A}_p\prec\tilde{B}_p$.
Case (iii) If $I^{\alpha}_{\mathrm{T}}(\tilde{A}_p)=I^{\alpha}_{\mathrm{T}}(\tilde{B}_p)$ then $\tilde{A}_p\sim\tilde{B}_p$.

Remark 4.1 For any two arbitrary generalised fuzzy numbers with different left height and right height, $\tilde{A}_p$ and $\tilde{B}_p$, we have

$$I^{\alpha}_{\mathrm{T}}(\tilde{A}_p+\tilde{B}_p)=I^{\alpha}_{\mathrm{T}}(\tilde{A}_p)+I^{\alpha}_{\mathrm{T}}(\tilde{B}_p)$$

Proposition 4.1 *Let $\tilde{A}=(a,b,c,d;h_{AL},h_{AR})$ and $\tilde{B}=(a,e,d;h_B)$ be GFN with different left height and right height and non-normal triangular fuzzy number, respectively, such that $-\infty<a\le b\le e\le c\le d<\infty$. Then*

(i) $I_{\mathrm{L}}(\tilde{A})\le I_{\mathrm{L}}(\tilde{B})$,
(ii) $I_{\mathrm{R}}(\tilde{A})\ge I_{\mathrm{R}}(\tilde{B})$,
(iii) $I_{\mathrm{M}}(\tilde{A})=I_{\mathrm{M}}(\tilde{B})$, *if* $b+c=2e$,
(iv) $I_{\mathrm{M}}(\tilde{A})>I_{\mathrm{M}}(\tilde{B})$, *if* $h_1<h_2$ *and* $b+c>2e$ *or* $h_1>h_2$ *and* $b+c<2e$,
(v) $I_{\mathrm{M}}(\tilde{A})<I_{\mathrm{M}}(\tilde{B})$, *if* $h_1<h_2$ *and* $b+c<2e$ *or* $h_1>h_2$ *and* $b+c>2e$,
(vi) $I^{\alpha}_{\mathrm{T}}(\tilde{A})>I^{\alpha}_{\mathrm{T}}(\tilde{B})$ *if* $\alpha h_2(c-e)+(1-\alpha)h_1(b-e)+(h_2-h_1)(b+c-2e)>0$,
(vii) $I^{\alpha}_{\mathrm{T}}(\tilde{A})<I^{\alpha}_{\mathrm{T}}(\tilde{B})$ *if* $\alpha h_2(c-e)+(1-\alpha)h_1(b-e)+(h_2-h_1)\ (b+c-2e)<0$ *and*
(viii) $I^{\alpha}_{\mathrm{T}}(\tilde{A})=I^{\alpha}_{\mathrm{T}}(\tilde{B})$ *if* $\alpha h_2(c-e)+(1-\alpha)h_1(b-e)+(h_2-h_1)\ (b+c-2e)=0$.

Proof From Eqs. (13), (14), (15) and (19), on appropriate substitutions of the variables the following could be easily obtained:

$$\begin{aligned} &I_{\mathrm{L}}(\tilde{A})=\frac{h_1(a+b)}{2}, \quad I_{\mathrm{R}}(\tilde{A})=\frac{h_2(d+c)}{2}, \\ &I_{\mathrm{M}}(\tilde{A})=\frac{(h_2-h_1)(b+c)}{2}, I_{\mathrm{L}}(\tilde{B})=\frac{h_1(a+e)}{2}, \\ &I_{\mathrm{R}}(\tilde{B})=\frac{h_2(e+d)}{2}, \quad I_{\mathrm{M}}(\tilde{B})=e(h_2-h_1), \\ &I^{\alpha}_{\mathrm{T}}(\tilde{A})=\frac{1}{2}\{\alpha h_2(c+d)+(1-\alpha)h_1(a+b) \\ &\qquad +(h_2-h_1)(c+b)\} \quad \text{and} \\ &I^{\alpha}_{\mathrm{T}}(\tilde{B})=\frac{1}{2}\{\alpha h_2(e+d)+(1-\alpha)h_1(a+e)+2e(h_2-h_1)\} \end{aligned}$$

where $h_1=\min(h_{AL},h_B)$ and $h_2=\min(h_{AR},h_B)$.

