Math Sci (2014) 8:116
DOI 10.1007/s40096-014-0116-7

ORIGINAL RESEARCH

Ideal extension of semigroups and their compactifications

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Received: 6 January 2014 / Accepted: 1 February 2014 / Published online: 8 March 2014

Abstract In this paper we consider compactification spaces of ideal extension for topological semigroups. As a consequence, we characterize compactification spaces for Brandt λ-extension of topological semigroups.

Keywords Semigroup compactification · Ideal extension · Brandt λ-extension

Mathematics Subject Classification (2010) 43A15 · 43A60

Introduction

Ideal extension for semigroups was studied by Clifford and Preston in [2]. Afterward, ideal extension for topological semigroup was considered by Chiristoph in [3]. He showed that if S and T are two disjoint topological semigroups such that T has a zero, then $H = \bar{T}^* \cup (0 \times S)$ is an ideal extension of S by T where $\bar{T^*} = \{(t, f(t)) : t \in T \setminus \{0\}\}$. Now, the natural question is: if H is an ideal extension of topological semigroup S by T and H', S' and T' are compactifications of H, S and T respectively, can H' be naturally characterized by S' and T'? In this paper we investigate ideal extension for topological semigroups using congruences of semigroups, then we apply this method to characterize compactification spaces of this structure.

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Preliminaries

Throughout, we use the notations introduced in [1]. For terms which are not introduced here, the reader may refer to [1, 2, 5, 6]. Let $\mathscr{B}(S)$ be the C^*-algebra of all bounded complex valued functions on S, $\mathfrak{F}$ be a unital C^*-subalgebra of $\mathscr{B}(S)$, $S^{\mathscr{F}}$ be the set of all multiplicative means on $\mathscr{F}$ and $\varepsilon : S \to S^{\mathscr{F}}$ be the evaluation mapping. $\mathscr{F}$ is called m-admissible if $T_\mu(\mathscr{F}) \subseteq \mathscr{F}$ for all $\mu \in S^{\mathscr{F}}$, where $T_\mu(f)(s) = \mu(L_s(f))$, $s \in S, f \in \mathscr{F}$. Now, $S^{\mathscr{F}}$ with the Gelfand topology and multiplication $\mu\nu(f) = \mu(T_\nu(f))$, $\mu, \nu \in S^{\mathscr{F}}$ is a compact Hausdorff right topological semigroup. Also if (ψ, X) is a compactification of S, then $\psi^*(C(X))$ is an m-admissible subalgebra of $C(S)$. Conversely, if $\mathscr{F}$ is an m-admissible subalgebra of $C(S)$, then there exists a unique (up to isomorphism) compactification (ψ, X) of S such that $\psi^*(C(X)) = \mathscr{F}$. The compactification corresponding to the m-admissible subalgebra $\mathscr{F}$ is $(\varepsilon, S^{\mathscr{F}})$ and $\varepsilon^*(C(S^{\mathscr{F}})) = \mathscr{F}$. A compactification with a given property $\mathscr{P}$ is called a $\mathscr{P}$-compactification. A universal $\mathscr{P}$-compactification of S is a $\mathscr{P}$-compactification of which every $\mathscr{P}$-compactification of S is a factor. Universal $\mathscr{P}$-compactifications, if they exist, are unique (up to isomorphism). We denote the universal $\mathscr{P}$-compactification of S by $S^{\mathscr{P}}$.

Compactifications of ideal extensions of semigroup

In this paper S and $T^* = T - \{0\}$ are semigroups with identities 1_S,1_T respectively.

By a partial homomorphism we mean a mapping $A \mapsto \bar{A}$ of $T^* = T - \{0\}$ into S such that $\overline{AB} = \bar{A}\bar{B}$, whenever $AB \neq 0$ and $\overline{1_T} = 1_S$. It is known that a partial homomorphism $A \to \bar{A}$ of the semigroup T^* into S determines an extension Ω of S by T as follows. For $A, B \in T$ and $s, t \in S$,

$$(P1)\quad AoB = \begin{cases} AB & \text{if} \quad AB \neq 0 \\ \overline{AB} & \text{if} \quad AB = 0 \end{cases}$$

$$(P2)\quad Aos = \overline{A}s, \qquad (P3)\quad soA = s\overline{A}, \qquad (P4)\quad sot = st.$$

and every extension can be so constructed [2, 4.19].

