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ORIGINAL RESEARCH

A simple higher order shear deformation theory for mechanical behavior of laminated composite plates

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Abstract In the present study, the static, buckling, and free vibration of laminated composite plates is examined using a refined shear deformation theory and developed for a bending analysis of orthotropic laminated composite plates. These models take into account the parabolic distribution of transverse shear stresses and satisfy the condition of zero shear stresses on the top and bottom surfaces of the plates. The most interesting feature of this theory is that it allows for parabolic distributions of transverse shear stresses across the plate thickness and satisfies the conditions of zero shear stresses at the top and bottom surfaces of the plate without using shear correction factors. The number of independent unknowns in the present theory is four, as against five in other shear deformation theories. In the analysis, the equation of motion for simply supported thick laminated rectangular plates is obtained through the use of Hamilton's principle. The accuracy of the analysis presented is demonstrated by comparing the results with solutions derived from other higher order models and with data found in the literature. It can be concluded that the proposed theory is accurate and simple in solving the static, the buckling, and free vibration behaviors of laminated composite plates.

Keywords Higher-order theories · Shear deformation theory of plates · Laminated composite plate

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Introduction

The use of composite material for the structure/component design has grown significantly over the past few decades because their response characteristics can be tailored to meet specific design requirements. Furthermore, composite structures possess high specific stiffness and high specific strength which leads to overall reduction of weight, by increasing the efficiency of the structure. Laminated composite plates are widely used in industry and new fields of technology. Due to the high degrees of anisotropy and the low rigidity in transverse shear of the plates, the Kirchhoff hypothesis as a classical theory is no longer adequate. This hypothesis states that the normal to the midplane of a plate remains straight and normal after deformation because of the negligible transverse shear effects. Refined theories without this assumption have been used recently. The free vibration frequencies calculated by using the classical theory of thin plates are higher than those obtained by the Mindlin theory of plates (Mindlin 1951), in which the transverse shear and rotary inertia effects are included.

A number of shear deformation theories have been proposed to date. The first such theory for laminated isotropic plates was proposed apparently by Stavski (1965). This theory was generalized to laminated anisotropic plates in Yang et al. (1966), Ambartsumyan (1969) and Ambartsumyan and Gnuni (1961). It was shown in Srinivas and Rao (1970), Whitney and Sun (1973) and Bert et al. (1974) that the Yang–Norris–Stavski (YNS) theory (Yang et al. 1966) is adequate for predicting the flexural vibration response of laminated anisotropic plates in the first few modes. In Whitney and Pagano (1970), the YNS theory was employed to study the cylindrical bending of antisymmetric cross-ply and angle-ply plate-strips under sinusoidal loading and the free vibration of antisymmetric

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angle-ply plate-strips (see also (Fortier and Rossettos 1973; Shinha and Rath 1975). Using the YNS theory, a closed-form solution for the free vibration of simply supported rectangular plates of antisymmetric angle-ply laminates was obtained in (Bert and Chen 1978). In Noor (1973) were also presented exact three-dimensional elasticity solutions for the free vibration of isotropic and anisotropic composite laminated plates, which serve as benchmark solutions for comparison by many researchers. The free vibration of antisymmetric angle-ply laminated plates, with reference to transverse shear deformations, was investigated in Reddy (1979) using the finite-element method The author also derived a set of variationally consistent equilibrium equations for the kinematic models originally proposed by Levinson and Murthy (Reddy 1984). In Reddy and Khdeir (1989), analytical and finite- element solutions for the vibration and buckling of laminated composite plates were found using various theories of plates to prove the necessity for shear deformation theories to predict the behavior of composite laminates. Using a higher order shear deformation theory, finite-element solutions for free vibration analysis of laminated composite plates were also obtained in (Shankara and Iyengar 1996). The complete set of linear equations of a second-order theory was derived in Khdeir and Reddy (1999) to analyze the free vibration behavior of cross-ply and antisymmetric angle-ply laminated plates. In Singh et al. (2001), the natural frequencies of composite plates with random material properties were determined using a higher order shear deformation theory (including the rotatory inertia effect). The natural frequencies of laminated composite plates were also found in Rastgaar et al. (2006) by employing a third-order shear deformation theory. In Simsek (2010a), the dynamic deflections and the stresses of a functionally graded simply supported beam subjected to a moving mass were investigated using the Euler–Bernoulli, Timoshenko, and the parabolic shear deformation theory of beams. In Simsek (2010b), the free vibration of functionally graded beams with different boundary conditions was examined by using the classical, first-order, and different higher order shear deformation theories of beams. A stress analysis of a functionally graded plate subjected to thermal and mechanical loads was performed in Matsunaga (2009) using a two-dimensional higher order theory. A new trigonometric shear deformation theory for isotropic and composite laminated and sandwich plates was developed recently in Mantari et al. (2012), El and Chulkov (1973), where displacements of the middle surface were expanded in terms of tangential trigonometric functions of the thickness coordinate, and the transverse displacements were assumed to be constant across the thickness.

In this paper, a refined and simple theory of plates is presented and applied to the investigation of static,

buckling, and free vibration behavior of laminated composite plates. This theory is based on the assumption that the in-plane and transverse displacements consist of bending and shear components where the bending components do not contribute to shear forces, and likewise, the shear components do not contribute to bending moments. The most interesting feature of this theory is that it allows for parabolic distributions of transverse shear stresses across the plate thickness and satisfies zero shear stress conditions at the top and bottom surfaces of the plate without using shear correction factors. The equations of motion are derived using Hamilton's principle. The fundamental frequencies are found by solving an Eigen value equation. The results obtained by the present method are compared with solutions and results of the first-order and the other higher-order theories.

Theoretical formulations

Basic assumptions

Consider a rectangular plate of total thickness h composed of n orthotropic layers with the coordinate system as shown in Fig. 1. The assumptions of the refined plate's theory are as follows:

- The displacements are small in comparison with the plate thickness and, therefore, strains involved are infinitesimal.
- The transverse displacement w includes three components of bending w_b and shear w_s. These components are functions of coordinates x, y, and time t only.

$$w(x, y, z, t) = w_b(x, y, t) + w_s(x, y, t) \tag{1}$$
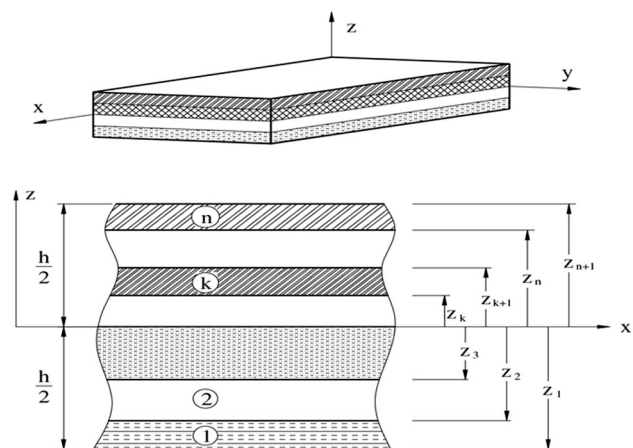
Fig. 1 Coordinate system and layer numbering used for a typical laminated plate

- The transverse normal stress σ_z is negligible in comparison with in-plane stresses σ_x and σ_y.
- The displacements U in x-direction and V in y-direction consist of extension, bending, and shear components:

$$U = u + u_b + u_s, \quad V = v + v_b + v_s \tag{2}$$

- The bending components u_b and v_b are assumed to be similar to the displacements given by the classical plate theory. Therefore, the expression for u_b and v_b can be given as

$$u_b = -z\frac{\partial w_b}{\partial x}, \quad v_b = -z\frac{\partial w_b}{\partial y} \tag{3}$$

- The shear components u_s and v_s give rise, in conjunction with w_s, to the parabolic variations of shear strains γ_{xz}, γ_{yz} and hence to shear stresses σ_{xz}, σ_{yz} through the thickness of the plate in such a way that shear stresses σ_{xz}, σ_{yz} are zero at the top and bottom faces of the plate. Consequently, the expression for u_s and v_s can be given as

$$u_s = f(z)\frac{\partial w_s}{\partial x}, \quad v_s = f(z)\frac{\partial w_s}{\partial y} \tag{4}$$

Kinematics

Based on the assumptions made in the preceding section, the displacement field can be obtained using Eqs. (1)–(4):

$$u(x, y, z, t) = u_0(x, y, t) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x}$$
$$v(x, y, z, t) = v_0(x, y, t) - z\frac{\partial w_b}{\partial y} - f(z)\frac{\partial w_s}{\partial y} \tag{5a}$$
$$w(x, y, z, t) = w_b(x, y, t) + w_s(x, y, t),$$

where u_0 and v_0 are the mid-plane displacements of the plate in the x and y direction, respectively; w_b and w_s are the bending and shear components of transverse displacement, respectively, while $f(z)$ represents shape functions determining the distribution of the transverse shear strains and stresses along the thickness. This function ensures zero transverse shear stresses at the top and bottom surfaces of the plate. The parabolic distributions of transverse shear stresses through the plate thickness are taken into account for the analysis, by means of the hyperbolic and exponential function of the assumed displacement field.