Now $I_{\mathrm{L}}(\tilde{A})-I_{\mathrm{L}}(\tilde{B})=\frac{h_1}{2}(b-e)\le 0$ as $b\le e\le c$, hence inequality (i) is deduced. Similarly, inequality (ii) could be deduced. Again, we have $I_{\mathrm{M}}(\tilde{A})-I_{\mathrm{M}}(\tilde{B})=\frac{h_2-h_1}{2}(b+c-2e)$ hence the inequalities (iii), (iv) and (v) follow immediately. Also we have

$$\begin{aligned} I^{\alpha}_{\mathrm{T}}(\tilde{A})-I^{\alpha}_{\mathrm{T}}(\tilde{B}) &= \frac{1}{2}\{\alpha h_2(c-e)+(1-\alpha)h_1(b-e) \\ &\quad +(h_2-h_1)(c+b-2e)\}. \end{aligned}$$

Hence the inequalities (vi), (vii) and (viii) can be deduced easily. □

Corollary 4.1 [9] *Let $\tilde{A}=(a,b,c,d;h_A)$ and $\tilde{B}=(a,e,d;h_B)$ be non-normal trapezoidal and triangular fuzzy numbers, respectively, where $-\infty<a\le b\le e\le c\le d<\infty$. Then*

(i) $I_{\mathrm{L}}(\tilde{A})\le I_{\mathrm{L}}(\tilde{B})$,
(ii) $I_{\mathrm{R}}(\tilde{A})\ge I_{\mathrm{R}}(\tilde{B})$,
(iii) $I^{\alpha}_{\mathrm{T}}(\tilde{A})>I^{\alpha}_{\mathrm{T}}(\tilde{B})$ *if* $e<c\alpha+(1-\alpha)b$,
(iv) $I^{\alpha}_{\mathrm{T}}(\tilde{A})=I^{\alpha}_{\mathrm{T}}(\tilde{B})$ *if* $e=c\alpha+(1-\alpha)b$ *and*
(v) $I^{\alpha}_{\mathrm{T}}(\tilde{A})<I^{\alpha}_{\mathrm{T}}(\tilde{B})$ *if* $e>c\alpha+(1-\alpha)b$.

These inequalities are particular case of the inequalities in the Proposition 4.1. These can be obtained by appropriate substitutions on the inequalities in the Proposition 4.1.

Proposition 4.2 *Let $\tilde{A} = (a, b, c, d; h_{AL}, h_{AR})$ and $\tilde{B}_2 = (a, b, c, d; h_{BL}, h_{BR})_2$ be GFN and 2-norm GFN with different left height and right height, respectively. Then*

(i) $I_{\mathrm{L}}(\tilde{A}) \geq I_{\mathrm{L}}(\tilde{B}_2)$,
(ii) $I_{\mathrm{L}}(\tilde{A}) \leq I_{\mathrm{L}}(\tilde{B}_2)$,
(iii) $I_{\mathrm{M}}(\tilde{A}) \geq (\leq) I_{\mathrm{M}}(\tilde{B}_2)$ *if* $h_1 < (>) h_2$,
(iv) $I^{\alpha}_{\mathrm{M}}(\tilde{A}) > I^{\alpha}_{\mathrm{M}}(\tilde{B}_2)$ *if* $\alpha h_2(d - c) + h_1(1 - \alpha)(a - b) + (h_2 - h_1)(b - c) < 0$,
(v) $I^{\alpha}_{\mathrm{M}}(\tilde{A}) < I^{\alpha}_{\mathrm{M}}(\tilde{B}_2)$ *if* $\alpha h_2(d - c) + h_1(1 - \alpha)(a - b) + (h_2 - h_1)(b - c) > 0$ *and*
(vi) $I^{\alpha}_{\mathrm{M}}(\tilde{A}) = I^{\alpha}_{\mathrm{M}}(\tilde{B}_2)$ *if* $\alpha h_2(d - c) + h_1(1 - \alpha)(a - b) + (h_2 - h_1)(b - c) = 0$.

