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# Global well-posedness of the two-dimensional stochastic viscous nonlinear wave equations

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**Abstract**

We study well-posedness of viscous nonlinear wave equations (vNLW) on the two-dimensional torus with a stochastic forcing. In particular, we prove pathwise global well-posedness of the stochastic defocusing vNLW with an additive stochastic forcing $D^\alpha \xi$, where $\alpha < \frac{1}{2}$ and $\xi$ denotes the space–time white noise.

**Keywords** Viscous nonlinear wave equation · Stochastic viscous nonlinear wave equation · Global well-posedness

**Mathematics Subject Classification** 35L05 · 35L71 · 35R60 · 60H15

## Contents



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# 1 Introduction

## 1.1 Viscous nonlinear wave equations

In this paper, we consider the following nonlinear wave equation (NLW) on the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$, augmented by viscous effects:

$$\begin{cases} \partial_t^2 u + (1-\Delta)u + D\partial_t u + |u|^{p-1}u = D^\alpha \xi \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \tag{1.1}$$

where $p > 1$, $D = |\nabla| = \sqrt{-\Delta}$, $\alpha < \frac{1}{2}$, and $\xi$ denotes the (Gaussian) space–time white noise on $\mathbb{R}_+ \times \mathbb{T}^2$. Our main goal in this paper is to prove pathwise global well-posedness of (1.1) in $C(\mathbb{R}_+; H^s(\mathbb{T}^2))$ for some $\alpha \le \alpha_p$ and $s \ge s_p$, where $H^s(\mathbb{T}^2)$ is the $L^2$-based Sobolev space on $\mathbb{T}^2$ with regularity $s$ (see Sect. 2 for more details).

In [21], Kuan–Čanić proposed the following viscous NLW on $\mathbb{R}^2$:

$$\partial_t^2 u - \Delta u + 2\mu D\partial_t u = F(u), \tag{1.2}$$

where $\mu > 0$ and $F(u)$ is a general external forcing. This equation typically shows up in fluid–structure interaction problems, such as the interaction between a stretched membrane and a viscous fluid. The viscosity term $2\mu D\partial_t u$ in (1.2) comes from the Dirichlet–Neumann operator typically arising in fluid–structure interaction problems in three dimensions. See [21, 23] for the derivation of (1.2). It is easy to see that, when $\mu \ge 1$, the Eq. (1.2) is purely parabolic (see [23, 26]). On the other hand, when $0 < \mu < 1$, the viscous NLW (1.2) exhibits an interesting mixture of dispersive effects and parabolic smoothing. Since the precise value of $0 < \mu < 1$ does not play an important role, we simply set $\mu = \frac{1}{2}$. In addition, we consider a defocusing power-type nonlinearity of the form

$$F(u) = -|u|^{p-1}u,$$

for positive real numbers $p > 1$. This power-type nonlinearity has been studied extensively for nonlinear dispersive equations (see, for example, [40]). With $\mu = \frac{1}{2}$ and $F(u) = -|u|^{p-1}u$, the general form of vNLW (1.2) becomes the following version of vNLW:

$$\partial_t^2 u - \Delta u + D\partial_t u + |u|^{p-1}u = 0 \tag{1.3}$$

We now consider the analytical aspects of vNLW (1.3). Note that as in the case of the usual NLW:

$$\partial_t^2 u - \Delta u + |u|^{p-1}u = 0,$$

the viscous NLW (1.3) on $\mathbb{R}^2$ enjoys the following scaling symmetry: $u(t,x) \mapsto u_\lambda(t,x) := \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x)$. Namely, if $u$ is a solution to (1.3), then $u_\lambda$ is also a solution to (1.3) for any $\lambda > 0$ with rescaled initial data. This scaling symmetry induces the scaling critical Sobolev regularity $s_{\text{scaling}}$ on $\mathbb{R}^2$ given by

$$s_{\text{scaling}} = 1 - \frac{2}{p-1}$$

such that under this scaling symmetry, the homogeneous Sobolev norm on $\mathbb{R}^2$ remains invariant. While there is no scaling symmetry on $\mathbb{T}^2$, the scaling critical regularity $s_{\text{scaling}}$ still plays an important role in studying nonlinear partial differential equations in the periodic setting, especially for dispersive equations. Namely, in both periodic and non-periodic settings, a dispersive equation is usually well-posed in $H^s$ for $s > s_{\text{scaling}}$ and is usually ill-posed in $H^s$ for $s < s_{\text{scaling}}$. On the one hand, there is a good local well-posedness theory for dispersive equations above the scaling regularities (see [3, 31, 38] for the references therein). Moreover, we show in this paper that vNLW (1.3) is locally well-posed in $H^s(\mathbb{T}^2)$ for all $s \geq s_{\text{crit}}$ (with a strict inequality when $p = 3$), where $s_{\text{crit}}$ is defined by

$$s_{\text{crit}} := \max(s_{\text{scaling}}, 0) = \max\left(1 - \frac{2}{p-1}, 0\right), \tag{1.4}$$

for a given $p > 1$. See "Appendix A". Here, the second regularity restriction 0 is required to make sense of powers of $u$. On the other hand, many dispersive equations are known to be ill-posed below the scaling critical regularity. Among these ill-posedness results, many of them are in the form of *norm inflation* (see [6, 8–10, 13, 20, 31, 33, 36, 37, 41]), which is a stronger notion of ill-posedness. In [21], Kuan–Čanić proved norm inflation for vNLW (1.3) in $\mathcal{H}^s(\mathbb{R}^d)$ for $0 < s < s_{\text{scaling}}$ and any odd integers $p \geq 3$. Moreover, they pointed out that the viscous term has the potential to slow down the growth of the $H^s$ norm, i.e. to slow down the speed of the norm inflation. For details, see [21]. Also, it is of interest to see if norm inflation for vNLW holds in negative Sobolev spaces. See [12].

Let us now turn our attention to the viscous NLW with a stochastic forcing. In [22], Kuan–Čanić studied the stochastic viscous wave equation with a multiplicative noise on $\mathbb{R}^d$, $d = 1, 2$:

$$\partial_t^2 u - \Delta u + D\partial_t u = f(u)\xi,$$

where $f$ is Lipschitz and $\xi$ is the (Gaussian) space–time white noise on $\mathbb{R}_+ \times \mathbb{R}^2$. In [26], Oh and the author studied (the renormalized version of) SvNLW (1.1) with $\alpha = \frac{1}{2}$. When $\alpha = \frac{1}{2}$, the solution is not a function but is only a distribution and thus a renormalization on the nonlinearity is required to give a proper meaning to the dynamics (which in particular forces us to consider $|u|^{p-1}u$ only for $p \in 2\mathbb{N} + 1$ or $u^k$ for an integer $k \geq 2$). See [26] for details. In the cubic case, we proved pathwise global well-posedness. For an odd integer $p \geq 5$, we also used an invariant measure argument to prove almost sure global well-posedness with suitable random initial data.[1] In this paper, our goal is to investigate further well-posedness of SvNLW (1.1) with an additive forcing $D^\alpha\xi$ and, in particular, prove pathwise global well-posedness for any $p > 1$, where the range of $\alpha < \frac{1}{2}$ depends on the degree $p > 1$ of the nonlinearity.

### 1.2 SvNLW with an additive stochastic forcing

We say that $u$ is a solution to SvNLW (1.1) if $u$ satisfies the following Duhamel formulation of (1.1):

$$u(t) = V(t)(u_0, u_1) - \int_0^t S(t-t')\big(|u|^{p-1}u\big)(t')dt' + \Psi. \tag{1.5}$$

Here, $V(t)$ is the linear propagator defined by

$$\begin{aligned} V(t)(u_0, u_1) = e^{-\frac{D}{2}t}\bigg(\cos(t[\![D]\!]) + \frac{D}{2[\![D]\!]}\sin(t[\![D]\!])\bigg)u_0 \\ +e^{-\frac{D}{2}t}\frac{\sin(t[\![D]\!])}{[\![D]\!]}u_1 \end{aligned} \tag{1.6}$$

and $S(t)$ is defined by

$$S(t) = e^{-\frac{D}{2}t}\frac{\sin(t[\![D]\!])}{[\![D]\!]}, \tag{1.7}$$

where

$$[\![D]\!] = \sqrt{1-\tfrac{3}{4}\Delta},$$

and $\Psi$ denotes the stochastic convolution defined by

$$\Psi := \Psi_\alpha = \int_0^t S(t-t')D^\alpha\xi(dt'). \tag{1.8}$$

A standard argument shows that $\Psi$ belongs to $C(\mathbb{R}_+; W^{\frac{1}{2}-\alpha-\varepsilon,\infty}(\mathbb{T}^2))$ almost surely, where $\varepsilon > 0$ can be arbitrarily small; see Lemma 2.5 below. In particular, when $\alpha < \frac{1}{2}$, $\Psi$ is a well-defined function on $\mathbb{R}_+ \times \mathbb{T}^2$.

We first state a local well-posedness result for SvNLW (1.1).

**Theorem 1.1** *Let $p > 1$ and $\alpha < \frac{1}{2}$. Define $q$, $r$, and $\sigma$ as follows.*

(i) *When $1 < p < 2$, set $q = 2+\delta$, $r = \frac{4+2\delta}{1+\delta}$, and $\sigma = 0$, for some sufficiently small $\delta > 0$.*

[1] Strictly speaking, almost sure global well-posedness holds for the noise $\sqrt{2}D^{\frac{1}{2}}\xi$, which makes the Gibbs measure for the standard NLW invariant under the SvNLW dynamics. For pathwise global well-posedness, a precise coefficient in front of the noise $D^{\frac{1}{2}}\xi$ does not play any role.

(ii) *When $p \geq 2$, set $q = p + \delta$, $r = 2p$, and $\sigma = 1 - \frac{1}{p+\delta} - \frac{1}{p}$ for some arbitrary $\delta > 0$.*

*Let $s \geq \sigma$. Then, SvNLW (1.1) is pathwise locally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$. More precisely, given any $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$, there exists $T = T_\omega(u_0, u_1)$ (which is positive almost surely) and a unique solution $u$ to (1.1) with $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ in the class*

$$\Psi + C([0, T]; H^\sigma(\mathbb{T}^2)) \cap L^q([0, T]; L^r(\mathbb{T}^2)).$$

We present the proof of Theorem 1.1 in Sect. 3. The proof of Theorem 1.1 is based on the following first order expansion [5, 11, 28]:

$$u = v + \Psi, \tag{1.9}$$

where the residual term $v$ satisfies the following equation:

$$\begin{cases} \partial_t^2 v + (1 - \Delta)v + D\partial_t v + |v + \Psi|^{p-1}(v + \Psi) = 0 \\ (v, \partial_t v)|_{t=0} = (u_0, u_1). \end{cases} \tag{1.10}$$

See Proposition 3.1 for the pathwise local well-posedness result at the level of the residual term $v$ using the homogeneous Strichartz estimates for the viscous wave equation (Lemma 2.8).

The main idea of the proof of pathwise local well-posedness of SvNLW (1.1) comes from [23]. Note that the nonlinearity $|u|^{p-1}u$ in SvNLW (1.1) is not necessarily algebraic for general $p > 1$, which creates a difficulty for obtaining the difference estimate when applying the contraction argument. To deal with this issue, we apply the idea from Oh–Okamoto–Pocovnicu [32] using the fundamental theorem of calculus.

***Remark 1.2*** (i) Using the same argument, the proof of Theorem 1.1 works for both the defocusing case (with the nonlinearity $|u|^{p-1}u$) and the focusing case (with the nonlinearity $-|u|^{p-1}u$, i.e. with the negative sign).
The proof of Theorem 1.1 also works for SvNLW with nonlinearity $u^k$, where $k \geq 2$ is an integer. In fact, a simple argument based on Sobolev's inequality can be applied to prove local well-posedness of SvNLW with nonlinearity $u^k$ in the class $\Psi + C([0, T]; \mathcal{H}^s(\mathbb{T}^2))$ for $s \geq 1$. See, for example, Proposition 3.1 in [26].

(ii) As it is written in Theorem 1.1, we point out that the regularity of initial data can be lowered to the subcritical case, i.e. $s \geq s_{\text{crit}}$ (with a strict inequality when $p = 3$), where $s_{\text{crit}}$ is the critical regularity as defined in (1.4) (note that $s_{\text{crit}} \leq \sigma$ with $\sigma$ defined in Theorem 1.1). See Theorem A.6 and Remark A.7 for details.

