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ORIGINAL RESEARCH

Multiple Heteroclinic solutions of bilateral difference systems with Laplacian operators

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Abstract Sufficient conditions guaranteeing the existence of three Heteroclinic solutions of a class of bilateral difference systems are established using a fixed point theorem. It is the purpose of this paper to show that the approach to get Heteroclinic solutions of BVPs using multi-fixed-point theorems can be extended to treat the bilateral difference systems with the nonlinear operators $x \to \Delta[p\phi(\Delta x)]$ and $y \to \Delta[q\psi(\Delta y)]$.

Keywords Laplacian operator · Bilateral difference system · Heteroclinic solution · Fixed point theorem

Mathematics Subject Classification 34B10 · 34B15 · 35B10

Introduction

Difference equations appear naturally as analogues and as numerical solutions of differential and delay differential equations having applications in applied digital control [1–3], biology, ecology, economics, physics and so on. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solution [25]. In recent years, there have been many papers interested in proving the existence of positive solutions of the boundary value problems (BVPs for short) for the finite difference equations since these BVPs have extensive applications, see the papers [5–17] and the references therein.

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Contrary to the case of boundary value problems in compact domains, for which a very wide literature has been produced, in the framework of unbounded intervals many questions are still open and the theory presents some critical aspects. One of the main difficulties consists in the lack of good priori estimates and appropriate compact embedding theorems for the usual Sobolev spaces.

Recently, the authors [18–22] studied the existence of solutions of the boundary value problems for infinite difference equations. In [19], the existence of multiple positive solutions of the boundary value problems for second-order discrete equations

$$\begin{cases} \Delta^2 x(n-1) - p\Delta x(n-1) - qx(n-1) + f(n, x(n)) = 0, n \in N, \\ \alpha x(0) - \beta\Delta x(0) = 0, \\ \lim_{n\to+\infty} x(n) = 0 \end{cases} \tag{1.1}$$

was investigated using the cone compression and expansion and fixed point theorems in Frechet spaces with application, where $N = \{0, 1, 2, \ldots\}$ the set of all non-negative integers, $\alpha > 0, \beta > 0,\ p > 0, q > 0$ and f is a continuous function and $\Delta x(n) = x(n+1) - x(n)$.

In paper [22], it was considered the existence of solutions of a class of the infinite time scale boundary value problems. It is easy to see that the results in [22] can be applied to the following BVP for the infinite difference equation

$$\begin{cases} \Delta^2 x(n) + f(n, x(n)) = 0, \quad n \in N, \\ x(0) = 0, \\ x(n) \text{ is bounded}. \end{cases} \tag{1.2}$$

The methods used in [22] are based upon the growth argument and the upper and lower solutions methods.

In [23], motivated by some models arising in hydrodynamics, Rachunek and Rachunkoa studied the second-order non-autonomous difference equation

$$\Delta x(n) = \left(\frac{n}{n+1}\right)^2 \left(\Delta x(n-1) + h^2 f(x(n))\right), \quad n \in N,$$

which can be transformed to the following form:

$$\Delta(n^2 \Delta x(n-1)) = h^2 (n+1)^2 f(x(n)), \quad n \in N,$$

where $h > 0$ is a parameter and f is Lipschitz continuous and has three real zeros $L_0 < 0 < L$, conditions for f under which for each sufficiently small $h > 0$ there exists a homoclinic solution of the above equation were presented. The homoclinic solution is a sequence $\{x(n)\}_{n=0}^{\infty}$ satisfying the equation and such that $\{x(n)\}_{n=0}^{\infty}$ is increasing, $x(0) = x(1) \in (L_0, 0)$ and $\lim_{n\to\infty} x(n) = L$.

We note that the difference equations discussed in [19, 22, 23] are those ones defined on $N = \{0, 1, 2, \cdots\}$. The existence of homoclinic solutions for second-order discrete Hamiltonian systems have been studied in [24, 26] by using fountain theorem.

Motivated by above mentioned papers, the purpose of this paper was to investigate the following boundary value problem of the second-order bilateral difference system using a different method

$$\begin{cases} \Delta[p(n)\phi(\Delta x(n))] + f(n, x(n), y(n)) = 0, & n \in Z, \\ \Delta[q(n)\psi(\Delta y(n))] + g(n, x(n), y(n)) = 0, & n \in Z, \\ \lim\limits_{n\to-\infty} x(n) - \sum\limits_{n=-\infty}^{+\infty} \alpha_n x(n) = 0, \\ \lim\limits_{n\to-\infty} y(n) - \sum\limits_{n=-\infty}^{+\infty} \gamma_n y(n) = 0, \\ \lim\limits_{n\to+\infty} \phi^{-1}(p(n))\Delta x(n) - \sum\limits_{n=-\infty}^{+\infty} \beta_n \Delta x(n) = 0, \\ \lim\limits_{n\to+\infty} \psi^{-1}(q(n))\Delta y(n) - \sum\limits_{n=-\infty}^{+\infty} \delta_n \Delta y(n) = 0, \end{cases} \tag{1.3}$$

where

(a) Z denotes the set of all integers, $\Delta x(n) = x(n+1) - x(n)$,

(b) $p(n), q(n) > 0$ for all $n \in Z$ satisfying

$$\sum_{s=-\infty}^{+\infty} \frac{1}{\phi^{-1}(p(s))} = +\infty, \quad \sum_{s=-\infty}^{0} \frac{1}{\phi^{-1}(p(s))} < +\infty,$$
$$\sum_{s=-\infty}^{+\infty} \frac{1}{\psi^{-1}(q(s))} = +\infty, \quad \sum_{s=-\infty}^{0} \frac{1}{\psi^{-1}(q(s))} < +\infty,$$

(c) $\alpha_n, \beta_n, \gamma_n, \delta_n \geq 0$ for all $n \in Z$ and satisfy

$$\sum_{n=-\infty}^{+\infty} \alpha_n < 1,$$
$$\sum_{n=-\infty}^{+\infty} \alpha_n \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))} < +\infty,$$
$$\sum_{n=-\infty}^{+\infty} \frac{\beta_n}{\phi^{-1}(p(n))} < \frac{1}{\phi^{-1}(1+\beta)} \text{ with } \beta > 0,$$

and

$$\sum_{n=-\infty}^{+\infty} \gamma_n < 1,$$
$$\sum_{n=-\infty}^{+\infty} \gamma_n \sum_{s=-\infty}^{n-1} \frac{1}{\psi^{-1}(q(s))} < +\infty,$$
$$\sum_{n=-\infty}^{+\infty} \frac{\delta_n}{\psi^{-1}(q(n))} < \frac{1}{\psi^{-1}(1+\delta)} \text{ with } \delta > 0,$$

(d) $f, g : Z \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$, both f and g are Caratheodory functions (see Definition 1 in "Main results"), and for each $n_0 \in Z, f(n, 0, 0)^2 + g(n, 0, 0)^2 \not\equiv 0$ for $n \leq n_0$,

(e) ϕ is defined by $\phi(x) = |x|^{s-2}x$ with $s > 1$, and $\psi(x) = |x|^{t-2}x$ with $t > 1$, their inverse functions are denoted by ϕ^{-1} and ψ^{-1}, respectively.

A pair of bilateral sequences $\{(x(n), y(n))\}$ is called a Heteroclinic solution of BVP (1.3) if $x(n), y(n)$ satisfy all equations in (1.3), $x(n) \geq 0, y(n) \geq 0$ for all $n \in Z$ and either $x(n) > 0$ for all $n \in Z$ or $y(n) > 0$ for all $n \in Z$.

We establish sufficient conditions for the existence of at least three Heteroclinic solutions of BVP (1.3). This paper may be the first one to study the solvability the boundary value problems of bilateral difference systems. The most interesting part in this article is to construct the nonlinear operator and the cone; this constructing method is not found in known papers.

The remainder of this paper is organized as follows: in "Main results", we first give some lemmas, then the main result (Theorem 1 in "Main results") and its proof are presented. An example is given in "An example" to illustrate the main result.

Main results

In this section, we present some background definitions in Banach spaces, state an important three fixed point theorem [4] and prove some technical lemmas. Then the main result is given and proved.

Denote

$$P_n = 1 + \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}, Q_n = 1 + \sum_{s=-\infty}^{n-1} \frac{1}{\psi^{-1}(q(s))}.$$

Definition 1 F is called a Caratheodory function if it satisfies that

$$(x, y) \to F(n, P_n x, Q_n y)$$

is continuous, and for each $r > 0$ there exists a nonnegative bilateral real number sequence $\{\phi_r(n)\}$ with $\sum_{n=-\infty}^{+\infty} \phi_r(n) < +\infty$ such that

$$|F(n, P_n x, Q_n y)| \le \phi_r(n)$$

for all $n \in Z, |x| \le r, |y| \le r$.

