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ORIGINAL RESEARCH

Three variable symmetric identities involving Carlitz-type q-Euler polynomials

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Abstract In this paper, we derive several identities of symmetry in three variables related to Carlitz-type q-Euler polynomials and alternating q-power sums. These and most of their identities are new, since there have been results only about identities of symmetry in two variables. The derivations of identities are based on the fermionic p-adic q-integral expressions of the generating functions for the Carlitz-type q-Euler polynomials.

Keywords Carlitz-type q-Euler polynomial · Fermionic p-adic q-integral

Mathematics Subject Classification 05A19 · 11B65 · 11B68

Introduction

Let p be a prime number with $p \equiv 1 \pmod 2$. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $|\cdot|_p$ be the normalized p-adic absolute value with $|p|_p = \frac{1}{p}$ and let q be an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. For a continuous function $f : \mathbb{Z}_p \to \mathbb{C}_p$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be [9–17]

$$\begin{aligned} I_{-q}(f) &= \int_{\mathbb{Z}_p} f(x)\mathrm{d}\mu_{-q}(x) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \end{aligned} \tag{1}$$

where $[x]_{-q} = \frac{1-(-q)^x}{1+q}$ and $[x]_q = \frac{1-q^x}{1-q}$.

From (1), we have

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \tag{2}$$

where $f_1(x) = f(x+1)$.

In general, one derives that

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l), \tag{3}$$

where $f_n(x) = f(x+n)$, $(n \geq 1)$.

So, for $n \equiv 1 \pmod 2$,

$$q^n I_{-q}(f_n) + I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l f(l); \tag{4}$$

for $n \equiv 0 \pmod 2$,

$$q^n I_{-q}(f_n) - I_{-q}(f) = -[2]_q \sum_{l=0}^{n-1} (-1)^l q^l f(l). \tag{5}$$

In particular, for $q = 1$, we have

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\mathrm{d}\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \tag{6}$$

and

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$$I_{-1}(f_1) + I_{-1}(f) = 2f(0). \tag{7}$$

As is well known, the ordinary Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \tag{8}$$

(see [1–24]). When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers.

From (5) and (6), we can easily derive

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{9}$$

By (7), we get

$$E_0 = 1, \quad (E+1)^n + E_n = 2\delta_{0,n}, \tag{10}$$

[11, 16, 17] with the usual convention about replacing E^n by E_n.

From (9), we note that

$$\begin{aligned} E_n(x) &= \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \int_{\mathbb{Z}_p} y^l d\mu_{-1}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_l. \end{aligned} \tag{11}$$

In light of (10), the Carlitz-type q-Euler numbers are given by

$$q(q\mathcal{E}_q + 1)^n + \mathcal{E}_{n,q} = [2]_q \delta_{n,0}, \quad (n \geq 0), \tag{12}$$

with the usual convention about replacing $\mathcal{E}_q^n$ by $\mathcal{E}_{n,q}$ (see [10, 12, 15]).

The q-Euler polynomials are defined by (see [10, 16])

$$\begin{aligned} \mathcal{E}_{n,q}(x) &= \left(q^x \mathcal{E}_q + [x]_q\right)^n \\ &= \sum_{l=0}^{n} \binom{n}{l} q^{lx} \mathcal{E}_{l,q} [x]_q^{n-l}. \end{aligned} \tag{13}$$

From (2), we derive

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{E}_{n,q} \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} e^{[y]_q t} d\mu_{-q}(y) \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[m]_q t}, \end{aligned} \tag{14}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-q}(y) \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[m+x]_q t}. \end{aligned} \tag{15}$$

By (13) and (15), we get

$$\begin{aligned} \mathcal{E}_{n,q}(x+y) &= \sum_{l=0}^{n} \binom{n}{l} q^{lx} \mathcal{E}_{l,q}(y) [x]_q^{n-l} \\ &= \sum_{l=0}^{n} \binom{n}{l} q^{(n-l)x} \mathcal{E}_{n-l,q}(y) [x]_q^{l}. \end{aligned} \tag{16}$$