Proof $\tilde{A}$ and $\tilde{B}_2$ are GFN and 2-norm GFN with different left height and right height. Hence by the proposed method $I_{\mathrm{L}}(\tilde{A})$, $I_{\mathrm{R}}(\tilde{A})$, $I_{\mathrm{M}}(\tilde{A})$, $I^{\alpha}_{\mathrm{T}}(\tilde{A})$ $I_{\mathrm{L}}(\tilde{B}_2)$, $I_{\mathrm{R}}(\tilde{B}_2)$, $I_{\mathrm{M}}(\tilde{B}_2)$ and $I^{\alpha}_{\mathrm{T}}(\tilde{B}_2)$ are obtained by using Eqs. (13), (14), (15) and (19) as:

$$I_{\mathrm{L}}(\tilde{A}) = \frac{h_1(a+b)}{2}, \quad I_{\mathrm{R}}(\tilde{A}) = \frac{h_2(d+c)}{2},$$
$$I_{\mathrm{M}}(\tilde{A}) = \frac{(h_2 - h_1)(b+c)}{2}, I_{\mathrm{L}}(\tilde{B}_2) = bh_1 + \frac{a-b}{4} h_1 \pi,$$
$$I_{\mathrm{R}}(\tilde{B}_2) = ch_2 + \frac{d-c}{4} h_2 \pi, \quad I_{\mathrm{M}}(\tilde{B}_2) = \frac{b+3c}{4}(h_2 - h_1)\pi,$$
$$I^{\alpha}_{\mathrm{T}}(\tilde{A}) = \frac{1}{2}\{\alpha h_2(c+d) + (1-\alpha)h_1(a+b) + (h_2 - h_1)(c+b)\} \quad \text{and}$$
$$I^{\alpha}_{\mathrm{T}}(\tilde{B}_2) = \frac{1}{4}[(h_2 - h_1)\{4c + \pi(b-c)\} + (1-\alpha) \times h_1\{4b + \pi(a-b)\} + \alpha h_2\{4c + \pi(d-c)\}]$$

where $h_1 = \min(h_{AL}, h_{BL})$ and $h_2 = \min(h_{AR}, h_{BR})$.

Now, $I_{\mathrm{L}}(\tilde{A}) - I_{\mathrm{L}}(\tilde{B}_2) = \frac{2-\pi}{4} h_1(a-b) \geq 0$ and $I_{\mathrm{R}}(\tilde{A}) - I_{\mathrm{R}}(\tilde{B}_2) = \frac{2-\pi}{4} h_2(d-c) \leq 0$. Thus the desired inequalities (i) and (ii) are obtained. $(b-c)(2-\pi)$ is always greater than or equal to zero, thus $I_{\mathrm{M}}(\tilde{A}) - I_{\mathrm{M}}(\tilde{B}_2) = (h_2 - h_1)\frac{(b-c)(2-\pi)}{4} \geq (\leq) 0$ if $h_1 < (>) h_2$, which prove the inequality (iii). For the inequalities (iv), (v) and (iv), we have

$$I^{\alpha}_{\mathrm{M}}(\tilde{A}) - I^{\alpha}_{\mathrm{M}}(\tilde{B}_2) = \frac{2-\pi}{4}\{\alpha h_2(d-c) + (1-\alpha)h_1(a-b) + (h_2 - h_1)(b-c)\}.$$

Hence the inequalities (iv), (v) and (vi) follow immediately. □

Corollary 4.2 [9] *Let $\tilde{A} = (a, b, c, d; h_A)$ and $\tilde{B}_2 = (a, b, c, d; h_B)$ be non-normal trapezoidal fuzzy number and non-normal 2-norm trapezoidal fuzzy number, respectively, then*

(i) $I_{\mathrm{L}}(\tilde{A}) \geq I_{\mathrm{L}}(\tilde{B}_2)$,
(ii) $I_{\mathrm{L}}(\tilde{A}) \leq I_{\mathrm{L}}(\tilde{B}_2)$,
(iii) $I^{\alpha}_{\mathrm{T}} < I^{\alpha}_{\mathrm{T}}(\tilde{B})$ *if* $\alpha(d-c) + (1-\alpha)(a-b) > 0$,
(iv) $I^{\alpha}_{\mathrm{T}} = I^{\alpha}_{\mathrm{T}}(\tilde{B})$ *if* $\alpha(d-c) + (1-\alpha)(a-b) = 0$ *and*
(v) $I^{\alpha}_{\mathrm{T}} > I^{\alpha}_{\mathrm{T}}(\tilde{B})$ *if* $\alpha(d-c) + (1-\alpha)(a-b) < 0$.