As usual, let E be a real Banach space. The nonempty convex closed subset P of E is called a cone in E if $ax \in P$ and $x + y \in P$ for all $x, y \in P$ and $a \ge 0$, and $x \in E$ and $-x \in E$ imply $x = 0$. A map $\varphi : P \to [0, +\infty)$ is a nonnegative continuous concave (or convex) functional map provided φ is nonnegative, continuous and satisfies

$$\varphi(tx + (1-t)y) \ge (\text{ or } \le)\ t\varphi(x) + (1-t)\varphi(y) \text{ for all } x, y \in P, t \in [0, 1].$$

An operator $T; E \to E$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Let E be a real Banach space, P be a cone of E, $\varphi : P \to P$ be a nonnegative convex continuous functional. Denote the sets by

$$P_c = \{x \in P : ||x|| < c\}, \quad \overline{P}_c = \{x \in P : ||x|| \le c\}$$

and

$$P(\varphi; b, d) = \{x \in P : \varphi(x) \ge b, ||x|| \le d\}.$$

Lemma 1 Suppose that E is a Banach space and P is a cone of E. Let $T : \overline{P}_c \to \overline{P}_c$ be a completely continuous operator and let φ be a nonnegative continuous concave functional on P. Suppose that there exist $0 < a < b < d \le c$ such that $\varphi(y) \le ||y||$ for all $y \in \overline{P}_c$ and

(C1) $\{y \in P(\varphi; b, d) | \varphi(y) > b\} \ne \emptyset$ and $\varphi(Ty) > b$ for $y \in P(\varphi; b, d)$;
(C2) $||Ty|| < a$ for $||y|| \le a$;
(C3) $\varphi(Ty) > b$ for $y \in P(\varphi; b, c)$ with $||Ty|| > d$.

Then T has at least three fixed points y_1, y_2 and y_3 such that $||y_1|| < a$, $\varphi(y_2) > b$ and $||y_3|| > a$ with $\varphi(y_3) < b$.

Choose

$$X = \left\{ \{x(n) : n \in Z\} : \begin{array}{c} x(n) \in R, n \in Z, \\ \text{there exist the limits} \\ \lim\limits_{n \to +\infty} \dfrac{x(n)}{P_n}, \quad \lim\limits_{n \to -\infty} \dfrac{x(n)}{P_n} \end{array} \right\}.$$

Define the norm

$$||x||_X = ||x|| = \sup_{n \in Z} \frac{|x(n)|}{P_n}, \quad x \in X.$$

It is easy to see that X is a real Banach space.

Choose

$$Y = \left\{ \{y(n) : n \in Z\} : \begin{array}{c} y(n) \in R, n \in Z, \\ \text{there exist the limits} \\ \lim\limits_{n \to +\infty} \dfrac{y(n)}{Q_n}, \quad \lim\limits_{n \to -\infty} \dfrac{y(n)}{Q_n} \end{array} \right\}.$$

Define the norm

$$||y||_Y = ||y|| = \sup_{n \in Z} \frac{|y(n)|}{Q_n}, \quad y \in Y.$$

It is easy to see that Y is a real Banach space.

Let $E = X \times Y$ be endowed with the norm

$$||(x, y)|| = \max\{||x||, ||y||\}, \quad (x, y) \in E.$$

Then E is a real Banach space.

Let $h(n) \ge 0$ for every $n \in Z$ be a bilateral sequence with $\sum_{n=-\infty}^{+\infty} h(n)$ converging. Consider the following BVP:

$$\begin{cases} \Delta[p(n)\phi(\Delta x(n))] + h(n) = 0, \quad n \in Z, \\ \lim\limits_{n \to -\infty} x(n) - \sum_{n=-\infty}^{+\infty} \alpha_n x(n) = 0, \\ \lim\limits_{n \to +\infty} \phi^{-1}(p(n))\Delta x(n) - \sum_{n=-\infty}^{+\infty} \beta_n \Delta x(n) = 0. \end{cases} \tag{2.1}$$

Lemma 2 Suppose that (b), (c) and (e) hold. Then $x \in X$ is a solution of BVP (2.1) if and only if

$$x(n) = \frac{1}{1 - \sum\limits_{n=-\infty}^{+\infty} \alpha_n} \sum_{-\infty}^{+\infty} \alpha_n \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))} \phi^{-1}\left(A_h + \sum_{t=s}^{+\infty} h(t)\right) + \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))} \phi^{-1}\left(A_h + \sum_{t=s}^{+\infty} h(t)\right), \tag{2.2}$$

where $A_h \in \left[0, \frac{1}{\beta} \sum_{s=-\infty}^{+\infty} h(s)\right]$ such that

$$\phi^{-1}(A_h) = \sum_{n=-\infty}^{+\infty} \frac{\beta_n}{\phi^{-1}(p(n))} \phi^{-1}\left(A_h + \sum_{s=n}^{+\infty} h(s)\right). \tag{2.3}$$

Proof **Step 1.** We prove that there a unique $A_h \in \left[0, \frac{1}{\beta}\sum_{s=-\infty}^{+\infty} h(s)\right]$ such that (2.3) holds. In fact, let

$$G(u) = 1 - \sum_{n=-\infty}^{+\infty} \frac{\beta_n}{\phi^{-1}(p(n))} \phi^{-1}\left(1 + \frac{1}{u}\sum_{s=n}^{+\infty} h(s)\right).$$

It is easy to see that G is continuous and increasing on $(-\infty, 0)$ and $(0, +\infty)$ respectively.

One sees from (c) that

$$\lim_{u\to-\infty} G(u) = 1 - \sum_{n=-\infty}^{+\infty} \frac{\beta_n}{\phi^{-1}(p(n))} > 0,$$

$$\lim_{u\to0^-} G(u) = +\infty,$$

$$\lim_{u\to0^+} G(u) = -\infty,$$

$$G\left(\frac{1}{\beta}\sum_{s=-\infty}^{+\infty} h(s)\right) \geq 1 - \sum_{n=-\infty}^{+\infty} \frac{\beta_n}{\phi^{-1}(p(n))}\phi^{-1}(1+\beta) \geq 0.$$

Then there is a unique $A_h \in \left[0, \frac{1}{\beta}\sum_{s=-\infty}^{+\infty} h(s)\right]$ such that (2.3) holds.

Step 2. Prove that x satisfies (2.2)–(2.3) if x is a solution of (2.1).

If x is a solution of (2.3), then there exist the limits

$$\lim_{n\to+\infty} \phi^{-1}(p(n))\Delta x(n) = c, \quad \lim_{n\to-\infty} \phi^{-1}(p(n))\Delta x(n)$$

and

$$\phi^{-1}(p(n))\Delta x(n) = \phi^{-1}\left(c + \sum_{s=n}^{+\infty} h(s)\right).$$

So

$$\Delta x(n) = \frac{1}{\phi^{-1}(p(n))}\phi^{-1}\left(c + \sum_{s=n}^{+\infty} h(s)\right).$$

Since $\sum_{n=-\infty}^{0} \frac{1}{\phi^{-1}(p(n))} < +\infty$, then $\lim_{n\to-\infty} x(n) = d \in R$ such that

$$x(n) = d + \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(c + \sum_{t=s}^{+\infty} h(t)\right). \tag{2.4}$$

From the boundary conditions in (2.1), we get

$$d = \frac{1}{1 - \sum\limits_{n=-\infty}^{+\infty} \alpha_n} \sum_{n=-\infty}^{+\infty} \alpha_n \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(c + \sum_{t=s}^{+\infty} h(t)\right) \tag{2.5}$$

and

$$\phi^{-1}(c) - \sum_{n=-\infty}^{+\infty} \beta_n \phi^{-1}\left(c + \sum_{t=n}^{+\infty} h(t)\right) = 0.$$

By Step 1, we see that $c = A_h$. We now prove that

$$\lim_{n\to+\infty} \frac{x(n)}{1 + \sum_{s=-\infty}^{n-1} \dfrac{1}{\phi^{-1}(p(s))}} = c. \tag{2.6}$$

In fact, if $c = 0$, then for any $\epsilon > 0$ there exists $H > 0$ such that

$$\phi^{-1}(p(n))|\Delta x(n)| < \frac{\epsilon}{2}, \quad n \geq H.$$

It follows that

$$|x(n)| \leq |x(H)| + \sum_{s=H}^{n-1} |\Delta x(s)| \leq |x(H)| + \frac{\epsilon}{2}\sum_{s=H}^{n-1} \frac{1}{\phi^{-1}(p(s))}, \quad n \geq H.$$

Then

$$\begin{aligned} &\frac{|x(n)|}{1 + \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\ &\leq \frac{|x(H)|}{1 + \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}} + \frac{1}{1 + \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \frac{\epsilon}{2}\sum_{s=H}^{n-1} \frac{1}{\phi^{-1}(p(s))} \\ &< \frac{|x(H)|}{1 + \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}} + \frac{\epsilon}{2}, \quad n \geq H. \end{aligned}$$

Since $\sum_{s=-\infty}^{\infty} \frac{1}{\phi^{-1}(p(s))} = \infty$, we can choose $H' > H$ large enough so that

$$\frac{|x(n)|}{1 + \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \leq \frac{|x(H)|}{1 + \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}} + \frac{\epsilon}{2} < \epsilon, \quad n \geq H',$$

which implies that

$$\lim_{n\to\infty} \frac{x(n)}{1 + \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}} = 0.$$

If $c \neq 0$, then $\lim_{n\to\infty}\left(\phi^{-1}(p(n))\Delta x(n) - c\right) = 0$. It follows that

$$\lim_{n\to\infty} \phi^{-1}(p(n))\Delta\left[x(n) - c\sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}\right] = 0.$$

Then we get similarly that

$$\lim_{n\to\infty} \frac{x(n) - c\sum_{s=-\infty}^{n-1} \dfrac{1}{\phi^{-1}(p(s))}}{1 + \sum_{s=-\infty}^{n-1} \dfrac{1}{\phi^{-1}(p(s))}} = 0.$$

Together with $\sum_{s=-\infty}^{\infty} \frac{1}{\phi^{-1}(p(s))} = \infty$, it follows that (2.6) holds. Then $\lim_{n\to-\infty} x(n) = d$, (2.4), (2.5) and (2.6) imply that $x \in X$ and x satisfies (2.2) and (2.3).