Explicit expressions for Carlitz-type q-Euler numbers can be obtained, for example, from their generating functions :

$$\mathcal{E}_{n,q}(x) = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{1}{1+q^{l+1}}, \tag{17}$$

and

$$\mathcal{E}_{n,q} = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{l+1}}. \tag{18}$$

In [10, 14, 15], Kim introduced the polynomials $\mathcal{E}_{n,q}^{(h,k)}(x)$ in terms of the following multiple fermionic p-adic q-integral on $\mathbb{Z}_p$:

$$\begin{aligned} \mathcal{E}_{n,q}^{(h,k)}(x) = &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{l=1}^{k}(h-l)y_l} \\ &\times [x + y_1 + \cdots + y_k]_q^n d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_k). \end{aligned} \tag{19}$$

In particular, if $k = 1$, $\mathcal{E}_{n,q}^{(h,1)}(x)$ will be simply denoted by $\mathcal{E}_{n,q}^{(h)}(x)$ so that

$$\mathcal{E}_{n,q}^{(h)}(x) = \int_{\mathbb{Z}_p} q^{(h-1)y} [x+y]_q^n d\mu_{-q}(y). \tag{20}$$

One can derive the following explicit expression of $\mathcal{E}_{n,q}^{(h,k)}(x)$:

$$\mathcal{E}_{n,q}^{(h,k)}(x) = \frac{[2]_q^k}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} q^{lx} \frac{1}{(-q^{l+h}; q^{-1})_k}, \tag{21}$$

where $(x:q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$ (see [10, 14, 15]).

The following simple facts will be used over and over again :

$$[a+b]_q = [a]_q + q^a [b]_q. \tag{22}$$

From (22), one can show

$$[a+b+c]_q = [a]_q + q^a [b]_q + q^{a+b} [c]_q, \tag{23}$$

$$[ab]_q = [a]_q [b]_{q^a}.$$

In this paper, we give several identities of symmetry in three variables related to Carlitz-type q-Euler polynomials and alternating q-power sums which are derived from the triple fermionic p-adic q-integral on $\mathbb{Z}_p$. These and most of their identities are new, since there have been results only about identities of symmetry in two variables.

Symmetric identities of q-Euler polynomials

First, we will consider the following triple integral which is obviously invariant under any permutations of w_1, w_2, w_3. This simple observation is the philosophy that underlies this paper.

Let

$$I = \int_{\mathbb{Z}_p^3} e^{[w_2w_3x_1+w_1w_3x_2+w_1w_2x_3+w_1w_2w_3(y_1+y_2+y_3)]_q t}. \times d\mu_{-q^{w_2w_3}}(x_1)d\mu_{-q^{w_1w_3}}(x_2)d\mu_{-q^{w_1w_2}}(x_3). \tag{24}$$

It is not difficult to show that

$$\begin{aligned}&[w_2w_3x_1 + w_1w_3x_2 + w_1w_2x_3 + w_1w_2w_3(y_1 + y_2 + y_3)]_q \\ &= [w_2w_3]_q[x_1 + w_1y_1]_{q^{w_2w_3}} + q^{w_2w_3(x_1+w_1y_1)}[w_1w_3]_q[x_2 + w_2y_2]_{q^{w_1w_3}} \\ &\quad + q^{w_2w_3(x_1+w_1y_1)+w_1w_3(x_2+w_2y_2)}[w_1w_2]_q[x_3 + w_3y_3]_{q^{w_1w_2}}.\end{aligned} \tag{25}$$