(iii) One can also directly prove local well-posedness of (1.1) for $u \in L^q([0, T]; L^r(\mathbb{T}^2))$ for some appropriate $q, r \geq 2$. Specifically, in the Duhamel formulation (1.5), the linear term $V(t)(u_0, u_1)$ can be estimated by the Strichartz estimate (Lemma 2.8), the nonlinear perturbation term $\int_0^t S(t - t')(|u|^{p-1}u)(t')dt'$ can be estimated by the Schauder estimate (Lemma 2.7) along with Young's convolution inequality, and the stochastic convolution $\Psi$ can also be bounded in

$L^q([0,T]; L^r(\mathbb{T}^2))$ (Lemma 2.5). This approach yields a stronger uniqueness result since the solution does not depend on any specific structure such as (1.9). Nevertheless, this paper is meant to be a continuation of the work in [26], and so we choose to study (1.1) from a dispersive point of view. Due to the assumption that the initial data lies in the Sobolev space $\mathcal{H}^s(\mathbb{T}^2)$ for some $s \geq 0$, it is more natural to construct the solution in $C([0,T]; H^s(\mathbb{T}^2))$ for $T > 0$. The Strichartz spaces $L^q([0,T]; L^r(\mathbb{T}^2))$ can be viewed as "helper" spaces that allow us to show local well-posedness for rough initial data (i.e. with $s \geq 0$ as small as possible).

We now turn our attention to pathwise global well-posedness of SvNLW (1.1), and we restrict our attention to the defocusing case. Our pathwise global well-posedness result reads as follows.

**Theorem 1.3** *Let $p > 1$ and $\alpha < \min(\frac{1}{2}, \frac{2}{p-1} - \frac{1}{2})$. Let $\sigma = \max(0, 1 - \frac{1}{p+\delta} - \frac{1}{p})$ for some arbitrary $\delta > 0$ and let $s \geq \sigma$. Then, SvNLW (1.1) is pathwise globally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$. More precisely, given any $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^s)$, there exists a unique global-in-time solution $u$ to (1.1) with $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ in the class*

$$\Psi + C(\mathbb{R}_+; H^\sigma(\mathbb{T}^2)).$$

In Theorem 1.3, the uniqueness holds in the following sense. For any $t_0 \in \mathbb{R}_+$, there exists a time interval $I(t_0) \ni t_0$ such that the solution $u$ to (1.1) is unique in

$$\Psi + C(I(t_0); H^\sigma(\mathbb{T}^2)) \cap L^q(I(t_0); L^r(\mathbb{T}^2)),$$

where $q, r \geq 2$ are as in Theorem 1.1.

As stated in Theorem 1.3, when $1 < p \leq 3$, we have the condition $\alpha < \frac{1}{2}$; when $p > 3$, we have the condition $\alpha < \frac{2}{p-1} - \frac{1}{2}$. As $p \to \infty$, the condition for $\alpha$ becomes $\alpha \leq -\frac{1}{2}$. Note that when $1 < p < 5$, we can prove pathwise global well-posedness of SvNLW (1.1) with the space–time white noise (i.e. $\alpha = 0$).

We prove Theorem 1.3 by studying (1.10) for the residual term $v$ in Sect. 4. From the proof of Theorem 1.1, we see that pathwise global well-posedness follows once we control the $\mathcal{H}^1$-norm of $\vec{v}(t) := (v(t), \partial_t v(t))$. For this purpose, we study the evolution of the energy

$$E(\vec{v}) = \frac{1}{2}\int_{\mathbb{T}^2}(v^2 + |\nabla v|^2)dx + \frac{1}{2}\int_{\mathbb{T}^2}(\partial_t v)^2 dx + \frac{1}{p+1}\int_{\mathbb{T}^2}|v|^{p+1}dx, \quad (1.11)$$

which is conserved under the (deterministic) usual NLW:

$$\partial_t^2 u + (1-\Delta)u + |u|^{p-1}u = 0.$$

Note that for our problem, we proceed with the first order expansion (1.9), where the residual term $v = \Psi - u$ only satisfies (1.10). In this case, the energy $E(\vec{v})$ is not conserved under the Eq. (1.10) because of the perturbative term $|v + \Psi|^{p-1}(v +$

$\Psi) - |v|^{p-1}v$. For our problem, we first follow the globalization argument by Burq–Tzvetkov [7] and establish an exponential growth bound on $E(\vec{v})$, which works in the sub-cubic case $1 < p \le 3$. For the super-cubic case $p > 3$, this argument no longer works due to the high homogeneity of the non-linearity. When $3 < p \le 5$, we use an integration by parts trick introduced by Oh–Pocovnicu [34]. In the super-quintic case $p > 5$, we use a trick involving the Taylor expansion, where the idea comes from Latocca [24].

One important prerequisite for studying the evolution of the energy $E(\vec{v})$ is that the local-in-time solution $\vec{v}$ lies in $\mathcal{H}^1(\mathbb{T}^2)$, which is not guaranteed by the pathwise local well-posedness result (Theorem 1.1) as it is written. Nonetheless, due to the dissipative nature of the equation, we show that $\vec{v}(t)$ indeed belongs to $\mathcal{H}^1(\mathbb{T}^2)$ for any $t > 0$ by using the Schauder estimate (Lemma 2.7) along with Theorem 1.1. See Sect. 4 for details.

We conclude our introduction by stating several remarks.

***Remark 1.4*** (i) We point out that Theorem 1.1 and 1.3 also hold if we have $-\Delta$ instead of $1 - \Delta$ in (1.1) by using an essentially identical proof.

(ii) In Oh [26] and the author studied SvNLW (1.1) with $\alpha = \frac{1}{2}$. In this case, due to $\alpha = \frac{1}{2}$, the stochastic term $\Psi$ defined in (1.8) turns out to be merely a distribution, so that we studied a renormalized version of (1.1) and proved pathwise global well-posedness in the cubic case. Because of the singular nature of the stochastic convolution in this setting, the standard Gronwall argument does not work, and so we used a Yudovich-type argument to bound the corresponding energy.

In the same paper, we also proved almost sure global well-posedness of (1.1) with $p \in 2\mathbb{N}+1$ and with random initial data, using the formal invariance of the Gibbs measure. However, the argument only works for $\alpha = \frac{1}{2}$, so it does not apply to our problem with $\alpha < \frac{1}{2}$ in this paper. Instead, in this paper, we establish pathwise global well-posedness of SvNLW (1.1).

(iii) We can also consider the vNLW with randomized initial data:

$$\begin{cases} \partial_t^2 u + (1-\Delta)u + D\partial_t u + |u|^{p-1}u = 0 \\ (u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega). \end{cases} \tag{1.12}$$

Here, the randomization $(u_0^\omega, u_1^\omega)$ of the initial data $(u_0, u_1)$ is defined by

$$(u_0^\omega, u_1^\omega) := \bigg( \sum_{n\in\mathbb{Z}^2} g_{n,0}(\omega)\widehat{u_0}(n)e^{in\cdot x}, \sum_{n\in\mathbb{Z}^2} g_{n,1}(\omega)\widehat{u_1}(n)e^{in\cdot x} \bigg), \tag{1.13}$$

where for $j = 0, 1$, $\widehat{u_j}(-n) = \overline{\widehat{u_j}(n)}$ for all $n \in \mathbb{Z}^2$ and $\{g_{n,j}\}_{n\in\mathbb{Z}^2}$ is a sequence of mean zero complex-valued random variables such that $g_{-n,j} = \overline{g_{n,j}}$ for all $n \in \mathbb{Z}^2$. Moreover, we assume that $g_{0,j}$ is real-valued for $j = 0, 1$, $\{g_{0,j}, \Re g_{n,j}, \Im g_{n,j}\}_{n\in\mathcal{I}, j=0,1}$ are independent with $\mathcal{I} = (\mathbb{Z}_+ \times \{0\}) \cup (\mathbb{Z} \times \mathbb{Z}_+)$, and there exists a constant $c > 0$ such that on the probability distributions $\mu_{n,j}$ of

$g_{n,j}$, we have

$$\int e^{\gamma \cdot x} d\mu_{n,j}(x) \leq e^{c|\gamma|^2}, \quad j = 0, 1 \tag{1.14}$$

for all $\gamma \in \mathbb{R}^2$ when $n \in \mathbb{Z}^2 \backslash \{0\}$ and all $\gamma \in \mathbb{R}$ when $n = 0$. Note that (1.14) is satisfied for standard complex-valued Gaussian random variables, standard Bernoulli random variables, and any random variables with compactly supported distributions.

The randomization (1.13) allows us to consider almost sure global well-posedness of (1.12) for $(u_0, u_1)$ living in negative Sobolev spaces. For almost sure local well-posedness, we consider the following first order expansion similar to (1.9):

$$u = v + z,$$

where $z$ is the solution of the linear viscous wave equation with initial data $(u_0^\omega, u_1^\omega)$:

$$z(t) = z^\omega(t) := V(t)(u_0^\omega, u_1^\omega)$$

with $V(t)$ defined as in (1.6). By using the Schauder estimate (Lemma 2.7), we can establish similar (but stronger) probabilistic Strichartz estimates for $z$ and $\langle \nabla \rangle^{-1} \partial_t z$ as in [34, 35]. This enables us to prove almost sure local well-posedness of (1.12) using a similar argument as for proving Theorem 1.1, as long as $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$ with $s > -\frac{1}{p}$. On the other hand, the proof for almost sure global well-posedness of (1.12) is much simpler than that for Theorem 1.3, since $z(t)$ is smooth for $t > 0$ thanks to the parabolic smoothing. We omit details since this is not the main focus in this paper.

## 2 Preliminary lemmas

In this section, we discuss some notations and lemmas that are necessary for proving our well-posedness results.

We use $A \lesssim B$ to denote $A \leq CB$ for some constant $C > 0$, and we write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. Also, we use $a+$ (and $a-$) to denote $a + \varepsilon$ (and $a - \varepsilon$, respectively) for arbitrarily small $\varepsilon > 0$. In addition, we use short-hand notations to work with space–time function spaces. For example, $C_T H_x^s = C([0, T]; H^s(\mathbb{T}^d))$.

### 2.1 Sobolev spaces and Besov spaces

Let $s \in \mathbb{R}$. We denote $H^s(\mathbb{T}^d)$ as the $L^2$-based Sobolev space with the norm:

$$\|u\|_{H^s(\mathbb{T}^d)} = \|\langle n \rangle^s \widehat{u}(n)\|_{\ell_n^2(\mathbb{Z}^d)},$$

where $\widehat{u}(n)$ is the Fourier coefficient of $u$ and $\langle \cdot \rangle = (1 + |\cdot|)^{\frac{1}{2}}$. We then define $\mathcal{H}^s(\mathbb{T}^d)$ as

$$\mathcal{H}^s(\mathbb{T}^d) = H^s(\mathbb{T}^d) \times H^{s-1}(\mathbb{T}^d).$$

Also, we denote $W^{s,p}(\mathbb{T}^d)$ as the $L^p$-based Sobolev space with the norm:

$$\|u\|_{W^{s,p}(\mathbb{T}^d)} = \big\|\mathcal{F}^{-1}(\langle n\rangle^s \widehat{u}(n))\big\|_{L^p(\mathbb{T}^d)},$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform on $\mathbb{T}^d$. When $p = 2$, we have $H^s(\mathbb{T}^d) = W^{s,2}(\mathbb{T}^d)$.

Let $\varphi : \mathbb{R} \to [0, 1]$ be a bump function such that $\varphi \in C_c([-\frac{8}{5}, \frac{8}{5}])$ and $\varphi \equiv 1$ on $[-\frac{5}{4}, \frac{5}{4}]$. For $\xi \in \mathbb{R}^d$, we define $\varphi_0(\xi) = \varphi(|\xi|)$ and

$$\varphi_j(\xi) = \varphi\big(\tfrac{|\xi|}{2^j}\big) - \varphi\big(\tfrac{|\xi|}{2^{j-1}}\big)$$

for $j \in \mathbb{Z}_+$. Note that

$$\sum_{j\in\mathbb{Z}_{\geq 0}} \varphi_j(\xi) = 1 \tag{2.1}$$

for any $\xi \in \mathbb{R}^d$. For $j \in \mathbb{Z}_{\geq 0}$, we define the Littlewood-Paley projector $\mathbf{P}_j$ as

$$\mathbf{P}_j u = \mathcal{F}^{-1}(\varphi_j \widehat{u}).$$

Due to (2.1), we have

$$u = \sum_{j=0}^{\infty} \mathbf{P}_j u. \tag{2.2}$$

We also recall the definition of Besov spaces $B^s_{p,q}(\mathbb{T}^d)$ equipped with the norm:

$$\|u\|_{B^s_{p,q}(\mathbb{T}^d)} = \big\|2^{sj}\|\mathbf{P}_j u\|_{L^p_x(\mathbb{T}^d)}\big\|_{\ell^q_j(\mathbb{Z}_{\geq 0})}.$$

Note that $H^s(\mathbb{T}^d) = B^s_{2,2}(\mathbb{T}^d)$.