Step 3. Prove that $x \in X$ and is a solution of (2.1) if x satisfies (2.2) and (2.3). The proof is simple and is omitted. The proof is complete. □

Let $h(n) \geq 0$ for every $n \in Z$ be a bilateral sequence with $\sum_{n=-\infty}^{+\infty} h(n)$ converging. Consider the following BVP:

$$\begin{cases} \Delta[q(n)\psi(\Delta y(n))] + h(n) = 0, \quad n \in Z, \\ \lim_{n\to-\infty} y(n) - \sum_{n=-\infty}^{+\infty} \gamma_n y(n) = 0, \\ \lim_{n\to+\infty} \psi^{-1}(q(n))\Delta y(n) - \sum_{n=-\infty}^{+\infty} \delta_n \Delta y(n) = 0. \end{cases} \tag{2.7}$$

Lemma 3 Suppose that (b), (c) and (e) hold. Then $y \in Y$ is a solution of BVP (2.7) if and only if

$$y(n) = \frac{1}{1 - \sum_{n=-\infty}^{+\infty} \gamma_n} \sum_{n=-\infty}^{+\infty} \gamma_n \sum_{s=-\infty}^{n-1} \frac{1}{\psi^{-1}(q(s))} \psi^{-1}\left(B_h + \sum_{t=s}^{+\infty} h(t)\right) + \sum_{s=-\infty}^{n-1} \frac{1}{\psi^{-1}(q(s))} \psi^{-1}\left(B_h + \sum_{t=s}^{+\infty} h(t)\right), \tag{2.8}$$

where $B_h \in \left[0, \frac{1}{\delta}\sum_{s=-\infty}^{+\infty} h(s)\right]$ such that

$$\psi^{-1}(B_h) = \sum_{n=-\infty}^{+\infty} \frac{\delta_n}{\psi^{-1}(q(n))} \psi^{-1}\left(B_h + \sum_{s=n}^{+\infty} h(s)\right). \tag{2.9}$$

Proof The proof is similar to that of Lemma 2 and is omitted. □

Let $k_1, k_2 \in Z$ with $k_1 + 2 < k_2$. Denote

$$\mu = \min\left\{\frac{P_{k_1}}{P_{k_2}}, \frac{1}{P_{k_2}}, \frac{1}{\phi^{-1}(p(k_1-1))P_{k_2}}, \frac{Q_{k_1}}{Q_{k_2}}, \frac{1}{Q_{k_2}}, \frac{1}{\psi^{-1}(q(k_1-1))Q_{k_2}}\right\}. \tag{2.10}$$

Choose

$$P = \left\{ (x, y) \in X : \begin{array}{c} x(n) \geq 0, \quad y(n) \geq 0 \quad \text{for all } n \in Z, \\ \min_{n\in[k_1,k_2]} \frac{x(n)}{P_n} \geq \mu \sup_{n\in Z} \frac{x(n)}{P_n}, \\ \min_{n\in[k_1,k_2]} \frac{y(n)}{Q_n} \geq \mu \sup_{n\in Z} \frac{y(n)}{Q_n}. \end{array} \right\},$$

It is easy to see that P is a nontrivial cone in X.

Define the nonlinear operator T on P by

$$(T(x, y))(n) = (T_1(x, y)(n), T_2(x, y)(n))$$

with

$$\begin{aligned} T_1(x, y)(n) = &\frac{1}{1 - \sum_{n=-\infty}^{+\infty} \alpha_n} \sum_{n=-\infty}^{+\infty} \alpha_n \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))} \phi^{-1} \\ &\times \left(A_f + \sum_{t=s}^{+\infty} f(t, x(t), y(t))\right) \\ &+ \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))} \phi^{-1}\left(A_f + \sum_{t=s}^{+\infty} f(t, x(t), y(t))\right), \\ T_2(x, y)(n) = &\frac{1}{1 - \sum_{n=-\infty}^{+\infty} \gamma_n} \sum_{n=-\infty}^{+\infty} \gamma_n \sum_{s=-\infty}^{n-1} \frac{1}{\psi^{-1}(q(s))} \psi^{-1} \\ &\times \left(B_g + \sum_{t=s}^{+\infty} g(t, x(t), y(t))\right) \\ &+ \sum_{s=-\infty}^{n-1} \frac{1}{\psi^{-1}(q(s))} \psi^{-1}\left(B_g + \sum_{t=s}^{+\infty} g(t, x(t), y(t))\right), \end{aligned}$$

where $A_f \in \left[0, \frac{1}{\beta} \sum_{s=-\infty}^{+\infty} f(s, x(s), y(s))\right]$ such that

$$\phi^{-1}(A_f) = \sum_{n=-\infty}^{+\infty} \frac{\beta_n}{\phi^{-1}(p(n))} \phi^{-1}\left(A_f + \sum_{s=n}^{+\infty} f(s, x(s), y(s))\right) \tag{2.11}$$

and $B_g \in \left[0, \frac{1}{\delta}\sum_{s=-\infty}^{+\infty} g(s, x(s), y(s))\right]$ such that

$$\psi^{-1}(B_g) = \sum_{n=-\infty}^{+\infty} \frac{\delta_n}{\psi^{-1}(q(n))} \psi^{-1}\left(B_g + \sum_{s=n}^{+\infty} g(s, x(s), y(s))\right). \tag{2.12}$$

Lemma 4 Suppose that (b)–(e) hold. Then $T : P \to P$ is well defined, $(x, y) \in P$ is a positive solution of BVP (1.3) if (x, y) is a fixed point of T, and T is completely continuous.

Proof For $(x, y) \in P$, we know that there exist $r > 0$ such that

$$0 \leq \frac{x(n)}{1 + \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}} = \frac{x(n)}{P_n} \leq r, \quad n \in Z,$$
$$0 \leq \frac{y(n)}{\sum_{s=-\infty}^{n-1} \frac{1}{\psi^{-1}(q(s))} + 1} = \frac{y(n)}{Q_n} \leq r, \quad n \in Z.$$

Since f and g are nonnegative Caratheodory functions, we know that there exists a nonnegative sequence $\phi_n(n)$ with $\sum_{n=-\infty}^{+\infty} \phi_r(n) < +\infty$ such that

$$0 \leq f(n, x(n), y(n)) = f\left(n, P_n\frac{x(n)}{P_n}, Q_n\frac{y(n)}{Q_n}\right) \leq \phi_r(n), \quad n \in Z,$$
$$0 \leq g(n, x(n), y(n)) = g\left(n, P_n\frac{x(n)}{P_n}, Q_n\frac{y(n)}{Q_n}\right) \leq \phi_r(n), \quad n \in Z.$$

By the definitions of T_1 and T_2, we get that

$$T_1(x,y)(n) \geq 0,\ T_2(x,y)(n) \geq 0, \qquad n \in Z \tag{2.13}$$

and

$$\begin{cases} \Delta[p(n)\phi(\Delta T_1(x,y)(n))] + f(n,x(n),y(n)) = 0, & n \in Z, \\ \Delta[q(n)\psi(\Delta T_2(x,y)(n))] + g(n,x(n),y(n)) = 0, & n \in Z, \\ \lim_{n\to-\infty} T_1(x,y)(n) - \sum_{n=-\infty}^{+\infty} \alpha_n T_1(x,y)(n) = 0, \\ \lim_{n\to-\infty} T_2(x,y)(n) - \sum_{n=-\infty}^{+\infty} \gamma_n T_2(x,y)(n) = 0, \\ \lim_{n\to+\infty} \phi^{-1}(p(n))\Delta T_1(x,y)(n) - \sum_{n=-\infty}^{+\infty} \beta_n \Delta T_1(x,y)(n) = 0, \\ \lim_{n\to+\infty} \psi^{-1}(q(n))\Delta T_2(x,y)(n) - \sum_{n=-\infty}^{+\infty} \delta_n \Delta T_2(x,y)(n) = 0. \end{cases} \tag{2.14}$$

Since $\Delta[p(n)\phi(\Delta T_1(x,y)(n))] = -f(n,x(n),y(n)) \leq 0$ for all $n \in Z$, we see that $p(n)\phi(\Delta T_1(x,y)(n))$ is decreasing. Then $\phi^{-1}(p(n))\Delta T_1(x,y)(n)$ is decreasing. It is easy to see that

$$\begin{aligned} \lim_{n\to+\infty} \phi^{-1}(p(n))\Delta T_1(x,y)(n) &= \sum_{n=-\infty}^{+\infty} \beta_n \Delta T_1(x,y)(n) \\ &= \sum_{n=-\infty}^{+\infty} \frac{\beta_n}{\phi^{-1}(p(n))} \phi^{-1}(p(n))\Delta T_1(x,y)(n) \\ &\geq \sum_{n=-\infty}^{+\infty} \frac{\beta_n}{\phi^{-1}(p(n))} \lim_{n\to+\infty} \phi^{-1}(p(n))\Delta T_1(x,y)(n). \end{aligned}$$