So the integrand of I is

$$\begin{aligned}&e^{[w_2w_3x_1+w_1w_3x_2+w_1w_2x_3+w_1w_2w_3(y_1+y_2+y_3)]_q t} \\ &= e^{[w_2w_3]_q[x_1+w_1y_1]_{q^{w_2w_3}} t} e^{q^{w_2w_3(x_1+w_1y_1)}[w_1w_3]_q[x_2+w_2y_2]_{q^{w_1w_3}} t} \\ &\quad \times e^{q^{w_2w_3(x_1+w_1y_1)+w_1w_3(x_2+w_2y_2)}[w_1w_2]_q[x_3+w_3y_3]_{q^{w_1w_2}} t} \\ &= \sum_{n=0}^{\infty}\Bigg\{\sum_{k+l+m=n}\binom{n}{k,l,m}[w_2w_3]_q^k[w_1w_3]_q^l[w_1w_2]_q^m q^{w_1w_2w_3(l+m)y_1} \\ &\quad \times q^{w_1w_2w_3my_2}q^{w_2w_3(l+m)x_1}[x_1 + w_1y_1]_{q^{w_2w_3}}^k q^{w_1w_3mx_2}[x_2 + w_2y_2]_{q^{w_1w_3}}^l \\ &\quad \times [x_3 + w_3y_3]_{q^{w_1w_2}}^m\Bigg\}\frac{t^n}{n!}.\end{aligned} \tag{26}$$

Thus the integral in (24) is

$$\begin{aligned}I &= \sum_{n=0}^{\infty}\Bigg\{\sum_{k+l+m=n}\binom{n}{k,l,m}[w_2w_3]_q^k[w_1w_3]_q^l[w_1w_2]_q^m q^{w_1w_2w_3(l+m)y_1} \\ &\quad \times q^{w_1w_2w_3my_2}\int_{\mathbb{Z}_p} q^{w_2w_3(l+m)x_1}[x_1 + w_1y_1]_{q^{w_2w_3}}^k d\mu_{-q^{w_2w_3}}(x_1) \\ &\quad \times \int_{\mathbb{Z}_p} q^{w_1w_3mx_2}[x_2 + w_2y_2]_{q^{w_1w_3}}^l d\mu_{-q^{w_1w_3}}(x_2) \\ &\quad \times \int_{\mathbb{Z}_p}[x_3 + w_3y_3]_{q^{w_1w_2}}^m d\mu_{-q^{w_1w_2}}(x_3)\Bigg\}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Bigg\{\sum_{k+l+m=n}\binom{n}{k,l,m}[w_2w_3]_q^k[w_1w_3]_q^l[w_1w_2]_q^m q^{w_1w_2w_3(l+m)y_1} \\ &\quad \times q^{w_1w_2w_3my_2}\mathcal{E}_{k,q^{w_2w_3}}^{(l+m+1)}(w_1y_1)\mathcal{E}_{l,q^{w_1w_3}}^{(m+1)}(w_2y_2)\mathcal{E}_{m,q^{w_1w_2}}(w_3y_3)\Bigg\}\frac{t^n}{n!}.\end{aligned} \tag{27}$$

Thus, we get the following theorem.

Theorem 1 *Let w_1, w_2, w_3 be any positive integers, n any nonnegative integer. Then the following expression is invariant under any permutations of w_1, w_2, w_3, so that it gives us six symmetries:*