These inequalities are particular case of the inequalities in the Proposition 4.2. These can be obtained by appropriate substitutions on the inequalities in the Proposition 4.2.

Proposition 4.3 *Let $\tilde{A} = (a, a, a, a; 1, 1)$ and $\tilde{B} = (b, b, b, b; 1, 1)$ be GFNs with height 1. Then*

(i) $I_{\mathrm{L}}(\tilde{A}) \geq (\leq) I_{\mathrm{L}}(\tilde{B})$, *if* $a \geq (\leq) b$,
(ii) $I_{\mathrm{M}}(\tilde{A}) \geq (\leq) I_{\mathrm{M}}(\tilde{B}_2)$ *if* $a \geq (\leq) b$ *and*
(iii) $I^{1}_{\mathrm{T}}(\tilde{A}) > (<) I^{1}_{\mathrm{T}}(\tilde{B})$ *if* $a > (<) b$.

The Proposition 4.3 validates that the proposed method can also be applied for real numbers.

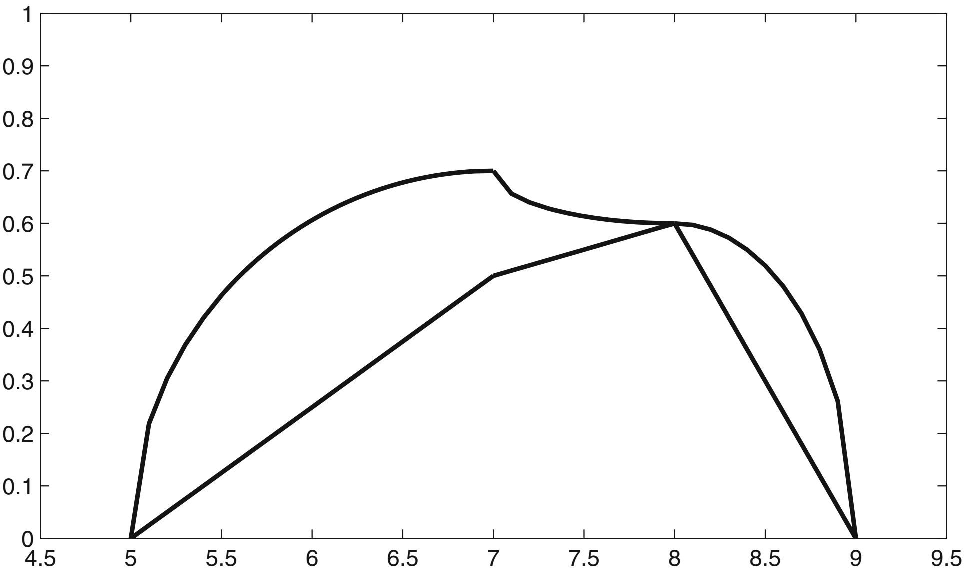
Example 4.1 Let $\tilde{A} = (5, 7, 8, 9; 0.5, 0.6)$ and $\tilde{B}_2 = (5, 7, 8, 9; 0.7, 0.6)_2$ be GFN and 2-norm GFN with different left height and right height, which are depicted in Fig. 1. But, according to the proposed modified method $h_1 = \min(0.5, 0.6)$ and $h_2 = \min(0.7, 0.6)$. Also, $I^{0.5}_{\mathrm{T}}(\tilde{A}) = 4.8000$ and $I^{0.5}_{\mathrm{T}}(\tilde{B}_2) = 4.7144$. Thus, $I^{0.5}_{\mathrm{T}}(\tilde{A}) > I^{0.5}_{\mathrm{T}}(\tilde{B}_2)$.

Example 4.2 Let $\tilde{A} = (0.1659, 0.2803, 0.7463, 1.154; 0.5, 0.6)$, $\tilde{B} = (0.1611, 0.2475, 0.5696, 0.8187; 0.4, 0.5)$ and $\tilde{C} = (0.1645, 0.2445, 0.5869, 0.8894; 0.5, 0.6)$, are GFNs with different left height and right height. Figure 2 depicts the membership function of the above fuzzy numbers.