Then (c) implies that

$$\lim_{n\to+\infty} \phi^{-1}(p(n))\Delta T_1(x,y)(n) \geq 0.$$

Hence

$$\phi^{-1}(p(n))\Delta T_1(x,y)(n) \geq 0, \quad n \in Z.$$

It follows that $\Delta T_1(x,y)(n) \geq 0$ for all $n \in Z$. So $T_1(x,y)(n)$ is increasing. We consider two cases:

Case 1: there is $n_0 \in Z$ such that

$$\sup_{n\in Z} \frac{T_1(x,y)(n)}{P_n} = \frac{T_1(x,y)(n_0)}{P_{n_0}}.$$

For $n_1, n, n_2 \in Z$ with $n_1 < n < n_2$, Since $\phi^{-1}(p(n))\Delta T_1(x,y)(n)$ is decreasing, we get

$$\phi^{-1}(p(s))\Delta T_1(x,y)(s) \leq \phi^{-1}(p(k))\Delta T_1(x,y)(k)$$

for all $s \geq k$. So there there is λ such that

$$\phi^{-1}(p(s))\Delta T_1(x,y)(s) \leq \lambda \leq \phi^{-1}(p(k))\Delta T_1(x,y)(k), \quad s \geq n > k.$$

Then we get

$$\begin{aligned} &(P_n - P_{n_1})\frac{T_1(x,y)(n_2) - T_1(x,y)(n)}{P_{n_2} - P_n} \\ &= \frac{(P_n - P_{n_1})\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}\phi^{-1}(p(s))\Delta T_1(x,y)(s)}{P_{n_2} - P_n} \\ &= \frac{\sum_{s=n_1}^{n-1} \frac{1}{\phi^{-1}(p(s))}\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}\phi^{-1}(p(s))\Delta T_1(x,y)(s)}{\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}} \\ &\leq \frac{\lambda\sum_{s=n_1}^{n-1} \frac{1}{\phi^{-1}(p(s))}\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}}{\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}} \\ &= \frac{\lambda\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}\sum_{s=n_1}^{n-1} \frac{1}{\phi^{-1}(p(s))}}{\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}} \\ &\leq \frac{\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}\sum_{s=n_1}^{n-1} \frac{1}{\phi^{-1}(p(s))}\phi^{-1}(p(s))\Delta T_1(x,y)(s)}{\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}} \\ &= T_1(x,y)(n) - T_1(x,y)(n_1). \end{aligned}$$

So

$$(P_n - P_{n_1})\frac{T_1(x,y)(n_2) - T_1(x,y)(n)}{P_{n_2} - P_n} + T_1(x,y)(n_1) - T_1(x,y)(n) \leq 0.$$

It follows that

$$T_1(x,y)(n) \geq \frac{P_{n_2} - P_n}{P_{n_2} - P_{n_1}} T_1(x,y)(n_1) + \frac{P_n - P_{n_1}}{P_{n_2} - P_{n_1}} T_1(x,y)(n_2). \tag{2.15}$$

If $n_0 = k_1$, we get

$$\begin{aligned} \min_{n\in[k_1,k_2]} \frac{T_1(x,y)(n)}{P_n} &\geq \frac{T_1(x,y)(k_1)}{P_{k_2}} \\ &= \frac{T_1(x,y)(n_0)}{1 + \sum_{s=-\infty}^{n_0-1} \frac{1}{\phi^{-1}(p(s))}} \frac{1 + \sum_{s=-\infty}^{k_1-1} \frac{1}{\phi^{-1}(p(s))}}{1 + \sum_{s=-\infty}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}} \\ &\geq \frac{P_{k_1}}{P_{k_2}} \sup_{n\in Z} \frac{T_1(x,y)(n)}{1 + \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\ &\geq \mu \sup_{n\in Z} \frac{T_1(x,y)(n)}{1 + \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}}. \end{aligned}$$

If $n_0 > k_1$, choose $n_1 = k_1 - 1$, $n = k_1$ and $n_2 = n_0$, by using (2.15) we have

$$\min_{n\in[k_1,k_2]} \frac{y(n)}{P_n} \geq \frac{T_1(x,y)(k_1)}{P_{k_2}}$$
$$\geq \frac{\frac{P_{n_0}-P_{k_1}}{P_{n_0}-P_{k_1-1}} T_1(x,y)(k_1-1) + \frac{P_{k_1}-P_{k_1-1}}{P_{n_0}-P_{k_1-1}} T_1(x,y)(n_0)}{P_{k_2}}$$
$$\geq \frac{\frac{P_{k_1}-P_{k_1-1}}{P_{n_0}-P_{k_1-1}} T_1(x,y)(n_0)}{P_{k_2}} = \frac{P_{n_0}\frac{P_{k_1}-P_{k_1-1}}{P_{n_0}-P_{k_1-1}}}{P_{k_2}} \frac{T_1(x,y)(n_0)}{P_{n_0}}$$
$$\geq \frac{1}{\phi^{-1}(p(k_1-1))P_{k_2}} \frac{T_1(x,y)(n_0)}{P_{n_0}} \geq \mu \sup_{n\in N_0} \frac{T_1(x,y)(n)}{P_n}.$$

If $n_0 < k_1$, we have

$$\min_{n\in[k_1,k_2]} \frac{T_1(x,y)(n)}{P_n} \geq \frac{T_1(x,y)(k_1)}{P_{k_2}}$$
$$\geq \frac{T_1(x,y)(n_0)}{P_{k_2}} = \frac{P_{n_0}}{P_{k_2}} \frac{T_1(x,y)(n_0)}{P_{n_0}}$$
$$\geq \frac{1}{P_{k_2}} \frac{T_1(x,y)(n_0)}{P_{n_0}} \geq \mu \sup_{n\in N_0} \frac{T_1(x,y)(n)}{P_n}.$$

Case 2: $\sup_{n\in Z} \frac{T_1(x,y)(n)}{P_n} = \lim_{n\to+\infty} \frac{T_1(x,y)(n)}{P_n}$. Choose $n' > k_2$, similarly to Case 1 we can prove that

$$\min_{n\in[k_1,k_2]} \frac{T_1(x,y)(n))}{P_n} \geq \mu \frac{T_1(x,y)(n')}{P_{n'}}.$$

Let $n' \to +\infty$, one sees

$$\min_{n\in[k_1,k_2]} \frac{T_1(x,y)(n))}{P_n} \geq \mu \sup_{n\in N_0} \frac{T_1(x,y)(n)}{P_n}.$$

Case 3: $\sup_{n\in Z} \frac{T_1(x,y)(n)}{P_n} = \lim_{n\to-\infty} \frac{T_1(x,y)(n)}{P_n}$. Choose $n' < k_1$, similarly to Case 1 we can prove that

$$\min_{n\in[k_1,k_2]} \frac{T_1(x,y)(n))}{P_n} \geq \mu \frac{T_1(x,y)(n')}{P_{n'}}.$$

Let $n' \to -\infty$, one sees

$$\min_{n\in[k_1,k_2]} \frac{T_1(x,y)(n))}{P_n} \geq \mu \sup_{n\in N_0} \frac{T_1(x,y)(n)}{P_n}.$$

From Cases 1, 2 and 3, we get

$$\min_{n\in[k_1,k_2]} \frac{T_1(x,y)(n))}{P_n} \geq \mu \sup_{n\in N_0} \frac{T_1(x,y)(n)}{P_n}. \tag{2.16}$$

Similarly we can prove that

$$\min_{n\in[k_1,k_2]} \frac{T_2(x,y)(n))}{Q_n} \geq \mu \sup_{n\in N_0} \frac{T_2(x,y)(n)}{Q_n}. \tag{2.17}$$

From (2.13), (2.16) and (2.17), we know that $T(x,y) \in P$. Thus $T : P \to P$ is well defined.

From (2.14), we get $\Delta T_1(x,y)(n) \geq 0$ for all $n \in Z$. So $T_1(x,y)(n)$ is increasing. Then

$$\lim_{n\to-\infty} T_1(x,y)(n) = \sum_{n=-\infty}^{+\infty} \alpha_n T_1(x,y)(n)$$
$$\geq \sum_{n=-\infty}^{+\infty} \alpha_n \lim_{n\to-\infty} T_1(x,y)(n).$$

It follows that

$$\left(1 - \sum_{n=-\infty}^{+\infty} \alpha_n\right) \lim_{n\to-\infty} T_1(x,y)(n) \geq 0.$$

So the assumption (c) implies that $\lim_{n\to-\infty} T_1(x,y)(n) \geq 0$. Similarly, we can prove that $\lim_{n\to-\infty} T_2(x,y)(n) \geq 0$. If there exists n_1, n_2 such that $T_1(x,y)(n_1) = 0$ and $T_2(x,y)(n_2) = 0$, then $T_1(x,y)(n) = T_2(x,y)(n) = 0$ for all $n \leq \min\{n_1, n_2\}$. Hence (2.14) shows us that $f(n,0,0) = 0$ and $g(n,0,0) = 0$ for all $n \leq \min\{n_1,n_2\}$, a contradiction to assumption (d). Hence we know that $(x,y) \in P$ is a positive solution of BVP (1.3) if and only if $(x,y) \in P$ is a fixed point of T.