We then recall the definition of paraproducts introduced by Bony [4]. For details, see [1, 16]. For given functions $u$ and $v$ on $\mathbb{T}^d$ of regularities $s_1$ and $s_2$, respectively. By (2.2), we can write the product $uv$ as

$$\begin{aligned} uv &= u \mathbin{\textcircled{\scriptsize <}} u + u \mathbin{\textcircled{\scriptsize =}} v + u \mathbin{\textcircled{\scriptsize >}} v \\ &:= \sum_{j<k-2} \mathbf{P}_j u \mathbf{P}_k v + \sum_{|j-k|\leq 2} \mathbf{P}_j u \mathbf{P}_k v + \sum_{k<j-2} \mathbf{P}_j u \mathbf{P}_k v. \end{aligned}$$

The term $u \mathbin{\textcircled{\scriptsize <}} v$ (and the term $u \mathbin{\textcircled{\scriptsize >}} v$) is called the paraproduct of $v$ by $u$ (and the paraproduct of $u$ by $v$, respectively), and it is well defined as a distribution of regularity $\min(s_2, s_1+s_2)$ (and $\min(s_1, s_1+s_2)$, respectively). The term $u \mathbin{\textcircled{\scriptsize =}} v$ is called the resonant product of $u$ and $v$, and it is well defined in general only if $s_1 + s_2 > 0$.

With these definitions in hand, we recall some basic properties of Besov spaces.

**Lemma 2.1** (i) *Let $s_1, s_2 \in \mathbb{R}$ and $1 \le p, p_1, p_2, q \le \infty$ which satisfies $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, we have*

$$\|u \otimes v\|_{B^{s_2}_{p,q}(\mathbb{T}^d)} \lesssim \|u\|_{L^{p_1}(\mathbb{T}^d)} \|v\|_{B^{s_2}_{p_2,q}(\mathbb{T}^d)}. \tag{2.3}$$

*When $s_1 + s_2 > 0$, we have*

$$\|u \circleddash v\|_{B^{s_1+s_2}_{p,q}(\mathbb{T}^d)} \lesssim \|u\|_{B^{s_1}_{p_1,q}(\mathbb{T}^d)} \|v\|_{B^{s_2}_{p_2,q}(\mathbb{T}^d)}. \tag{2.4}$$

(ii) *Let $s_1 < s_2$ and $1 \le p, q \le \infty$. Then, we have*

$$\|u\|_{B^{s_1}_{p,q}(\mathbb{T}^d)} \lesssim \|u\|_{W^{s_2,p}(\mathbb{T}^d)}. \tag{2.5}$$

*In particular, when $q = \infty$, we have*

$$\|u\|_{B^{s_1}_{p,\infty}(\mathbb{T}^d)} \lesssim \|u\|_{W^{s_1,p}(\mathbb{T}^d)}. \tag{2.6}$$

See [1, 30] for the proofs of (2.3) and (2.4) in the $\mathbb{R}^d$ setting, which can be easily extended to the $\mathbb{T}^d$ setting. The embedding (2.5) follows from the $L^p$ boundedness of $\mathbf{P}_j$ and the $\ell^q$-summability of $\{2^{(s_1-s_2)j}\}_{j\in\mathbb{Z}_{\ge 0}}$, and the embedding (2.6) follows easily from the $L^p$ boundedness of $\mathbf{P}_j$.

Using (2.3) and (2.4), we get the following product estimate.

**Corollary 2.2** *Let $s > 0$, $1 \le p, q \le \infty$ and $1 \le p_1, p_2, q_1, q_2 \le \infty$ satisfying*

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{p}.$$

*Then,*

$$\|uv\|_{B^s_{p,q}(\mathbb{T}^d)} \lesssim \|u\|_{B^s_{p_1,q}(\mathbb{T}^d)} \|v\|_{L^{q_1}(\mathbb{T}^d)} + \|u\|_{L^{p_2}(\mathbb{T}^d)} \|v\|_{B^s_{q_2,q}(\mathbb{T}^d)}.$$

Next, we recall the following chain rule estimates.

**Lemma 2.3** *Let $u$ be a smooth function on $\mathbb{T}^d$, $s \in (0, 1)$, $r \ge 2$. Let $F$ denote the function $F(u) = |u|^{r-1}u$ or $F(u) = |u|^r$.*
(i) *Let $1 < p, p_1 < \infty$ and $1 < p_2 \le \infty$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, we have*

$$\|F(u)\|_{W^{s,p}(\mathbb{T}^d)} \lesssim \|u\|_{W^{s,p_1}(\mathbb{T}^d)} \big\||u|^{r-1}\big\|_{L^{p_2}(\mathbb{T}^d)}. \tag{2.7}$$

(ii) *Let $1 \le p, q \le \infty$ and $1 \le p_1, p_2 \le \infty$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, we have*

$$\|F(u)\|_{B^s_{p,q}(\mathbb{T}^d)} \lesssim \|u\|_{B^s_{p_1,q}(\mathbb{T}^d)} \big\||u|^{r-1}\big\|_{L^{p_2}(\mathbb{T}^d)}. \tag{2.8}$$

The estimate (2.7) follows immediately from the fractional chain rule on $\mathbb{T}^d$ in [14]. For the proof of (2.8), see, for example, Lemma 3.5 in [24] in the $\mathbb{R}^d$ setting, which can be easily extended to the $\mathbb{T}^d$ setting.

Lastly, we recall the following Gagliardo-Nirenberg interpolation inequality.

**Lemma 2.4** *Let $p_1, p_2 \in (1, \infty)$ and $s_1, s_2 > 0$. Let $p > 1$ and $\theta \in (0, 1)$ satisfying*

$$-\frac{s_1}{d} + \frac{1}{p} = (1-\theta)\left(\frac{1}{p_1} - \frac{s_2}{d}\right) + \frac{\theta}{p_2} \quad \text{and} \quad s_1 \le (1-\theta)s_2.$$

*Then, for $u \in W^{s_2,p_1}(\mathbb{T}^d) \cap L^{p_2}(\mathbb{T}^d)$, we have*

$$\|u\|_{W^{s_1,p}(\mathbb{T}^d)} \lesssim \|u\|^{1-\theta}_{W^{s_2,p_1}(\mathbb{T}^d)} \|u\|^{\theta}_{L^{p_2}(\mathbb{T}^d)}.$$

This inequality follows from a direct application of Sobolev's inequality on $\mathbb{T}^d$ (see [2]) and then interpolation.

### 2.2 On the stochastic term

In this subsection, we discuss the regularity properties of the stochastic term $\Psi$ defined in (1.8). Given $N \in \mathbb{N}$, we denote $\Psi_N = \pi_N \Psi$ as the truncated stochastic convolution, where $\pi_N$ is the frequency cutoff onto the spatial frequencies $\{|n| \le N\}$. Then, we have the following regularity result for $\Psi$.

**Lemma 2.5** *For any $\varepsilon > 0$ and $T > 0$, $\Psi_N$ converges to $\Psi$ in $C([0, T]; W^{1-2\alpha-\varepsilon,\infty}(\mathbb{T}^2))$ almost surely. In particular, we have*

$$\Psi \in C([0, T]; W^{\frac{1}{2}-\alpha-\varepsilon,\infty}(\mathbb{T}^2))$$

*almost surely.*

The proof of Lemma 2.5 follows from a straightforward modification of the proof in [18, Lemma 3.1], and so we omit details. See also [17, Proposition 2.1].

***Remark 2.6*** One can use an integration by parts to write

$$\widehat{\Psi}(t, n) = -\int_0^t B_n(t') \frac{d}{ds}\bigg|_{s=t'} \left(e^{-\frac{|n|}{2}(t-s)} \frac{\sin((t-s)[\![n]\!])}{[\![n]\!]} |n|^\alpha\right) dt'$$

almost surely, which allows us to compute that

$$\begin{aligned}\partial_t \widehat{\Psi}(t, n) = \int_0^t \bigg( &-\frac{|n|}{2[\![n]\!]} e^{-\frac{|n|}{2}(t-t')} \sin((t-t')[\![n]\!]) \\ &+ e^{-\frac{|n|}{2}(t-t')} \cos((t-t')[\![n]\!])\bigg) |n|^\alpha dB_n(t')\end{aligned}$$

almost surely. Using a similar argument as in [18, Lemma 3.1] or [17, Proposition 2.1], we have $\partial_t \Psi \in C([0,T]; W^{-\frac{1}{2}-\alpha-,\infty}(\mathbb{T}^2))$ almost surely. This will be useful in the proof of pathwise global well-posedness of SvNLW (1.1) in Sect. 4.2.

### 2.3 Linear estimates

In this subsection, we show some relevant linear estimates and the Strichartz estimates that are used to prove our well-posedness results.

Let

$$P(t) = e^{-\frac{D}{2}t}$$

be the Poisson kernel with a parameter $\frac{t}{2}$, which appears in the viscous wave linear propagator $V(t)$ defined in (1.6). We first recall the following Schauder-type estimate for the Poisson kernel $P(t)$. For a proof, see Lemma 2.3 in [26].

**Lemma 2.7** *Let $1 \le p \le q \le \infty$ and $\beta \ge 0$. Then, we have*

$$\| D^\beta P(t)\phi \|_{L^q(\mathbb{T}^d)} \lesssim t^{-\beta - d(\frac{1}{p}-\frac{1}{q})} \|\phi\|_{L^p(\mathbb{T}^d)}$$

*for any $0 < t \le 1$.*

Next, we turn our attention to the Strichartz estimates for the homogeneous linear viscous wave equation on $\mathbb{T}^d$. We recall that the linear propagator $V(t)$ is defined in (1.6).

**Lemma 2.8** *Given $s \ge 0$, suppose that $2 < q \le \infty$, $2 \le r \le \infty$ satisfy the following scaling condition:*

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s. \tag{2.9}$$

*Then, we have*

$$\| V(t)(\phi_0, \phi_1) \|_{C([0,T]; \mathcal{H}^{s-1}(\mathbb{T}^d))} \lesssim \|(\phi_0, \phi_1)\|_{\mathcal{H}^s(\mathbb{T}^d)} \tag{2.10}$$

*and*

$$\| V(t)(\phi_0, \phi_1) \|_{L^q([0,T]; L^r_x(\mathbb{T}^d))} \lesssim \|(\phi_0, \phi_1)\|_{\mathcal{H}^s(\mathbb{T}^d)} \tag{2.11}$$

*for all $0 < T \le 1$.*

***Proof*** The bound (2.10) can be immediately seen from the definition of the $\mathcal{H}^s$-norm and the fact that $e^{-\frac{|n|}{2}t} \le 1$, $|\cos(t[\![n]\!])| \le 1$, and $|\sin(t[\![n]\!])| \le 1$.

To prove (2.11), we use the $TT^*$ method. We first consider the case when $s = 0$. Let

$$V_1(t) = e^{-\frac{D}{2}t}\cos(t[\![D]\!]), \quad V_2(t) = e^{-\frac{D}{2}t}\frac{D}{2[\![D]\!]}\sin(t[\![D]\!]),$$

so that

$$V(t)(\phi_0, \phi_1) = V_1(t)\phi_0 + V_2(t)\phi_0 + S(t)\phi_1.$$

Let $L : L^2(\mathbb{T}^d) \to L^q_T L^r_x(\mathbb{T}^d)$ be the linear operator given by $L\phi = V_1(t)\phi$. Note that $L^*$ is the linear operator given by

$$L^* f = \int_0^T V_1(t') f(t') dt'$$

for any space–time Schwartz function $f$. By Minkowski's integral inequality, the Schauder estimate (Lemma 2.7) twice, the scaling condition (2.9), and the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
\|LL^* f\|_{L^q_T L^r_x} &\le \bigg\| \int_0^T \big\| e^{-\frac{D}{2}(t+t')}\cos(t[\![D]\!])\cos(t'[\![D]\!]) f(t') \big\|_{L^r_x} dt' \bigg\|_{L^q_T} \\
&\lesssim \bigg\| \int_0^T \frac{1}{|t-t'|^{2(\frac{1}{2}-\frac{1}{r})}} \big\| e^{-\frac{D}{4}(t+t')}\cos(t[\![D]\!])\cos(t'[\![D]\!]) f(t') \big\|_{L^2_x} dt' \bigg\|_{L^q_T} \\
&\lesssim \bigg\| \int_0^T \frac{1}{|t-t'|^{2(\frac{1}{r'}-\frac{1}{r})}} \| f(t') \|_{L^{r'}_x} dt' \bigg\|_{L^q_T} \\
&= \bigg\| \int_0^\infty \frac{1}{|t-t'|^{2/q}} \| \mathbf{1}_{[0,T]} f(t') \|_{L^{r'}_x} dt' \bigg\|_{L^q_T} \\
&\lesssim \|f\|_{L^{q'}_T L^{r'}_x}.
\end{aligned}$$

Thus, by a standard duality argument, we obtain

$$\|V_1(t)\phi_0\|_{L^q_T L^r_x(\mathbb{T}^d)} \lesssim \|\phi_0\|_{L^2(\mathbb{T}^d)}.$$

By using similar arguments, we obtain

$$\|V_2(t)\phi_0\|_{L^q_T L^r_x(\mathbb{T}^d)} \lesssim \|\phi_0\|_{L^2}, \quad \|S(t)\phi_1\|_{L^q_T L^r_x} \lesssim \|\phi_0\|_{H^{-1}(\mathbb{T}^d)},$$

so that we have

$$\|V(t)(\phi_0, \phi_1)\|_{L^q_T L^r_x(\mathbb{T}^d)} \lesssim \|(\phi_0, \phi_1)\|_{\mathcal{H}^0(\mathbb{T}^d)}. \tag{2.12}$$

When $s > 0$, by Sobolev's inequality, the scaling condition (2.9), and (2.12), we obtain

$$\begin{aligned}\|V(t)(\phi_0,\phi_1)\|_{L^q_T L^r_x} &\lesssim \|V(t)(\langle\nabla\rangle^s\phi_0,\langle\nabla\rangle^s\phi_1)\|_{L^q_T L_x^{1/(\frac{s}{d}+\frac{1}{r})}}\\ &= \|V(t)(\langle\nabla\rangle^s\phi_0,\langle\nabla\rangle^s\phi_1)\|_{L^q_T L_x^{d/(\frac{d}{2}-\frac{1}{q})}}\\ &\lesssim \|(\langle\nabla\rangle^s\phi_0,\langle\nabla\rangle^s\phi_1)\|_{\mathcal{H}^0}\\ &= \|(\phi_0,\phi_1)\|_{\mathcal{H}^s},\end{aligned}$$

as desired. □

***Remark 2.9*** (i) Compared to the Strichartz estimates for the usual linear wave equations [15, 17, 19, 25], the Strichartz estimates for the homogeneous linear viscous wave equation on $\mathbb{T}^d$ hold for a larger class of pairs $(q, r)$, thanks to the parabolic smoothing effect.