Here $I^{0.5}_{\mathrm{T}}(\tilde{A}) = 0.3335$, $I^{0.5}_{\mathrm{T}}(\tilde{B}) = 0.2553$ and $I^{0.5}_{\mathrm{T}}(\tilde{C}) = 0.2670$, and $I^{1}_{\mathrm{T}}(\tilde{A}) = 0.1406$, $I^{1}_{\mathrm{T}}(\tilde{B}) = 0.1226$ and $I^{1}_{\mathrm{T}}(\tilde{C}) = 0.1234$. Thus a moderate decision and a pessimistic decision maker rank them as $\tilde{A} > \tilde{C} > \tilde{B}$.

Example 4.3 Let $\tilde{A}_2 = (-2, -1, 0, 1; 0.5, 0.5)_2$, $\tilde{B}_2 = (-1.5, -0.5, 0.5, 1.5; 0.5, 0.6)_2$ and $\tilde{C}_2 = (1, 1.5, 2, 2.5; 0.6, 0.5)_2$, are GFNs with different left height and right height. The membership functions of the fuzzy numbers are depicted in Fig. 3. Here, $I^{0.5}_{\mathrm{T}}(\tilde{A}_2) = -0.2500$, $I^{0.5}_{\mathrm{T}}(\tilde{B}_2) = 0.000$ and $I^{0.5}_{\mathrm{T}}(\tilde{C}_2) = 0.6787$, and $I^{0}_{\mathrm{T}}(\tilde{A}_2) = -0.8927$, $I^{0}_{\mathrm{T}}(\tilde{B}_2) = -0.6427$ and $I^{0}_{\mathrm{T}}(\tilde{C}_2) = 0.5537$. A moderate decision maker ($\alpha = 0.5$) ranks $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ as $\tilde{C} > \tilde{B} > \tilde{A}$ and also a pessimistic decision maker ranks them in the same order.

Fig. 1 Fuzzy numbers $\tilde{A} = (5, 7, 8, 9; 0.5, 0.6)$ and $\tilde{B}_2 = (5, 7, 8, 9; 0.7, 0.6)_2$



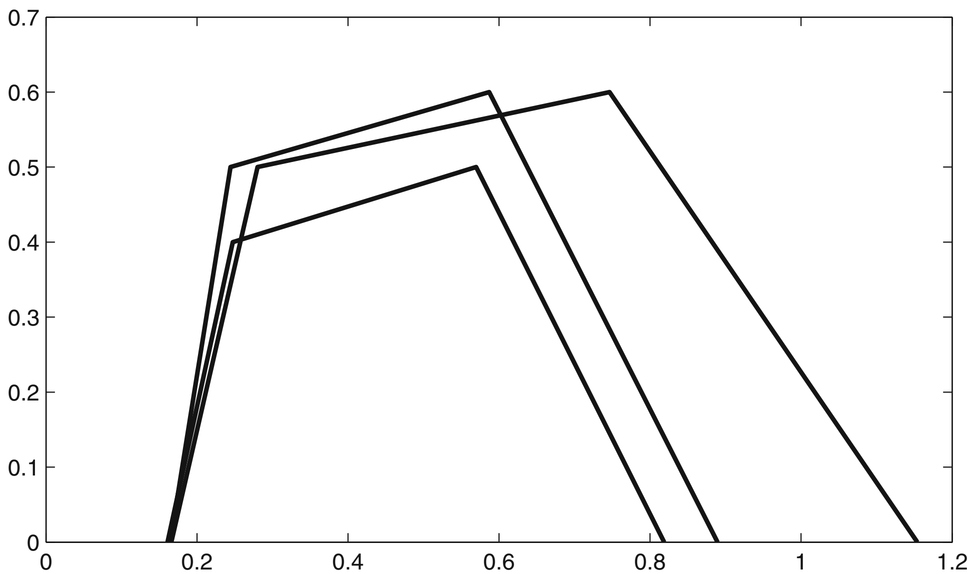


Fig. 2 Fuzzy numbers $\tilde{A} = (0.1659, 0.2803, 0.7463, 1.154; 0.5, 0.6)$, $\tilde{B} = (0.1611, 0.2475, 0.5696, 0.8187; 0.4, 0.5)$ and $\tilde{C} = (0.1645, 0.2445, 0.5869, 0.8894; 0.5, 0.6)$

Example 4.4 Let $\tilde{A} = (1, 1, 1, 1; 1, 1)$ and $\tilde{B} = (2, 2, 2, 2; 1, 1)$ be GFNs which are actually real numbers. Now by Proposition 4.3 $\tilde{B} > \tilde{A}$ trivially.