Present model 1 HSDT The function $f(z)$ is an hyperbolic shape function (Hassaine Daouadji et al. 2012, 2013) (Hyperbolic Shear Deformation Theory):

$$f(z) = z\left[1 + \frac{3\pi}{2}\sec h^2\left(\frac{1}{2}\right)\right] - \frac{3\pi}{2}h\tanh\left(\frac{z}{h}\right) \tag{5b}$$

Present model 2 ESDT The function $f(z)$ is an exponential shape function (Karama et al. 2003) (Exponential Shear Deformation Theory):

$$f(z) = z - ze^{\left(-2\frac{z^2}{h^2}\right)} \tag{5c}$$

The strains associated with the displacements in Eq. (5a), (5b), (5c) are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z\begin{Bmatrix} k_x^b \\ k_y^b \\ k_{xy}^b \end{Bmatrix} + f(z)\begin{Bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{Bmatrix},$$
$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = g(z)\begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix}, \tag{6a}$$

where

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\[6pt] \dfrac{\partial v_0}{\partial y} \\[6pt] \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix}, \begin{Bmatrix} k_x^b \\ k_y^b \\ k_{xy}^b \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_b}{\partial x^2} \\[6pt] -\dfrac{\partial^2 w_b}{\partial y^2} \\[6pt] -2\dfrac{\partial^2 w_b}{\partial x\partial y} \end{Bmatrix},$$
$$\begin{Bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_s}{\partial x^2} \\[6pt] -\dfrac{\partial^2 w_s}{\partial y^2} \\[6pt] -2\dfrac{\partial^2 w_s}{\partial x\partial y} \end{Bmatrix}, \begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial w_s}{\partial y} \\[6pt] \dfrac{\partial w_s}{\partial x} \end{Bmatrix} \tag{6b}$$

and: $g(z) = 1 - f'(z), f'(z) = \dfrac{\mathrm{d}f(z)}{\mathrm{d}z}$.

Constitutive equations

The stress state in each layer is given by Hooke's law

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}, \tag{7a}$$

where Q_{ij} are the stiffnesses, which are defined in terms of engineering constants in the material axes of the layer:

$$Q_{11} = \frac{E_{11}}{1 - v_{12}v_{21}}, Q_{22} = \frac{E_{22}}{1 - v_{12}v_{21}}, Q_{12}$$
$$= \frac{v_{12}E_{22}}{1 - v_{12}v_{21}}, Q_{66} = G_{12}, Q_{44} = G_{23}, Q_{55} = G_{13}. \tag{7b}$$

Since the laminate is made of several orthotropic layers with their material axes oriented arbitrarily with respect to laminate coordinates, the constitutive equations of each layer must be transformed to the laminate coordinates $x, y,$ and z. The stress–strain relations in the laminate coordinates of a kth layer are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^{(k)},$$

(7c)

where $\bar{Q}_{ij}$ are the transformed material constants, which are given in (Karama et al. 2003) as

$$\bar{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta)$$

$$\bar{Q}_{22} = Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\cos^3\theta$$
$$\qquad + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\cos\theta$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta$$
$$\qquad + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\cos^3\theta$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\cos^2\theta$$
$$\qquad + Q_{66}(\sin^4\theta + \cos^4\theta)$$

$$\bar{Q}_{44} = Q_{44}\cos^2\theta + Q_{55}\sin^2\theta$$

$$\bar{Q}_{45} = (Q_{55} - Q_{44})\cos\theta\sin\theta$$

$$\bar{Q}_{55} = Q_{55}\cos^2\theta + Q_{44}\sin^2\theta \qquad (7d)$$

In which θ is the angle between the global x-axis and the local x-axis of each layer.

Governing equations

Using Hamilton's energy principle, we derive the equation of motion of the laminated composite plate

$$\delta \int_{t_1}^{t_2} (U - V - T)\mathrm{d}t = 0, \qquad (8a)$$

where U is the strain energy, T is the kinetic energy of the plate, and V is the work of external forces. Employing the principle of minimum total energy leads to the general equation of motion and boundary conditions. Taking the variation of the above equation and integrating by parts, we obtain

$$\int_{t_2}^{t_1} \left[\int_V (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{xz}\delta\gamma_{xz}) \right.$$
$$- \rho(\ddot{u}_0\delta u_0 + \ddot{v}_0\delta v_0 + (\ddot{w}_b + \ddot{w}_s)\delta(w_b + w_s))\mathrm{d}V - \int_A q\delta(w_b + w_s)\mathrm{d}A$$
$$\left. + \int_A \left[N_x^0 \frac{\partial^2(w_b + w_s)}{\partial x^2} + N_y^0 \frac{\partial^2(w_b + w_s)}{\partial y^2} + 2N_{xy}^0 \frac{\partial^2(w_b + w_s)}{\partial x \partial y} \right]\mathrm{d}A \right]\mathrm{d}t = 0,$$

(8b)

where q and N_x^0, N_y^0, N_{xy}^0 are transverse and in-plane distributed force, respectively.

Where two points above a variable means the second derivative with respect to time. Using the combination of Eqs. (6a), (6b), (8a) and (8b) which takes the form

$$\int_{t_2}^{t_1} \left[\int_A \left(-\delta u_0 N_{x,x} - \delta v_0 N_{y,y} - \delta u_0 N_{xy,y} - \delta v_0 N_{xy,x} - \delta w_b M_{x,xx}^b - \delta w_b M_{y,yy}^b - 2\delta w_b M_{xy,xy}^b \right. \right.$$

$$\left. -\delta w_s M_{x,xx}^s - \delta w_s M_{y,yy}^s - 2\delta w_s M_{xy,xy}^s - \delta w_s S_{xz,x}^s - \delta w_s S_{yz,y}^s \right)\mathrm{d}A - \int_A q\delta(w_b + w_s)\mathrm{d}A$$

$$+ \int_A \left[N_x^0(w_{b,xx} + w_{s,xx}) + N_y^0(w_{b,yy} + w_{s,yy}) + 2N_{xy}^0(w_{b,xy} + w_{s,xy}) \right]\mathrm{d}A$$

(9)

$$- \int_A \left\{ \delta u_0(I_1\ddot{u}_0 - I_2\ddot{w}_{b,x} - I_4\ddot{w}_{s,x}) + \delta v_0(I_1\ddot{v}_0 - I_2\ddot{w}_{b,y} - I_4\ddot{w}_{s,y}) \right.$$

$$+ \delta w_b\left[I_1(\ddot{w}_b + \ddot{w}_s) + I_2(\ddot{u}_{0,x} + \ddot{v}_{o,y}) - I_3(\ddot{w}_{b,xx} + \ddot{w}_{b,yy}) - I_5(\ddot{w}_{s,xx} + \ddot{w}_{s,yy}) \right]$$

$$\left. \left. + \delta w_s\left[I_1(\ddot{w}_b + \ddot{w}_s) + I_4(\ddot{u}_{0,x} + \ddot{v}_{o,y}) - I_5(\ddot{w}_{b,xx} + \ddot{w}_{b,yy}) - I_6(\ddot{w}_{s,xx} + \ddot{w}_{s,yy}) \right] \right\}\mathrm{d}A \right]\mathrm{d}t = 0$$

The stress resultants N, M, and S are defined as

$$(N_x, N_y, N_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) \, \mathrm{d}z$$

$$= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} (\sigma_x, \sigma_y, \tau_{xy}) \, \mathrm{d}z \qquad (10a)$$

$$(M_x^b, M_y^b, M_{xy}^b) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) \, z \, \mathrm{d}z$$

$$= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} (\sigma_x, \sigma_y, \tau_{xy}) \, z \, \mathrm{d}z \qquad (10b)$$

$$(M_x^s, M_y^s, M_{xy}^s) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) f(z) \, \mathrm{d}z$$

$$= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} (\sigma_x, \sigma_y, \tau_{xy}) f(z) \, \mathrm{d}z \qquad (10c)$$

$$(S_{xz}^s, S_{yz}^s) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) \, g(z) \, \mathrm{d}z$$

$$= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} (\tau_{xz}, \tau_{yz}) \, g(z) \, \mathrm{d}z \qquad (10d)$$

Inserting Eqs. (7a), (7b), (7c), (7d) into Eqs. (10a), (10b), (10c), (10d) and integrating across the thickness of the plate, the stress resultants are obtained:

$$\begin{Bmatrix} N \\ M^b \\ M^s \end{Bmatrix} = \begin{bmatrix} A & B & B^s \\ B & D & D^s \\ B^s & D^s & H^s \end{bmatrix} \begin{Bmatrix} \varepsilon \\ k^b \\ k^s \end{Bmatrix},$$

$$\begin{Bmatrix} S_{yz}^s \\ S_{xz}^s \end{Bmatrix} = \begin{bmatrix} A_{44}^s & A_{45}^s \\ A_{45}^s & A_{55}^s \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix}, \qquad (11a)$$

where

$$N = \{N_x, N_y, N_{xy}\}^t, \ M^b = \left\{ M_x^b, M_y^b, M_{xy}^b \right\}^t, M^s$$

$$= \left\{ M_x^s, M_y^s, M_{xy}^s \right\}^t \qquad (11b)$$

$$\varepsilon = \left\{ \varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0 \right\}, k^b = \left\{ k_x^b, k_y^b, k_{xy}^b \right\}, k^s = \left\{ k_x^s, k_y^s, k_{xy}^s \right\}$$
$$(11c)$$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}, B = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix},$$

$$D = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \qquad (11d)$$

$$B^s = \begin{bmatrix} B_{11}^s & B_{12}^s & B_{16}^s \\ B_{12}^s & B_{22}^s & B_{26}^s \\ B_{16}^s & B_{26}^s & B_{66}^s \end{bmatrix}, D^s = \begin{bmatrix} D_{11}^s & D_{12}^s & D_{16}^s \\ D_{12}^s & D_{22}^s & D_{26}^s \\ D_{16}^s & D_{26}^s & D_{66}^s \end{bmatrix},$$

$$H^s = \begin{bmatrix} H_{11}^s & H_{12}^s & H_{16}^s \\ H_{12}^s & H_{22}^s & H_{26}^s \\ H_{16}^s & H_{26}^s & H_{66}^s \end{bmatrix} \qquad (11e)$$