Now, we prove that T is completely continuous. It suffices to prove that both $T_1 : P \to X$ and $T_2 : P \to Y$ are completely continuous. So we need to prove that both T_1 and T_2 are continuous on P, map bounded subsets into relatively compact sets. We divide the proof into three steps:

Step 1: Prove that both T_1 and T_2 are continuous. For $X_k \in E(k = 0,1,2,\cdots)$ with $X_k \to X_0$ as $k \to +\infty$, we prove that $T(X_k) \to X_0$ as $k \to +\infty$. Suppose that $X_k(n) = (x_k(n), y_k(n))$. Then

$$T_1(X_k)(n) = \frac{\sum_{n=-\infty}^{+\infty} \alpha_n \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))} \phi^{-1}\left(A_{fk} + \sum_{t=s}^{+\infty} f(t, x_k(t), y_k(t))\right)}{1 - \sum_{n=-\infty}^{+\infty} \alpha_n}$$
$$+ \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))} \phi^{-1}\left(A_{fk} + \sum_{t=s}^{+\infty} f(t, x_k(t), y_k(t))\right),$$
$$T_2(X_k)(n) = \frac{\sum_{n=-\infty}^{+\infty} \gamma_n \sum_{s=-\infty}^{n-1} \frac{1}{\psi^{-1}(q(s))} \psi^{-1}\left(B_{gk} + \sum_{t=s}^{+\infty} g(t, x_k(t), y_k(t))\right)}{1 - \sum_{n=-\infty}^{+\infty} \gamma_n}$$
$$+ \sum_{s=-\infty}^{n-1} \frac{1}{\psi^{-1}(q(s))} \psi^{-1}\left(B_{gk} + \sum_{t=s}^{+\infty} g(t, x_k(t), y_k(t))\right),$$

where $A_{fk} \in \left[0, \frac{1}{\beta} \sum_{s=-\infty}^{+\infty} f(s, x_k(s), y_k(s))\right]$ such that

$$\phi^{-1}(A_{fk}) = \sum_{n=-\infty}^{+\infty} \frac{\beta_n \phi^{-1}\left(A_{fk} + \sum_{s=n}^{+\infty} f(s, x_k(s), y_k(s))\right)}{\phi^{-1}(p(n))},$$
$$k = 0, 1, 2, \ldots \tag{2.18}$$

and $B_{gk} \in \left[0, \frac{1}{\delta} \sum_{s=-\infty}^{+\infty} g(s, x_k(s), y_k(s))\right]$ such that

$$\psi^{-1}(B_{gk}) = \sum_{n=-\infty}^{+\infty} \frac{\delta_n \psi^{-1}\left(B_{gk} + \sum_{s=n}^{+\infty} g(s, x_k(s), y_k(s))\right)}{\psi^{-1}(q(n))},$$
$$k = 0, 1, 2, \ldots. \tag{2.19}$$

We know that there exist $r > 0$ such that

$$0 \le \frac{x_k(n)}{1 + \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}} = \frac{x_k(n)}{P_n} \le r, \quad k = 0, 1, 2, \ldots, n \in Z,$$
$$0 \le \frac{y_k(n)}{\sum_{s=-\infty}^{n-1} \frac{1}{\psi^{-1}(q(s))} + 1} = \frac{y_k(n)}{Q_n} \le r, \quad k = 0, 1, 2, \ldots, \quad n \in Z.$$

Since f and g are nonnegative Caratheodory functions, we know that there exists a nonnegative sequence $\phi_n(n)$ with $\sum_{n=-\infty}^{+\infty} \phi_r(n) < +\infty$ such that

$$0 \le f(n, x_k(n), y_k(n)) = f\left(n, P_n \frac{x_k(n)}{P_n}, Q_n \frac{y_k(n)}{Q_n}\right) \le \phi_r(n), \quad n \in Z,$$
$$0 \le g(n, x_k(n), y_k(n)) = g\left(n, P_n \frac{x_k(n)}{P_n}, Q_n \frac{y_k(n)}{Q_n}\right) \le \phi_r(n), \quad n \in Z.$$

We first prove that $A_{fk} \to A_{f0}$ as $k \to +\infty$ and $B_{gk} \to B_{g0}$ as $k \to +\infty$. It is easy to show that

$$0 \le A_{fk} \le \frac{1}{\beta} \sum_{s=-\infty}^{+\infty} f(s, x_k(s), y_k(s))$$
$$\le \frac{1}{\beta} \sum_{s=-\infty}^{+\infty} \phi_r(n), \quad k = 0, 1, 2, \ldots, \quad n \in Z.$$

Without loss of generality, suppose that $A_{fk} \to \overline{A} \ne A_{f0}$. Then there exist two subsequences $A^1_{fk_i}$ and $A^2_{fk_i}$ with $A^1_{fk_i} \to A_1$ and $A^2_{fk_i} \to A_2$ as $i \to +\infty$. From

$$\phi^{-1}(A^j_{fk_i}) = \sum_{n=-\infty}^{+\infty} \frac{\beta_n}{\phi^{-1}(p(n))} \phi^{-1}\left(A^j_{fk_i} + \sum_{s=n}^{+\infty} f(s, x_{k_i}(s), y_{k_i}(s))\right)$$
$$\le \phi^{-1}\left(\frac{1}{\beta} \sum_{s=-\infty}^{+\infty} \phi_r(n) + \sum_{s=-\infty}^{+\infty} \phi_r(s)\right) \sum_{n=-\infty}^{+\infty} \frac{\beta_n}{\phi^{-1}(p(n))}.$$

Let $i \to +\infty$, we get

$$\phi^{-1}(A_j) = \sum_{n=-\infty}^{+\infty} \frac{\beta_n}{\phi^{-1}(p(n))} \phi^{-1}\left(A_j + \sum_{s=n}^{+\infty} f(s, x_0(s), y_0(s))\right).$$

Together with (2.18), we get $A_1 = A_2 = A_{f0}$. Then $A_{fk} \to A_{f0}$ as $k \to +\infty$. Similarly, we can prove that $B_{gk} \to B_{g0}$ as $k \to +\infty$. These together with the continuous property of f imply that T_1 is continuous at X_0. Similarly, we can prove that T_2 is continuous at X_0. So T is continuous at X_0.

Step 2: For each bounded subset $\Omega \subset P$, prove that $T\Omega$ is bounded. In fact, for each bounded subset $\Omega \subseteq D$, and $(x, y) \in \Omega$. Then there exists $r > 0$ satisfying

$$||(x, y)|| = \max\left\{\sup_{n \in Z} \frac{|x(n)|}{P_n}, \sup_{n \in Z} \frac{|y(n)|}{Q_n}\right\} \le r.$$

Since f and g are nonnegative Caratheodory functions, we know that there exists a nonnegative sequence $\phi_n(n)$ with $\sum_{n=-\infty}^{+\infty} \phi_r(n) < +\infty$ such that

$$0 \le f(n, x(n), y(n)) = f\left(n, P_n \frac{x(n)}{P_n}, Q_n \frac{y(n)}{Q_n}\right) \le \phi_r(n), \quad n \in Z,$$
$$0 \le g(n, x(n), y(n)) = g\left(n, P_n \frac{x(n)}{P_n}, Q_n \frac{y(n)}{Q_n}\right) \le \phi_r(n), \quad n \in Z.$$

The method used in Step 1 implies that there exist constants $M > 0$ such that $|A_f| \le M$ for all $(x, y) \in \Omega$. Then

$$\frac{|T_1(x, y)(n)|}{P_n} = \frac{1}{P_n} \frac{1}{1 - \sum_{n=-\infty}^{+\infty} \alpha_n}$$
$$\times \sum_{n=-\infty}^{+\infty} \alpha_n \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))} \phi^{-1}\left(A_f + \sum_{t=s}^{+\infty} f(t, x(t), y(t))\right)$$
$$+ \frac{1}{P_n} \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))} \phi^{-1}\left(A_f + \sum_{t=s}^{+\infty} f(t, x(t), y(t))\right)$$
$$\le \frac{1}{P_n} \frac{1}{1 - \sum\limits_{n=-\infty}^{+\infty} \alpha_n}$$
$$\times \sum_{n=-\infty}^{+\infty} \alpha_n \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))} \phi^{-1}\left(M + \sum_{t=-\infty}^{+\infty} \phi_r(t)\right)$$
$$+ \frac{1}{P_n} \sum_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))} \phi^{-1}\left(M + \sum_{t=-\infty}^{+\infty} \phi_r(t)\right)$$
$$\le \frac{1}{1 - \sum_{n=-\infty}^{+\infty} \alpha_n} \phi^{-1}\left(M + \sum_{t=-\infty}^{+\infty} \phi_r(t)\right).$$

Similarly, one has that

$$\frac{|T_2(x, y)(n)|}{Q_n} \le \frac{1}{1 - \sum\limits_{n=-\infty}^{+\infty} \gamma_n} \psi^{-1}\left(M + \sum_{t=-\infty}^{+\infty} \phi_r(t)\right).$$

It follows that $T\Omega$ is bounded.