Let S and T be disjoint topological semigroups, with T having a zero element 0. A topological semigroup Ω is called an ideal extension of S by T if Ω contains S as an ideal and the Rees factor semigroup $\frac{\Omega}{S}$ is topologically isomorphic to T. The existence of ideal extension of topological semigroups was expressed in [3]. In the next Theorem we introduce the ideal extension of topological semigroups using congruences technique on semigroups which is our main tool in the following.

Theorem 1 *Let S and T be disjoint topological semigroups such that T has a zero and Ω be ideal extension of S by T. Then there exists a congruence ρ on Ω such that $\frac{\Omega}{\rho} \simeq \frac{\Omega}{S} \simeq T$.*

Proof We regard $\Omega \times \Omega$ with the product topology. Let τ be the equivalence relation generated by $\{(u, su') \mid s \in S, u, u' \in \Omega\}$ and $\rho_\Omega = \{(x, y) \in \Omega \times \Omega \mid (uxv, uyv) \in \tau, \text{for all } u, v \in \Omega\}$. By Proposition 1.5.10 [5], ρ_Ω is the largest congruence on $\Omega \times \Omega$ contained in τ. We use the techniques of Proposition 8.1.8 [5] to show that if $u_1 \rho_\Omega u_2$, then $u_1 = u_2$ or there exists $s \in S$ such that $u_1 = su_2$. Since Ω is a topological semigroup, ρ_Ω is closed congruence on Ω. Thus, $\frac{\Omega}{\rho_\Omega}$ is a topological semigroup with quotient topology. Let $\pi : \Omega \to \frac{\Omega}{\rho_\Omega}$ be the natural quotient map. If $v \in ker(\pi) = [1]_{\rho_\Omega}$, then $v = s.1 = s$. Hence, $ker(\pi) = \{u \in \Omega \mid [u] = 1\} = S$. This implies that $\frac{\Omega}{\rho_\Omega} \simeq \frac{\Omega}{S} \simeq T$.

Let S and T be disjoint topological semigroups such that T has a zero and Ω be an ideal extension of S by T. Let (ψ, X) be a topological semigroup compactification of Ω and τ_X be the equivalence relation generated by $\{(x, \psi(s)y) \mid x, y \in X, s \in S\}$ and ρ_X be the closure of the largest congruence on $X \times X$ contained in τ_X. We fixed these notations for the rest of this paper. □

Theorem 2 *Let S, T be disjoint topological semigroups such that T has a zero and Ω be an ideal extension of S by T. Let (ψ, X) be a topological semigroup compactification of Ω. Then $\frac{X}{\rho_X}$ is a topological semigroup compactification of $\frac{\Omega}{S} \simeq T$.*

Proof Let $\sigma_1 \rho_\Omega \sigma_2$, then $\psi(\sigma_1) \rho_X \psi(\sigma_2)$. Thus ψ preserves congruence. This implies that there exists a continuous homomorphism $\hat{\psi} : \frac{\Omega}{\rho_\Omega} \to \frac{X}{\rho_X}$ such that $\hat{\pi} \circ \psi = \hat{\psi} \circ \pi$, where $\pi : \Omega \to \frac{\Omega}{\rho_\Omega}$, $\hat{\pi} : X \to \frac{X}{\rho_X}$. Since ρ_X is closed and X is a compact Hausdorff topological semigroup, $\frac{X}{\rho_X}$ is a compact Hausdorff topological semigroup. We have $\overline{\hat{\psi}\left(\frac{\Omega}{\rho_\Omega}\right)} = \overline{\hat{\psi} o \pi} \quad (\Omega) = \overline{\hat{\pi} o \psi(\Omega)} \supseteq \hat{\pi}(\overline{\psi(\Omega)}) = \hat{\pi}(X) = \frac{X}{\rho_X}$. Also $\hat{\psi}\left(\frac{\Omega}{\rho_\Omega}\right) = \hat{\psi} o \pi(\Omega) = \hat{\pi} o \psi(\Omega) \subseteq \hat{\pi}(\Lambda(X)) = \Lambda(\hat{\pi}(X)) = \left(\frac{X}{\rho_X}\right)$. Therefore, $\frac{X}{\rho_X}$ is a topological semigroup compactification of $\frac{\Omega}{\rho_\Omega} \simeq T$. □