Example 4.5 Consider the following set of fuzzy numbers,

Set A $\tilde{A} = (0.4, 0.5, 0.5, 0.6; 1, 1)$,
$\tilde{B} = (0.2, 0.4, 0.6, 0.8; 1, 1)$.

Set B $\tilde{A} = (0.2, 0.3, 0.3, 0.6; 1, 1)$,
$\tilde{B} = (0.2, 0.4, 0.4, 0.6; 1, 1)$,
$\tilde{C} = (0.2, 0.5, 0.5, 0.6; 1, 1)$.

Set C $\tilde{A} = (0.2, 0.4, 0.4, 0.6; 0.9, 0.9)$,
$\tilde{B} = (0.2, 0.4, 0.4, 0.6; 1, 1)$,
$\tilde{C} = (0.2, 0.4, 0.4, 0.6; 0.5, 0.5)$.

Set D $\tilde{A} = (0.4, 0.5, 0.5, 0.6; 0.5, 0.5)$,
$\tilde{B} = (0.2, 0.4, 0.6, 0.8; 0.6, 0.6)$.

Set E $\tilde{A} = (0.4, 0.5, 0.5, 0.6; 0.6, 0.7)$,
$\tilde{B} = (0.2, 0.4, 0.6, 0.8; 0.5, 0.6)$.

Set F $\tilde{A} = (0.4, 0.5, 0.5, 0.6; 0.6, 0.7)$,
$\tilde{B} = (0.2, 0.4, 0.6, 0.8; 0.5, 0.6)_2$.

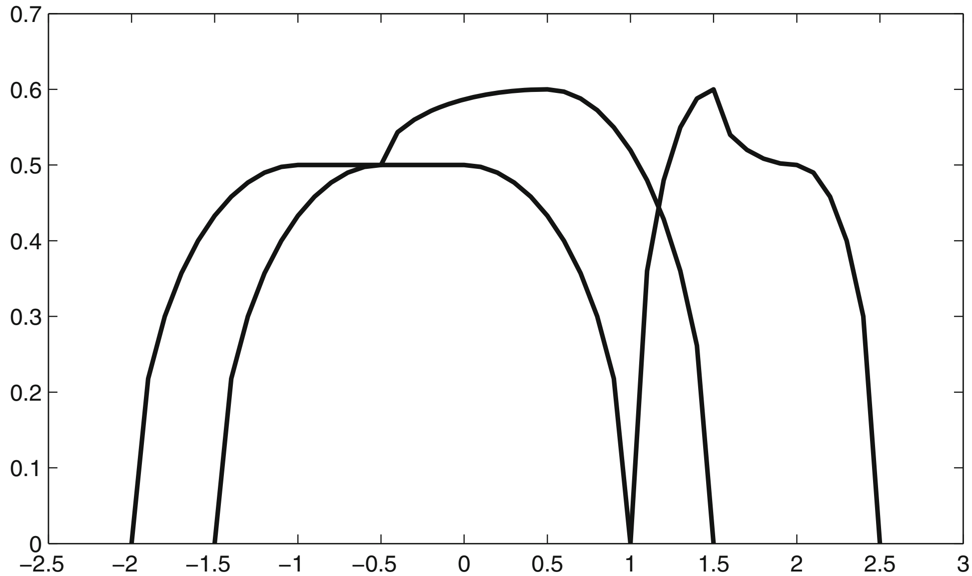


Fig. 3 Fuzzy numbers $\tilde{A}_2 = (-2, -1, 0, 1; 0.5, 0.5)_2$, $\tilde{B}_2 = (-1.5, -0.5, 0.5, 1.5; 0.5, 0.6)_2$ and $\tilde{C}_2 = (1, 1.5, 2, 2.5; 0.6, 0.5)_2$

Table 1 Comparison of the proposed method with Kim and Park [8] and Kumar et al. [9] methods