Step 3: For each bounded subset $\Omega \subset P$, prove that $T\Omega$ is relatively compact. We need to prove that both $\{T_1(x, y)(n) : (x, y) \in \Omega\}$ and $\{T_2(x, y)(n) : (x, y) \in \Omega\}$ are uniformly equi-convergent as $n \to \pm\infty$. We have

$$\left|\frac{T_1(x,y)(n)}{P_n}-\frac{\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\right|$$
$$\leq\left|\frac{1}{P_n}\frac{1}{1-\sum_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)\right.$$
$$\left.-\frac{1}{1-\sum_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)\right|$$
$$+\frac{1}{P_n}\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)$$
$$\leq\frac{1-P_n}{P_n}\frac{1}{1-\sum_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(M+\sum_{t=-\infty}^{+\infty}\phi_r(t)\right)$$
$$+\phi^{-1}\left(M+\sum_{t=-\infty}^{+\infty}\phi_r(t)\right)\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}$$
$$=\phi^{-1}\left(M+\sum_{t=-\infty}^{+\infty}\phi_r(t)\right)\left[\frac{\sum\limits_{n=-\infty}^{+\infty}\alpha_n\sum\limits_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}}{1-\sum_{n=-\infty}^{+\infty}\alpha_n}+1\right]\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}$$
$\to 0$ uniformly as $n\to\infty$.

Furthermore, we have

$$\left|\frac{T_1(x,y)(n)}{P_n}-\phi^{-1}(A_f)\right|$$
$$\leq\frac{1}{P_n}\frac{1}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}$$
$$\times\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)$$

$$+\left|\frac{1}{P_n}\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)-\phi^{-1}(A_f)\right|$$
$$\leq\frac{1}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(M+\sum_{t=-\infty}^{+\infty}\phi_r(t)\right)\frac{1}{P_n}$$
$$+\frac{1}{P_n}\sum_{s=-\infty}^{n-1}\frac{\left|\phi^{-1}\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)-\phi^{-1}(A_f)\right|}{\phi^{-1}(p(s))}$$
$$+\phi^{-1}(A_f)\left|\frac{\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}}{P_n}-1\right|$$
$$\leq\frac{1}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(M+\sum_{t=-\infty}^{+\infty}\phi_r(t)\right)\frac{1}{P_n}$$
$$+\frac{1}{P_n}\sum_{s=-\infty}^{n-1}\frac{\left|\phi^{-1}\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)-\phi^{-1}(A_f)\right|}{\phi^{-1}(p(s))}+\phi^{-1}(M)\frac{1}{P_n}.$$

Since $|A_f|\leq M$, $\left|A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right|\leq M$ and ϕ^{-1} is uniformly continuous on $[-M,M]$, then for any $\epsilon>0$ there exists $\sigma>0$ such that $u_1,u_2\in[-M,M]$ and $|u_1-u_2|<\sigma$ imply that $|\phi^{-1}(u_1)-\phi^{-1}(u_2)|<\epsilon$. Since

$$\left|A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))-A_f\right|\leq\sum_{t=s}^{+\infty}\phi_r(t)\to 0$$
uniformly as $s\to+\infty$,

then there exists $S>0$ such that

$$\left|A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))-A_f\right|<\sigma, s>S,(x,y)\in\Omega.$$

So

$$\left|\phi^{-1}\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)-\phi^{-1}(A_f)\right|<\epsilon, s>S,$$
$(x,y)\in\Omega$.

It follows that

$$\frac{1}{P_n}\sum_{s=-\infty}^{n-1}\frac{\left|\phi^{-1}\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)-\phi^{-1}(A_f)\right|}{\phi^{-1}(p(s))}$$
$$\leq\frac{1}{P_n}\left[\sum_{s=S+1}^{n-1}\frac{\epsilon}{\phi^{-1}(p(s))}+\sum_{s=-\infty}^{S}\frac{\left|\phi^{-1}\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)-\phi^{-1}(A_f)\right|}{\phi^{-1}(p(s))}\right]$$
$$\leq\frac{1}{P_n}\left[\sum_{s=S+1}^{n-1}\frac{\epsilon}{\phi^{-1}(p(s))}+2\phi^{-1}(M)\sum_{s=-\infty}^{S}\frac{1}{\phi^{-1}(p(s))}\right]$$
$$\leq\epsilon+\frac{2\phi^{-1}(M)\sum_{s=-\infty}^{S}\frac{1}{\phi^{-1}(p(s))}}{P_n}\to 0\text{ uniformly as }n\to+\infty.$$

Hence

$$\left|\frac{T_1(x,y)(n)}{P_n}-\phi^{-1}(A_f)\right|\to 0\text{ uniformly as }n\to\infty.$$

One knows that $T_1(\Omega)$ is relatively compact. Similarly we can prove that $T_2(\Omega)$ is relatively compact. Hence $T(\Omega)$ is relatively compact.

From Steps 1, 2 and 3, we know that T is completely continuous. The proof is ended. □

For positive constants a,b,c,d and integers k_1,k_2 with $k_1<k_2$, denote

$$Q=\min\left\{\phi\left(\frac{c\left(1-\sum_{n=-\infty}^{+\infty}\alpha_n\right)}{1-\sum_{n=-\infty}^{+\infty}\alpha_n+\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}}\right)\frac{\delta}{3+3\delta},\right.$$
$$\left.\psi\left(\frac{c\left(1-\sum_{n=-\infty}^{+\infty}\gamma_n\right)}{1-\sum_{n=-\infty}^{+\infty}\gamma_n+\sum_{n=-\infty}^{+\infty}\gamma_n\sum_{s=-\infty}^{n-1}\frac{1}{\psi^{-1}(q(s))}}\right)\frac{\beta}{3+3\beta}\right\};$$

$$W=\max\left\{\phi\left(\frac{bP_{k_2}}{\sum_{s=-\infty}^{k_1-1}\frac{1}{\phi^{-1}(p(s))}}\right)\frac{1}{\sum_{t=k_1}^{k_2}\frac{1}{2^{|t|}}},\ \psi\left(\frac{bQ_{k_2}}{\sum_{s=-\infty}^{k_1-1}\frac{1}{\psi^{-1}(q(s))}}\right)\frac{1}{\sum_{t=k_1}^{k_2}\frac{1}{2^{|t|}}}\right\};$$
$$E=\min\left\{\phi\left(\frac{a\left(1-\sum_{n=-\infty}^{+\infty}\alpha_n\right)}{1-\sum_{n=-\infty}^{+\infty}\alpha_n+\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}}\right)\frac{\delta}{3+3\delta},\right.$$
$$\left.\psi\left(\frac{a\left(1-\sum_{n=-\infty}^{+\infty}\gamma_n\right)}{1-\sum_{n=-\infty}^{+\infty}\gamma_n+\sum_{n=-\infty}^{+\infty}\gamma_n\sum_{s=-\infty}^{n-1}\frac{1}{\psi^{-1}(q(s))}}\right)\frac{\beta}{3+3\beta}\right\}.$$

Theorem 1 Suppose that (b)–(e) hold. Choose $k_1,k_2\in N$ with $k_1<k_2$. Let μ be defined by (2.10). Furthermore, suppose that there exist $0<a<b<\frac{b}{\mu}<c$ such that

(A1): $f(n,P_nu,Q_nv) \le \frac{Q}{2^{|n|}}$ for all $n \in Z, u,v \in [0,c]$; $g(n,P_nu,Q_nv) \le \frac{Q}{2^{|n|}}$ for all $n \in Z, u,v \in [0,c]$;

(A2): $f(n,P_nu,Q_nv) \ge \frac{W}{2^{|n|}}$ for all $n \in [k_1,k_2]$, $u,v \in [b,\frac{b}{\mu}]$; $g(n,P_nu,Q_nv) \ge \frac{W}{2^{|n|}}$ for all $n \in [k_1,k_2]$, $u,v \in [b,\frac{b}{\mu}]$;

(A3): $f(n,P_nu,Q_nv) \le \frac{E}{2^{|n|}}$ for all $n \in Z, u,v \in [0,a]$; $g(n,P_nu,Q_nv) \le \frac{E}{2^{|n|}}$ for all $n \in Z, u,v \in [0,a]$.

Then BVP (1.3) has at least three positive solutions x_1,x_2,x_3 such that

$$\begin{aligned}
&\sup_{n\in Z}\frac{x_1(n)}{P_n} < a,\quad \sup_{n\in Z}\frac{y_1(n)}{Q_n} < a,\\
&\min_{n\in[k_1,k_2]}\frac{x_2(n)}{P_n} > b,\quad \min_{n\in[k_1,k_2]}\frac{y_2(n)}{Q_n} > b,\\
&\text{either } \sup_{n\in Z}\frac{x_3(n)}{P_n} > a \text{ or } \sup_{n\in Z}\frac{y_3(n)}{Q_n} > a,\\
&\text{either } \min_{n\in[k_1,k_2]}\frac{x_3(n)}{P_n} < b \text{ or } \min_{n\in[k_1,k_2]}\frac{y_3(n)}{Q_n} < b.
\end{aligned} \tag{2.20}$$

Proof Let E, P and T be defined above. We complete the proof using Lemma 1. Define the functional on $\varphi : P \to R$ by

$$\varphi(x,y) = \min\left\{\min_{n\in[k_1,k_2]}\frac{x(n)}{P_n},\ \min_{n\in[k_1,k_2]}\frac{y(n)}{Q_n}\right\},\ (x,y)\in P.$$

It is easy to see that ϕ is a nonnegative continuous convex functional on the cone P. Choose $d = \frac{b}{\mu}$. Then $0<a<b<d<c$. Now we prove all assumptions in Lemma 1 are satisfied.