And the stiffness components and inertias are given as

$$\left(A_{ij}, B_{ij}, D_{ij}, B_{ij}^s, D_{ij}^s, H_{ij}^s \right) = \int_{-h/2}^{h/2} \bar{Q}_{ij}(1, z, z^2, f(z), zf(z), f^2(z)) \mathrm{d}z ,$$

$$(i,j) = (1, 2, 6) \qquad (12a)$$

$$A_{ij}^s = \int_{-h/2}^{h/2} \bar{Q}_{ij} [g(z)]^2 \, \mathrm{d}z, \ (i,j) = (4, 5) \qquad (12b)$$

$$(I_1, I_2, I_3, I_4, I_5, I_6) = \int_{-h/2}^{h/2} \rho(1, z, z^2, f(z), zf(z), [f(z)]^2) \mathrm{d}z.$$
$$(12c)$$

Collecting the coefficients of δu_0, δv_0, δw_b, and δw_s in Eq. (9), the equations of motion are obtained as

$$\delta u_0 : N_{x,x} + N_{xy,y} = I_1 \ddot{u}_0 - I_2 \ddot{w}_{b,x} - I_4 \ddot{w}_{s,x}$$

$$\delta v_0 : N_{xy,x} + N_{y,y} = I_1 \ddot{v}_0 - I_2 \ddot{w}_{b,y} - I_4 \ddot{w}_{s,y}$$

$$\delta w_b : M_{x,xx}^b + 2M_{xy,xy}^b + M_{y,yy}^b + q + N$$

$$= I_1(\ddot{w}_b + \ddot{w}_s) + I_2(\ddot{u}_{0,x} + \ddot{v}_{0,y})$$

$$- I_3(\ddot{w}_{b,xx} + \ddot{w}_{b,yy}) - I_5(\ddot{w}_{s,xx} + \ddot{w}_{s,yy}),$$

$$\delta w_s : M_{x,xx}^s + 2M_{xy,xy}^s + M_{y,yy}^s + S_{xz,x}^s + S_{yz,y}^s + q + N$$

$$= I_1(\ddot{w}_b + \ddot{w}_s) + I_4(\ddot{u}_{0,x} + \ddot{v}_{0,y})$$

$$- I_5(\ddot{w}_{b,xx} + \ddot{w}_{b,yy}) - I_6(\ddot{w}_{s,xx} + \ddot{w}_{s,yy}) \qquad (13a)$$

where N is defined by

$$N = N_x^0 \frac{\partial^2 (w_b + w_s)}{\partial x^2} + N_y^0 \frac{\partial^2 (w_b + w_s)}{\partial y^2}$$

$$+ 2N_{xy}^0 \frac{\partial^2 (w_b + w_s)}{\partial x \partial y}, \qquad (13b)$$

Clearly, when the effect of transverse shear deformation is neglected ($w_s = 0$), Eqs. (13a), and (13b) yield the equations of motion of a composite plate based on the classical theory of plates.

Analytical solutions for simply supported rectangular laminates

For antisymmetric cross-ply laminates

The Navier solutions can be developed for rectangular laminates with two sets of simply supported boundary conditions. For antisymmetric cross-ply laminates, the following plate stiffnesses are identically zero:

$$A_{16} = A_{26} = D_{16} = D_{26} = D_{16}^s = D_{26}^s = H_{16}^s = H_{26}^s = 0$$
$$B_{12} = B_{26} = B_{16} = B_{66} = B_{12}^s = B_{16}^s = B_{26}^s = B_{66}^s = A_{45}^s = 0$$
$$B_{22} = -B_{11}, B_{22}^s = -B_{11}^s \qquad (14)$$

The following boundary conditions for antisymmetric cross-ply laminates can be written as

$$v(0,y) = w_b(0,y) = w_s(0,y) = \frac{\partial w_b}{\partial y}(0,y) = \frac{\partial w_s}{\partial y}(0,y) = 0$$

$$v(a,y) = w_b(a,y) = w_s(a,y) = \frac{\partial w_b}{\partial y}(a,y) = \frac{\partial w_s}{\partial y}(a,y) = 0$$

$$N_x(0,y) = M_x^b(0,y) = M_x^s(0,y) = N_x(a,y) = M_x^b(a,y)$$
$$= M_x^s(a,y) = 0$$

$$u(x,0) = w_b(x,0) = w_s(x,0) = \frac{\partial w_b}{\partial x}(x,0) = \frac{\partial w_s}{\partial x}(x,0) = 0$$

$$u(x,b) = w_b(x,b) = w_s(x,b) = \frac{\partial w_b}{\partial x}(x,b) = \frac{\partial w_s}{\partial x}(x,b) = 0$$

$$N_y(x,0) = M_y^b(x,0) = M_y^s(x,0) = N_y(x,b) = M_y^b(x,b)$$
$$= M_y^s(x,b) = 0 \qquad (15)$$

The boundary conditions in Eq. (15) are satisfied by the following expansions:

$$u_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn}\, e^{i\omega t} \cos(\lambda x)\sin(\mu y)$$

$$v_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn}\, e^{i\omega t} \sin(\lambda x)\cos(\mu y)$$

$$w_b = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{bmn}\, e^{i\omega t} \sin(\lambda x)\sin(\mu y) \qquad (16)$$

$$w_s = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn}\, e^{i\omega t} \sin(\lambda x)\sin(\mu y)$$

where U_{mn}, V_{mn}, W_{bmn}, and W_{smn} unknown parameters must be determined, ω is the Eigen frequency associated with (m, n) the Eigen-mode, and $\lambda = \frac{m\pi}{a}$ and $\mu = \frac{n\pi}{b}$.

The transverse load q is also expanded in the double-Fourier sine series as follows:

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn}\sin(\lambda x)\sin(\mu y) \qquad (17)$$

The coefficients Q_{mn} are given below for some typical loads:

$$Q_{mn} = \frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin(\lambda x)\sin(\mu y)\mathrm{d}x\mathrm{d}y,$$
$$\text{and}\begin{cases} Q_{mn} = q_0 \\ Q_{mn} = \dfrac{16q_0}{mn\pi^2} \end{cases} \qquad (18)$$

Substituting Eqs. (14), (16), and (17) into Eqs. (13a), (13b), the Navier solution of antisymmetric cross-ply laminates can be determined from equations

$$\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33}+S & a_{34}+S \\ a_{14} & a_{24} & a_{34}+S & a_{44}+S \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{34} & ms_{44} \end{bmatrix}\right)$$
$$\begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{bmn} \\ W_{smn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q \\ q \end{Bmatrix}, \qquad (19)$$

where

$$a_{11} = A_{11}\lambda^2 + A_{66}\mu^2, a_{12} = \lambda\mu(A_{12} + A_{66}),$$
$$a_{13} = -B_{11}\lambda^3, a_{14} = -B_{11}^s\lambda^3$$
$$a_{22} = A_{66}\lambda^2 + A_{22}\mu^2, a_{23} = B_{11}\mu^3, a_{24} = B_{11}^s\mu^3$$
$$a_{33} = D_{11}\lambda^4 + 2(D_{12} + 2D_{66})\lambda^2\mu^2 + D_{22}\mu^4$$
$$a_{34} = D_{11}^s\lambda^4 + 2(D_{12}^s + 2D_{66}^s)\lambda^2\mu^2 + D_{22}^s\mu^4$$
$$a_{44} = H_{11}^s\lambda^4 + 2(H_{12}^s + 2H_{66}^s)\lambda^2\mu^2 + H_{22}^s\mu^4 + A_{55}^s\lambda^2 + A_{44}^s\mu^2$$
$$m_{11} = m_{22} = I_1, \ m_{33} = I_1 + I_3(\lambda^2 + \mu^2)$$
$$m_{34} = I_1 + I_5(\lambda^2 + \mu^2), m_{44} = I_1 + I_6(\lambda^2 + \mu^2), S = N_x^0\lambda^2 + N_y^0\mu^2 \qquad (20)$$

For antisymmetric angle-ply laminates

For antisymmetric angle-ply laminates, the following plate stiffnesses are identically zero:

$$A_{16} = A_{26} = D_{16} = D_{26} = D_{16}^s = D_{26}^s = H_{16}^s = H_{26}^s = 0$$
$$B_{11} = B_{12} = B_{22} = B_{66} = B_{11}^s = B_{12}^s = B_{22}^s = B_{66}^s = A_{45}^s = 0. \qquad (21)$$