Theorem 3 *Let S and T be disjoint topological semigroups such that T has a zero and Ω be an ideal extension of S by T. Let $(\varepsilon_T, T^{\mathscr{P}})$ and $(\varepsilon_\Omega, \Omega^{\mathscr{P}})$ be the universal $\mathscr{P}$-compactifications of T and Ω respectively. Then $T^{\mathscr{P}} \simeq \frac{\Omega^{\mathscr{P}}}{\rho_{\Omega^{\mathscr{P}}}}$ if*

1. *$\mathscr{P}$ is invariant under homomorphism,*
2. *universal $\mathscr{P}$-compactification is a topological semigroup.*

Proof By Theorem 2, $\left(\hat{\varepsilon}_\Omega, \frac{\Omega^{\mathscr{P}}}{\rho_{\Omega^{\mathscr{P}}}}\right)$ is a compactification of $\frac{\Omega}{S} \simeq T$. By universal property of $\mathscr{P}$-compactification $(\varepsilon_T, T^{\mathscr{P}})$ of T [1, 1.4.10], there exists a continuous homomorphism $\phi_1 : T^{\mathscr{P}} \longrightarrow \frac{\Omega^{\mathscr{P}}}{\rho_{\Omega^{\mathscr{P}}}}$ such that $\varphi_1 \circ \varepsilon_T = \hat{\varepsilon_\Omega}$. Also homomorphism $\eta = \varepsilon_T \circ \pi : \Omega \to \frac{\Omega}{S} \simeq T \to T^{\mathscr{P}}$ provides a continuous homomorphism $\varphi_2 : \Omega^{\mathscr{P}} \longrightarrow T^{\mathscr{P}}$ such that $\varphi_2 \circ \varepsilon_\Omega = \eta$. Let $\hat{\sigma}_1 \rho_{\Omega^{\mathscr{P}}} \hat{\sigma}_2$ $(\hat{\sigma}_1, \hat{\sigma}_2 \in \Omega^{\mathscr{P}})$. Choose nets $\{u_\alpha\}$, $\{v_\alpha\}$ in Ω such that $\lim_\alpha \varepsilon_\Omega(u_\alpha) = \hat{\sigma}_1$, $\lim_\alpha \varepsilon_\Omega(v_\alpha) = \hat{\sigma}_2$. We have $\hat{\sigma}_1 = \hat{s}\hat{\sigma}_2$, where $\hat{s} = \varepsilon_\Omega(s)$ for some $s \in S$. Thus,

$$\begin{aligned}
\varphi_2(\hat{\sigma}_1) = \varphi_2(\hat{s}\hat{\sigma}_2) &= \varphi_2(\varepsilon_\Omega(s) \lim_\alpha \varepsilon_\Omega(v_\alpha)) \\
&= \lim_\alpha \varphi_2 o \varepsilon_\Omega(sv_\alpha) = \lim_\alpha \eta(sv_\alpha) \\
&= \lim_\alpha \eta(s)\eta(v_\alpha) = \lim_\alpha \varphi_2 o \varepsilon_\Omega(v_\alpha) \\
&= \varphi_2(\hat{\sigma}_2)
\end{aligned}$$