(ii) In Kuan–Čanić [21] proved the Strichartz estimates for the homogeneous linear viscous wave equation on $\mathbb{R}^d$. They used the method from Keel–Tao [19], so that their result requires $(q, r)$ to be $\sigma$-admissible for some $\sigma > 0$, i.e. $(q, r, \sigma) \neq (2, \infty, 1)$ and

$$\frac{2}{q} + \frac{2\sigma}{r} \leq \sigma.$$

We point out that the $TT^*$ method we use in the proof also works on $\mathbb{R}^d$ and does not have this $\sigma$-admissible restriction on $q$ and $r$. However, our proof works only for $s \geq 0$.

We complete this subsection by establishing the following inhomogeneous linear estimates.

**Lemma 2.10** *Let $p \geq 2$ and let $S(t)$ be as in (1.7). Then, given $\delta > 0$, we have*

$$\Big\| \int_0^t S(t-t')F(t')dt' \Big\|_{L^{p+\delta}([0,T];L_x^{2p}(\mathbb{T}^2))} \lesssim \|F\|_{L^1([0,T];L^2(\mathbb{T}^2))} \tag{2.13}$$

*for any* $0 < T \leq 1$.

***Proof*** We let

$$s = 1 - \frac{1}{p+\delta} - \frac{2}{2p} = 1 - \frac{1}{p+\delta} - \frac{1}{p},$$

so that $(p+\delta, 2p, s)$ satisfies the scaling condition in Lemma 2.8. By Minkowski's integral inequality and Lemma 2.8, we obtain

$$\begin{aligned}\Big\| \int_0^t S(t-t')F(t')dt' \Big\|_{L_T^{p+\delta}L_x^{2p}} &\lesssim \int_0^T \|\mathbf{1}_{[0,t]}(t')S(t-t')F(t')\|_{L_T^{p+\delta}L_x^{2p}}dt'\\ &\lesssim \int_0^T \|F(t')\|_{H_x^{s-1}}dt'\end{aligned}$$

$$\leq \|F\|_{L^1_T L^2_x},$$

so that (2.13) follows. □

**Lemma 2.11** *Let $S(t)$ be as in (1.7). Then, given $s \leq 1$, we have*

$$\left\| \int_0^t S(t-t')F(t')dt' \right\|_{C([0,T];H^s_x(\mathbb{T}^2))} \lesssim \|F\|_{L^1([0,T];L^2_x(\mathbb{T}^2))}, \tag{2.14}$$

$$\left\| \partial_t \int_0^t S(t-t')F(t')dt' \right\|_{C([0,T];H^{s-1}_x(\mathbb{T}^2))} \lesssim \|F\|_{L^1([0,T];L^2_x(\mathbb{T}^2))} \tag{2.15}$$

*for any* $0 < T \leq 1$.

***Proof*** The estimate (2.14) follows from (1.7) and Minkowski's integral inequality. The estimate (2.15) follows similarly by noting that

$$\partial_t \int_0^t S(t-t')F(t')dt' = \int_0^t \partial_t S(t-t')F(t')dt',$$

where

$$\partial_t S(t) = e^{-\frac{D}{2}t}\left(\cos(t[\![D]\!]) - \frac{D}{2[\![D]\!]}\sin(t[\![D]\!])\right).$$

□

## 3 Local well-posedness of SvNLW

In this section, we prove Theorem 1.1, pathwise local well-posedness for SvNLW (1.1). As mentioned in Sect. 1.2, we consider the following vNLW:

$$\begin{cases} \partial_t^2 v + (1-\Delta)v + D\partial_t v + F(v+\Psi) = 0 \\ (v, \partial_t v)|_{t=0} = (u_0, u_1) \end{cases} \tag{3.1}$$

for given initial data $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$, $F(u) = |u|^{p-1}u$, and $\Psi$ is the stochastic convolution defined in (1.8). By Lemma 2.5, we can fix a good $\omega \in \Omega$ such that $\Psi = \Psi(\omega) \in C([0,T]; W^{\frac{1}{2}-\alpha-\varepsilon,\infty}(\mathbb{T}^2))$ for $\alpha < \frac{1}{2}$ and sufficiently small $\varepsilon > 0$, so that (3.1) becomes a deterministic equation. Then, we have the following pathwise local well-posedness of (3.1).

**Proposition 3.1** *Let $p > 1$ and $\alpha < \frac{1}{2}$. Define $q$, $r$, and $\sigma$ as follows.*

(i) *When $1 < p < 2$, set $q = 2 + \delta$, $r = \frac{4+2\delta}{1+\delta}$, and $\sigma = 0$, for some sufficiently small $\delta > 0$.*

(ii) *When* $p \geq 2$, *set* $q = p + \delta$, $r = 2p$, *and* $\sigma = 1 - \frac{1}{p+\delta} - \frac{1}{p}$, *for some arbitrary* $\delta > 0$.

*Let* $s \geq \sigma$. *Then,* (3.1) *is pathwise locally well-posed in* $\mathcal{H}^s(\mathbb{T}^2)$. *More precisely, given any* $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$, *there exists* $0 < T = T_\omega(u_0, u_1) \leq 1$ *and a unique solution* $\vec{v} = (v, \partial_t v)$ *to* (3.1) *in the class*

$$(v, \partial_t v) \in C([0, T]; \mathcal{H}^\sigma(\mathbb{T}^2)) \quad \textit{and} \quad v \in L^q([0, T]; L^r(\mathbb{T}^2)).$$

Note that Theorem 1.1 follows immediately from Proposition 3.1. The main idea of the proof of Proposition 3.1 comes from [23].

***Proof*** We first consider the case when $p \geq 2$. We write (3.1) in the Duhamel formulation:

$$v(t) = \Gamma(v) := V(t)(u_0, u_1) - \int_0^t S(t - t')F(v + \Psi)(t')dt', \tag{3.2}$$

where $V(t)$ and $S(t)$ are as defined in (1.6) and (1.7), respectively. Let $\vec{\Gamma}(v) = (\Gamma(v), \partial_t \Gamma(v))$ and $\vec{v} = (v, \partial_t v)$. Given $0 < T \leq 1$, we define the space $\mathcal{X}(T)$ as

$$\mathcal{X}^\sigma(T) = \mathcal{X}_1^\sigma(T) \times \mathcal{X}_2^\sigma(T),$$

where

$$\begin{aligned}
\mathcal{X}_1^\sigma(T) &:= C([0, T]; H^\sigma(\mathbb{T}^2)) \cap L^{p+\delta}([0, T]; L^{2p}(\mathbb{T}^2)),\\
\mathcal{X}_2^\sigma(T) &:= C([0, T]; H^{\sigma-1}(\mathbb{T}^2)).
\end{aligned}$$

Here, $\delta > 0$ is arbitrary and $\sigma = 1 - \frac{1}{p+\delta} - \frac{1}{p}$. Note that this choice of $\sigma$ along with the $L_T^{p+\delta} L_x^{2p}$ norm satisfies the scaling condition in Lemma 2.8. Our goal is to show that $\vec{\Gamma}$ is a contraction on a ball in $\mathcal{X}^\sigma(T)$ for some $0 < T \leq 1$.

By (3.2), Lemma 2.8, (1.6), Lemmas 2.10 and 2.11, and Sobolev's inequality with the fact that $|\mathbb{T}^2| = 1$, we have

$$\begin{aligned}
\|\vec{\Gamma}(v)\|_{\mathcal{X}^\sigma(T)} &\lesssim \|(u_0, u_1)\|_{\mathcal{H}^\sigma} + \big\| |v + \Psi|^p \big\|_{L_T^1 L_x^2}\\
&\lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + T^\theta \Big( \|v\|_{L_T^{p+\delta} L_x^{2p}}^p + \|\Psi\|_{L_T^{p+\delta} L_x^{2p}}^p \Big)\\
&\lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + T^\theta \Big( \|\vec{v}\|_{\mathcal{X}^\sigma(T)}^p + \|\Psi\|_{C_T W_x^{\frac{1}{2}-\alpha-\varepsilon,\infty}}^p \Big)
\end{aligned} \tag{3.3}$$

for some $\theta > 0$ and sufficiently small $\varepsilon > 0$.

For the difference estimate, we use the idea from Oh–Okamoto–Pocovnicu [32]. Noticing that $F'(u) = p|u|^{p-1}$, we use (3.2), Lemmas 2.8, 2.10 and 2.11, the fundamental theorem of calculus, Minkowski's integral inequality, Hölder's inequality, and Sobolev's inequality to obtain

$$\begin{aligned}\|\vec{\Gamma}(v)-\vec{\Gamma}(w)\|_{\mathcal{X}^\sigma(T)} &\lesssim \|F(v+\Psi)-F(w+\Psi)\|_{L^1_T L^2_x}\\ &= \bigg\| \int_0^1 F'(w+\Psi+\tau(v-w))(v-w)d\tau \bigg\|_{L^1_T L^2_x}\\ &\lesssim \int_0^1 \|w+\Psi+\tau(v-w)\|^{p-1}_{L^p_T L^{2p}_x} \|v-w\|_{L^p_T L^{2p}_x} d\tau\\ &\lesssim T^\theta\Big(\|v\|^{p-1}_{L^{p+\delta}_T L^{2p}_x}+\|w\|^{p-1}_{L^{p+\delta}_T L^{2p}_x}+\|\Psi\|^{p-1}_{L^{p+\delta}_T L^{2p}_x}\Big)\|v-w\|_{L^{p+\delta}_T L^{2p}_x}\\ &\lesssim T^\theta\Big(\|\vec{v}\|^{p-1}_{\mathcal{X}^\sigma(T)}+\|\vec{w}\|^{p-1}_{\mathcal{X}^\sigma(T)}+\|\Psi\|^{p-1}_{C_T W_x^{\frac{1}{2}-\alpha-\varepsilon,\infty}}\Big)\|\vec{v}-\vec{w}\|_{\mathcal{X}^\sigma(T)}\end{aligned} \tag{3.4}$$

for some $\theta > 0$ and sufficiently small $\varepsilon > 0$.

Thus, by choosing $T = T_\omega(\|(u_0, u_1)\|_{\mathcal{H}^s}) > 0$ small enough, we obtain that $\vec{\Gamma}$ is a contraction on the ball $B_R \subset \mathcal{X}^\sigma(T)$ of radius $R \sim 1 + \|(u_0, u_1)\|_{\mathcal{H}^s}$. Note that at this point, the uniqueness of the solution $v$ only holds in the ball $B_R$, but we can use a standard continuity argument to extend the uniqueness of $v$ to the entire $\mathcal{X}^\sigma(T)$.