Test sets	Index	Kim and Park			Kumar's method			Proposed method		
		$\tilde{A}$	$\tilde{B}$	$\tilde{C}$	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$
Set A	$\alpha = 1.0$	0.5714	0.75	–	0.55	0.70	–	0.55	0.70	–
	$\alpha = 0.5$	0.5000	0.50	–	0.50	0.50	–	0.50	0.50	–
	$\alpha = 0.0$	0.4286	0.25	–	0.45	0.30	–	0.45	0.30	–
Set B	$\alpha = 1.0$	0.5714	0.6666	0.8000	0.45	0.50	0.55	0.45	0.50	0.55
	$\alpha = 0.5$	0.3857	0.4999	0.6143	0.35	0.40	0.45	0.35	0.40	0.45
	$\alpha = 0.0$	0.2000	0.3333	0.4286	0.25	0.30	0.35	0.25	0.30	0.35
Set C	$\alpha = 1.0$	–	–	–	0.45	0.50	0.25	0.45	0.50	0.25
	$\alpha = 0.5$	–	–	–	0.36	0.40	0.20	0.36	0.40	0.20
	$\alpha = 0.0$	–	–	–	0.27	0.30	0.15	0.27	0.30	0.15
Set D	$\alpha = 1.0$	–	–	–	0.275	0.35	–	0.275	0.35	–
	$\alpha = 0.5$	–	–	–	0.250	0.25	–	0.250	0.25	–
	$\alpha = 0.0$	–	–	–	0.225	0.15	–	0.225	0.15	–
Set E	$\alpha = 1.0$	–	–	–	–	–	–	0.3800	0.470	–
	$\alpha = 0.5$	–	–	–	–	–	–	0.3275	0.335	–
	$\alpha = 0.0$	–	–	–	–	–	–	0.2750	0.200	–
Set F	$\alpha = 1.0$	–	–	–	–	–	–	0.3800	0.4985	–
	$\alpha = 0.5$	–	–	–	–	–	–	0.3275	0.3321	–
	$\alpha = 0.0$	–	–	–	–	–	–	0.2750	0.1658	–

The results of the sets A, B, C, D and E are depicted in the Table 1. Sets A and B consist of normal fuzzy numbers, hence the ranking order by the three methods is same. Sets C and D consist of non-normal fuzzy numbers, Kim and Park [8] give no option for ranking such type of fuzzy number. The method of Kumar et al. [9] and the proposed method's ranking order are same for the sets C and D. However, sets E and F which consist of p-norm generalised fuzzy numbers with different left height and right height can be ranked only by the proposed method.

Validation of the proposed modified ranking method

For the validation of the proposed ranking method, the following reasonable axioms that Wang and Kerre [13] have proposed for fuzzy numbers' ranking are considered. Let *RM* be an ordering method, *S* the set of fuzzy numbers for which the method *RM* can be applied, and $\mathcal{A}$ and $\mathcal{A}'$ finite subsets of *S*. The statements of two elements $\tilde{A}_p$ and $\tilde{B}_p$ in $\mathcal{A}$ satisfy that $\tilde{A}_p$ has a higher ranking than $\tilde{B}_p$ when *RM* is applied to the fuzzy numbers in $\mathcal{A}$ will be written as $\tilde{A}_p \succ \tilde{B}_p$ by RM on $\mathcal{A}$. $\tilde{A}_p \sim \tilde{B}_p$ by RM on $\mathcal{A}$, and $\tilde{A}_p \succeq \tilde{B}_p$ by RM on $\mathcal{A}$ are similarly interpreted. The following axioms show the reasonable properties of the ordering approach RM.

