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ORIGINAL RESEARCH

Common fixed points of f-contraction mappings in complex valued metric spaces

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Abstract In this paper, some common fixed point theorems have been established for two Banach pairs of mappings with f-contraction defined on a complex valued metric space satisfying contractive condition involving product. Some consequences which are associated with properties P and Q are also obtained.

Keywords Fixed point · Complex valued metric space · Periodic point · f-contraction

Mathematics Subject Classification 47H10 · 54H25

Introduction

Fixed point theory which has important applications such as game theory, military, economics, statistics and medicine is one of the most famous theory in mathematics. It is well known that the Banach's contraction principle is a fundamental result in fixed point theory. There are many generalizations of this principle. A new generalization of contraction mapping has been introduced and called f-contraction mappings on metric spaces which are related with another function by Beiranvand [2]. In 2009, Morales and Rojas [7, 8] have extended f-contraction mappings to cone metric spaces by proving fixed point theorems for f-Kannan, f-Chatterjea, f-Zamfirescu, f-weakly contraction mappings. Besides, Sintunavarat and Kumam [15, 16] use different types of f-weak contractions to generalize some contractions which are existing in the literature. Subrahmanyam has initiated the concept of Banach operator of type k in complete metric spaces. Afterwards, Chen and Li [4] have introduced the notion of Banach operator pairs as a new class of non-commuting maps and have proved various best approximation results using some common fixed point theorems for f-nonexpansive mappings.

Many authors have generalized and established the notion of a metric spaces in the recent past such as rectangular metric spaces, semi-metric spaces, quasi-metric spaces, quasi-semi metric spaces, pseudo-metric spaces, 2-metric spaces, D-metric spaces, G-metric spaces, K-metric spaces, cone metric spaces and etc. Recently, Azam et al. [1]. have introduced the complex valued metric spaces which is a generalization of the metric space and also have obtained some fixed point results for a pair of mappings for contraction condition satisfying rational expressions which are not meaningful in cone metric spaces. Therefore, many results of analysis can not be generalized to cone metric spaces. Later, one can study the progresses of a host of results of analysis involving divisions in the framework of complex valued metric spaces. There exist various paper on complex valued metric spaces such as [1, 3, 9, 12, 18] and [13, 14].

The purpose of this paper is to prove common fixed point theorems for two Banach pairs of mappings satisfying contractive condition including product in complex valued metric space using f- contraction. Also, we obtain some

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consequences related with P and Q properties which are defined by Jeong and Rhoades [6].

Basic facts and definitions

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

$z_1 \preceq z_2$ if and only if $\mathrm{Re}(z_1) \leq \mathrm{Re}(z_2),\ \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$.

It follows that

$z_1 \preceq z_2$

if one of the following conditions is satisfied:

i. $\mathrm{Re}\ (z_1) = \mathrm{Re}\ (z_2),\ \mathrm{Im}\ (z_1) < \mathrm{Im}\ (z_2)$,
ii. $\mathrm{Re}\ (z_1) < \mathrm{Re}\ (z_2),\ \mathrm{Im}\ (z_1) = \mathrm{Im}\ (z_2)$,
iii. $\mathrm{Re}\ (z_1) < \mathrm{Re}\ (z_2),\ \mathrm{Im}\ (z_1) < \mathrm{Im}\ (z_2)$,
iv. $\mathrm{Re}\ (z_1) = \mathrm{Re}\ (z_2),\ \mathrm{Im}\ (z_1) = \mathrm{Im}\ (z_2)$.

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \preceq z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_2|,$$
$$z_1 \preceq z_2,\ z_2 \prec z_3 \Longrightarrow z_1 \prec z_3.$$

Definition [18] Let $z_1, z_2 \in \mathbb{C}$ and the 'max' function for the partial order relation $\preceq$ is defined on $\mathbb{C}$ by:

i. $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \preceq z_2$;
ii. $z_1 \preceq \max\{z_2, z_3\} \Rightarrow z_1 \preceq z_2$ or $z_1 \preceq z_3$;
iii. $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \preceq z_2$ or $|z_1| \leq |z_2|$.