Then, φ_2 preserves congruence. Thus there exists a continuous homomorphism $\varphi_3 : \frac{\Omega^{\mathscr{P}}}{\rho_{\Omega^{\mathscr{P}}}} \longrightarrow T^{\mathscr{P}}$ such that $\varphi_3 \circ \pi_{\Omega^{\mathscr{P}}} = \varphi_2$, where $\pi_{\Omega^{\mathscr{P}}} : \Omega^{\mathscr{P}} \to \frac{\Omega^{\mathscr{P}}}{\rho_{\Omega^{\mathscr{P}}}}$. We show that $\varphi_1 o \varphi_3 = \text{id}_{\frac{\Omega^{\mathscr{P}}}{\rho_{\Omega^{\mathscr{P}}}}}$. If $\pi_{\Omega^{\mathscr{P}}}(t) \in \frac{\Omega^{\mathscr{P}}}{\rho_{\Omega^{\mathscr{P}}}}$, then we can find a net $\{\sigma_\alpha\}$ in Ω such that $\lim_\alpha \varepsilon_\Omega(\sigma_\alpha) = t$. we have

$$\begin{aligned}
\varphi_1 o \varphi_3(\pi_{\Omega^{\mathscr{P}}}(t)) = \varphi_1 o \varphi_2(t) &= \lim_\alpha \varphi_1 o \varphi_2(\varepsilon_\Omega(\sigma_\alpha)) \\
&= \lim_\alpha \varphi_1 o \eta(\sigma_\alpha) = \lim_\alpha \varphi_1 o \varepsilon_T o \pi(\sigma_\alpha) \\
&= \lim_\alpha \hat{\varepsilon_\Omega} o \pi(\sigma_\alpha) = \lim_\alpha \pi_{\Omega^{\mathscr{P}}}(\varepsilon_\Omega(\sigma_\alpha)) \\
&= \pi_{\Omega^{\mathscr{P}}}(\lim_\alpha \varepsilon_\Omega(\sigma_\alpha)) = \pi_{\Omega^{\mathscr{P}}}(t).
\end{aligned}$$

Similarly, $\varphi_3 o \varphi_1 = id_{T^{\mathscr{P}}}$. Therefore, $T^{\mathscr{P}} \simeq \frac{\Omega^{\mathscr{P}}}{\rho_{\Omega^{\mathscr{P}}}}$. □

Corollary 1 *Let Ω be an ideal extension of topological semigroup S by topological semigroup T. Let $(\varepsilon_s, S^{\text{sap}}), (\varepsilon_\Omega, \Omega^{\text{sap}})$ [resp. $(\varepsilon_s, S^{\text{ap}}), (\varepsilon_\Omega, \Omega^{\text{ap}})$] be the strongly almost periodic compactifications [resp. almost periodic compactifications] of S and Ω, respectively. Then $T^{\text{sap}} \simeq \frac{\Omega^{\text{sap}}}{\rho_{\Omega^{\text{sap}}}}$ [resp. $T^{\text{ap}} \simeq \frac{\Omega^{\text{ap}}}{\rho_{\Omega^{\text{ap}}}}$], where $\hat{S} = \rho_{\Omega^{\text{sap}}}$ [resp. where $\hat{S} = \rho_{\Omega^{\text{ap}}}$].*

Example 1 Let $S = \mathscr{M}^0(G, P, I, J)$ be the Rees matrix semigroup where G is a topological group, I and J are arbitrary nonempty sets and $P = (p_{ji})$ is a $J \times I$ matrix with entries in $G^0 = G \cup \{0\}$. In [7], it is shown that there is a continuous partial homomorphism $\theta : S \to G$; then there exists an extension Ω of G by S and $\frac{\Omega}{G} \simeq \frac{\Omega}{\rho} \simeq S$ where $\rho_\Omega = \{(u, v) \in \Omega \times \Omega \mid u = gv \text{ for some } g \in G\}$. Also, $S^{\text{ap}} \simeq \frac{\Omega^{\text{ap}}}{\rho_{\Omega^{\text{ap}}}}$ and $S^{\text{sap}} \simeq \frac{\Omega^{\text{sap}}}{\rho_{\Omega^{\text{sap}}}}$ □

Theorem 4 *Let S and T be disjoint topological semigroups such that T has a zero and Ω be ideal extension of S by T. Let (ψ_S, X_S) and (ψ_T, X_T) be topological semigroup compactifications of S and T, respectively, such that $X_S \cap X_T = \emptyset$. Then the following assertion holds.*