For the case when $1 < p < 2$, we may have $p + \delta < 2$, so that Lemma 2.8 may not work for the $L^{p+\delta}_T L^{2p}_x$ norm. Instead, we consider the $L^q_T L^r_x$ norm with $q = 2+\delta$ and $r = \frac{4+2\delta}{1+\delta}$, where $\delta > 0$ is small enough so that $r$ is close enough to 4. We also set $\sigma = 0$, so that that this choice of $\sigma$ along with this $L^q_T L^r_x$ norm satisfies the scaling condition in Lemma 2.8. Note that we also need to modify the definition of $\mathcal{X}^\sigma_1(T)$ using this $L^q_T L^r_x$ norm. We then modify (3.3) as follows. By (3.2), Lemma 2.8, (1.6), Lemmas 2.10 and 2.11, and Sobolev's inequality, we have

$$\begin{aligned}\|\vec{\Gamma}(v)\|_{\mathcal{X}^0(T)} &\lesssim \|(u_0,u_1)\|_{\mathcal{H}^0} + \big\||v+\Psi|^p\big\|_{L^1_T L^2_x}\\ &\lesssim \|(u_0,u_1)\|_{\mathcal{H}^s} + T^\theta\Big(\|v\|^p_{L^{2+\delta}_T L^{2p}_x}+\|\Psi\|^p_{L^{p+\delta}_T L^{2p}_x}\Big)\\ &\lesssim \|(u_0,u_1)\|_{\mathcal{H}^s} + T^\theta\Big(\|\vec{v}\|^p_{\mathcal{X}^0(T)}+\|\Psi\|^p_{C_T W_x^{\frac{1}{2}-\alpha-\varepsilon,\infty}}\Big)\end{aligned}$$

for some $\theta > 0$. Here, we can ensure that $2p \le r = \frac{4+2\delta}{1+\delta}$ for any $1 < p < 2$ by choosing $\delta = \delta(p) > 0$ small enough. A similar modification can be applied to (3.4) to obtain a difference estimate, which then allows us to close the contraction argument. □

***Remark 3.2*** We point out that the local well-posedness result of vNLW (3.1) can be improved using the inhomogeneous Strichartz estimates. In particular, we can show that (3.1) is locally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$ as long as $s \ge s_{\text{crit}}$ (with a strict inequality when $p = 3$), where $s_{\text{crit}}$ is the critical regularity as defined in (1.4). For details, see Theorem A.6 and Remark A.7.

## 4 Global well-posedness of SvNLW

In this section, we aim to prove Theorem 1.3, i.e. pathwise global well-posedness of SvNLW (1.1). As mentioned in Sect. 1.2, we prove Theorem 1.3 by studying the Eq. (1.10) for $v$ with $(v, \partial_t v)|_{t=0} = (u_0, u_1)$, for given initial data $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$ of (1.1).

Fix an arbitrary $T \geq 1$. In view of Proposition 3.1, in order to show well-posedness of (3.1) on $[0, T]$, it suffices to show that the $\mathcal{H}^\sigma$-norm of the solution $\vec{v}(t) = (v(t), \partial_t v(t))$ to (3.1) remains finite on $[0, T]$, where $\sigma$ is as defined in Proposition 3.1. This will allow us to iteratively apply the pathwise local well-posedness result in Proposition 3.1.

In fact, we show that the solution $\vec{v}(t)$ belongs to $\mathcal{H}^1(\mathbb{T}^2)$. Let $0 < t \leq 1$. From Lemma 2.7, we have

$$\|V(t)(u_0, u_1)\|_{\mathcal{H}^1} \lesssim (1 + t^{-1+\sigma})\|(u_0, u_1)\|_{\mathcal{H}^\sigma}. \tag{4.1}$$

Then, let $0 < T_0 \leq 1$ be the local existence time as in the proof of Proposition 3.1. Thus, given $s \geq \sigma$, by (3.2), (4.1), Lemma 2.11, Hölder's inequality, and Sobolev's inequality, we have that for $0 < t \leq T_0$,

$$\begin{aligned}\|\vec{v}(t)\|_{\mathcal{H}^1} &\lesssim (1 + t^{-1+\sigma})\|(u_0, u_1)\|_{\mathcal{H}^\sigma} + \|(v + \Psi)^p\|_{L^1_{T_0} L^2_x} \\ &\lesssim (1 + t^{-1+\sigma})\|(u_0, u_1)\|_{\mathcal{H}^s} + T_0^\theta \Big(\|v\|^p_{L^q_{T_0} L^r_x} + \|\Psi\|^p_{C_{T_0} W_x^{\frac{1}{2}-\alpha-\varepsilon,\infty}}\Big),\end{aligned} \tag{4.2}$$

where $\delta > 0$, $\varepsilon > 0$ are sufficiently small, $\theta > 0$, and $q, r$ are as defined in the statement of Proposition 3.1. Here, due to Lemma 2.5, we can fix a good $\omega \in \Omega$ such that $\Psi = \Psi(\omega) \in C([0, T_0]; W^{\frac{1}{2}-\alpha-\varepsilon,\infty}(\mathbb{T}^2))$ for $\alpha < \frac{1}{2}$ and sufficiently small $\varepsilon > 0$, so that we know from (4.2) that $\|\vec{v}(t)\|_{\mathcal{H}^1} < \infty$. A standard argument then shows that $\vec{v} \in C((0, T_0]; \mathcal{H}^1(\mathbb{T}^2))$. Thus, our main goal is to control the $\mathcal{H}^1$-norm of $\vec{v}(t)$ on $[0, T]$ by bounding the energy $E(\vec{v})$ defined in (1.11).

For the following computation, we need to work with the smooth solution $(v_N, \partial_t v_N)$ to the truncated equation with initial data $(\pi_N v_0, \pi_N v_1)$, where $\pi_N$ is the frequency truncation onto the frequencies $\{|n| \leq N\}$. After establishing an upper bound for $E(\vec{v}(t))$ with the implicit constant independent of $N$, we can take $N \to \infty$ by using Proposition 3.1 (specifically, the continuous dependence of a solution on the initial data). Here, we omit details and work with $(v, \partial_t v)$ instead for simplicity. See, for example, [34] for a standard argument.

### 4.1 Case $1 < p \leq 3$

In this case, we follow the globalization argument by Burq–Tzvetkov [7]. For simplicity of notation, we set $E(t) = E(\vec{v}(t))$.

Given $T > 0$, we fix $0 < t \leq T$. By (1.11) and (1.10), we have

$$\begin{aligned}\partial_t E(t) &= \int_{\mathbb{T}^2} \partial_t v\big(\partial_t^2 v + (1-\Delta)v + |v|^{p-1}v\big)dx \\ &\leq -\int_{\mathbb{T}^2} \partial_t v\big(|v+\Psi|^{p-1}(v+\Psi) - |v|^{p-1}v\big)dx.\end{aligned} \tag{4.3}$$

Let $F(u) = |u|^{p-1}u$, so that we can compute $F'(u) = p|u|^{p-1}$. Thus, by the fundamental theorem of calculus, we have

$$\begin{aligned}|v+\Psi|^{p-1}(v+\Psi) - |v|^{p-1}v &= F(v+\Psi) - F(v) \\ &= \Psi\int_0^1 F'(v+\tau\Psi)d\tau \\ &\lesssim |\Psi||v|^{p-1} + |\Psi|^p.\end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4) and then applying the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\partial_t E(t) &\lesssim \|\Psi\|_{L^\infty_x}\int_{\mathbb{T}^2}|\partial_t v||v|^{p-1}dx + \int_{\mathbb{T}^2}|\partial_t v||\Psi|^p dx \\ &\leq \|\Psi\|_{L^\infty_x}\bigg(\int_{\mathbb{T}^2}(\partial_t v)^2 dx\bigg)^{\frac{1}{2}}\bigg(\int_{\mathbb{T}^2}|v|^{2(p-1)}dx\bigg)^{\frac{1}{2}} \\ &\quad + \|\Psi\|^{2p}_{L^\infty_x}\bigg(\int_{\mathbb{T}^2}(\partial_t v)^2 dx\bigg)^{\frac{1}{2}} \\ &\leq C(\Psi)E(t),\end{aligned} \tag{4.5}$$

as long as $2(p-1) \leq p+1$, or equivalently, $p \leq 3$. By Gronwall's inequality on (4.5), we get

$$E(t) \lesssim e^{C(\Psi)t}$$

for any $0 < t \leq T$.

### 4.2 Case $3 < p \leq 5$

In this case, we follow the idea introduced by Oh–Pocovnicu [34]. See also [27, 29, 39] for similar arguments. In this setting, we let $\alpha < \frac{2}{p-1} - \frac{1}{2}$, the reason of which will become clear in the following steps.

By (1.11), (1.10), and Taylor's theorem, we have

$$\begin{aligned}\partial_t E(t) &= \int_{\mathbb{T}^2} \partial_t v\big(\partial_t^2 v + (1-\Delta)v + |v|^{p-1}v\big)dx \\ &= -\int_{\mathbb{T}^2} \partial_t v\big(|v+\Psi|^{p-1}(v+\Psi) - |v|^{p-1}v\big)dx - \int_{\mathbb{T}^2} \big(D^{\frac{1}{2}}\partial_t v\big)^2 dx \\ &\le -p\int_{\mathbb{T}^2} \partial_t v \cdot |v|^{p-1}\Psi dx - \frac{p(p-1)}{2}\int_{\mathbb{T}^2} \partial_t v \cdot |v+\theta\Psi|^{p-3}(v+\theta\Psi)\Psi^2 dx \\ &=: A_1 + A_2, \end{aligned} \tag{4.6}$$

where $\theta \in (0,1)$. To estimate $A_2$, by the Cauchy-Schwartz inequality and Cauchy's inequality, we have

$$\begin{aligned}|A_2| &\lesssim \int_{\mathbb{T}^2} |\partial_t v|\big(|v|^{p-2}\Psi^2 + \Psi^p\big)dx \\ &\lesssim \left(\int_{\mathbb{T}^2} (\partial_t v)^2 dx\right)^{1/2}\left(\|\Psi\|_{L_x^\infty}^4 \int_{\mathbb{T}^2} |v|^{2(p-2)}dx + \|\Psi\|_{L_x^{2p}}^{2p}\right)^{1/2} \\ &\lesssim (1+\|\Psi\|_{L_x^\infty}^4)E(t) + \|\Psi\|_{L_x^{2p}}^p, \end{aligned} \tag{4.7}$$

where in the last inequality, we need $2(p-2) \le p+1$, which is equivalent to $p \le 5$. To estimate $A_1$, for $0 < t_1 \le t_2 \le T$, by integration by parts and Young's inequality, we have

$$\begin{aligned}\int_{t_1}^{t_2} A_1 dt' &= -\int_{t_1}^{t_2}\int_{\mathbb{T}^2} \partial_t(|v|^{p-1}v)\Psi dx dt' \\ &= -\int_{\mathbb{T}^2} |v(t_2)|^{p-1}v(t_2)\Psi(t_2)dx + \int_{\mathbb{T}^2} |v(t_1)|^{p-1}v(t_1)\Psi(t_1)dx \\ &\quad + \int_{t_1}^{t_2}\int_{\mathbb{T}^2} |v|^{p-1}v(\partial_t\Psi)dxdt' \\ &\lesssim \varepsilon\|v(t_2)\|_{L_x^{p+1}}^{p+1} + \frac{1}{\varepsilon}\|\Psi(t_2)\|_{L_x^{p+1}}^{p+1} + \|v(t_1)\|_{L_x^{p+1}}^{p+1} + \|\Psi(t_1)\|_{L_x^{p+1}}^{p+1} \\ &\quad + \int_{t_1}^{t_2}\int_{\mathbb{T}^2} |v|^{p-1}v(\partial_t\Psi)dxdt', \end{aligned} \tag{4.8}$$

where $0 < \varepsilon < 1$. We see in Remark 2.6 that $\partial_t\Psi \in C([0,T]; W^{-\frac{1}{2}-\alpha-,\infty}(\mathbb{T}^2))$. By duality, Hölder's inequality, Lemma 2.3 (i), and Lemma 2.4, we obtain

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{\mathbb{T}^2} |v|^{p-1}v(\partial_t\Psi)dxdt' \\ &= \int_{t_1}^{t_2}\int_{\mathbb{T}^2} \langle\nabla\rangle^{\frac{1}{2}+\alpha+}(|v|^{p-1}v)\langle\nabla\rangle^{-\frac{1}{2}-\alpha-}(\partial_t\Psi)dxdt' \\ &\lesssim \int_{t_1}^{t_2} \|v\|_{L_x^{p+1}}^{p-1}\big\|\langle\nabla\rangle^{\frac{1}{2}+\alpha+}v(t')\big\|_{L_x^{\frac{p+1}{2}}}\big\|\langle\nabla\rangle^{-\frac{1}{2}-\alpha-}(\partial_t\Psi)(t')\big\|_{L_x^\infty}dt'\end{aligned}$$

$$\lesssim \|\partial_t \Psi\|_{C_T W_x^{-\frac{1}{2}-\alpha-,\infty}} \int_{t_1}^{t_2} E(t')^{\frac{p-1}{p+1}} \|\langle\nabla\rangle v\|_{L_x^2}^{\frac{2}{p-1}} \|v\|_{L_x^{p+1}}^{\frac{p-3}{p-1}} dt'$$
$$\lesssim \|\partial_t \Psi\|_{C_T W_x^{-\frac{1}{2}-\alpha-,\infty}} \int_{t_1}^{t_2} E(t')dt', \tag{4.9}$$

where we require that

$$\frac{1}{2} + \alpha + = \frac{2}{p-1},$$

which is equivalent to $\alpha < \frac{2}{p-1} - \frac{1}{2}$. By combining (4.6), (4.7), (4.8), and (4.9), we have

$$E(t_2) \leq (1 + C_1(\Psi)) \int_{t_1}^{t_2} E(t')dt' + C_2(\Psi, v(t_1)).$$

By Gronwall's inequality, we get

$$E(t) \lesssim e^{C(\Psi)t}$$

for any $0 < t \leq T$.