Using the previous definition, we have the following lemma:

Lemma 2.1 [18] Let $z_1, z_2, z_3, \ldots \in \mathbb{C}$ and the partial order relation $\preceq$ is defined on $\mathbb{C}$. Then, the following statements are obvious:

i. If $z_1 \preceq \max\{z_2, z_3\}$, then $z_1 \preceq z_2$ if $z_3 \preceq z_2$;
ii. If $z_1 \preceq \max\{z_2, z_3, z_4\}$, then $z_1 \preceq z_2$, if $\max\{z_3, z_4\} \preceq z_2$;
iii. If $z_1 \preceq \max\{z_2, z_3, z_4, z_5\}$, then $z_1 \preceq z_2$ if $\max\{z_3, z_4, z_5\} \preceq z_2$ and so on.

Now, we give the definition of complex valued metric space which has been introduced by Azam et al. [1].

Definition Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies:

C_1. $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
C_2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
C_3. $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then, d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Let (X, d) be a complex valued metric space and $\{x_n\}_{n\geq 1}$ be a sequence in X and $x \in X$. We say that the sequence $\{x_n\}_{n\geq 1}$ converges to x if for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$. The sequence $\{x_n\}_{n\geq 1}$ is Cauchy sequence if for every $c \in \mathbb{C}$, with $\theta \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$, $d(x_n, x_m) \prec c$. The metric space (X, d) is a complete complex valued metric space, if every Cauchy sequence is convergent.

Now, we give some lemmas which we require to prove the main results.

Lemma 2.2 [1] Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 2.3 [1] Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ is Cauchy sequence if and only if $|d(x_n, x_m)| \to 0$ as $m, n \to \infty$.

Here, some essential notions are given about f-contraction.

Definition [2] Let (X, d) be a metric space and $T, f : X \to X$ be two functions. A mapping T is said to be f-contraction if there exists $k \in [0, 1)$ such that

$$d(fTx, fTy) \leq kd(fx, fy)$$

for all $x, y \in X$.

Example 1 [10] Let $X = [0, \infty)$ be with the usual metric. Let define two mappings as $T, f : X \to X$

$$Tx = \beta x, \quad \beta > 1$$
$$fx = \frac{\alpha}{x^2}, \quad \alpha \in \mathbb{R}$$

It is clear that, T is not a contraction but it is $f-$contraction since,

$$d(fTx, fTy) = \left|\frac{\alpha}{\beta^2 x^2} - \frac{\alpha}{\beta^2 y^2}\right| = \frac{1}{\beta^2}|fx - fy|$$

Definition [2] Let (X, d) be a metric space. Then,

i. A mapping $T : X \to X$ is said to be sequentially convergent, if the sequence $\{y_n\}$ in X is convergent whenever $\{Ty_n\}$ is convergent.
ii. $T : X \to X$ is said to be sub-sequentially convergent, if the sequence $\{y_n\}$ has a convergent subsequence whenever $\{Ty_n\}$ is convergent.

Definition [17] Let f be a self-mapping of a normed space X. Then, f is called a Banach operator of type k if

$$\|f^2x - fx\| \le k\|fx - x\|$$

for some $k \ge 0$ and for all $x \in X$.

Definition [4] Let f and T be self-mappings of a non-empty subset M of a normed linear space X. Then, (f, T) is a Banach operator pair, if any one of the following conditions is satisfied:

i. $f[F(T)] \subseteq F(T)$,
ii. $Tfx = fx$ for each $x \in F(T)$,
iii. $fTx = Tfx$ for each $x \in F(T)$,
iv. $\|fTx - Tx\| \le k\|Tx - x\|$ for some $k \ge 0$.

Now, we give an example which illustrates Banach operator pairs.