$$\begin{aligned}&\sum_{k+l+m=n}\binom{n}{k,l,m}[w_2w_3]_q^k[w_1w_3]_q^l[w_1w_2]_q^m q^{w_1w_2w_3(l+m)y_1} \\ &\quad \times q^{w_1w_2w_3my_2}\mathcal{E}_{k,q^{w_2w_3}}^{(l+m+1)}(w_1y_1)\mathcal{E}_{l,q^{w_1w_3}}^{(m+1)}(w_2y_2)\mathcal{E}_{m,q^{w_1w_2}}(w_3y_3) \\ &= \sum_{k+l+m=n}\binom{n}{k,l,m}[w_1w_3]_q^k[w_2w_3]_q^l[w_1w_2]_q^m q^{w_1w_2w_3(l+m)y_1} \\ &\quad \times q^{w_1w_2w_3my_2}\mathcal{E}_{k,q^{w_1w_3}}^{(l+m+1)}(w_2y_1)\mathcal{E}_{l,q^{w_2w_3}}^{(m+1)}(w_1y_2)\mathcal{E}_{m,q^{w_1w_2}}(w_3y_3) \\ &= \sum_{k+l+m=n}\binom{n}{k,l,m}[w_1w_3]_q^k[w_1w_2]_q^l[w_2w_3]_q^m q^{w_1w_2w_3(l+m)y_1} \\ &\quad \times q^{w_1w_2w_3my_2}\mathcal{E}_{k,q^{w_1w_3}}^{(l+m+1)}(w_2y_1)\mathcal{E}_{l,q^{w_1w_2}}^{(m+1)}(w_3y_2)\mathcal{E}_{m,q^{w_2w_3}}(w_1y_3) \\ &= \sum_{k+l+m=n}\binom{n}{k,l,m}[w_2w_3]_q^k[w_1w_2]_q^l[w_1w_3]_q^m q^{w_1w_2w_3(l+m)y_1} \\ &\quad \times q^{w_1w_2w_3my_2}\mathcal{E}_{k,q^{w_2w_3}}^{(l+m+1)}(w_1y_1)\mathcal{E}_{l,q^{w_1w_2}}^{(m+1)}(w_3y_2)\mathcal{E}_{m,q^{w_1w_3}}(w_2y_3) \\ &= \sum_{k+l+m=n}\binom{n}{k,l,m}[w_1w_2]_q^k[w_2w_3]_q^l[w_1w_3]_q^m q^{w_1w_2w_3(l+m)y_1} \\ &\quad \times q^{w_1w_2w_3my_2}\mathcal{E}_{k,q^{w_1w_2}}^{(l+m+1)}(w_3y_1)\mathcal{E}_{l,q^{w_2w_3}}^{(m+1)}(w_1y_2)\mathcal{E}_{m,q^{w_1w_3}}(w_2y_3) \\ &= \sum_{k+l+m=n}\binom{n}{k,l,m}[w_1w_2]_q^k[w_1w_3]_q^l[w_2w_3]_q^m q^{w_1w_2w_3(l+m)y_1} \\ &\quad \times q^{w_1w_2w_3my_2}\mathcal{E}_{k,q^{w_1w_2}}^{(l+m+1)}(w_3y_1)\mathcal{E}_{l,q^{w_1w_3}}^{(m+1)}(w_2y_2)\mathcal{E}_{m,q^{w_2w_3}}(w_1y_3).\end{aligned}$$

We define, for nonnegative integers n, m, w, $K_{n,m}(w|q)$ as

$$K_{n,m}(w|q) = \sum_{i=0}^{w}(-1)^i q^{ni}[i]_q^m. \tag{28}$$

In particular, for $w = 0$, or $m = 0$, we have

$$K_{n,m}(0|q) = \begin{cases}1, & \text{if } m = 0 \\ 0, & \text{if } m > 0,\end{cases} \tag{29}$$

and

$$K_{n,0}(w|q) = [w+1]_{-q^n}. \tag{30}$$

We now apply the formula of (3) as follows :

$$q^n I_{-q}(f_n) + (-1)^{n-1}I_{-q}(f) = [2]_q\sum_{l=0}^{n-1}(-1)^{n-1-l}q^l f(l), \tag{31}$$

with

$$f(x) = e^{[w_1w_2x]_q t}, "q = q^{w_1w_2}", n = w_3.$$

From (35), we have

$$
\begin{aligned}
&(q^{w_1w_2})^{w_3}\int_{\mathbb{Z}_p} e^{[w_1w_2(x+w_3)]_q t}d\mu_{-q^{w_1w_2}}(x)+(-1)^{w_3-1}\int_{\mathbb{Z}_p} e^{[w_1w_2x]_q t}d\mu_{-q^{w_1w_2}}(x)\\
&=[2]_{q^{w_1w_2}}\sum_{l=0}^{w_3-1}(-1)^{w_3-1-l}q^{w_1w_2l}e^{[w_1w_2l]_q t}\\
&=(-1)^{w_3-1}[2]_{q^{w_1w_2}}\sum_{m=0}^{\infty}K_{1,m}(w_3-1|q^{w_1w_2})\frac{\left([w_1w_2]_q t\right)^m}{m!}\\
&=\sum_{m=0}^{\infty}\Big\{[w_1w_2]_q^m\\
&\quad\times\int_{\mathbb{Z}_p}\left(q^{w_1w_2w_3}[x_1+w_3]^m_{q^{w_1w_2}}+(-1)^{w_3-1}[x]^m_{q^{w_1w_2}}\right)d\mu_{-q^{w_1w_2}}(x)\Big\}\frac{t^m}{m!}.
\end{aligned}
\tag{32}
$$

Thus, by (32), we obtain the following lemma.