(1): Prove that $\varphi(x,y) \le ||(x,y)||$ for all $(x,y)\in \overline{P}_c$. It is easy to see that $\varphi(x,y) \le ||(x,y)||$ for all $(x,y)\in \overline{P}_c$.

(2): Prove that $T(\overline{P}_c) \subseteq \overline{P}_c$. For $(x,y)\in\overline{P}_c$, we have $||(x,y)|| \le c$, then

$$\max\left\{\sup_{n\in Z}\frac{x(n)}{P_n},\ \sup_{n\in Z}\frac{y(n)}{Q_n}\right\} \le c.$$

Then

$$0 \le \frac{x(n)}{P_n} \le c,\ 0 \le \frac{y(n)}{Q_n} \le c,\ n\in Z.$$

From (A1), we get

$$f(n,x(n),y(n)) = f\left(n,P_n\frac{x(n)}{P_n},Q_n\frac{y(n)}{Q_n}\right) \le \frac{Q}{2^{|n|}}, n\in Z,$$

$$g(n,x(n),y(n)) = g\left(n,P_n\frac{x(n)}{P_n},Q_n\frac{y(n)}{Q_n}\right) \le \frac{Q}{2^{|n|}}, n\in Z.$$

So

$$\begin{aligned}
\frac{T_1(x,y)(n)}{P_n} &= \frac{1}{P_n}\frac{1}{1-\sum_{n=-\infty}^{+\infty}\alpha_n}\\
&\times\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)\\
&+\frac{1}{P_n}\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)\\
&\le \frac{1}{P_n}\frac{1}{1-\sum_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\\
&\quad\times\left(\frac{1+\delta}{\delta}\sum_{t=-\infty}^{+\infty}Q2^{-|t|}\right)+\frac{1}{P_n}\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\\
&\quad\times\left(\frac{1+\delta}{\delta}\sum_{t=-\infty}^{+\infty}Q2^{-|t|}\right)\\
&< \left[1+\frac{1}{1-\sum_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\right]\\
&\phi^{-1}\left(\frac{1+\delta}{\delta}\sum_{t=-\infty}^{+\infty}Q2^{-|t|}\right) \le c.
\end{aligned}$$

Similarly we get

$$\begin{aligned}
\frac{T_2(x,y)(n)}{Q_n} &= \frac{1}{Q_n}\frac{1}{1-\sum_{n=-\infty}^{+\infty}\gamma_n}\\
&\times\sum_{n=-\infty}^{+\infty}\gamma_n\sum_{s=-\infty}^{n-1}\frac{1}{\psi^{-1}(q(s))}\psi^{-1}\left(B_g+\sum_{t=s}^{+\infty}g(t,x(t),y(t))\right)\\
&+\frac{1}{Q_n}\sum_{s=-\infty}^{n-1}\frac{1}{\psi^{-1}(q(s))}\psi^{-1}\left(B_g+\sum_{t=s}^{+\infty}g(t,x(t),y(t))\right) \le c.
\end{aligned}$$

Hence $T(x,y)\in\overline{P}_c$. Then $T(\overline{P}_c)\subseteq\overline{P}_c$.

(3): $\{(x,y)\in P(\varphi;b,d)|\varphi(x,y)>b\}\ne\emptyset$ and $\varphi(T(x,y))>b$ for $(x,y)\in P(\varphi;b,d)$. Since $\frac{b}{\mu}>b$, one sees that $\{(x,y)\in P(\varphi;b,d)$ $|\varphi(x,y)>b\}\ne\emptyset$. For $(x,y)\in P(\varphi;b,d)$, we have

$$\max\left\{\sup_{n\in Z}\frac{x(n)}{P_n},\ \sup_{n\in Z}\frac{y(n)}{Q_n}\right\} \le d = \frac{b}{\mu},$$

and

$$\min\left\{\min_{n\in[k_1,k_2]}\frac{x(n)}{P_n},\ \min_{n\in[k_1,k_2]}\frac{y(n)}{Q_n}\right\} \ge b.$$

Then

$$b \le \frac{x(n)}{P_n},\ \frac{y(n)}{Q_n} \le \frac{b}{\mu},\ n\in[k_1,k_2].$$

It follows from (A2) that

$$f(n,x(n),y(n)) = f\left(n,P_n\frac{x(n)}{P_n},Q_n\frac{y(n)}{Q_n}\right) \ge \frac{W}{2^{|n|}}, n\in[k_1,k_2],$$

$$g(n,x(n),y(n)) = \left(n,P_n\frac{x(n)}{P_n},Q_n\frac{y(n)}{Q_n}\right) \ge \frac{W}{2^{|n|}}, n\in[k_1,k_2].$$

Hence

$$\min_{n\in[k_1,k_2]}\frac{T_1(x,y)(n)}{P_n}\geq\frac{T_1(x,y)(k_1)}{P_{k_2}}$$
$$=\frac{1}{P_{k_2}}\frac{1}{1-\sum_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}$$
$$\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)$$
$$+\frac{1}{P_{k_2}}\sum_{s=-\infty}^{k_1-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)$$
$$\geq\frac{1}{P_{k_2}}\sum_{s=-\infty}^{k_1-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)$$
$$>\frac{1}{P_{k_2}}\sum_{s=-\infty}^{k_1-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(\sum_{t=k_1}^{k_2}f(t,x(t),y(t))\right)$$
$$\geq\frac{1}{P_{k_2}}\sum_{s=-\infty}^{k_1-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(\sum_{t=k_1}^{k_2}\frac{W}{2^{|t|}}\right)\geq b.$$

Similarly, we have

$$\min_{n\in[k_1,k_2]}\frac{T_2(x,y)(n)}{Q_n}>\frac{1}{Q_{k_2}}\sum_{s=-\infty}^{k_1-1}\frac{1}{\psi^{-1}(q(s))}\psi^{-1}$$
$$\times\left(\sum_{t=k_1}^{k_2}g(t,x(t),y(t))\right)$$
$$\geq\frac{1}{Q_{k_2}}\sum_{s=-\infty}^{k_1-1}\frac{1}{\psi^{-1}(q(s))}\psi^{-1}\left(\sum_{t=k_1}^{k_2}\frac{W}{2^{|t|}}\right)\geq b.$$

Hence $\varphi(T(x,y))>b$ for $(x,y)\in P(\varphi;b,d)$.

(4): $||T(x,y)||<a$ for $||(x,y)||\leq a$. For $||(x,y)||\leq a$, we have

$$\max\left\{\sup_{n\in Z}\frac{x(n)}{P_n},\ \sup_{n\in Z}\frac{y(n)}{Q_n}\right\}\leq a.$$

Then

$$0\leq\frac{x(n)}{P_n}\leq a,\ 0\leq\frac{y(n)}{Q_n}\leq a,\ n\in Z.$$

From (A3), we get

$$f(n,x(n),y(n))=f\left(n,P_n\frac{x(n)}{P_n},Q_n\frac{y(n)}{Q_n}\right)\leq\frac{E}{2^{|n|}},n\in Z,$$
$$g(n,x(n),y(n))=g\left(n,P_n\frac{x(n)}{P_n},Q_n\frac{y(n)}{Q_n}\right)\leq\frac{E}{2^{|n|}},n\in Z.$$

So

$$\frac{T_1(x,y)(n)}{P_n}=\frac{1}{P_n}\frac{1}{1-\sum_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}$$
$$\times\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)$$
$$+\frac{1}{P_n}\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(A_f+\sum_{t=s}^{+\infty}f(t,x(t),y(t))\right)$$
$$\leq\frac{1}{P_n}\frac{1}{1-\sum_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}$$
$$\times\left(\frac{1+\delta}{\delta}\sum_{t=-\infty}^{+\infty}E2^{-|t|}\right)$$
$$+\frac{1}{P_n}\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}\left(\frac{1+\delta}{\delta}\sum_{t=-\infty}^{+\infty}E2^{-|t|}\right)$$
$$<\left[1+\frac{1}{1-\sum_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}\right]\phi^{-1}$$
$$\times\left(\frac{1+\delta}{\delta}\sum_{t=-\infty}^{+\infty}E2^{-|t|}\right)\leq a.$$

Similarly, we get

$$\frac{T_2(x,y)(n)}{Q_n}=\frac{1}{Q_n}\frac{1}{1-\sum_{n=-\infty}^{+\infty}\gamma_n}\times$$
$$\sum_{n=-\infty}^{+\infty}\gamma_n\sum_{s=-\infty}^{n-1}\frac{1}{\psi^{-1}(q(s))}\psi^{-1}\left(B_g+\sum_{t=s}^{+\infty}g(t,x(t),y(t))\right)$$
$$+\frac{1}{Q_n}\sum_{s=-\infty}^{n-1}\frac{1}{\psi^{-1}(q(s))}\psi^{-1}\left(B_g+\sum_{t=s}^{+\infty}g(t,x(t),y(t))\right)<a.$$

Hence $||T(x,y)||<a$.