Example 2 Let $X = \mathbb{C}$ be the set of complex numbers and define $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = e^{it}|z_1 - z_2|,$$

where $t \in [0, \frac{\pi}{2}]$ and for all $z_1, z_2 \in X$. Then, (X, d) is complex valued metric space. Suppose that f, S and T be self-mappings of (X, d) as follows:

$$fz = 2z - 1, \quad Sz = 4z - 3 \quad \text{and} \quad \mathrm{Tz} = 3\mathrm{z} - 2.$$

Note that $F(T) = \{1\}$, then we have the following conditions:

i. $f[F(T)] = f(\{1\}) = \{1\} \subseteq F(T)$,
ii. $Tfz = fz$ for $z = 1 \in F(T)$,
iii. $fTz = Tfz$ where $z = 1 \in F(T)$,
iv. for $k = \frac{3}{2}$,

$$d(fTz, Tz) = d(6z - 5, 3z - 2) \le \frac{3}{2}d(3z - 2, z) = \frac{3}{2}d(Tz, z).$$

Hence, (f, T) is a Banach pair. And also (f, S) is said to be a Banach pair with $k = \frac{4}{3}$. Moreover, the unique common fixed point of f, S and T is 1 in X.

However, we give the following definitions which are P and Q properties.

Definition [6] If a map f satisfies $F(f) = F(f^n)$ for each $n \in \mathbb{N}$, then it is said to have property P, where $F(f)$ is the set of fixed points of the mapping f. If $F(f^n) \cap F(g^n) = F(f) \cap F(g)$ for each $n \in \mathbb{N}$, then we say that f and g have property Q.

It is obvious that, if f is a map which has a fixed point z, then z is also fixed point of $F(f^n)$ for every natural number n. However, the converse is false and the related examples were given in [6].

Some fixed point and periodic point results

Theorem 3.1 Let f, S and T be continuous self-mappings of a complete complex valued metric space (X, d). Assume that the mapping f is an injective and sub-sequentially convergent. If the mappings f, S and T satisfy

$$\begin{aligned} [d(fSx, fTSy)]^2 \preceq{}& \alpha \max\{d(fx, fSx)d(fSy, fTSy), \\ & d(fx, fTSy)d(fSy, fSx)\} \\ &+ \beta \max\{d(fx, fSx)d(fSy, fSx), \\ & d(fx, fTSy)d(fSy, fTSy)\} \end{aligned} \tag{1}$$

for all $x, y \in X$ where $\alpha, \beta \in [0, 1)$ with $\alpha + 2\beta < 1$, then

i. T and S have a unique common fixed point,
ii. if (f, T) and (f, S) are Banach pairs, then f, T and S have a unique common fixed point in X,
iii. T and S have property Q.

Proof (i)Take $x_0 \in X$ as an arbitrary element and define the sequences

$$x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1},$$

for all $n \ge 0$. Then,

$$[d(fx_{2n+1}, fx_{2n})]^2 = [d(fSx_{2n}, fTx_{2n-1})]^2 = [d(fSx_{2n}, fTSx_{2n-2})]^2$$

using (1) and triangle inequality

$$\begin{aligned} [d(fx_{2n+1}, fx_{2n})]^2 &= [d(fSx_{2n}, fTSx_{2n-2})]^2 \\ &\preceq \alpha \max\{d(fx_{2n}, fSx_{2n})d(fSx_{2n-2}, fTSx_{2n-2}), \\ &\qquad d(fx_{2n}, fTSx_{2n-2})d(fSx_{2n-2}, fSx_{2n})\} \\ &\quad + \beta \max\{d(fx_{2n}, fSx_{2n})d(fSx_{2n-2}, fSx_{2n}), \\ &\qquad d(fx_{2n}, fTSx_{2n-2})d(fSx_{2n-2}, fTSx_{2n-2})\} \\ &\preceq \alpha \max\{d(fx_{2n}, fx_{2n+1})d(fx_{2n-1}, fx_{2n}), \\ &\qquad d(fx_{2n}, fx_{2n})d(fx_{2n-1}, fx_{2n+1})\} \\ &\quad + \beta \max\{d(fx_{2n}, fx_{2n+1})d(fx_{2n-1}, fx_{2n+1}), \\ &\qquad d(fx_{2n}, fx_{2n})d(fx_{2n-1}, fx_{2n})\} \end{aligned}$$

which implies that

$$\begin{aligned} [|d(fx_{2n+1}, fx_{2n})|]^2 \le{}& \alpha|d(fx_{2n}, fx_{2n+1})||d(fx_{2n-1}, fx_{2n})| \\ &+ \beta|d(fx_{2n}, fx_{2n+1})||d(fx_{2n-1}, fx_{2n+1})| \end{aligned}$$

and

$$|d(fx_{2n+1}, fx_{2n})|^2 \le [\alpha|d(fx_{2n-1}, fx_{2n})| + \beta|d(fx_{2n-1}, fx_{2n+1})|]|d(fx_{2n}, fx_{2n+1})|.$$