Lemma 2 *Let w_1, w_2 be any positive integers.*

(i) *For $w_3 \equiv 1$ (mod 2), we have*

$$
\begin{aligned}
&q^{w_1w_2w_3}\int_{\mathbb{Z}_p} e^{[w_1w_2(x+w_3)]_q t}d\mu_{-q^{w_1w_2}}(x)+\int_{\mathbb{Z}_p} e^{[w_1w_2x]_q t}d\mu_{-q^{w_1w_2}}(x)\\
&=\sum_{m=0}^{\infty}\left\{[w_1w_2]_q^m\int_{\mathbb{Z}_p}\left(q^{w_1w_2w_3}[x+w_3]^m_{q^{w_1w_2}}+[x]^m_{q^{w_1w_2}}\right)d\mu_{-q^{w_1w_2}}(x)\right\}\frac{t^m}{m!}\\
&=[2]_{q^{w_1w_2}}\sum_{i=0}^{w_3-1}(-1)^i q^{w_1w_2i}e^{[w_1w_2i]_q t}\\
&=[2]_{q^{w_1w_2}}\sum_{m=0}^{\infty}K_{1,m}(w_3-1|q^{w_1w_2})\frac{\left([w_1w_2]_q t\right)^m}{m!}.
\end{aligned}
$$

(ii) *For $w_3 \equiv 0$ (mod 2), we have*

$$
\begin{aligned}
&q^{w_1w_2w_3}\int_{\mathbb{Z}_p} e^{[w_1w_2(x+w_3)]_q t}d\mu_{-q^{w_1w_2}}(x)-\int_{\mathbb{Z}_p} e^{[w_1w_2x]_q t}d\mu_{-q^{w_1w_2}}(x)\\
&=\sum_{m=0}^{\infty}\left\{[w_1w_2]_q^m\int_{\mathbb{Z}_p}\left(q^{w_1w_2w_3}[x+w_3]^m_{q^{w_1w_2}}-[x]^m_{q^{w_1w_2}}\right)d\mu_{-q^{w_1w_2}}(x)\right\}\frac{t^m}{m!}\\
&=-[2]_{q^{w_1w_2}}\sum_{i=0}^{w_3-1}(-1)^i q^{w_1w_2i}e^{[w_1w_2i]_q t}\\
&=-[2]_{q^{w_1w_2}}\sum_{m=0}^{\infty}K_{1,m}(w_3-1|q^{w_1w_2})\frac{\left([w_1w_2]_q t\right)^m}{m!}.
\end{aligned}
$$

Consider the following sum of triple integrals.

$$
\begin{aligned}
I_1&=q^{w_1w_2w_3}\int_{\mathbb{Z}_p^3} e^{[w_2w_3x_1+w_1w_3x_2+w_1w_2x_3+w_1w_2w_3(y_1+y_2+1)]_q t}\\
&\quad\times d\mu_{-q^{w_2w_3}}(x_1)d\mu_{-q^{w_1w_3}}(x_2)d\mu_{-q^{w_1w_2}}(x_3)\\
&\quad+\int_{\mathbb{Z}_p^3} e^{[w_2w_3x_1+w_1w_3x_2+w_1w_2x_3+w_1w_2w_3(y_1+y_2)]_q t}\\
&\quad\times d\mu_{-q^{w_2w_3}}(x_1)d\mu_{-q^{w_1w_3}}(x_2)d\mu_{-q^{w_1w_2}}(x_3),
\end{aligned}
\tag{33}
$$

which is obviously invariant under any permutations of w_1, w_2, w_3.