- A_1 For $\tilde{A}_p \in \mathcal{A}$, $\tilde{A}_p \preceq \tilde{A}_p$ by RM on $\mathcal{A}$.
- A_2 For $(\tilde{A}_p, \tilde{B}_p) \in \mathcal{A}^2$, $\tilde{A}_p \preceq \tilde{B}_p$ and $\tilde{B}_p \preceq \tilde{A}_p$ by RM on $\mathcal{A}$, we should have $\tilde{A}_p \sim \tilde{B}_p$ by RM on $\mathcal{A}$.
- A_3 For $(\tilde{A}_p, \tilde{B}_p, \tilde{C}_p) \in \mathcal{A}^3$, $\tilde{A}_p \preceq \tilde{B}_p$ and $\tilde{B}_p \preceq \tilde{C}_p$ by RM on $\mathcal{A}$, we should have $\tilde{A}_p \preceq \tilde{C}_p$ by RM on $\mathcal{A}$.
- A_4 For $(\tilde{A}_p, \tilde{B}_p) \in \mathcal{A}^2$, $\inf \operatorname{supp}(\tilde{B}_p) > \sup \operatorname{supp}(\tilde{A}_p)$, we should have $\tilde{A}_p \preceq \tilde{B}_p$ by RM on $\mathcal{A}$.
- A'_4 For $(\tilde{A}_p, \tilde{B}_p) \in \mathcal{A}^2$, $\inf \operatorname{supp}(\tilde{B}_p) > \sup \operatorname{supp}(\tilde{A}_p)$, we should have $\tilde{A}_p \prec \tilde{B}_p$ by RM on $\mathcal{A}$.
- A_5 Let $(\tilde{A}_p, \tilde{B}_p) \in (\mathcal{A} \cap \mathcal{A}')^2$. We obtain the ranking order $\tilde{A}_p \preceq \tilde{B}_p$ by RM on $\mathcal{A}'$ if and only if $\tilde{A}_p \preceq \tilde{B}_p$ by RM on $\mathcal{A}$.
- A_6 Let $\tilde{A}_p, \tilde{B}_p, \tilde{A}_p + \tilde{C}_p$ and $\tilde{B}_p + \tilde{C}_p$ be elements of S. If $\tilde{A}_p \succeq \tilde{B}_p$ by RM on $\{\tilde{A}_p, \tilde{B}_p\}$, then $\tilde{A}_p + \tilde{C}_p \succeq \tilde{B}_p + \tilde{C}_p$ by RM on $\{\tilde{A}_p + \tilde{C}_p, \tilde{B}_p + \tilde{C}_p\}$.
- A'_6 Let $\tilde{A}_p, \tilde{B}_p, \tilde{A}_p + \tilde{C}_p$ and $\tilde{B}_p + \tilde{C}_p$ be elements of S. If $\tilde{A}_p \succ \tilde{B}_p$ by RM on $\{\tilde{A}_p, \tilde{B}_p\}$, then $\tilde{A}_p + \tilde{C}_p \succ \tilde{B}_p + \tilde{C}_p$ by RM on $\{\tilde{A}_p + \tilde{C}_p, \tilde{B}_p + \tilde{C}_p\}$ when $\tilde{C}_p \neq \phi$.

Proposition 4.4 *The proposed ranking method RM has the properties* A_1, A_2, A_3, A_4, A'_4, A_5, A_6 *and* A'_6.

Proof It is easy to verify that properties A_1–A_5 are hold. For the proof of A_6, consider the generalised fuzzy numbers with different left height and right height as $\tilde{A}_p = (a, b, c, d; h_L, h_R)_p$, $\tilde{B}_p = (q, r, s, t; h'_L, h'_R)_p$ and $\tilde{C}_p = (l, m, n, o, ; h''_L, h''_R)_p$. Let $\tilde{A}_p \succeq \tilde{B}_p$ by RM, hence

$$I_T^\alpha(\tilde{A}_p) \geq I_T^\alpha(\tilde{B}_p),$$

by adding $I_T^\alpha(\tilde{C}_p)$

$$I_T^\alpha(\tilde{A}_p) + I_T^\alpha(\tilde{C}_p) \geq I_T^\alpha(\tilde{B}_p) + I_T^\alpha(\tilde{C}_p),$$

and by Remark 4.1

$$I_T^\alpha(\tilde{A}_p + \tilde{C}_p) \geq I_T^\alpha(\tilde{B}_p + \tilde{C}_p).$$

Therefore, $\tilde{A}_p + \tilde{C}_p \succeq \tilde{B}_p + \tilde{C}_p$. Similarly A'_6 also holds. □

Conclusions

In this paper, ranking of p-norm GFNs with different left height and right height is proposed. The proposed method is generalization of Kumar's approach. Kumar's approach can only deal with non-normal p-norm trapezoidal fuzzy numbers. The proposed method can handle non-normal p-norm trapezoidal fuzzy numbers as well as p-norm GFNs with different left height and right height.

Acknowledgments The authors would like to thank the anonymous referees for their valuable comments and suggestions which improved the paper form technical as well as clarity point of view. The author RC would like to thank Indian Institute of Technology, Guwahati for funding the research work.

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