(a) *Ideal extension X_Ω of X_S by X_T exist.*
(b) *Topological center $\Lambda(\Omega)$ is an ideal extension of $\Lambda(S)$ by $\Lambda(T)$.*
(c) *(ψ_Ω, X_Ω) is a topological semigroup compactification of Ω where $\psi_\Omega|_T = \psi_T, \psi_\Omega|_S = \psi_S$.*

Proof (a) First, we note that if 0 be zero element of T, then $\psi_T(0)$ is zero element of X_T. It is enough to show that there is a continuous partial homomorphism $\hat{\theta} : X_T^* = X_T - \{0\} \to X_S$. Let $x_T \in X_T^*$ then there exists net $\{u_\alpha\}$ in T such that $\psi_T(u_\alpha) \to x_T$. Now $\{\psi_S \circ \theta(u_\alpha)\}$ is a net in X_S and by compactness of X_S, there exists $x_s \in X_S$ such that $\psi_S \circ \theta(u_\alpha) \to x_S$. Let $\hat{\theta} : X_T^* \to X_S$ by $\hat{\theta}(x_T) = x_S$. Obviously, $\hat{\theta}$ is well defined. Suppose $x_T, y_T \in X_T$ and $\{u_\alpha\}$, $\{v_\alpha\}$ are nets in T such that $\lim_\alpha(\psi_S \circ \theta(u_\alpha)) = \hat{\theta}(x_T)$ and $\lim_\alpha(\psi_S \circ \theta(v_\alpha)) = \hat{\theta}(y_T)$. We have

$$\begin{aligned}\hat{\theta}(x_T)\hat{\theta}(y_T) &= \lim_\alpha(\psi_S(\theta(u_\alpha))\psi_S(\theta(v_\alpha))) \\ &= \lim_\alpha \psi_S(\theta(u_\alpha v_\alpha)) \\ &= \hat{\theta}(x_T y_T)\end{aligned}$$

Clearly, $\hat{\theta}$ is continuous. Thus by Theorem 1, ideal extension X_Ω of X_S by X_T exist.

(b) Obviously, $\Lambda(T) \cap \Lambda(S) = \emptyset$. Define $\theta' : \Lambda(T)^* = \Lambda(T) - \{0\} \to \Lambda(S)$ by $\theta'(\lambda_t) = \lambda_{\theta(t)} (t \in T)$. Now θ' is a continuous partial homomorphism then there exists an ideal extension ω of $\Lambda(S)$ by $\Lambda(T)$. Let $\lambda_\sigma \in \Lambda(\Omega)$. Then, if $\sigma \in S$ so $\lambda_\sigma \in \Lambda(S)$ and if $\sigma \in T$ so $\lambda_\sigma \in \Lambda(T)$. Thus $\Lambda(\Omega) \subseteq \omega$. Obviously, $\omega \subseteq \Lambda(\Omega)$. Then $\Lambda(\Omega) = \omega$.

(c) By (a) ideal extension X_Ω of X_S by X_T exist. Suppose $x \in X_\Omega = X_S \cup X_T^*$, then there exists $\{u_\alpha\} \in \Omega = S \cup T^*$ such that $\psi_\Omega(u_\alpha) \to x$. Thus $\overline{\psi_\Omega(\Omega)} = X_\Omega$. Also,

$$\begin{aligned}\psi_\Omega(S) &= \psi_\Omega|_S(S) = \psi_S(S) \subseteq \Lambda(X_S) \\ \psi_\Omega(T) &= \psi_\Omega|_T(T) = \psi_T(T) \subseteq \Lambda(X_T)\end{aligned}$$

Now by (b), $\psi_\Omega(\Omega) \subseteq \Lambda(X_\Omega)$.