We deduce that

$$|d(fx_{2n+1}, fx_{2n})| \le \frac{\alpha + \beta}{1 - \beta}|d(fx_{2n}, fx_{2n-1})|,$$

where $\lambda = \frac{\alpha+\beta}{1-\beta} < 1$. Continuing in this way, we have

$$|d(fx_{n+1}, fx_n)| \le \lambda|d(fx_n, fx_{n-1})| \le \ldots \le \lambda^n|d(fx_1, fx_0)|$$

for all $n \geq 0$.

So for any $n > m$, we obtain

$$\begin{aligned}|d(fx_n,fx_m)| &\leq |d(fx_n,fx_{n-1})| + |d(fx_{n-1},fx_{n-2})| + \ldots + |d(fx_{m+1},fx_m)| \\ &\leq \left[\lambda^{n-1} + \lambda^{n-2} + \ldots + \lambda^m\right] |d(fx_1,fx_0)| \\ &\leq \frac{\lambda^m}{1-\lambda} |d(fx_1,fx_0)|\end{aligned}$$

and so,

$$|d(fx_n,fx_m)| \leq \frac{\lambda^m}{1-\lambda} |d(fx_1,fx_0)| \to 0, \quad \text{as} \quad \mathrm{m,n} \to \infty.$$

This implies that $\{fx_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} fx_n = z$. Since f is sub-sequentially convergent, $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $\lim_{n\to\infty} x_m = u$. As f is continuous,

$$\lim_{n\to\infty} fx_m = fu.$$

By the uniqueness of the limit, $z = fu$. Since T and S are continuous, $\lim_{m\to\infty} Tx_m = Tu$ and $\lim_{m\to\infty} Sx_m = Su$. Again since f is continuous,

$$\lim_{m\to\infty} fTx_m = fTu \quad \text{and} \quad \lim_{m\to\infty} \mathrm{fSx_m} = \mathrm{fSu}.$$

Now, we prove $fz = fSz$. Assume that m is even, then

$$\lim_{n\to\infty} fSx_{2n} = fSu.$$

By (1), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}[d(fSz,fz)]^2 &\leq [d(fSz,fx_{2n+2}) + d(fx_{2n+2},fz)]^2 \\ &= [d(fSz,fTSx_{2n}) + d(fx_{2n+2},fz)]^2 \\ &\leq [\alpha \max\{d(fz,fSz)d(fSx_{2n},fTSx_{2n}), \\ &\qquad d(fz,fTSx_{2n})d(fSx_{2n},fSz)\} \\ &\quad + \beta \max\{d(fz,fSz)d(fSx_{2n},fSz), \\ &\qquad d(fz,fTSx_{2n})d(fSx_{2n},fTSx_{2n})\} + d(fx_{2n+2},fz)]^2 \\ &= [\alpha \max\{d(fz,fSz)d(fx_{2n+1},fx_{2n+2}), \\ &\qquad d(fz,fx_{2n+2})d(fx_{2n+1},fSz)\} \\ &\quad + \beta \max\{d(fz,fSz)d(fx_{2n+1},fSz), \\ &\qquad d(fz,fx_{2n+2})d(fx_{2n+1},fx_{2n+2})\} + d(fx_{2n+2},fz)]^2.\end{aligned}$$