The following boundary conditions for antisymmetric angle-ply laminates can be written as

$$u(0,y) = w_b(0,y) = w_s(0,y) = \frac{\partial w_b}{\partial y}(0,y) = \frac{\partial w_s}{\partial y}(0,y) = 0$$

$$u(a,y) = w_b(a,y) = w_s(a,y) = \frac{\partial w_b}{\partial y}(a,y) = \frac{\partial w_s}{\partial y}(a,y) = 0$$

$$N_{xy}(0,y) = M_x^b(0,y) = M_x^s(0,y) = N_{xy}(a,y) = M_x^b(a,y)$$
$$= M_x^s(a,y) = 0$$

$$v(x,0) = w_b(x,0) = w_s(x,0) = \frac{\partial w_b}{\partial x}(x,0) = \frac{\partial w_s}{\partial x}(x,0) = 0$$

$$v(x,b) = w_b(x,b) = w_s(x,b) = \frac{\partial w_b}{\partial x}(x,b) = \frac{\partial w_s}{\partial x}(x,b) = 0$$

$$N_{xy}(x,0) = M_y^b(x,0) = M_y^s(x,0) = N_{xy}(x,b) = M_y^b(x,b)$$
$$= M_y^s(x,b) = 0. \tag{22}$$

The boundary conditions in Eq. (22) are satisfied by the following expansions:

$$u_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} e^{i\omega t} \sin(\lambda x) \cos(\mu y)$$

$$v_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} e^{i\omega t} \cos(\lambda x) \sin(\mu y)$$

$$w_b = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{bmn} e^{i\omega t} \sin(\lambda x) \sin(\mu y) \tag{23}$$

$$w_s = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{smn} e^{i\omega t} \sin(\lambda x) \sin(\mu y).$$

Substituting Eqs. (21), (17), and (23) into Eqs. (13a), (13b), the equations of the form in Eq. (19) are obtained with the following coefficients:

$$a_{11} = A_{11}\lambda^2 + A_{66}\mu^2; a_{12} = \lambda\mu(A_{12} + A_{66});$$

$$a_{13} = -(3B_{16}\lambda^2\mu + B_{26}\mu^3), a_{22} = A_{66}\lambda^2 + A_{22}\mu^2$$

$$a_{14} = -(3B_{16}^s\lambda^2\mu + B_{26}^s\mu^3); a_{23} = -(B_{16}\lambda^3 + 3B_{26}\lambda\mu^2);$$

$$a_{24} = -(B_{16}^s\lambda^3 + 3B_{26}^s\lambda\mu^2)$$

$$a_{33} = D_{11}\lambda^4 + 2(D_{12} + 2D_{66})\lambda^2\mu^2 + D_{22}\mu^4;$$

$$a_{34} = D_{11}^s\lambda^4 + 2(D_{12}^s + 2D_{66}^s)\lambda^2\mu^2 + D_{22}^s\mu^4$$

$$a_{44} = H_{11}^s\lambda^4 + 2(H_{12}^s + 2H_{66}^s)\lambda^2\mu^2 + H_{22}^s\mu^4 + A_{55}^s\lambda^2 + A_{44}^s\mu^2;$$

$$S = N_x^0\lambda^2 + N_y^0\mu^2 m_{11} = m_{22} = I_1, m_{33} = I_1 + I_3(\lambda^2 + \mu^2);$$

$$m_{34} = I_1 + I_5(\lambda^2 + \mu^2), m_{44} = I_1 + I_6(\lambda^2 + \mu^2) \tag{24}$$

Numerical results and discussion

In this study, various numerical examples are described and discussed for verifying the accuracy of the present models in predicting the static bending, critical buckling load, and free vibration behaviors of simply supported antisymmetric cross-ply and angle-ply laminates. To verify, the results achieved by current models are compared with those of

Reddy (1984) and exact solution of elasticity in three dimensions (Pagano 1970). The effectiveness of these present′s theories is use of the extension component of transverse displacement. The following lamina properties are used:

Material 1 (Noor 1975): $E_1 = 40E_2$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $v_{12} = 0.25$

Material 2 (Ren 1990): $E_1 = 40E_2$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.6E_2$, $v_{12} = 0.25$

Material 3 (Pagano 1970): $E_1 = 25E_2$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, $v_{12} = 0.25$

For convenience, the following nondimensionalizations are used in presenting the numerical results in graphical and tabular forms:

$$\bar{w} = \frac{100h^3E_2}{q_0a^4}w(a/2,b/2), \bar{\sigma}_x = \frac{h^2}{q_0a^2}\sigma_x(a/2,b/2),$$

$$\bar{\sigma}_y = \frac{h^2}{q_0a^2}\sigma_y(a/2,b/2)$$

$$\bar{\tau}_{xy} = \frac{h^2}{q_0a^2}\tau_{xy}(0,0), \bar{\tau}_{xz} = \frac{h}{q_0a}\tau_{xz}(0,b/2),$$

$$\bar{\omega} = \omega\frac{a^2}{h}\sqrt{\frac{\rho}{E_2}} \bar{N} = N_{cr}\left(\frac{a^2}{E_2h^3}\right)$$

$$\tag{25}$$

Numerical results for bending analysis

The static bending solution obtained by setting the time derivative terms and in-plane forces to zero and simplified as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{bmn} \\ W_{smn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \\ Q_{mn} \end{Bmatrix}. \tag{26}$$

A simply supported two-layer antisymmetric angle-ply (45°/−45°) laminate under sinusoidal transverse load is considered. Material set 2 is used. The numerical results of nondimensionalized deflection for the square and rectangular plates are shown in Table 1. In the case of thick plates, a considerable difference exists between the results obtained using the various models and the values reported by Ren's model (Ren 1990). For a / h ratio equal to 4, the deflections predicted by Reddy's (Reddy 1984), both these theories are 20–25 % lower for a square flat, and 15 % and 20 % lower for a rectangular flat as compared to the values, therefore, obtained by Ren model's (Ren 1990). The results computed using all the five models are in good agreement with those reported by Reddy (Reddy 1984) and Ren (Ren 1990) for thin plates (a/h = 100). The nondimensionalized deflections of two-layer (45°/−45°) square laminates under sinusoidal transverse load are presented in Fig. 1 for various ratio of modulus E_1/E_2 ($G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.6E_2$, $m_{12} = 0.25$, a/h = 10).

Table 1 Nondimensionalized deflections of simply supported two-layer (45°/−45°) square and rectangular laminates under sinusoidal transverse load

| a/h	Theory	$\bar{w}$	
		Square plate ($a = b$)	Rectangular plate ($b = 3a$)
4	Model-Ren (Ren 1990)	1.4471	3.9653
	Model-HSDT (Reddy 1984)	1.0203	3.1560
	Present model 1	1.0220	3.0995
	Present model 2	1.0203	3.0971
10	Model-Ren (Ren 1990)	0.6427	2.3953
	Model-HSDT (Reddy 1984)	0.5581	2.2439
	Present model 1	0.5583	2.2328
	Present model 2	0.5581	2.2325
100	Model-Ren (Ren 1990)	0.4685	2.0686
	Model-HSDT (Reddy 1984)	0.4676	2.0671
	Present model 1	0.4676	2.0670
	Present model 2	0.4676	2.0670

Table 2 Nondimensionalized deflections and stresses in two-layer (0°/90°) simply supported square laminated plate under sinusoidal transverse load

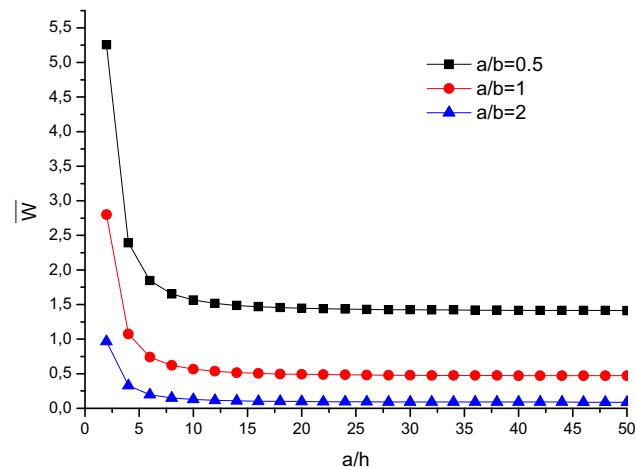| a/h	Theory	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$
2	Model-elasticity (Pagano 1970)	4.9362	−0.9070	1.4480	−0.0964
	Model-Reddy (Reddy 1984)	4.5619	−1.4277	1.4277	−0.0719
	Present model 1	4.5728	−1.4256	1.4256	−0.0719
	Present model 2	4.5619	−1.4277	1.4277	−0.0719
5	Model-elasticity (Pagano 1970)	1.7287	−0.7723	0.8036	−0.0586
	Model-Reddy (Reddy 1984)	1.6670	−0.8385	0.8385	−0.0558
	Present model 1	1.6680	−0.8380	0.8380	−0.0558
	Present model 2	1.6670	−0.8385	0.8385	−0.0558
10	Model-elasticity (Pagano 1970)	1.2318	−0.7317	0.7353	−0.0540
	Model-Reddy (Reddy 1984)	1.2161	−0.7468	0.7468	−0.0533
	Present model 1	1.2164	−0.7467	0.7467	−0.0533
	Present model 2	1.2161	−0.7468	0.7468	−0.0533
20	Model-elasticity (Pagano 1970)	1.1060	−0.7200	0.7206	−0.0529
	Model-Reddy (Reddy 1984)	1.1018	−0.7235	0.7235	−0.0527
	Present model 1	1.1019	−0.7235	0.7235	−0.0527
	Present model 2	1.1018	−0.7235	0.7235	−0.0527
100	Model-elasticity (Pagano 1970)	1.0742	−0.7219	0.7219	−0.0529
	Model-Reddy (Reddy 1984)	1.0651	−0.7161	0.7161	−0.0525
	Present model 1	1.0651	−0.7161	0.7161	−0.0525
	Present model 2	1.0651	−0.7161	0.7161	−0.0525