### 4.3 Case $p > 5$

In this case, we follow the idea by Latocca [24]. In this setting, we also let $\alpha < \frac{2}{p-1} - \frac{1}{2}$.

We need the following lemma to close the energy estimates in the Gronwall argument. We define $\beta_p := \lceil \frac{p-3}{2} \rceil$, $F(u) := |u|^{p-1}u$, and $s_p := \frac{p-3}{p-1}$.

**Lemma 4.1** *For any $0 < t \leq T$ and every integer $1 \leq k \leq \beta_p$, we have*

$$\left| \int_{\mathbb{T}^2} F^{(k-1)}(v(t))\Psi(t)^{k-1}\partial_t\Psi(t)dx \right|$$
$$\lesssim g\big(\|\Psi\|_{L^\infty([0,T];X)}, \|\langle\nabla\rangle^{-1}\partial_t\Psi\|_{L^\infty([0,T];Y)}\big)(1 + E(t)),$$

*where $g$ is a polynomial with positive coefficients, and*

$$X := L^\infty(\mathbb{T}^2) \cap B^{1-s_p}_{\frac{p+1}{2},1}(\mathbb{T}^2) \quad \textit{and} \quad Y := L^\infty(\mathbb{T}^2) \cap B^{s_p}_{\infty,1}(\mathbb{T}^2).$$

Note that given $\alpha < \frac{2}{p-1} - \frac{1}{2}$, by Lemma 2.5, Remark 2.6, and Lemma 2.1 (ii), we have

$$g\big(\|\Psi\|_{L^\infty([0,T];X)}, \|\langle\nabla\rangle^{-1}\partial_t\Psi\|_{L^\infty([0,T];Y)}\big) < \infty$$

almost surely.

Let us first assume Lemma 4.1 and work on the energy bound. As in the case when $p > 3$, we can compute that for $0 < t \le T$,

$$\partial_t E(t) \le -\int_{\mathbb{T}^2} \partial_t v(F(v+\Psi) - F(v))dx. \tag{4.10}$$

For our convenience we compute that for $k \in \mathbb{Z}_+$,

$$F^{(k)}(u) = \begin{cases} C_{p,k}|u|^{p-k-1}u & \text{for} \quad k \text{ even,} \\ C_{p,k}|u|^{p-k} & \text{for} \quad k \text{ odd.} \end{cases}$$

By Taylor's formula at the point $v(t,x)$ with integral remainder up to the order $\beta_p = \lceil \frac{p-3}{2} \rceil$, we have

$$F(v+\Psi) - F(v) = \sum_{k=1}^{\beta_p} \frac{1}{k!} F^{(k)}(v)\Psi^k + \int_v^{v+\Psi} \frac{F^{(\beta_p+1)}(\tau)}{\beta_p!}(v+\Psi-\tau)^{\beta_p} d\tau.$$

Let $0 < t_1 \le t_2 \le T$. By integrating (4.10) from $t_1$ to $t_2$, we can write

$$E(t_2) \le E(t_1) + \sum_{k=1}^{\beta_p} C_k I_k + C_p R, \tag{4.11}$$

where

$$\begin{aligned} I_k &:= -\int_{t_1}^{t_2} \int_{\mathbb{T}^2} \partial_t v F^{(k)}(v) \Psi^k dx dt' \quad \text{for} \quad 1 \le k \le \beta_p, \\ R &:= -\int_{t_1}^{t_2} \int_{\mathbb{T}^2} \int_v^{v+\Psi} \partial_t v F^{(\beta_p+1)}(\tau)(v+\Psi-\tau)^{\beta_p} d\tau dx dt'. \end{aligned}$$

We first estimate $R$. Note that for $\tau \in [v, v+\Psi]$, we have

$$|F^{(\beta_p+1)}(\tau)| \lesssim |v|^{p-\beta_p-1} + |\Psi|^{p-\beta_p-1}.$$

Thus, by Hölder's inequality and Young's inequality, we have

$$\begin{aligned} R &\lesssim \int_{t_1}^{t_2} \int_{\mathbb{T}^2} \partial_t v (|v|^{p-\beta_p-1}|\Psi|^{\beta_p+1} + |\Psi|^p) dx dt' \\ &\lesssim \int_{t_1}^{t_2} \|\partial_t v(t')\|_{L^2_x} \|v(t')\|_{L^{p+1}_x}^{p-\beta_p-1} \|\Psi(t')\|_{L_x^{rp(\beta_p+1)}}^{\beta_p+1} dt' \\ &\quad + \int_{t_1}^{t_2} \|\partial_t v(t')\|_{L^2_x}^2 dt' + \|\Psi\|_{L_T^{2p} L_x^{2p}}^{2p} \\ &\le \|\Psi\|_{L_T^{2p} L_x^{2p}}^{2p} + \left(1 + \|\Psi\|_{L_T^\infty L_x^{rp(\beta_p+1)}}^{\beta_p+1}\right) \end{aligned}$$

$$\times \int_{t_1}^{t_2} \max \left\{ E(t'), E(t')^{\frac{1}{2}+\frac{p-\beta_p-1}{p+1}} \right\} dt', \tag{4.12}$$

where $r_p$ satisfies $\frac{1}{2} + \frac{p-\beta_p-1}{p+1} + \frac{1}{r_p} = 1$. Since $\beta_p = \lceil \frac{p-3}{2} \rceil \geq \frac{p-3}{2}$, we have $\frac{p-\beta_p-1}{p+1} \leq \frac{1}{2}$, so that

$$R \lesssim \|\Psi\|^{2p}_{L^{2p}_T L^{2p}_x} + \left( 1 + \|\Psi\|^{\beta_p+1}_{L^\infty_T L^{r_p(\beta_p+1)}_x} \right) \int_{t_1}^{t_2} (1 + E(t')) dt'.$$

We now estimate $I_k$. By Fubini's theorem and integration by parts in time, we have

$$\begin{aligned}
|I_k| &= \left| - \int_{\mathbb{T}^2} \int_{t_1}^{t_2} \partial_t (F^{(k-1)}(v)) \Psi^k dt' dx \right| \\
&\leq \left| \int_{\mathbb{T}^2} F^{(k-1)}(v(t_2)) \Psi^k(t_2) dx \right| + \left| \int_{\mathbb{T}^2} F^{(k-1)}(v(t_1)) \Psi^k(t_1) dx \right| \\
&\quad + \left| k \int_{\mathbb{T}^2} \int_{t_1}^{t_2} F^{(k-1)}(v(t')) \Psi(t')^{k-1} \partial_t \Psi(t') dt' dx \right| \\
&\lesssim \int_{\mathbb{T}^2} |v(t_2)|^{p-k+1} |\Psi(t_2)|^k + |v(t_1)|^{p-k+1} |\Psi(t_1)|^k dx \\
&\quad + \left| \int_{t_1}^{t_2} \int_{\mathbb{T}^2} F^{(k-1)}(v(t')) \Psi(t')^{k-1} \partial_t \Psi(t') dx dt' \right| \\
&= J_k + K_k.
\end{aligned} \tag{4.13}$$

To handle $J_k$, by Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}
J_k &\leq E(t_2)^{\frac{p-k+1}{p+1}} \|\Psi(t_2)\|^k_{L^{p+1}_x} + E(t_1)^{\frac{p-k+1}{p+1}} \|\Psi(t_1)\|^k_{L^{p+1}_x} \\
&\leq \varepsilon E(t_2) + C_1 E(t_1) + C_2 \|\Psi\|^{p+1}_{L^\infty_T L^{p+1}_x},
\end{aligned} \tag{4.14}$$

where $0 < \varepsilon < 1$. To deal with $K_k$, by Lemma 4.1,

$$K_k \lesssim g\big( \|\Psi\|_{L^\infty([0,T];X)}, \|\langle \nabla \rangle^{-1} \partial_t \Psi\|_{L^\infty([0,T];Y)} \big) \left( 1 + \int_{t_1}^{t_2} E(t') dt' \right). \tag{4.15}$$

By combining (4.11), (4.12), (4.13), (4.14), (4.15), we obtain

$$\begin{aligned}
E(t_2) &\lesssim \left( 1 + \|\Psi\|^{\beta_p+1}_{L^\infty_T L^{r_p(\beta_p+1)}_x} + g\big( \|\Psi\|_{L^\infty([0,T];X)}, \|\langle \nabla \rangle^{-1} \partial_t \Psi\|_{L^\infty([0,T];Y)} \big) \right) \\
&\quad \times \left( 1 + \int_{t_1}^{t_2} E(t') dt' \right) + \|\Psi\|^{2p}_{L^{2p}_T L^{2p}_x} + \|\Psi\|^{p+1}_{L^\infty_T L^{p+1}_x} + E(t_1).
\end{aligned}$$

We can then use Gronwall's inequality to get the desired bound.

We now provide the proof of Lemma 4.1.

***Proof of Lemma 4.1*** Recall that $s_p = \frac{p-3}{p-1}$. We first consider the case when $k \geq 2$. By the Fourier–Plancherel theorem, we have

$$\begin{aligned}
&\left| \int_{\mathbb{T}^2} F^{(k-1)}(v(t))\Psi(t)^{k-1}\partial_t \Psi(t)dx \right| \\
&= \left| \sum_{j'=-1}^{1} \sum_{j\geq 0} \int_{\mathbb{T}^2} \mathbf{P}_j(F^{(k-1)}(v(t))\Psi(t)^{k-1})\mathbf{P}_{j+j'}(\partial_t \Psi(t))dx \right| \\
&\lesssim \sum_{j>2} \int_{\mathbb{T}^2} |\mathbf{P}_j(F^{(k-1)}(v(t))\Psi(t)^{k-1})||\mathbf{P}_j(\partial_t \Psi(t))|dx \\
&\quad + \sum_{j=0}^{2} \int_{\mathbb{T}^2} |\mathbf{P}_j(F^{(k-1)}(v(t))\Psi(t)^{k-1})||\mathbf{P}_j(\partial_t \Psi(t))|dx \\
&=: I_1 + I_2.
\end{aligned} \tag{4.16}$$

Let $r_k := \frac{(k-1)(p+1)}{k}$. To estimate $I_2$, by Hölder's inequality, Bernstein's inequality, and Young's inequality, we have

$$\begin{aligned}
I_2 &\lesssim \|\Psi(t)\|_{L_x^{r_k}}^{k-1} \|v(t)\|_{L_x^{p+1}}^{p-k+1} \sum_{j=0}^{2} \|\mathbf{P}_j \partial_t \Psi(t)\|_{L_x^\infty} \\
&\lesssim \|\Psi(t)\|_{L_x^{r_k}}^{k-1} \|\langle\nabla\rangle^{-1}\partial_t \Psi(t)\|_{L_x^\infty} E(t)^{\frac{p-k+1}{p+1}} \\
&\lesssim E(t) + \|\Psi\|_{L_T^\infty L_x^{r_k}}^{r_k} \|\langle\nabla\rangle^{-1}\partial_t \Psi\|_{L_T^\infty L_x^\infty}^{\frac{p+1}{k}}.
\end{aligned}$$

It remains to estimate $I_1$. By Hölder's inequality, Bernstein's inequality, and then Hölder's inequality for series,

$$\begin{aligned}
I_1 &\lesssim \sum_{j>2} 2^{j(1-s_p)} \|\mathbf{P}_j(F^{(k-1)}(v(t))\Psi(t)^{k-1})\|_{L_x^1} 2^{js_p} \|\mathbf{P}_j(\langle\nabla\rangle^{-1}\partial_t \Psi(t))\|_{L_x^\infty} \\
&\leq \|F^{(k-1)}(v(t))\Psi(t)^{k-1}\|_{B_{1,\infty}^{1-s_p}} \|\langle\nabla\rangle^{-1}\partial_t \Psi\|_{B_{\infty,1}^{s_p}}.
\end{aligned}$$

Then, by Corollary 2.2, we have

$$\begin{aligned}
\|F^{(k-1)}(v(t))\Psi(t)^{k-1}\|_{B_{1,\infty}^{1-s_p}} &\lesssim \|F^{(k-1)}(v(t))\|_{B^{1-s_p}_{\frac{p+1}{p+2-k},\infty}} \|\Psi(t)^{k-1}\|_{L^{\frac{p+1}{k-1}}} \\
&\quad + \||v(t)|^{p-k+1}\|_{L^{\frac{p+1}{p-k+1}}} \|\Psi(t)^{k-1}\|_{B^{1-s_p}_{p_k,\infty}} \\
&\lesssim \|F^{(k-1)}(v(t))\|_{B^{1-s_p}_{\frac{p+1}{p+2-k},\infty}} \|\Psi(t)\|_{L^{p+1}}^{k-1}
\end{aligned}$$

$$+E(t)^{\frac{p-k+1}{p+1}}\|\Psi(t)^{k-1}\|_{B^{1-s_p}_{p_k,\infty}}, \tag{4.17}$$

where $p_k$ satisfies $\frac{1}{p_k}+\frac{p-k+1}{p+1}=1$. By Lemma 2.3 (ii), we have

$$\begin{aligned}\|\Psi(t)^{k-1}\|_{B^{1-s_p}_{p_k,\infty}} &\lesssim \|\Psi(t)\|_{B^{1-s_p}_{p_k,\infty}}\|\Psi(t)\|^{k-2}_{L^\infty}\\ &\le \|\Psi(t)\|_{B^{1-s_p}_{\frac{p+1}{2},\infty}}\|\Psi(t)\|^{k-2}_{L^\infty}.\end{aligned} \tag{4.18}$$