Also, for every $n \in \mathbb{N}$, we can write

$$\begin{aligned}\left|[d(fSz,fz)]^2\right| &\leq |[\alpha \max\{d(fz,fSz)d(fx_{2n+1},fx_{2n+2}), \\ &\qquad d(fz,fx_{2n+2})d(fx_{2n+1},fSz)\} \\ &\quad + \beta \max\{d(fz,fSz)d(fx_{2n+1},fSz), \\ &\qquad d(fz,fx_{2n+2})d(fx_{2n+1},fx_{2n+1})\} + d(fx_{2n+2},fz)]^2|.\end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned}[|d(fSz,fz)|]^2 &\leq [\alpha \max\{|d(fz,fSz)||d(fz,fz)|, |d(fz,fz)||d(fz,fSz)|\} \\ &\quad + \beta \max\{|d(fz,fSz)||d(fz,fSz)|, |d(fz,fz)||d(fz,fz)|\} \\ &\quad + |d(fz,fz)|]^2,\end{aligned}$$

which amounts to say that

$$[|d(fSz,fz)|]^2 \leq \beta[|d(fSz,fz)|]^2.$$

Because $(1-\sqrt{\beta}) < 1$, $|d(fz,fSz)| = 0$, i.e., $fz = fSz$. As f is injective, $z = Sz$. Thus, z is the fixed point of S.

And to show $fz = fTz$ we suppose that m is odd, then

$$\lim_{n\to\infty} fTx_{2n+1} = fTz$$

Now, using (1) and triangle inequality, we get

$$\begin{aligned}[d(fz,fTz)]^2 &= [d(fSz,fTSz)]^2 \\ &\leq [\alpha \max\{d(fz,fSz)d(fSz,fTSz), \\ &\qquad d(fz,fTSz)d(fSz,fSz)\} \\ &\quad + \beta \max\{d(fz,fSz)d(fSz,fSz), \\ &\qquad d(fz,fTSz)d(fSz,fTSz)\}]^2 \\ &\leq [\alpha \max\{d(fz,fz)d(fz,fTz), d(fz,fTz)d(fz,fz)\} \\ &\quad + \beta \max\{d(fz,fz)d(fz,fz), d(fz,fTz)d(fz,fTz)\}]^2.\end{aligned}$$

Since $(1-\sqrt{\beta}) < 1$, we obtain $d(fz,fTz) = 0$. Hence, $fTz = fz$. As f is injective, $z = Tz$, i.e., z is the fixed point of S, too.

Now, we demonstrate that S and T have a unique common fixed point. For this, assume that $w \in X$ is an another common fixed point of S and T.

$$\begin{aligned}[d(fz,fw)]^2 &= [d(fSz,fTSw)]^2 \\ &\preceq [\alpha \max\{d(fz,fSz)d(fSw,fTSw), d(fz,fTSw)d(fSw,fSz)\} \\ &\quad + \beta \max\{d(fz,fSz)d(fSw,fSz), \\ &\qquad d(fz,fTSw)d(fSw,fTSw)\}]^2\end{aligned}$$

Thus, we have $[d(fz,fw)]^2 \leq \alpha[d(fz,fw)]^2$. Since $(1-\sqrt{\alpha}) < 1$, $d(fz,fw) = 0$ consequently provides that $fz = fw$. We know that f is injective, $z = w$ is the unique common fixed point of T and S.

(ii) Since we assume that (f,S) and (f,T) are Banach pairs; f, S and f, T commute at the fixed point of S and T, respectively. This implies that $fSz = Sfz$ for $z \in F(S)$. So, $fz = Sfz$ which gives that fz is another fixed point of S. It is true for T, too. By the uniqueness of fixed point of S, $fz = z$. Hence, $z = fz = Sz = Tz$, z is unique common fixed point of f, S and T in X.

(iii) By (i), S and T have a common fixed point in X. Define $fU = fTS$. Then, by (1)

$$\begin{aligned}[d(fSx,fUy)]^2 &\leq \alpha \max\{d(fx,fSx)d(fSy,fUy), d(fx,fUy)d(fSy,fSx)\} \\ &\quad + \beta \max\{d(fx,fSx)d(fSy,fSx), d(fx,fUy)d(fSy,fUy)\}\end{aligned}$$