A simply supported two-layer (0°/90°) antisymmetric square laminate under sinusoidal transverse load is considered. The layers have equal thickness. Material set 3 is used. Numerical values of nondimensionalized transverse displacement and in plane stresses are shown in Table 2. Three-dimensional elasticity results are obtained using the method given by (Pagano 1970). The results clearly indicate that the percentage error with respect to three-dimensional elasticity solution in predicting the transverse displacement and in-plane stresses is very much lesser in the case of present models and the prediction of in-plane normal stresses, $\bar{\sigma}_x$, $\bar{\sigma}_y$, is very poor.

To further illustrate the accuracy of present theory for wide range of thickness ratio a/h and material anisotropy E_1/E_2, the variations of dimensionless deflection with respect to thickness ratio and material anisotropy are illustrated in Figs. 2 and 3, respectively. The obtained results are compared with those predicted by (Reddy 1984). Again, the present's models and existing FSDT give almost identical solutions, whereas CPT underestimates deflections of thick laminates with $a/h < 20$ due to ignoring shear deformation effects (Fig. 3). The through thickness variations and corresponding values of the in-plane displacement, normal stresses ($\bar{\sigma}_x$, $\bar{\sigma}_y$), and shear
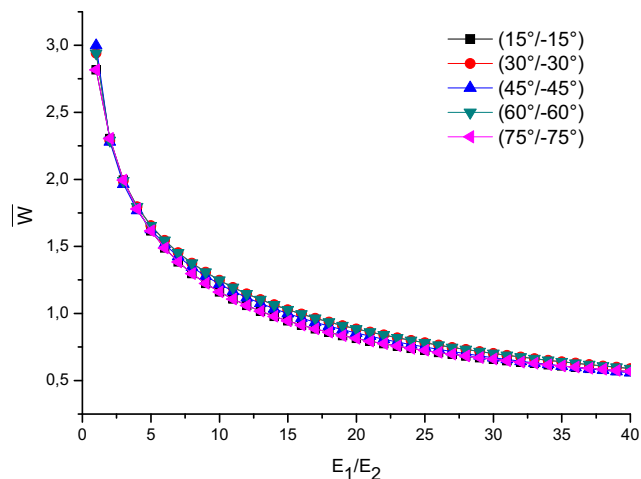
Fig. 2 The effect of modulus ratio on nondimensionalized deflection of simply supported two-layer $(\theta/-\theta)$ square laminates under sinusoidal transverse load $(a/h = 10)$
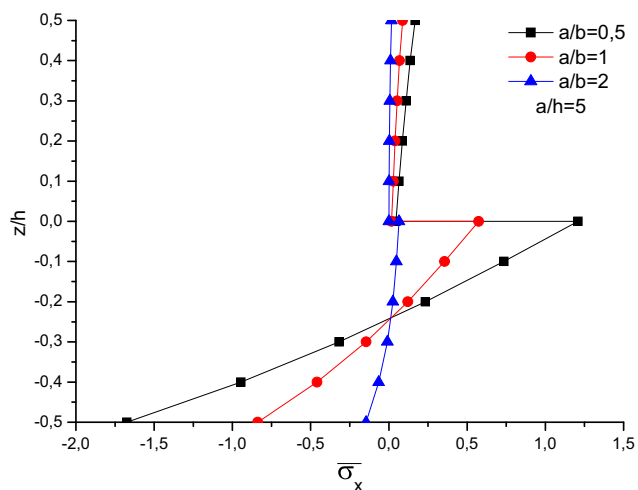

Fig. 4 Variation of normal stress $\bar{\sigma}_x$ through the thickness of simply supported two-layer (0°/90°) square plate for different values of the aspect ratio

Fig. 3 The effect of side-to-thickness ratio on nondimensionalized deflection of simply supported two-layer (45°/−45°) square laminates under sinusoidal transverse load

Fig. 5 Variation of normal stress $\bar{\sigma}_y$ through the thickness of simply supported antisymmetric two-layer (0°/90°) square plate for different values of the aspect ratio
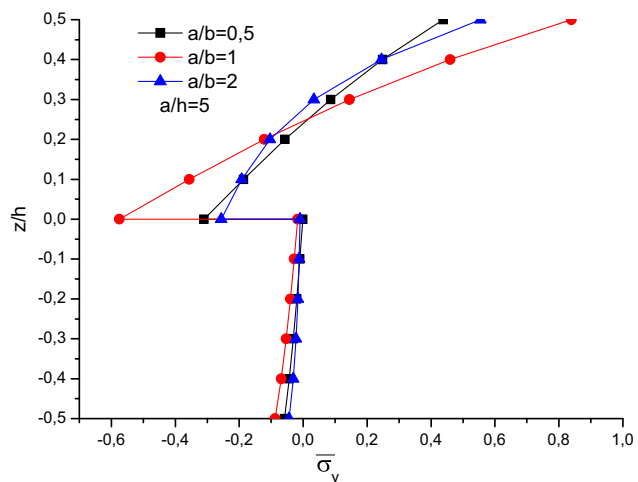
stresses $(\bar{\sigma}_{xy}, \bar{\sigma}_{xz})$ are also given in Figs. 4, 5, 6, and 7, respectively, for a moderately thick laminate with $a/h = 10$. An excellent agreement between the results predicted by the present theories and results of the first-order and the other higher order theories is found in the literature.

Numerical results for free vibration analysis

In the case of free vibration, the natural frequencies of the laminates can be obtained by setting the determinant of the coefficient of the following matrix to zero:
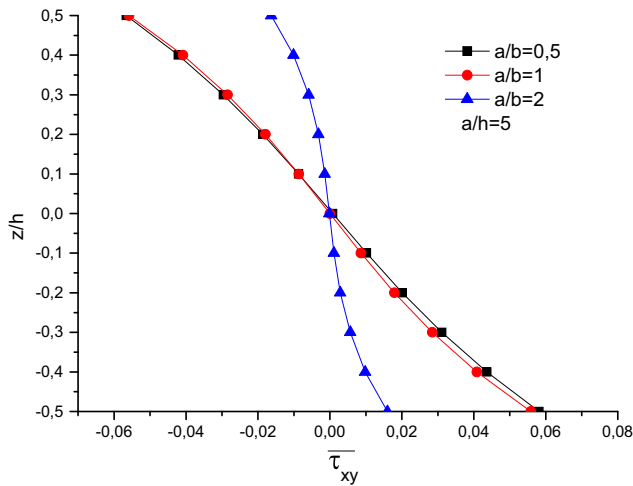
$$\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{34} & ms_{44} \end{bmatrix} \right)$$
$$\begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{bmn} \\ W_{smn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{27}$$

In Tables 3 and 4, the nondimensional fundamental frequencies of anti-symmetrically laminated cross-ply plates obtained using different shear deformation theories

Fig. 6 Variation of longitudinal tangential stress $\bar{\tau}_{xy}$ through the thickness of simply supported two-layer (0°/90°) square plate for different values of the aspect ratio
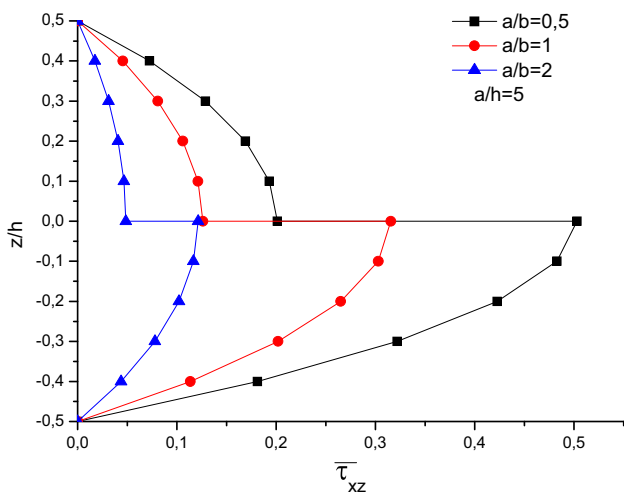
Fig. 7 Variation of tangential stress $\bar{\tau}_{xz}$ through the thickness of simply supported two-layer (0°/90°) square plate for different values of the aspect ratio

are shown for various values of *a/h* and modules ratios. It can be seen that, in general, the present model gives more accurate results in predicting the natural frequencies than the PSDT (Reddy 1984) and the three-dimensional elasticity solution given in (Noor 1973). It should be noted that unknown functions in the present model are four, while the unknown functions in the higher order shear deformation theories (Reddy 1984) are five. It can be concluded that the present model is not only accurate, but also simple in predicting the natural frequencies of laminated plates.