For simplicity, we put

$$
a=a(x_1)=q^{w_2w_3(x_1+w_1y_1)},\quad b=b(x_2)=q^{w_1w_3(x_2+w_2y_2)}.
\tag{34}
$$

Then, from (33), we note that

$$
\begin{aligned}
I_1&=\sum_{k,l=0}^{\infty}[w_2w_3]_q^k[w_1w_3]_q^l\frac{t^{k+l}}{k!l!}\int_{\mathbb{Z}_p^2}a^l[x_1+w_1y_1]^k_{q^{w_2w_3}}[x_2+w_2y_2]^l_{q^{w_1w_3}}\\
&\quad\times\left\{\sum_{m=0}^{\infty}\frac{[w_1w_2]_q^m(abt)^m}{m!}\right.\\
&\quad\times\left.\int_{\mathbb{Z}_p}\left(q^{w_1w_2w_3}[x_3+w_3]^m_{q^{w_1w_2}}+[x_3]^m_{q^{w_1w_2}}\right)d\mu_{-q^{w_1w_2}}(x_3)\right\}\\
&\quad\times d\mu_{-q^{w_2w_3}}(x_1)d\mu_{-q^{w_1w_3}}(x_2).
\end{aligned}
\tag{35}
$$

Assume now that $w_3 \equiv 1$ (mod 2). Then, by (i) of Lemma 2, we get

$$
\begin{aligned}
I_1&=[2]_{q^{w_1w_2}}\sum_{k,l,m=0}^{\infty}[w_2w_3]_q^k[w_1w_3]_q^l[w_1w_2]_q^m\frac{t^{k+l+m}}{k!l!m!}\\
&\quad\times K_{1,m}(w_3-1|q^{w_1w_2})\int_{\mathbb{Z}_p}a^{l+m}[x_1+w_1y_1]^k_{q^{w_2w_3}}d\mu_{-q^{w_2w_3}}(x_1)\\
&\quad\times\int_{\mathbb{Z}_p}b^m[x_2+w_2y_2]^l_{q^{w_1w_3}}d\mu_{-q^{w_1w_3}}(x_2).
\end{aligned}
\tag{36}
$$

Recovering $a=q^{w_2w_3(x_1+w_1y_1)}$, $b=q^{w_1w_3(x_2+w_2y_2)}$, I_1 can be rewritten as

$$
\begin{aligned}
I_1&=[2]_{q^{w_1w_2}}\sum_{k,l,m=0}^{\infty}[w_2w_3]_q^k[w_1w_3]_q^l[w_1w_2]_q^m\frac{t^{k+l+m}}{k!l!m!}\\
&\quad\times K_{1,m}(w_3-1|q^{w_1w_2})q^{w_1w_2w_3(l+m)y_1}q^{w_1w_2w_3my_2}\\
&\quad\times\int_{\mathbb{Z}_p}q^{w_2w_3(l+m)x_1}[x_1+w_1y_1]^k_{q^{w_2w_3}}d\mu_{-q^{w_2w_3}}(x_1)\\
&\quad\times\int_{\mathbb{Z}_p}q^{w_1w_3mx_2}[x_2+w_2y_2]^l_{q^{w_1w_3}}d\mu_{-q^{w_1w_3}}(x_2)\\
&=\sum_{n=0}^{\infty}\left\{[2]_{q^{w_1w_2}}\sum_{k+l+m=n}\binom{n}{k,l,m}[w_2w_3]_q^k[w_1w_3]_q^l[w_1w_2]_q^m\right.\\
&\quad\times K_{1,m}(w_3-1|q^{w_1w_2})q^{w_1w_2w_3(l+m)y_1}q^{w_1w_2w_3my_2}\\
&\quad\times\left.\mathcal{E}^{(l+m+1)}_{k,q^{w_2w_3}}(w_1y_1)\mathcal{E}^{(m+1)}_{l,q^{w_1w_3}}(w_2y_2)\right\}\frac{t^n}{n!}.
\end{aligned}
\tag{37}
$$