From (i), we know that

$$Sz = z = Tz = TSz = Uz.$$

Let $z \in F(S^n) \cap F(T^n)$. Then, $z \in F(U^n)$

$$[d(fSz,fz)]^2 = \left[d\left(fS^{n+1}z,fU^nz\right)\right]^2 = \left[d\left(fS(S^nz),fU\left(U^{n-1}z\right)\right)\right]^2 \\ \leq \alpha \max\left\{d\left(fS^nz,fS^{n+1}z\right)d\left(fSU^{n-1}z,fU^nz\right), \right. \\ \left. d(fS^nz,fU^nz)d\left(fSU^{n-1}z,fS^{n+1}z\right)\right\} \\ + \beta \max\left\{d\left(fS^nz,fS^{n+1}z\right)d\left(fSU^{n-1}z,fS^{n+1}z\right), \right. \\ \left. d(fS^nz,fU^nz)d\left(fSU^{n-1}z,fU^nz\right)\right\},$$

which implies that

$$\left|[d(fSz,z)]^2\right| \leq \left|\alpha \max\left\{d(fz,fSz)d\left(fSU^{n-1}z,fz\right), \right.\right. \\ \left. d(fz,fz)d\left(fSU^{n-1}z,fSz\right)\right\} \\ + \beta \max\left\{d(fz,fSz)d\left(fSU^{n-1}z,fSz\right), \right. \\ \left.\left. d(fz,fz)d\left(fSU^{n-1}z,fz\right)\right\}\right|$$

and

$$[|d(fSz,z)|]^2 \leq \alpha \max\left\{|d(fz,fSz)|\left|d\left(fSU^{n-1}z,fz\right)\right|, \right. \\ \left. |d(fz,fz)|\left|d\left(fSU^{n-1}z,fSz\right)\right|\right\} \\ + \beta \max\left\{|d(fSz,fSz)|\left|d\left(fSU^{n-1}z,fSz\right)\right|, \right. \\ \left. |d(fSz,fUz)|\left|d\left(fSU^{n-1}z,fz\right)\right|\right\}$$

and then

$$[|d(fSz,z)|]^2 \leq \left[\alpha\left|d\left(fSU^{n-1}z,fz\right)\right| + \beta\left|d\left(fSU^{n-1}z,fSz\right)\right|\right]|d(fSz,fz)|.$$

Then, the following step is obviously recognized

$$|d(fSz,z)| \leq \frac{\alpha+\beta}{1-\beta}\left|d\left(fSU^{n-1}z,fz\right)\right|$$
$$|d(fSz,z)| = |d(fSU^nz,fz)| \leq \lambda\left|d\left(fSU^{n-1}z,fz\right)\right|$$

where $\lambda = \frac{\alpha+\beta}{1-\beta} < 1$. Continuing in this way, we have

$$|d(fSz,z)| = |d(fSU^nz,fz)| \leq \lambda\left|d\left(fSU^{n-1}z,fz\right)\right| \\ \leq \lambda^2\left|d\left(fSU^{n-2}z,fz\right)\right| \leq \ldots \leq \lambda^n|d(fSz,z)|$$

for all $n \geq 0$. Since $\lambda^n \to 0$ as $n \to 0$, we get $|d(fSz,fz)| = 0$ and $fSz = fz$. By the injectivity of f, we get $Sz = z$. Using (i), we also obtain $z = Tz$ and consequently S and T have the property Q. □

Using the same technique as in Theorem 3.1, the following two theorems can be proved.

Theorem 3.2 Let $f,T,S : X \to X$ be continuous self-mappings on a complete complex valued metric (X,d) also f be injective and sub-sequentially convergent mapping satisfying the following inequality;

$$[d(fSx,fTy)]^2 \preceq \alpha \max\{d(fx,fSx)d(fy,fTy), \\ d(fx,fTy)d(fy,fSx)\} \\ + \beta \max\{d(fx,fSx)d(fy,fSx), \\ d(fx,fTy)d(fy,fTy)\} \tag{2}$$

for all $x, y \in X$ and $\alpha, \beta \in [0,1)$ with $\alpha + 2\beta < 1$. Then, the following the statements hold:

i. T and S have a unique common fixed point
ii. if (f,T) and (f,S) are Banach pairs, then f, T and S have a unique common fixed point in X.
iii. T and S have property Q.