The following theorem shows that topological semigroup compactifications of S and T can be constructed by topological semigroup compactification of their ideal extension. □

Theorem 5 *Let S and T be disjoint topological semigroups such that T has a zero and Ω be an ideal extension of S by T. Suppose (ψ_Ω, X_Ω) is a topological semigroup compactification of Ω. Then there are topological semigroups compactifications (ψ_S, X_S), (ψ_T, X_T) of S and T, respectively, such that X_Ω is an ideal extension of X_S by X_T.*

Proof Set $\psi_S = \psi_\Omega|_S : S \to X_\Omega$ and $\overline{\psi_S(S)} = X_S$. It is clear that $X_S \subseteq X_\Omega$ is a compact topological subsemigroup of X_Ω and $\psi(S) \subseteq \Lambda(X_S)$. Thus (ψ_S, X_S) is a topological semigroup compactification of S . Now we show that for every $x, x' \in X_\Omega$, $(xX_S)(x'X_S) \subseteq x''X_S$ for some $x'' \in X_\Omega$. Let $g \in (xX_S)(x'X_S)$ then there exist nets $\{u_\alpha\}$, $\{v_\alpha\}$ in Ω and u_1,v_1 in X_S such that $g = \lim_\alpha \psi_\Omega(u_\alpha)u_1\psi_\Omega(v_\alpha)v_1$. Also there exist nets $\{s_\alpha\}$, $\{t_\alpha\}$ in S such that $\lim_\alpha \psi_S(s_\alpha) \to u_1, \lim_\alpha \psi_S(t_\alpha) \to v_1$. Then,

$$\begin{aligned}g &= \lim_\alpha \psi_\Omega(u_\alpha)\psi_S(s_\alpha)\psi_\Omega(v_\alpha)\psi_S(t_\alpha) \\ &= \lim_\alpha \psi_\Omega(u_\alpha s_\alpha v_\alpha t_\alpha)\end{aligned}$$

On the other hand, $\frac{\Omega}{S} \simeq T$ so for every $a, b \in \Omega$, there exists $c \in \Omega$ such that $aS.bS = cS$. Thus for every α, there exists $\{w_\alpha\} \in \Omega$ and $q_\alpha \in S$ such that $u_\alpha s_\alpha v_\alpha t_\alpha = w_\alpha\ q_\alpha$. The compactness of X_Ω and X_S allows us to assume $g = x''\ q''$ for some $x'' \in X_\Omega, q'' \in X_S$. This implies that $\frac{X_\Omega}{X_S}$ is semigroup. Also, $\frac{X_\Omega}{X_S}$ is compact topological semigroup [1, 1.3.8]. Let $X_T = \frac{X_\Omega}{X_S}$, then X_Ω is a topological extension of X_S by X_T. Let $t \in T \simeq \frac{\Omega}{S}$ then $t = \pi(\sigma)$ for some $\sigma \in \Omega$. Define $\psi_T : T \to X_T$ by $\psi(t) = \pi' \circ \psi_\Omega(\sigma)$ where $\pi' : X_\Omega \to \frac{X_\Omega}{X_S} = X_T$. It remains to show that (ψ_T, X_T) is a topological semigroup compactification of T . We have

$$\overline{\psi_T(T)} = \overline{\pi' \circ \psi_\Omega(\Omega)} \supseteq \pi' \circ \overline{\psi_\Omega(\Omega)} = \pi'(X_\Omega) = \frac{X_\Omega}{X_S} = X_T$$

and

$$\psi_T(T) = \pi' \circ \psi_\Omega(\Omega) \subseteq \pi' \circ \Lambda(X_\Omega) = \Lambda(\pi'(X_\Omega)) = \Lambda\left(\frac{X_\Omega}{X_S}\right) = \Lambda(X_T).$$

□

Compactification of Brant λ-extensions

An important class of semigroups which has been considered from various points of view is completely 0-simple semigroup and Brandt λ-extension [see 2, 4, 5, 6, 7, for instance]. In this section we use topological extension technique to characterize compactification spaces of Brandt λ-extension.

Let $G^0 = G \cup \{0\}$ [resp. G] be a group with zero [resp. group] and, E and F be arbitrary nonempty sets. Let P be a $E \times F$ matrix over G^0 [resp. G]. The set $S = G \times E \times F \ \cup \ \{0\}$ [resp. $S = G \times E \times F$] is a semigroup under the composition

$$(i,a,j) \circ (l,b,k) = \begin{cases} (i, ap_{jl}b, k) & \text{if} \quad p_{jl} \neq 0 \\ o & \text{otherwise} \end{cases}$$

This semigroup is denoted by $S = M(G, P, E, F)$ and is called Rees $E \times F$ matrix semigroup over G^0 [resp. G] with the sandwich matrix P.