By Lemma 2.3 (ii), Lemma 2.1 (ii), and Lemma 2.4, we have

$$\begin{aligned}\|F^{(k-1)}(v(t))\|_{B^{1-s_p}_{\frac{p+1}{p+2-k},\infty}} &\lesssim \|v(t)\|_{B^{1-s_p}_{\frac{p+1}{2},\infty}}\big\||v(t)|^{p-k}\big\|_{L^{\frac{p+1}{p-k}}}\\ &\lesssim \|v(t)\|_{W^{1-s_p,\frac{p+1}{2}}}E(t)^{\frac{p-k}{p+1}}\\ &\lesssim \|\langle\nabla\rangle v(t)\|^{1-\beta}_{L^2}\|v(t)\|^{\beta}_{L^{p+1}}E(t)^{\frac{p-k}{p+1}},\end{aligned}$$

where $\beta\in[0,s_p]$ satisfies $\frac{2}{p+1}=\frac{1-s_p}{2}+\frac{\beta}{p+1}$, and so $\beta=\frac{p-3}{p-1}=s_p$. Thus, we obtain

$$\begin{aligned}\|F^{(k-1)}(v(t))\|_{B^{1-s_p}_{\frac{p+1}{p+2-k},\infty}} &\lesssim E(t)^{\frac{1-\beta}{2}+\frac{\beta}{p+1}+\frac{p-k}{p+1}} = E(t)^{\frac{2}{p+1}+\frac{p-k}{p+1}}\\ &\lesssim 1+E(t).\end{aligned} \tag{4.19}$$

By combining (4.17), (4.18), and (4.19), we obtain the desired bound for $I_1$.

For the case when $k=1$, after (4.16), we have the estimate $I_2\lesssim E(t)+\|\langle\nabla\rangle^{-1}\partial_t\Psi\|^{p+1}_{L^\infty_T L^\infty_x}$. For the term $I_1$, by the estimate in (4.19), we have

$$I_1\lesssim \|F(v(t))\|_{B^{1-s_p}_{1,\infty}}\|\langle\nabla\rangle^{-1}\partial_t\Psi\|_{B^{s_p}_{\infty,1}}\lesssim \|\langle\nabla\rangle^{-1}\partial_t\Psi\|_{B^{s_p}_{\infty,1}}(1+E(t)),$$

as desired. □

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## Appendix A: On local well-posedness of subcritical vNLW

In this appendix, we aim to show that the deterministic viscous NLW is locally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$ with $s \geq s_{\text{crit}}$, where we recall that $s_{\text{crit}}$ is defined by

$$s_{\text{crit}} := \max\left(1 - \frac{2}{p-1}, 0\right). \tag{A.1}$$

More precisely, we prove local well-posedness of the following subcritical vNLW:

$$\begin{cases} \partial_t^2 u + (1-\Delta)u + D\partial_t u \pm |u|^{p-1}u = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \tag{A.2}$$

where $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$ and $s \geq s_{\text{crit}}$ (with a strict inequality when $p = 3$). To achieve this, we will need the inhomogeneous Strichartz estimates for the linear viscous wave equation on $\mathbb{T}^2$.

### A.1 The inhomogeneous Strichartz estimates

In this subsection, we prove the Strichartz estimates for the inhomogeneous linear viscous wave equation on $\mathbb{T}^d$. To achieve this, we first establish the following estimate for the linear operator $S(t)$ defined in (1.7).

**Lemma A.1** *Let $1 \leq p \leq 2 \leq q \leq \infty$. Then, we have*

$$\|S(t)\phi\|_{L^q(\mathbb{T}^d)} \lesssim t^{1-d(\frac{1}{p}-\frac{1}{q})}\|\phi\|_{L^p(\mathbb{T}^d)}$$

*for any $0 < t \leq 1$*

***Proof*** By (1.7) and applying the Schauder estimate (Lemma 2.7) twice, we obtain

$$\begin{aligned} \|S(t)\phi\|_{L^q(\mathbb{T}^d)} &= \left\| e^{-\frac{D}{4}t} \frac{\sin(t[\![D]\!])}{[\![D]\!]} e^{-\frac{D}{4}t}\phi \right\|_{L^q(\mathbb{T}^d)} \\ &\lesssim t^{-d(\frac{1}{2}-\frac{1}{q})} \left\| \frac{\sin(t[\![D]\!])}{[\![D]\!]} e^{-\frac{D}{4}t}\phi \right\|_{L^2(\mathbb{T}^d)} \\ &\leq t^{1-d(\frac{1}{2}-\frac{1}{q})} \left\| e^{-\frac{D}{4}t}\phi \right\|_{L^2(\mathbb{T}^d)} \\ &\lesssim t^{1-d(\frac{1}{p}-\frac{1}{q})}\|\phi\|_{L^p(\mathbb{T}^d)}, \end{aligned}$$

as desired. □

We now establish the Strichartz estimates for the inhomogeneous linear viscous wave equation on $\mathbb{T}^d$. We say that $u$ is a solution to the following inhomogeneous linear viscous wave equation:

$$\begin{cases} \partial_t^2 u + (1-\Delta)u + D\partial_t u = f \\ (u, \partial_t u)|_{t=0} = (\phi_0, \phi_1), \end{cases} \tag{A.3}$$

if $u$ satisfies the following Duhamel formulation:

$$u(t) = V(t)(\phi_0, \phi_1) + \int_0^t S(t-t')f(t')dt',$$

where $V(t)$ and $S(t)$ are as defined in (1.6) and (1.7), respectively.

**Lemma A.2** *Given $s \geq 0$, suppose that $1 < \widetilde{q} \leq 2 < q < \infty$, $1 \leq \widetilde{r} \leq 2 \leq r \leq \infty$ satisfy the following scaling condition:*

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s = \frac{1}{\widetilde{q}} + \frac{d}{\widetilde{r}} - 2. \tag{A.4}$$

*Then, a solution $u$ to the inhomogeneous linear viscous wave Eq. (A.3) satisfies the following inequality:*

$$\|(u, \partial_t u)\|_{C_T \mathcal{H}_x^s(\mathbb{T}^d)} + \|u\|_{L_T^q L_x^r(\mathbb{T}^d)} \lesssim \|(\phi_0, \phi_1)\|_{\mathcal{H}^s(\mathbb{T}^d)} + \|f\|_{L_T^{\widetilde{q}} L_x^{\widetilde{r}}(\mathbb{T}^d)}, \tag{A.5}$$

*for all* $0 < T \leq 1$.

***Proof*** By (1.6), we have

$$\big\|\big(V(t)(\phi_0, \phi_1), \partial_t V(t)(\phi_0, \phi_1)\big)\big\|_{C_T \mathcal{H}_x^s(\mathbb{T}^d)} \lesssim \|(\phi_0, \phi_1)\|_{\mathcal{H}^s(\mathbb{T}^d)}. \tag{A.6}$$

By Lemma 2.8, we have

$$\|V(t)(\phi_0, \phi_1)\|_{L_T^q L_x^r(\mathbb{T}^d)} \lesssim \|(\phi_0, \phi_1)\|_{\mathcal{H}^s(\mathbb{T}^d)}. \tag{A.7}$$

We then use Lemma 3.5 in [21] (which is in the $\mathbb{R}^d$ setting, but the proof also works in the $\mathbb{T}^d$ setting with Lemma A.1 in hand) to obtain

$$\bigg\| \int_0^t S(t-t')f(t')dt' \bigg\|_{L_T^q L_x^r(\mathbb{T}^d)} \lesssim \|f\|_{L_T^{\widetilde{q}} L_x^{\widetilde{r}}(\mathbb{T}^d)}. \tag{A.8}$$

It remains to show

$$\bigg\| \int_0^t S(t-t')f(t')dt' \bigg\|_{C_T H_x^s(\mathbb{T}^d)} \lesssim \|f\|_{L_T^{\widetilde{q}} L_x^{\widetilde{r}}(\mathbb{T}^d)} \tag{A.9}$$

and

$$\left\| \partial_t \int_0^t S(t-t')f(t')dt' \right\|_{C_T H_x^{s-1}(\mathbb{T}^d)} \lesssim \|f\|_{L_T^{\widetilde{q}} L_x^{\widetilde{r}}(\mathbb{T}^d)}, \tag{A.10}$$

so that (A.5) follows from (A.6), (A.7), (A.8), (A.9), and (A.10).

To show that the inequality (A.9) holds, we use the Littlewood-Paley decomposition as in Lemma 3.6 in [21]. In view of the proof of Lemma 3.6 in [21], we know that it suffices to show (A.9) for all $f$ such that $\widehat{f}$ is supported in $\{n \in \mathbb{Z}^d : 2^{j-1} \le |n| \le 2^{j+1}\}$ for all $j \in \mathbb{Z}_+$ (the case for $\{n \in \mathbb{Z}^d : 0 \le |n| \le 2\}$ follows in a similar manner) with the underlying constant independent of $j$. Fix $0 < t < T$. By Minkowski's integral inequality, Hölder's inequality in $n$, Hausdorff-Young inequality, Hölder's inequality in $t'$ (along with the fact that the number of lattice points inside a ball of radius $R$ in $\mathbb{R}^d$ is $O(R^d)$), and a change of variable, we have

$$\begin{aligned}
&\left\| \int_0^t S(t-t')f(t')dt' \right\|_{C_T H_x^s} \\
&\lesssim \int_0^t \left( \sum_{n\in\mathbb{Z}^2} |n|^{2s} \left| e^{-\frac{|n|}{2}(t-t')} \frac{\sin((t-t')[\![n]\!])}{[\![n]\!]} \widehat{f}(t',n) \right|^2 \right)^{1/2} dt' \\
&\lesssim 2^{(j+1)s} \int_0^t (t-t') e^{2^{j-2}(t-t')} \left( \sum_{n\in\mathbb{Z}^2} \left|\widehat{f}(t',n)\right|^2 \right)^{1/2} dt' \\
&\lesssim 2^{(j+1)s} \int_0^t (t-t') e^{2^{j-2}(t-t')} \left(2^{(j+1)d}\right)^{\frac{\widetilde{r}'-2}{2\widetilde{r}'}} \|\widehat{f}(t',n)\|_{\ell_n^{\widetilde{r}'}} dt' \\
&\lesssim 2^{(j+1)(s+\frac{d}{\widetilde{r}}-\frac{d}{2})} \left( \int_0^t \left|(t-t') e^{2^{j-2}(t-t')}\right|^{\widetilde{q}'} dt' \right)^{1/\widetilde{q}'} \|f\|_{L_T^{\widetilde{q}} L_x^{\widetilde{r}}} \\
&\lesssim 2^{(j+1)(s+\frac{d}{\widetilde{r}}-\frac{d}{2}+\frac{1}{\widetilde{q}}-2)} \|f\|_{L_T^{\widetilde{q}} L_x^{\widetilde{r}}}.
\end{aligned}$$

By using the second equality in the scaling condition (A.4), we obtain the desired inequality with the underlying constant independent of $j$, and so the inequality (A.9) follows. The inequality (A.10) follows in a similar manner. □

***Remark A.3*** As in the case of the homogeneous Strichartz estimates (Lemma 2.8), the Strichartz estimates for the inhomogeneous linear viscous wave equation on $\mathbb{T}^d$ also hold for a larger class of pairs $(q, r)$ and $(\widetilde{q}, \widetilde{r})$ compared to the Strichartz estimates for the usual linear wave equations [15, 17, 19, 25]. Again, this is due to the parabolic smoothing effect. Note that this is also true on $\mathbb{R}^d$ (see [21]).

We complete this subsection by making the following observation. Recall that we are considering the viscous NLW on $\mathbb{T}^2$ with nonlinearity $|u|^{p-1}u$ for $p > 1$. Suppose that we can find pairs $(q, r)$ and $(\widetilde{q}, \widetilde{r})$ satisfying the scaling condition (A.4) such that

$$q > p\widetilde{q} \quad \text{and} \quad r \ge p\widetilde{r}.$$

Then, by Hölder's inequality and the fact that $|\mathbb{T}^2| = 1$, we have

$$\big\| |u|^{p-1}u \big\|_{L_T^{\widetilde{q}} L_x^{\widetilde{r}}} \le T^{\frac{1}{\widetilde{q}} - \frac{p}{q}} \|u\|^p_{L_T^q L_x^r}.$$

Note that the power of $T$ is positive when $q > p\widetilde{q}$. The following lemma shows that there exist such pairs $(q, r)$ and $(\widetilde{q}, \widetilde{r})$.