(5): $\varphi(T(x,y))>b$ for $(x,y)\in P(\varphi;b,c)$ with $||T(x,y)||>d$. For $(x,y)\in P(\varphi;b,c)$ with $||T(x,y)||>d$, we have $\varphi(x,y)\leq b$ and $||(x,y)||\leq$. Then

$$\min\left\{\min_{n\in[k_1,k_2]}\frac{x(n)}{P_n},\ \min_{n\in[k_1,k_2]}\frac{y(n)}{Q_n}\right\}\leq b,$$

$$\max\left\{\sup_{n\in Z}\frac{x(n)}{P_n},\ \sup_{n\in Z}\frac{y(n)}{Q_n}\right\}\leq c,$$

and

$$\max\left\{\sup_{n\in Z}\frac{T_1(x,y)(n)}{P_n},\ \sup_{n\in Z}\frac{T_2(x,y)(n)}{Q_n}\right\}>d.$$

So

$$\varphi(T(x,y)) = \min\left\{\min_{n\in[k_1,k_2]}\frac{x(n)}{P_n},\ \min_{n\in[k_1,k_2]}\frac{y(n)}{Q_n}\right\}$$
$$\geq \mu\max\left\{\sup_{n\in Z}\frac{T_1(x,y)(n)}{P_n},\ \sup_{n\in Z}\frac{T_2(x,y)(n)}{Q_n}\right\} > \mu d = b.$$

Then T has at least three fixed points (x_1,y_1), (x_2,y_2) and (x_3,y_3) such that $||(x_1,y_1)||<a$, $\psi(x_2,y_2)>b$ and $||(x_3,y_3)||>a$ with $\psi(x_3,y_3)<b$. Then (x_1,y_1), (x_2,y_2) and (x_3,y_3) satisfy (2.20). The proof is completed. □

An example

In this section, we present an example to illustrate efficiency of Theorem 1.

Example 1 Consider the following boundary value problem of the bilateral difference system:

$$\begin{cases}\Delta[p(n)\Delta x(n)]+f(n,x(n),y(n))=0, & n\in Z,\\ \Delta[q(n)\Delta y(n)]+g(n,x(n),y(n))=0, & n\in Z,\\ \lim_{n\to-\infty}x(n)=0,\\ \lim_{n\to-\infty}y(n)=0,\\ \lim_{n\to+\infty}p(n)\Delta x(n)=0,\\ \lim_{n\to+\infty}q(n)\Delta y(n)=0,\end{cases}\tag{3.1}$$

where $p(n)=q(n)=2^{-n}$, $f,g:Z\times[0,+\infty)^2\to[0,+\infty)$ are defined by

$$f(n,u,v)=2^{-|n|}[f_1(2^{-n}u)+f_2(2^{-n}v)]$$
$$g(n,u,v)=2^{-|n|}[g_1(2^{-n}u)+g_2(2^{-n}v)]$$

with

$$f_1(u)=f_2(u)=2^{-|n|}\begin{cases}\frac{1}{24}u, & u\in[0,96],\\ 4+\frac{235595-4}{140-96}(u-96), & u\in[96,140],\\ 235595, & u\in[140,3688200],\\ 235595\times e^{u-3688200}, & u\geq 3688200,\end{cases}$$

$$g_1(u)=g_2(u)=2^{-|n|}\begin{cases}\frac{1}{24}u, & u\in[0,96],\\ 4+\frac{235595-4}{140-96}(u-96), & u\in[96,140],\\ 235595, & u\in[140,3688200],\\ 235595\times e^{u-3688200}, & u\geq 3688200.\end{cases}$$

Then (3.1) has at least three positive solutions (x_1,y_1), (x_2,y_2) and (x_3,y_3) satisfying

$$\begin{aligned}&\sup_{n\in Z}\frac{x_1(n)}{2^n}<96,\quad \sup_{n\in Z}\frac{y_1(n)}{2^n}<96,\\ &\min_{n\in[10,12]}\frac{x_2(n)}{2^n}>140,\quad \min_{n\in[10,12]}\frac{y_2(n)}{2^n}>140,\\ &\text{either }\sup_{n\in Z}\frac{x_3(n)}{2^n}>96\text{ or }\sup_{n\in Z}\frac{y_3(n)}{2^n}>96\\ &\text{either }\min_{n\in[10,12]}\frac{x_3(n)}{2^n}<140\text{ or }\min_{n\in[10,12]}\frac{y_3(n)}{2^n}<140.\end{aligned}\tag{3.2}$$

Proof Corresponding to BVP (1.3), $p(n)=q(n)=2^{-n}$, $\alpha_i=\beta_i=\gamma_i=\delta_i=0$ for $i=1,2,\cdots,n$, $\phi(x)=\psi(x)=x$ with $\phi^{-1}(x)=\psi^{-1}(x)=x$, and

$$\sum_{n=-\infty}^{+\infty}\frac{\beta_n}{\phi^{-1}(p(n))}=0<\frac{1}{\phi^{-1}(1+\beta)}\ \text{ with }\beta=1>0,$$
$$\sum_{n=-\infty}^{+\infty}\frac{\delta_n}{\psi^{-1}(q(n))}=0<\frac{1}{\psi^{-1}(1+\delta)}\ \text{ with }\delta=1>0.$$

One sees that (b), (c), (d) and (e) hold.

By direct computation, we know that

$$P_n=Q_n=1+\sum_{s=-\infty}^{n-1}2^s=2^n.$$

Choose the constant $k_1=10, k_2=12$, $a=96, b=140$, $c=3688200$. It is easy to see that

$$\mu=\min\left\{\frac{P_{k_1}}{P_{k_2}},\frac{1}{P_{k_2}},\frac{1}{\phi^{-1}(p(k_1-1))P_{k_2}},\frac{Q_{k_1}}{Q_{k_2}},\frac{1}{Q_{k_2}},\frac{1}{\psi^{-1}(q(k_1-1))Q_{k_2}}\right\}=2^{-12},$$

$$Q=\min\left\{\phi\left(\frac{c(1-\sum_{n=-\infty}^{+\infty}\alpha_n)}{1-\sum_{n=-\infty}^{+\infty}\alpha_n+\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}}\right)\frac{\delta}{3+3\delta},\right.$$
$$\left.\psi\left(\frac{c(1-\sum_{n=-\infty}^{+\infty}\gamma_n)}{1-\sum_{n=-\infty}^{+\infty}\gamma_n+\sum_{n=-\infty}^{+\infty}\gamma_n\sum_{s=-\infty}^{n-1}\frac{1}{\psi^{-1}(q(s))}}\right)\frac{\beta}{3+3\beta}\right\}=614700;$$

$$W=\max\left\{\phi\left(\frac{bP_{k_2}}{\sum_{s=-\infty}^{k_1-1}\frac{1}{\phi^{-1}(p(s))}}\right)\frac{1}{\sum_{t=k_1}^{k_2}\frac{1}{2^{|t|}}},\ \psi\left(\frac{bQ_{k_2}}{\sum_{s=-\infty}^{k_1-1}\frac{1}{\psi^{-1}(q(s))}}\right)\frac{1}{\sum_{t=k_1}^{k_2}\frac{1}{2^{|t|}}}\right\}=10\times 2^{15};$$

$$E=\min\left\{\phi\left(\frac{a(1-\sum_{n=-\infty}^{+\infty}\alpha_n)}{1-\sum_{n=-\infty}^{+\infty}\alpha_n+\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{s=-\infty}^{n-1}\frac{1}{\phi^{-1}(p(s))}}\right)\frac{\delta}{3+3\delta},\right.$$
$$\left.\psi\left(\frac{a(1-\sum_{n=-\infty}^{+\infty}\gamma_n)}{1-\sum_{n=-\infty}^{+\infty}\gamma_n+\sum_{n=-\infty}^{+\infty}\gamma_n\sum_{s=-\infty}^{n-1}\frac{1}{\psi^{-1}(q(s))}}\right)\frac{\beta}{3+3\beta}\right\}=16.$$

So

$$f(n,P_nu,Q_nv)=2^{-|n|}[f_1(u)+f_2(v)],$$
$$g(n,P_nu,Q_nv)=2^{-|n|}[g_1(u)+g_2(v)].$$

It is easy to check that

(A1): $f(n, P_n u, Q_n v) \le \frac{307530}{2^{|n|}}$ for all $n \in Z, u, v \in [0, 3688200]$; $g(n, P_n u, Q_n v) \le \frac{307530}{2^{|n|}}$ for all $n \in Z, u, v \in [0, 3688200]$;

(A2): $f(n, P_n u, Q_n v) \ge \frac{5 \times 2^{15}}{2^{|n|}}$ for all $n \in [10, 12]$, $u, v \in [140, 573440]$; $f(n, P_n u, Q_n v) \ge \frac{5 \times 2^{15}}{2^{|n|}}$ for all $n \in [10, 12]$, $u, v \in [140, 573440]$;

(A3): $f(n, P_n u, Q_n v) \le \frac{8}{2^{|n|}}$ for all $n \in Z, u, v \in [0, 96]$; $f(n, P_n u, Q_n v) \le \frac{8}{2^{|n|}}$ for all $n \in Z, u, v \in [0, 96]$.

Then by Theorem 1, BVP (3.1) has at least three positive solutions $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) satisfying (3.2). The proof is completed. □

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