The variation of natural frequencies with respect to side-to-thickness ratio *a/h* is presented in Table 5. The natural

frequencies obtained using the present model is compared with Reddy's theory PSDT Reddy (1984), Swaminathan and Patil (2008), and FSDT. In the case of thick plates (*a/h* ratios 2, 4, 5, and 10) there is a considerable difference between the results computed using the present and the theories of Reddy (1984), Swaminathan and Patil (2008), and (Xiang et al. (2011). The variation of natural frequencies with respect to side-to-thickness ratio *a/h* for different E_1/E_2 ratio is presented. For a four-layered thick plate with *a/h* ratio equal to 2 and E_1/E_2 ratio equal to 3 and 10, the percentage difference in values predicted by present theory is 0.15 and 3.50 % lower as compared to Reddy's theory PSDT Reddy (1984) and Swaminathan and Patil (2008). At higher range of E_1/E_2 ratio equal to 20–40, the percentage difference in values between both the theories is very much higher, and Reddy's theory very much over-predicts the natural frequency values. For a four-layered thick plate with a/h ratio equal to 2 and E_1/E_2 ratio equal to 20, 30, and 40, the percentage differences in values predicted by present theory are 6, 8, and 9.50 % lower as compared to the theories of Reddy (1984), Swaminathan and Patil (2008), and Xiang et al. (2011). The difference between the models tends to reduce for thin and relatively thin plates. Irrespective of the number of layers, the percentage difference in values between the two theories increases with the increase in the degree of anisotropy. As the number of layer increases, the percentage difference in values between the two theories decreases significantly.

The obtained results of fundamental frequencies are compared with the exact 3D solutions reported by Reddy's theory (Reddy 1984). Here also the results obtained by the present theories are almost identical with those predicted by existing FSDT. This statement is also firmly demonstrated in Figs. 8 and 9 in which the results obtained by the present theory and FSDT are in excellent agreement for a wide range of thickness ratio *a/h*. According to Table 6 the present results are in good agreement with the results of Reddy PSDT Reddy (1984), Swaminathan and Patil (2008) and Xiang et al. (2011).

Numerical results for buckling analysis

For buckling analysis, the applied loads are assumed to be in-plane forces

$$N_x^0 = -N_0, \quad N_y^0 = \gamma N_0, \quad \gamma = \frac{N_x^0}{N_y^0}, \quad N_{xy}^0 = 0. \quad (28)$$

The buckling solution can be obtained from Eq. (19) by setting the time derivative terms and transverse forces to zero:

Table 3 nondimensional fundamental frequencies of antisymmetric square plates at various values of orthotropy ratio with $a/h = 5$

No of layers	Theory	E_1/E_2				
		3	10	20	30	40
$(0°/90°)_1$	Model-Exact (Noor 1973)	6.2578	6.9845	7.6745	8.1763	8.5625
	Present model 1	6.2168	6.9881	7.8198	8.5028	9.0841
	Present model 2	6.2169	6.9887	7.8210	8.5050	9.0871
	Model-PSDT (Reddy 1984)	6.2169	6.9887	7.8210	8.5050	9.0871
$(0°/90°)_2$	Model-exact (Noor 1973)	6.5455	8.1445	9.4055	10.1650	10.6790
	Present model 1	6.5009	8.1958	9.6273	10.5359	11.1728
	Present model 2	6.5008	8.1954	9.6265	10.5348	11.1716
	Model-PSDT (Reddy 1984)	6.5008	8.1954	9.6265	10.5348	11.1716
$(0°/90°)_3$	Model-exact (Noor 1973)	6.6100	8.4143	9.8398	10.6950	11.2720
	Present model 1	6.5558	8.4053	9.9182	10.8546	11.5009
	Present model 2	6.5558	8.4052	9.9181	10.8547	11.5012
	Model-PSDT (Reddy 1984)	6.5558	8.4052	9.9181	10.8547	11.5012
$(0°/90°)_5$	Model-exact (Noor 1973)	6.6458	8.5625	10.0843	11.0027	11.6245
	Present model 1	6.5842	8.5126	10.0671	11.0191	11.6721
	Present model 2	6.5842	8.5126	10.0674	11.0197	11.6730
	Model-PSDT (Reddy 1984)	6.5842	8.5126	10.0674	11.0197	11.6730

Table 4 Nondimensional fundamental frequencies of antisymmetric square plates at various values of a/h with $E_1/E_2 = 40$

No of layers	Theory	a/h					
		2	4	10	20	50	100
$(0°/90°)_1$	Present model 1	5.7100	8.3507	10.5669	11.1048	11.2750	11.3001
	Present model 2	5.7170	8.3546	10.5680	11.1052	11.2751	11.3002
	Model-PSDT (Reddy 1984)	5.7170	8.3546	10.5680	11.1052	11.2751	11.3002
$(0°/90°)_2$	Present model 1	5.7528	9.7366	14.8474	16.5737	17.1850	17.2784
	Present model 2	5.7546	9.7357	14.8463	16.5733	17.1849	17.2784
	Model-PSDT (Reddy 1984)	5.7546	9.7357	14.8463	16.5733	17.1849	17.2784
$(0°/90°)_3$	Present model 1	5.8702	9.9870	15.4635	17.3774	18.0644	18.1699
	Present model 2	5.8741	9.9878	15.4632	17.3772	18.0644	18.1698
	Model-PSDT (Reddy 1984)	5.8741	9.9878	15.4632	17.3772	18.0644	18.1698
$(0°/90°)_5$	Present model 1	5.9476	10.1226	15.7700	17.7743	18.4984	18.6097
	Present model 2	5.9524	10.1241	15.7700	17.7743	18.4984	18.6097
	Model-PSDT (Reddy 1984)	5.9524	10.1241	15.7700	17.7743	18.4984	18.6097

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} - N_0(\lambda^2 + \gamma\mu^2) & a_{34} - N_0(\lambda^2 + \gamma\mu^2) \\ a_{14} & a_{24} & a_{34} - N_0(\lambda^2 + \gamma\mu^2) & a_{44} - N_0(\lambda^2 + \gamma\mu^2) \end{bmatrix}$$
$$\begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{bmn} \\ W_{smn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{29}$$

Following the condensation of variables procedure to eliminate the in-plane displacements U_{mn} and V_{mn}, the following system is obtained:

$$\begin{bmatrix} \bar{a}_{33} - N_0(\lambda^2 + \gamma\mu^2) & \bar{a}_{34} - N_0(\lambda^2 + \gamma\mu^2) \\ \bar{a}_{43} - N_0(\lambda^2 + \gamma\mu^2) & \bar{a}_{44} - N_0(\lambda^2 + \gamma\mu^2) \end{bmatrix} \begin{Bmatrix} W_{bmn} \\ W_{smn} \end{Bmatrix}$$
$$= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{30}$$

where

$$\bar{a}_{33} = a_{33} - a_{13}\frac{b_1}{b_0} - a_{23}\frac{b_2}{b_0}, \bar{a}_{34} = a_{34} - a_{14}\frac{b_1}{b_0} - a_{24}\frac{b_2}{b_0},$$
$$\bar{a}_{43} = a_{34} - a_{13}\frac{b_3}{b_0} - a_{23}\frac{b_4}{b_0} \bar{a}_{44} = a_{44} - a_{14}\frac{b_3}{b_0} - a_{24}\frac{b_4}{b_0},$$
$$b_0 = a_{11}a_{22} - a_{12}^2, b_1 = a_{13}a_{22} - a_{12}a_{23}$$
$$b_2 = a_{11}a_{23} - a_{12}a_{13}, b_3 = a_{14}a_{22} - a_{12}a_{24}, b_4 = a_{11}a_{24} - a_{12}a_{14} \tag{31}$$

Table 5 Non-dimensionalized fundamental frequencies for a simply supported antisymmetric angle-ply square laminated plate