As the expression in (37) is invariant under any permutations of w_1, w_2, w_3 and it is equal to (37) provided that $w_3 \equiv 1$ (mod 2), we see that the expression in the curly bracket of (37) is invariant under any permutations of w_1, w_2, w_3, when $w_1 \equiv w_2 \equiv w_3 \equiv 1$ (mod 2). Instead of the sum of triple integrals in (33), we now consider their difference, namely,

$$q^{w_1w_2w_3}\int_{\mathbb{Z}_p^3} e^{[w_2w_3x_1+w_1w_3x_2+w_1w_2x_3+w_1w_2w_3(y_1+y_2+1)]_q t} \times d\mu_{-q^{w_2w_3}}(x_1)d\mu_{-q^{w_1w_3}}(x_2)d\mu_{-q^{w_1w_2}}(x_3) - \int_{\mathbb{Z}_p^3} e^{[w_2w_3x_1+w_1w_3x_2+w_1w_2x_3+w_1w_2w_3(y_1+y_2)]_q t} \times d\mu_{-q^{w_2w_3}}(x_1)d\mu_{-q^{w_1w_3}}(x_2)d\mu_{-q^{w_1w_2}}(x_3), \tag{38}$$

which is invariant under any permutations of w_1, w_2, w_3. Proceeding analogously to the above and using (ii) of Lemma 2, we see that (38) is equal to the negative of the expression in (37), provided that $w_3 \equiv 0 \pmod 2$. Thus, we see that the expression in curly bracket of (37) is invariant under any permutations of w_1, w_2, w_3, when $w_1 \equiv w_2 \equiv w_3 \equiv 0 \pmod 2$.

Thus, we obtain the following theorem.

Theorem 3 *Let w_1, w_2, w_3 be positive integers satisfying either $w_1 \equiv w_2 \equiv w_3 \equiv 1$ (mod 2) or $w_1 \equiv w_2 \equiv w_3 \equiv 0$ (mod 2). Then, for any nonnegative integer n, the following expressions*

$$[2]_{q^{w_{\sigma(1)\sigma(2)}}} \sum_{k+l+m=n} \binom{n}{k,l,m} \mathcal{E}^{(l+m+1)}_{k,q^{w_{\sigma(2)}w_{\sigma(3)}}}(w_{\sigma(1)}y_1) \times \mathcal{E}^{(m+1)}_{l,q^{w_{\sigma(1)}w_{\sigma(3)}}}(w_{\sigma(2)}y_2) K_{1,m}(w_{\sigma(3)}-1|q^{w_{\sigma(1)}w_{\sigma(2)}}) \times q^{w_1w_2w_3(l+m)y_1} q^{w_1w_2w_3my_2}[w_{\sigma(2)}w_{\sigma(3)}]_q^k \times [w_{\sigma(1)}w_{\sigma(3)}]_q^l [w_{\sigma(1)}w_{\sigma(2)}]_q^m$$

are all the same for any $\sigma \in S_3$.

Remark 1 We can obtain many interesting identities by letting $w_3 = 1$ or $w_2 = w_3 = 1$, in view of (29). However, writing those down requires much space and so we omit it.

With the same

$$a = q^{w_2w_3(x_1+w_1y_1)}, \quad b = q^{w_1w_3(x_2+w_2y_2)},$$

I_1 can be written as

$$I_1 = \int_{\mathbb{Z}_p^2} e^{[w_2w_3]_q[x_1+w_1y_1]_{q^{w_2w_3}} t} e^{[w_1w_3]_q[x_2+w_2y_2]_{q^{w_1w_3}} at} \times d\mu_{-q^{w_2w_3}}(x_1)d\mu_{-q^{w_1w_3}}(x_2) \times \int_{\mathbb{Z}_p}\left(q^{w_1w_2w_3}e^{[w_1w_2(x_3+w_3)]_q abt} + e^{[w_1w_2x_3]_q abt}\right)d\mu_{-q^{w_1w_2}}(x_3). \tag{39}$$

Let $w_3 \equiv 1 \pmod 2$. Then, from (i) of Lemma 2, the inner integral in (39) is

$$[2]_{q^{w_1w_2}} \sum_{i=0}^{w_3-1}(-1)^i q^{w_1w_2i} e^{[w_1w_2i]_q abt}. \tag{40}$$