In the special case, if $P = I$ is an identity matrix, $S = G^0$ is semigroup with zero, and $E = F = I_\lambda$ is a set of cardinality $\lambda \geq 1$. Define the semigroup operation on the set $B_\lambda(S) = M(S, I, I_\lambda, I_\lambda)$ by

$$(i,a,j) \circ (l,b,k) = \begin{cases} (i, ab, k) & \text{if}\, j = l \\ 0, & \text{if} \quad j \neq l \end{cases}$$

and $(i, a, j).0 = 0.(i, a, j) = 0.0 = 0$ for all $a, b \in S, i, j, l, k \in I_\lambda$. The semigroup $B_\lambda(S)$ is called Brandt λ-extension of S [4]. Now let $i \to u_i$ and $j \to v_j$ be mappings of E and F to S such that $u_k.u_k = 1_S, \forall k \in \lambda$. Then mapping $\theta : B_\lambda(S)^* = B_\lambda(S) - \{0\} \to S$ by $\theta\ (i, s, j) = u_i s\ u_j$ is a partial homomorphism.

Let S be a topological semigroup with zero and Brandt λ-extension of S , $B_\lambda(S)$ be equipped with product topology then $B_\lambda(S)$ is a topological semigroup. Now $\theta : B_\lambda(S)^* = B_\lambda(S) - \{0\} \to S^* = S - \{0\}$ by $\theta\ (i, s, j) = u_i s\ u_j$ is a continuous partial homomorphism. Then there exists an ideal extension Ω of S^* by $B_\lambda(S)$ and $\frac{\Omega}{S^*} \simeq B_\lambda(S)$.

The following Corollaries are immediately results of Theorems 3.4, 3.5, 3.6.

Corollary 2 *Let S be a topological semigroup with zero and Ω be an ideal extension of $S^* = S - \{0\}$ by $B_\lambda(S)$. Let $(\psi\ , X)$ be a topological semigroup compactification of topological semigroup Ω. Then, $\frac{X}{\rho_X}$ is a topological semigroup compactification of $B_\lambda(S)$.*

Corollary 3 *Let S be a topological semigroup with zero and Ω be an ideal extension of $S^* = S - \{0\}$ by $B_\lambda(S)$. Suppose $(\varepsilon_{B_\lambda(S)}, B_\lambda(S)^{\mathscr{P}})$ and $(\varepsilon_\Omega, \Omega^{\mathscr{P}})$ are the universal $\mathscr{P}$-compactifications of $B_\lambda(S)$ and Ω, respectively. Then $B_\lambda(S)^{\mathscr{P}} \simeq \frac{\Omega^{\mathscr{P}}}{\rho_{\Omega^{\mathscr{P}}}}$, if*

1. *$\mathscr{P}$ is invariant under homomorphism,*
2. *universal $\mathscr{P}$-compactification is a topological semigroup.*

Corollary 4 *Let S be a topological semigroup with zero and Ω be an ideal extension of $S^* = S - \{0\}$ by $B_\lambda(S)$. Let $(\varepsilon_{B_\lambda(S)}, B_\lambda(S)^{sap})$ [resp. $(\varepsilon_{B_\lambda(S)}, B_\lambda(S)^{ap})$] and $(\varepsilon_\Omega, \Omega^{sap})$ [resp. $(\varepsilon_\Omega, \Omega^{ap})$] be the strongly almost periodic compactifications [resp. almost periodic compactifications] of $B_\lambda(S)$ and Ω, respectively. Then $B_\lambda(S)^{sap} \simeq \frac{\Omega^{sap}}{\rho_{\Omega^{sap}}}$ [resp. $B_\lambda(S)^{ap} \simeq \frac{\Omega^{ap}}{\rho_{\Omega^{ap}}}$].*

Acknowledgments The author would like to sincerely thank the referee for his/her valuable comments and useful suggestions. Also, this research is supported by a grant of the Islamic Azad University, Central Tehran Branch.

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