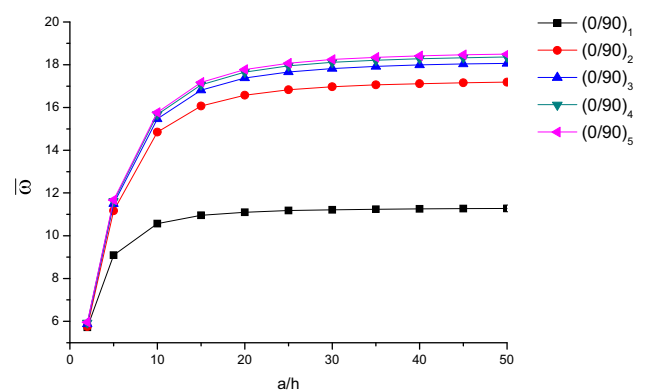| No of layers	Theory	a/h								
		2	4	5	10	12.5	20	25	50	100
(45°/−45°)₁	Present model 1	6.3247	9.7517	10.8336	13.2605	13.7058	14.2455	14.3823	14.5722	14.6211
	Present model 2	6.3368	9.7594	10.8398	13.2631	13.7040	14.2463	14.3828	14.5724	14.6212
	Model-PSDT (Reddy 1984)	6.2837	9.7593	10.8401	13.2630	13.7040	14.2463	14.3827	14.5723	14.6214
	Model-Swaminathan (Swaminathan and Patil 2008)	5.3325	8.8426	10.0350	12.9115	13.4690	14.1705	14.3500	14.6012	14.6668
(45°/−45°)₂	Present model 1	6.1019	10.6508	12.5342	18.3240	19.7645	21.8072	22.3804	23.2238	23.4508
	Present model 2	6.1068	10.6508	12.5332	18.3221	19.7621	21.8062	22.3798	23.2237	23.4508
	Model-PSDT (Reddy 1984)	6.1067	10.6507	12.5331	18.3221	19.7621	21.8063	22.3798	23.2236	23.4507
	Model-Swaminathan (Swaminathan and Patil 2008)	5.5674	10.0731	11.9465	17.8773	19.4064	21.6229	22.2554	23.1949	23.4499
(45°/−45°)₄	Present model 1	6.3049	10.9870	12.9697	19.2659	20.8885	23.2390	23.9092	24.9046	25.1745
	Present model 2	6.3140	10.9906	12.9720	19.2660	20.8885	23.2388	23.9091	24.9046	25.1745
	Model-PSDT (Reddy 1984)	6.2836	10.9905	12.9719	19.2659	20.8884	23.2388	23.9091	24.9046	25.1744
	Model-Swaminathan (Swaminathan and Patil 2008)	5.9234	10.7473	12.7523	19.1258	20.7784	23.1829	23.8713	24.8959	25.1741
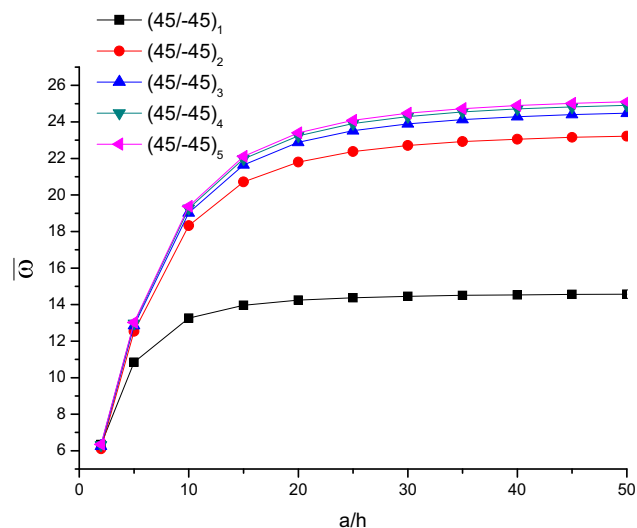
Fig. 8 Variation of dimensionless fundamental frequency of antisymmetric angle-ply (45/−45)ₙ square laminates versus thickness ratio (Material 3, $E_1/E_2 = 40$)

Fig. 9 Variation of dimensionless fundamental frequency of antisymmetric cross-ply (0/90)ₙ square laminates versus thickness ratio (Material 3, $E_1/E_2 = 40$)

For nontrivial solution, the determinant of the coefficient matrix in Eq. (30) must be zero. This gives the following expression for buckling load:

$$N_0 = \frac{1}{\lambda^2 + \gamma\mu^2} \times \frac{\bar{a}_{33}\bar{a}_{44} - \bar{a}_{34}\bar{a}_{43}}{\bar{a}_{33} + \bar{a}_{44} - \bar{a}_{34} - \bar{a}_{43}} \quad (32)$$

A simply supported anti-symmetric cross-ply (0/90)ₙ (n = 2, 3, 5) square laminate subjected to uniaxial compressive load is considered. Table 7 shows a comparison between the results obtained using the various models and the three-dimensional elasticity solutions given by Noor (1975). The results clearly indicate that the present model gives more accurate results in predicting the buckling loads when compared to Reddy (1984). Compared to the three-dimensional elasticity solution, the buckling loads predicted by present model, Reddy (1984), are 6–7 %, for four-layer antisymmetric cross-ply (0/90/0/90) square laminates. The effect of side-to-thickness ratio on buckling load of simply supported four-layer (0/90/0/90) square laminates is also presented in Figs. 10 and 11.

In Table 8, a simply supported two-layer anti-symmetric angle-ply (θ/ − θ) square laminate subjected to uniaxial compressive is considered. The numerical values of buckling. The results are compared with the values reported by Ren (1990). For all values of side-to-thickness ratio and fiber orientation, the buckling loads predicted by the present model and Reddy (1984) are almost identical. For

Table 6 The non-dimensional fundamental frequency of the simply supported square plate ($\theta/-\theta/\ldots$) ($E_1/E_2 = 40$)

| Layers	Theory	a/h			
		10	20	50	100
$(5/-5/5/-5/5/-5)$	Present model 1	15.9840	18.0774	18.8394	18.9568
	Present model 2	15.9841	18.0774	18.8393	18.9568
	Model-PSDT (Reddy 1984)	14.848	17.619	18.753	18.935
	Model-Xiang (Xiang et al. 2011)	15.405	17.943	18.942	19.206
$(30/-30/30/-30/30/-30)$	Present model 1	18.3356	21.7196	23.0815	23.2988
	Present model 2	18.3353	21.7194	23.0814	23.2988
	Model-PSDT (Reddy 1984)	18.170	21.648	23.067	23.295
	Model-Xiang (Xiang et al. 2011)	19.075	22.304	23.579	23.968
$(45/-45/45/-45/45/-45)$	Present model 1	19.0252	22.8770	24.4802	24.7392
	Present model 2	19.0249	22.8768	24.4802	24.7392
	Model-PSDT (Reddy 1984)	19.025	22.877	24.480	24.739
	Model-Xiang (Xiang et al. 2011)	20.027	23.623	25.061	25.478

Table 7 Nondimensional uniaxial buckling load of simply supported antisymmetric cross-ply ($0/90/\ldots$) square laminates ($a/h = 10$)

Number of layers	Theory	$\overline{N}$
4	Present model 1	22.5821
	Present model 2	22.5790
	Model-exact (Noor 1975)	21.2796
	Model-Reddy (Reddy 1984)	22.5790
6	Present model 1	24.4605
	Present model 2	24.4596
	Model-exact (Noor 1975)	23.6689
	Model-Reddy (Reddy 1984)	24.4596
10	Present model 1	25.4223
	Present model 2	25.4225
	Model-exact (Noor 1975)	24.9636
	Model-Reddy (Reddy 1984)	25.4225

Fig. 10 The effect of side-to-thickness ratio on nondimensionalized uniaxial buckling load of simply supported antisymmetric cross-ply $(0/90)_n$ square laminates

a/h ratio equal to 4 and the fiber orientation equal to 30°, the buckling load values predicted by Reddy (1984), and present model are 18–2 % lower as compared to the values obtained by Ren (1990). The results computed using all the five models are in a good agreement with those reported by Ren (1990) for thin plates ($a/h = 100$). The effect of modulus ratio on nondimensionalized uniaxial buckling load of simply supported two-layer (45/−45) square laminate is presented in Fig. 12 ($G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $v_{12} = 0.25$, $a/h = 10$).

Conclusions

A refined higher order shear deformation theory of plates has been successfully developed for the static, buckling, and free vibration of simply supported laminated plates. The theory allows for a square-law variation in the
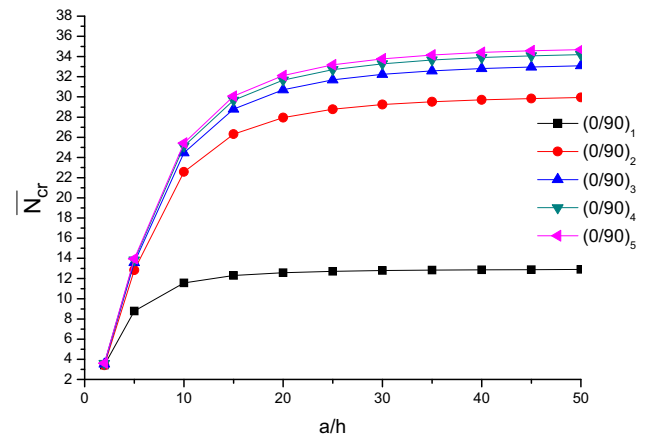
transverse shear strains across the plate thickness and satisfies the zero-traction boundary conditions on the top and bottom surfaces of the plate without using shear correction factors. The equations of motion were derived from Hamilton's principle. The accuracy and efficiency of the present models have been demonstrated for static and free vibration behaviors of anti-symmetric cross-ply and angleply laminates. The conclusions of this theory are as follows:

- The deflection load obtained using present models (a simpler version of present theory with four unknowns) and other higher-order theories found in the literature (five unknowns) are almost identical.
- Compared to the three-dimensional elasticity solution, the present models give more accurate results of static and dynamic load than the height order shear deformation theory.
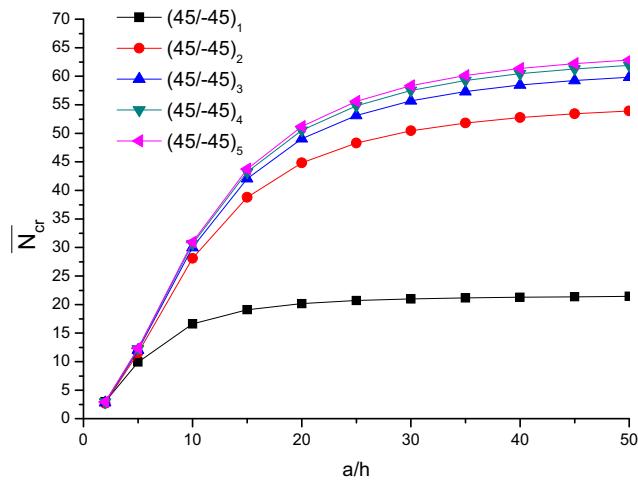
Fig. 11 The effect of side-to-thickness ratio on nondimensionalized uniaxial buckling load of simply supported antisymmetric angle-ply square laminates
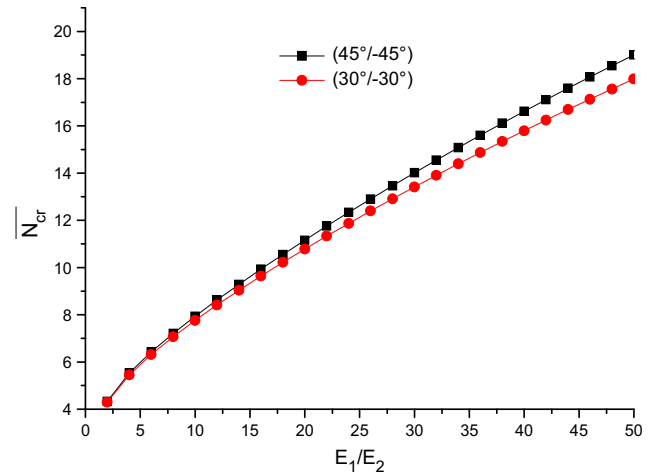
Fig. 12 The effect of modulus ratio on nondimensionalized uniaxial buckling load of simply supported two-layer antisymmetric angle-ply square laminates ($a/h = 10$)

Table 8 Nondimensional uniaxial buckling load of simply supported two-layer ($\theta/-\theta$) square laminates

a/h	Theory	$\overline{N}$	
		$\theta = 30°$	$\theta = 45°$
4	Present model 1	9.5554	9.9145
	Present model 2	9.5688	9.9302
	Model-Ren (Ren 1990)	9.5368	9.8200
	Model-Reddy (Reddy 1984)	9.3391	8.2377
10	Present model 1	17.2737	18.1473
	Present model 2	17.2795	18.1544
	Model-Ren (Ren 1990)	15.7517	16.4558
	Model-Reddy (Reddy 1984)	17.1269	18.1544
100	Present model 1	20.5040	21.6662
	Present model 2	20.5020	21.6663
	Model-Ren (Ren 1990)	20.4793	21.6384
	Model-Reddy (Reddy 1984)	20.5017	21.6663

- Compared to the three-dimensional elasticity solution, the present theories give more accurate results of deflection and dynamic load than the height order shear deformation theory found in the literature.
- The natural frequencies obtained by the proposed model with four unknowns are almost identical to those predicted by the shear deformation theories containing five unknowns.
- The buckling load obtained using present's model (a simpler version of present theory with four unknowns) and height order shear deformation Reddy's theory (Reddy 1984) (five unknowns) are comparable.
- Compared to the three-dimensional elasticity solution, the present model gives more accurate results of

buckling load than the height order shear deformation theory.

It can be concluded that the present models proposed prove to be accurate in solving the static, buckling, and dynamic behaviors of anti-symmetric cross-ply and angle-ply laminated composite plates and efficient in predicting the vibration responses of composite plates.

References

Ambartsumyan SA (1969) Theory of Anisotropic Plates. In: Cheron T (ed) Technomic, Stamford, CT

Ambartsumyan SA, Gnuni VTS (1961) On the dynamic stability of nonlinearly-elastic three-layered plates. Prikladnaya Matematika i Mekhanika 25(4):748–750 **(in Russian)**

Bert CW (1974) Structure design and analysis: Part I. In: Chamis CC (ed) Analysis of Plates, (Chapter 4). Academic Press, New York

Bert CW, Chen TLC (1978) Effect of shear deformation on vibration of antisymmetric angle-ply laminated rectangular plates. Int J Solids Struct 14:465–473

El G, Chulkov PP (1973) Stability and vibrations of three-layered ShelLY. Mashinostroenie, Moscow (in Russian)

Fortier RC, Rossettos JN (1973) On the vibration of shear-deformable curved anisotropic composite plates. J Appl Mech 40:299–301

Hassaine Daouadji T, Tounsi A, Hadji L, Abdelaziz HH, Adda bedia EA (2012) A theoretical analysis for static and dynamic behavior of functionally graded plates. Mater Phys Mech J. 14. ISSN 1605-8119 P 110-128 N0 2–02

Hassaine Daouadji T, Tounsi A, Adda bedia EA (2013) Analytical solution for bending analysis of functionally graded plates. Sci Iran

Trans B Mech Eng. revised 26 December 2012; accepted 1 February 2013 online 06 mai 2013. doi:10.1016/j.scient.2013.02.014

Karama M, Afaq KS, Mistou S (2003) Mechanical behavior of laminated composite beam by the new multi-layered laminated composite structures model with transverse shear stress continuity. Int J Solids Struct 40:1525–1546

Khdeir AA, Reddy JN (1999) Free vibration of laminated composite plates using second-order shear deformation theory. Compos Struct 71:617–626

Mantari JL, Oktem AS, Guedes Soares C (2012) A new trigonometric shear deformation theory for isotropic, laminated composite and sandwich plates. Int J Solids Struct 49:43–53

Matsunaga H (2009) Stress analysis of functionally graded plates subjected to thermal and mechanical loadings. Compos Struct 87:344–357

Mindlin RD (1951) Influence of rotary inertia and shear on flexural motions of isotropic, elastic plates. J Appl Mech 18:31–38

Noor AK (1973) Free vibrations of multilayered composite plates. AIAA J 11:1038–1039

Noor AK (1975) Stability of multilayered composite plate. Fibre Sci Technol 8:81–89

Pagano NJ (1970) Exact solutions for rectangular bidirectional composites and sandwich plates. J Compos Mater 4(1):20–34

Rastgaar M, Mahinfalah AM, Jazar GN (2006) Natural frequencies of laminated composite plates using third-order shear deformation theory. Compos Struct 72:273–279

Reddy JN (1979) Free vibration of antisymmetric angle-ply laminated plates including transverse shear deformation by the finite element method. J Sound Vib 66(4):565–576

Reddy JN (1984) A simple higher-order theory for laminated composite plates. ASME J Appl Mech 51:745–752

Reddy JN, Khdeir AA (1989) Buckling and vibration of laminated composite plates using various plate theories. AIAAJ 27(12): 1808–1817

Ren JG (1990) Bending, vibration and buckling of laminated plates. In: Cheremisinoff NP (ed) Handbook of ceramics and composites, vol 1. Marcel Dekker, New York, pp 413–450

Shankara CA, Iyengar NG (1996) A C0 element for the free vibration analysis of laminated composite plates. J Sound Vib 191(5): 721–738

Shinha PK, Rath AK (1975) Vibration and buckling of cross-ply laminated circular cylindrical panels. Aeronaut Q 26:211–218

Simsek M (2010a) Vibration analysis of a functionally graded beam under a moving mass by using different beam theories. Compos Struct 92:904–917

Simsek M (2010b) Fundamental frequency analysis of functionally graded beams by using different higher-order beam theories. Nucl Eng Des 240:697–705

Singh BN, Yadav D, Iyengar NGR (2001) Natural frequencies of composite plates with random material properties using higher-order shear deformation theory. Int J Mech Sci 43:2193–2214

Srinivas S, Rao AK (1970) Bending, vibration and buckling of simply supported thick orthotropic rectangular plates and laminates. Int J Solids Struct 6:1463–1481

Stavski Y (1965) On the theory of symmetrically heterogeneous plates having the same thickness variation of the elastic moduli. Topics in Applied Mechanics. American Elsevier, New York, p 105

Swaminathan K, Patil S (2008) Analytical solutions using a higher order refined computational model with 12 degrees of freedom for the free vibration analysis of antisymmetric angle-ply plates. Compos Struct 82(2):209–216

Whitney JM, Pagano NJ (1970) Shear deformation in heterogeneous anisotropic plates. J Appl Mech 37:1031–1036

Whitney JM, Sun CT (1973) A higher-order theory for extensional motion of laminated composites. J Sound Vib 30:85–97

Xiang S, Li G, Zhang W, Yang M (2011) A meshless local radial point collocation method for free vibration analysis of laminated composite plates. Compos Struct 93:280–286

Yang PC, Norris CH, Stavsky Y (1966) Elastic wave propagation in heterogeneous plates. Int J Solids Struct 2:665–684