**Lemma A.4** *Let $s_{crit}$ be as defined in (A.1). Given $s_{crit} < s < 1$, there exist $1 < \widetilde{q} \le 2 < q < \infty$, $1 \le \widetilde{r} \le 2 \le r \le \infty$ satisfying the scaling condition (A.4) such that*

$$q > p\widetilde{q} \quad \text{and} \quad r \ge p\widetilde{r}. \tag{A.11}$$

***Proof*** In view of Lemma 3.3 in [17], given $0 < s < 1$, we have

$$\min\left(\frac{q}{\widetilde{q}}, \frac{r}{\widetilde{r}}\right) \le \frac{3-s}{1-s},$$

and the equality holds by taking, for example,

$$(q, r) = \left(\frac{3-s}{1-s}\delta, \frac{2}{1-s-\frac{1-s}{(3-s)\delta}}\right) \quad \text{and} \quad (\widetilde{q}, \widetilde{r}) = \left(\delta, \frac{2}{3-s-\frac{1}{\delta}}\right), \tag{A.12}$$

where $\delta = \delta(s) > 1$ is sufficiently close to 1. Moreover, we note that $\frac{3-s}{1-s} > p$ if and only if $s > 1 - \frac{2}{p-1}$. Thus, as long as $s_{\text{crit}} < s < 1$, there exist pairs $(q, r)$ and $(\widetilde{q}, \widetilde{r})$ that satisfy (A.11). □

***Remark A.5*** In the case when $p > 3$ and $s = s_{\text{crit}} = 1 - \frac{2}{p-1} > 0$, we have

$$\min\left(\frac{q}{\widetilde{q}}, \frac{r}{\widetilde{r}}\right) \le \frac{3-s}{1-s} = p,$$

so that we can only find pairs $(q, r)$ and $(\widetilde{q}, \widetilde{r})$ that satisfy $q = p\widetilde{q}$ and $r = p\widetilde{r}$ instead of $q > p\widetilde{q}$ and $r \ge p\widetilde{r}$. Such pairs do exist. One can take, for example, $(q, r)$ and $(\widetilde{q}, \widetilde{r})$ as in (A.12).

In the case when $1 < p \le 3$ and $s = s_{\text{crit}} = 0$, there does not exist any pair $(\widetilde{q}, \widetilde{r})$ that satisfies $1 < \widetilde{q} \le 2$, $1 \le \widetilde{r} \le 2$, and the scaling condition (A.4) (with $d = 2$) simultaneously. In this case, the inhomogeneous Strichartz estimates (Lemma A.2) no longer applies, so that an alternative approach is needed to deal with this case.

### A.2: Local well-posedness of subcritical vNLW

In this subsection, we prove the following theorem for the local well-posedness result of vNLW (A.2).

**Theorem A.6** *Let $p > 1$ and let $s_{crit}$ be as in (A.1). Then, (A.2) is locally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$ for*

$$\text{(i) } p \neq 3\text{: } s \geq s_{crit} \quad \textit{or} \quad \text{(ii) } p = 3\text{: } s > s_{crit}.$$

*More precisely, given any* $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$*, there exists* $0 < T = T(u_0, u_1) \leq 1$ *and a unique solution* $\vec{u} = (u, \partial_t u)$ *to* (A.2) *in the class*

$$(u, \partial_t u) \in C([0, T]; \mathcal{H}^s(\mathbb{T}^2)) \quad \textit{and} \quad u \in L^q([0, T]; L^r(\mathbb{T}^2)),$$

*for some suitable* $q, r \geq 2$.

***Proof*** For the proof, we only consider the case $s < 1$. We first consider the case $s > s_{\text{crit}}$. We write (A.2) in the Duhamel formulation:

$$u(t) = \Gamma(u) := V(t)(u_0, u_1) - \int_0^t S(t - t')F(u)(t')dt', \tag{A.13}$$

where $F(u) = |u|^{p-1}u$, $V(t)$ is as defined in (1.6), and $S(t)$ is as defined in (1.7). Let $\vec{\Gamma}(u) = (\Gamma(u), \partial_t\Gamma(u))$ and $\vec{u} = (u, \partial_t u)$.

Let $(q, r)$ and $(\widetilde{q}, \widetilde{r})$ be as given in Lemma A.4, which guarantees that $q > p\widetilde{q}$ and $r \geq p\widetilde{r}$. Given $0 < T \leq 1$, we define the space $\mathcal{Y}(T)$ as

$$\mathcal{Y}^s(T) = \mathcal{Y}^s_1(T) \times \mathcal{Y}^s_2(T),$$

where

$$\begin{aligned} \mathcal{Y}^s_1(T) &:= C([0, T]; H^s(\mathbb{T}^2)) \cap L^q([0, T]; L^r(\mathbb{T}^2)), \\ \mathcal{Y}^s_2(T) &:= C([0, T]; H^{s-1}(\mathbb{T}^2)). \end{aligned}$$

Our goal is to show that $\vec{\Gamma}$ is a contraction on a ball in $\mathcal{Y}^s(T)$ for some $0 < T \leq 1$.

By (A.13), Lemma A.2, and Hölder's inequality, we have

$$\begin{aligned} \|\vec{\Gamma}(u)\|_{\mathcal{Y}^s(T)} &\lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + \big\| |u|^p \big\|_{L^{\widetilde{q}}_T L^{\widetilde{r}}_x} \\ &\lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + T^\theta \|u\|^p_{L^q_T L^r_x} \\ &\lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + T^\theta \|\vec{u}\|^p_{\mathcal{Y}^s(T)} \end{aligned} \tag{A.14}$$

for some $\theta > 0$.

For the difference estimate, we use the idea from Oh–Okamoto–Pocovnicu [32]. Noticing that $F'(u) = p|u|^{p-1}$, we use (A.13), Lemma A.2, the fundamental theorem of calculus, Minkowski's integral inequality, and Hölder's inequality to obtain

$$\begin{aligned} \|\vec{\Gamma}(u) - \vec{\Gamma}(v)\|_{\mathcal{Y}^s(T)} &\lesssim \|F(u) - F(v)\|_{L^{\widetilde{q}}_T L^{\widetilde{r}}_x} \\ &= \bigg\| \int_0^1 F'(v + \tau(u - v))(u - v)d\tau \bigg\|_{L^{\widetilde{q}}_T L^{\widetilde{r}}_x} \\ &\lesssim \int_0^1 \|v + \tau(u - v)\|^{p-1}_{L^{p\widetilde{q}}_T L^{p\widetilde{r}}_x} \|u - v\|_{L^{p\widetilde{q}}_T L^{p\widetilde{r}}_x} d\tau \end{aligned}$$

$$\begin{aligned}&\lesssim T^\theta\Big(\|u\|_{L^q_T L^r_x}^{p-1}+\|v\|_{L^q_T L^r_x}^{p-1}\Big)\|u-v\|_{L^q_T L^r_x}\\&\lesssim T^\theta\Big(\|\vec{u}\|_{\mathcal{Y}^s(T)}^{p-1}+\|\vec{v}\|_{\mathcal{Y}^s(T)}^{p-1}\Big)\|\vec{u}-\vec{v}\|_{\mathcal{Y}^s(T)}.\end{aligned}$$

for some $\theta > 0$.

Thus, by choosing $T = T(\|(u_0, u_1)\|_{\mathcal{H}^s}) > 0$ small enough, we obtain that $\vec{\Gamma}$ is a contraction on the ball $B_R \subset \mathcal{Y}^s(T)$ of radius $R \sim 1 + \|(u_0, u_1)\|_{\mathcal{H}^s}$.

In the case when $p > 3$ and $s = s_{\text{crit}} = 1 - \frac{2}{p-1} > 0$, we can only find pairs $(q, r)$ and $(\widetilde{q}, \widetilde{r})$ that satisfy $q = p\widetilde{q}$ and $r = p\widetilde{r}$ (see Remark A.5). In this case, we modify the argument as follows. By (A.13), Lemma A.2, and Hölder's inequality, we obtain

$$\begin{aligned}\|\Gamma(u)\|_{L^q_T L^r_x}&\lesssim \|V(t)(u_0,u_1)\|_{L^q_T L^r_x}+\big\||u|^p\big\|_{L^{\widetilde{q}}_T L^{\widetilde{r}}_x}\\&\lesssim \|V(t)(u_0,u_1)\|_{L^q_T L^r_x}+\|u\|^p_{L^q_T L^r_x}\end{aligned}$$

for some $\theta > 0$ and sufficiently small $\varepsilon > 0$. A difference estimate on $\Gamma(u) - \Gamma(v)$ also holds by a similar computation. By the dominated convergence theorem, we have $\|u\|^p_{L^q_T L^r_x} \to 0$ as $T \to 0$. Thus, we can choose $T = T(u_0, u_1) > 0$ sufficiently small such that $\|V(t)(u_0, u_1)\|_{L^q_T L^r_x} \le \frac{1}{2}\eta \ll 1$, so that we can show that $\Gamma$ is a contraction on the ball of radius $\eta$ in $L^q_T L^r_x$. Moreover, (A.14) gives

$$\|\vec{u}\|_{C_T\mathcal{H}^s_x}=\|\vec{\Gamma}(u)\|_{C_T\mathcal{H}^s_x}\lesssim \|(u_0,u_1)\|_{\mathcal{H}^s}+\|u\|^p_{L^q_T L^r_x}<\infty,$$

so that $\vec{u} = (u, \partial_t u) \in C_T\mathcal{H}^s_x$.

Lastly, we consider the case when $1 < p < 3$ and $s = s_{\text{crit}} = 0$. Note that $s = 0$ along with the $L^3_T L^3_x$ norm satisfies the scaling condition (2.9) in Lemma 2.8. By (A.13), Minkowski's integral inequality, Lemma 2.8, Sobolev's inequality, and Hölder's inequality, we obtain

$$\begin{aligned}\|\Gamma(u)\|_{L^3_T L^3_x}&\lesssim \|V(t)(u_0,u_1)\|_{L^3_T L^3_x}+\int_0^T\big\|\mathbf{1}_{[0,t]}(t')S(t-t')\big(|u|^{p-1}u\big)(t')\big\|_{L^3_T L^3_x}dt'\\&\lesssim \|(u_0,u_1)\|_{\mathcal{H}^0}+\int_0^T\big\|\big(|u|^{p-1}u\big)(t')\big\|_{H^{-1}_x}dt'\\&\lesssim \|(u_0,u_1)\|_{\mathcal{H}^0}+\big\||u|^p\big\|_{L^1_T L^{1+}_x}\\&\lesssim \|(u_0,u_1)\|_{\mathcal{H}^0}+T^\theta\|u\|_{L^3_T L^3_x}\end{aligned}$$

for some $\theta > 0$. Also, by (1.6) and (1.7), we easily obtain

$$\|\vec{\Gamma}(u)\|_{C_T\mathcal{H}^0_x}\lesssim \|(u_0,u_1)\|_{\mathcal{H}^0}+T^\theta\|u\|_{L^3_T L^3_x}.$$

Similar difference estimates also hold, so that we can conclude using the standard contraction argument. This finishes the proof. □

We finish this appendix by stating several remarks.

***Remark A.7*** (i) At this point, we do not know how to prove local well-posedness for the cubic vNLW (with $p = 3$) in $L^2(\mathbb{T}^2)$, i.e. with $s = s_{\text{crit}} = 0$. It would be of interest to investigate if spaces of functions of bounded $p$-variation (i.e. $U^p$- and $V^p$-spaces) such as those in [3, 32] can be applied to handle the cubic case.

(ii) A slight modification of the proof of Theorem A.6 yields local well-posedness of SvNLW (1.1) in $\mathcal{H}^s(\mathbb{T}^2)$ for all $s \geq s_{\text{crit}}$ (with a strict inequality when $p = 3$), which improves the local well-posedness result for SvNLW (1.1) in Theorem 1.1.

(iii) One can compare the local well-posedness result for vNLW (A.2) in Theorem A.6 with the local well-posedness result for the usual NLW (see Remark 1.4 in [17]):

$$\partial_t^2 u - \Delta u \pm |u|^{p-1} u = 0.$$

Note that vNLW enjoys a better local well-posedness result than does the usual NLW, thanks to the parabolic smoothing effect.

(iv) Note that the global well-posedness result of SvNLW (1.1) in Theorem 1.3 easily gives global well-posedness of vNLW (A.2) in the class $\mathcal{H}^s(\mathbb{T}^2)$ for $s \geq \max(0, 1 - \frac{1}{p+\delta} - \frac{1}{p})$, where $\delta > 0$ is arbitrary. However, at this point, we do not know how to prove global well-posedness of vNLW (A.2) in $\mathcal{H}^s(\mathbb{T}^2)$ for $s_{\text{crit}} \leq s < \max(0, 1 - \frac{1}{p+\delta} - \frac{1}{p})$. The main difficulty for this range of $s$ is showing $\vec{v}(t) \in \mathcal{H}^1(\mathbb{T}^2)$ for all small enough $t > 0$, which is needed to guarantee the finiteness of the energy $E(\vec{v})$ defined in (1.11).

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