Theorem 3.3 If S, T and an injective and sub-sequentially convergent mapping f are self-mappings defined on a complete complex valued metric space (X,d) satisfying

$$[d(fSx,fTy)]^2 \preceq \alpha \max\{d(fx,fSx)d(y,Ty), d(fy,fSx)d(fx,fSx)\} \\ + \beta \max\{d(fx,fTy)d(fy,fSx), d(fx,fTy)d(fy,fTy)\} \tag{3}$$

for all $x, y \in X$ and $0 \leq 3\alpha + \beta < 1$. Then, the following hold:

i. T and S have a unique common fixed point,
ii. if (f,T) and (f,S) are Banach pairs, then f, T and S have a unique common fixed point in X.
iii. T and S have property Q.

Theorem 3.4 Let f, T and S be continuous self-mappings of a complete complex valued metric space (X,d). Assume that f is an injective and sub-sequentially convergent mapping. If the mappings f, T and S satisfy

$$d(fSx,fTy) \preceq h \max\left\{d(fx,fy), d(fx,fSx), d(fy,fTy), \right. \\ \left. \frac{d(fx,fTy)+d(fy,fSx)}{2}\right\} \tag{4}$$

for all $x, y \in X$ and $0 < h < 1$. Then,

i. T and S have a unique common fixed point,
ii. if (f,T) and (f,S) are Banach pairs, then f, T and S have a unique common fixed point in X.
ii. T and S have property Q.

Proof Choose $x_0 \in X$ as an arbitrary element and define the sequences

$$x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1},$$

for each $n \geq 0$. Then, using (4) and triangle inequality

$$d(fx_{2n+1},fx_{2n}) = d(fSx_{2n},fTx_{2n-1}) \\ \preceq h \max\left\{d(fx_{2n},fx_{2n-1}), \right. \\ d(fx_{2n},fSx_{2n}), d(fx_{2n-1},fTx_{2n-1}), \\ \left. \frac{d(fx_{2n},fTx_{2n-1})+d(fx_{2n-1},fSx_{2n})}{2}\right\} \\ = h \max\left\{d(fx_{2n},fx_{2n-1}), \right. \\ d(fx_{2n},fx_{2n+1}), d(fx_{2n-1},fx_{2n}), \\ \left. \frac{d(fx_{2n},fx_{2n})+d(fx_{2n-1},fx_{2n+1})}{2}\right\}$$

then we get

$$|d(fx_{2n+1},fx_{2n})| \le h\max\{|d(fx_{2n},fx_{2n-1})|,|d(fx_{2n},fx_{2n+1})|,\frac{|d(fx_{2n-1},fx_{2n+1})|}{2}\Bigg\}$$

Case 1:Let $|d(fx_{2n},fx_{2n+1})|$ be maximum, then

$$|d(fx_{2n+1},fx_{2n})| \le h|d(fx_{2n},fx_{2n+1})|$$

which is a contradiction.

Case 2: Suppose that $\frac{|d(fx_{2n-1},fx_{2n+1})|}{2}$ be maximum, then it must be

$$|d(fx_{2n-1},fx_{2n})| \le \frac{|d(fx_{2n-1},fx_{2n+1})|}{2}$$

and using the triangle inequality, we have

$$|d(fx_{2n-1},fx_{2n})| \le |d(fx_{2n},fx_{2n+1})|$$

which is a contradiction too. Then, we have to investigate other case.

Case 3: Assume that $|d(fx_{2n},fx_{2n-1})|$ is maximum, then

$$|d(fx_{2n+1},fx_{2n})| \le h|d(fx_{2n},fx_{2n-1})|.$$

Thus, for all $n \in \mathbb{N}$, we get

$$\begin{aligned}|d(fx_{n+1},fx_n)| &\le h|d(fx_n,fx_{n-1})| \\ &\le \ldots \le h^n|d(fx_1,fx_0)|.\end{aligned}$$

Since $h^n \to 0$ as $n \to 0$, we can write $|d(fx_{2n+1},fx_{2n})| = 0$ for all $n \in \mathbb{N}$ where $0<h<1$.

So for any $n > m$, because $|d(fx_m,fx_n)| = 0$, using same procedure as in Theorem 3.1, we get $\{fx_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} fx_n = z$. Since f is sub-sequentially convergent, $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $\lim_{n\to\infty} x_m = u$. As f is continuous,

$$\lim_{n\to\infty} fx_m = fu.$$

By the uniqueness of the limit, $z = fu$. Since T and S are continuous, $\lim_{m\to\infty} Tx_m = Tu$ and $\lim_{m\to\infty} Sx_m = Su$. Again since f is continuous,

$$\lim_{m\to\infty} fTx_m = fTu \quad \text{and} \quad \lim_{m\to\infty} \mathrm{fSx_m} = \mathrm{fSu}.$$

Hence, the existence and uniqueness of common fixed point of S and T is proved using the same method in Theorem 3.1.