So (40) is equal to

$$\begin{aligned} I_1 &= [2]_{q^{w_1w_2}} \sum_{i=0}^{w_3-1}(-1)^i q^{w_1w_2i} \int_{\mathbb{Z}_p^2} e^{[w_2w_3]_q[x_1+w_1y_1]_{q^{w_2w_3}} t} \times e^{[w_1w_3]_q\left[x_2+w_2y_2+\frac{w_2}{w_3}i\right]_{q^{w_1w_3}} at} d\mu_{-q^{w_2w_3}}(x_1)d\mu_{-q^{w_1w_3}}(x_2) \\ &= [2]_{q^{w_1w_2}} \sum_{i=0}^{w_3-1}(-1)^i q^{w_1w_2i} \sum_{k,l=0} \frac{t^{k+l}}{k!l!} q^{w_1w_2w_3ly_1}[w_2w_3]_q^k[w_1w_3]_q^l \\ &\times \int_{\mathbb{Z}_p} q^{w_2w_3lx_1}[x_1+w_1y_1]^k_{q^{w_2w_3}} d\mu_{-q^{w_2w_3}}(x_1) \\ &\times \int_{\mathbb{Z}_p}\left[x_2+w_2y_2+\frac{w_2}{w_3}i\right]^l_{q^{w_1w_3}} d\mu_{-q^{w_1w_3}}(x_2) \\ &= \sum_{n=0}^{\infty}\left\{[2]_{q^{w_1w_2}} \sum_{k=0}^{n}\binom{n}{k} \mathcal{E}^{(n-k+1)}_{k,q^{w_2w_3}}(w_1y_1) q^{w_1w_2w_3(n-k)y_1}[w_2w_3]_q^k[w_1w_3]_q^{n-k} \right. \\ &\left. \times \sum_{i=0}^{w_3-1}(-1)^i q^{w_1w_2i}\mathcal{E}_{l,q^{w_1w_3}}\left(w_2y_2+\frac{w_2}{w_3}i\right)\right\}\frac{t^n}{n!}. \end{aligned} \tag{41}$$

Recalling that I_1 is invariant under any permutations of w_1, w_2, w_3 and it is equal to (41) for $w_3 \equiv 1 \pmod 2$, we see that the expression in the curly bracket of (41) is invariant under any permutations of w_1, w_2, w_3, when $w_1 \equiv w_2 \equiv w_3 \equiv 1 \pmod 2$. Also, starting from (39), using (ii) of Lemma 2, and proceeding analogous to the above, we see that the expression in the curly bracket of (41) is also invariant under any permutations of w_1, w_2, w_3, when $w_1 \equiv w_2 \equiv w_3 \equiv 0 \pmod 2$. Thus, we have the following theorem.

Theorem 4 *Let w_1, w_2, w_3 be positive integers satisfying either $w_1 \equiv w_2 \equiv w_3 \equiv 1$ (mod 2) or $w_1 \equiv w_2 \equiv w_3 \equiv 0$ (mod 2). Then, for any nonnegative integer n, the following expressions*

$$[2]_{q^{w_{\sigma(1)}w_{\sigma(2)}}} \sum_{k=0}^{n}\binom{n}{k} \mathcal{E}^{(n-k+1)}_{k,q^{w_{\sigma(2)}w_{\sigma(3)}}}(w_{\sigma(1)}y_1) q^{w_1w_2w_3(n-k)y_1}[w_{\sigma(2)}w_{\sigma(3)}]_q^k \times [w_{\sigma(1)}w_{\sigma(3)}]_q^{n-k} \sum_{i=0}^{w_{\sigma(3)}-1}(-1)^i q^{w_{\sigma(1)}w_{\sigma(2)}i}\mathcal{E}_{l,q^{w_{\sigma(1)}w_{\sigma(3)}}}\left(w_{\sigma(2)}y_2+\frac{w_{\sigma(2)}}{w_{\sigma(3)}}i\right)$$

are all the same for any $\sigma \in S_3$.

Remark 2 In view of (29), by specializing $w_3 = 1$ or $w_2 = w_3 = 1$, we can obtain many interesting identities. However, we will omit those, as this requires much space.

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