(ii) Now, we show that f, S and T have the unique common fixed point. we assume that (f,S) and (f,T) are Banach pairs; f,S and f,T commute at the fixed point of S and T, respectively. This implies that $fSz = Sfz$ for $z \in F(S)$. So, $fz = Sfz$ which gives that fz is another fixed point of S. It is true for T, too. By the uniqueness of fixed point of S, $fz = z$. Hence, $z = fz = Sz = Tz$, z is unique common fixed point of f, S and T in X.

(iii) S and T have the property Q, and is based on an argument similar to the process used in Theorem 3.1. □

Corollary 3.5 Let (X,d) be a complete complex valued metric space and $T : X \to X$. If there exists an injective and sub-sequentially mapping f such that for all $x,y \in X$ and $\alpha,\beta \in [0,1)$ with $\alpha + 2\beta < 1$:

$$\begin{aligned}[d(fTx,fTy)]^2 \preceq \alpha\max\{&d(fx,fTx)d(fy,fTy), \\ &d(fx,fTy)d(fy,fTx)\} \\ +\beta\max\{&d(fx,fTx)d(fy,fTx), \\ &d(fx,fTy)d(fy,fTy)\},\end{aligned} \tag{5}$$

then

i. T has a unique fixed point,
ii. if (f,T) is a Banach pair, then f, T have a unique common fixed point in X.
iii. T has property P.

Proof By taking $S = T$ in Theorem 3.2, (i) and (ii) can be obtained. Now, we prove that T has property P. Let $z \in F(T^n)$. Then,

$$[d(fz,\mathrm{fTz})]^2 = [d(fT^nz,\mathrm{fT^{n+1}z})]^2 = [d(fT(T^{n-1}z),\mathrm{fT(T^nz)})]^2$$

$$\preceq \alpha\max\{d(fT^{n-1}z,\mathrm{fT^nz})d(fT^nz,\mathrm{fT^{n+1}z}),\mathrm{d(fT^{n-1}z,fT^{n+1}z)d(fT^nz,fT^nz)}\}$$

$$+\beta\max\{d(fT^{n-1}z,\mathrm{fT^nz})d(fT^nz,\mathrm{fT^nz}),\mathrm{d(fT^{n-1}z,fT^{n+1}z)d(fT^nz,fT^{n+1}z)}\}$$

which implies that

$$\left|[d(fz,\mathrm{fTz})]^2\right| \preceq \alpha\left|d(fT^{n-1}z,\mathrm{fz})d(fz,\mathrm{fTz})\right| + \beta\left|d(fT^{n-1}z,\mathrm{fTz})d(fz,\mathrm{fTz})\right|$$

$$|d(fz,\mathrm{fTz})| \preceq \alpha\left|d(fT^{n-1}z,\mathrm{fz})\right| + \beta\left|d(fT^{n-1}z,\mathrm{fTz})\right|$$

$$|d(fz,\mathrm{fTz})| \preceq \frac{\alpha+\beta}{1-\beta}\left|d\left(fT^{n-1}z,\mathrm{fz}\right)\right|.$$

Then,

$$\begin{aligned}|d(fz,\mathrm{fTz})| = \left|d\left(fT^nz,\mathrm{fT^{n+1}z}\right)\right| &\preceq \lambda\left|d\left(fT^{n-1}z,\mathrm{fT^nz}\right)\right| \\ &\preceq \ldots \preceq \lambda^n|d(fz,\mathrm{fTz})|,\end{aligned}$$

where $0<\lambda = \frac{\alpha+\beta}{1-\beta} < 1$. Since $\lambda^n \to 0$ as $n \to \infty$, we get $|d(fz,\mathrm{fTz})| = 0$ and $fz = fTz$. By the injectivity of f, $z = Tz$, *i.e.*, T has property P. □

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