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Numerical approximation of nonlinear SPDE's

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Abstract

The numerical analysis of stochastic parabolic partial differential equations of the form

$$du + A(u)\,dt = f\,dt + g\,dW,$$

is surveyed, where A is a nonlinear partial operator and W a Brownian motion. This manuscript unifies much of the theory developed over the last decade into a cohesive framework which integrates techniques for the approximation of deterministic partial differential equations with methods for the approximation of stochastic ordinary differential equations. The manuscript is intended to be accessible to audiences versed in either of these disciplines, and examples are presented to illustrate the applicability of the theory.

Keywords SPDE's · Weak martingale solution · Fully discrete scheme · Numerical analysis

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1 Introduction

We consider the numerical approximation of solutions of stochastic partial differential equations (SPDE's) of the form

$$du + A(u)\,dt = f\,dt + g\,dW, \qquad u(0) = u^0. \tag{1}$$

The solution $u := \{u(t) \,|\, t \in [0, T]\}$ is a stochastic process taking values in a Banach space U. The operator $A : U \to U'$, processes f, g, and the random variable u^0 are specified later, and $W := \{W_t \,|\, t \geq 0\}$ is a Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{0\leq t\leq T}, \mathbb{P})$.

The existence theory for (1) was first developed for linear spatial operators and then extended in various directions. The analysis of numerical schemes to approximate solutions of (1) has paralleled this development within the last decade.

(i) The stochastic linear heat equation: $A(u) = -\Delta u$; [19, 41].
(ii) Problems with Lipschitz nonlinearities: $A(u) = -\Delta u + F(u)$; [11, 18, 24, 29].
(iii) Semi-linear equations which involve locally Lipschitz nonlinearities:
   (a) The Allen-Cahn equation: $A(u) = -\Delta u + (|u|^2 - 1)u$; [23, 30, 36],
   (b) The nonlinear Schrödinger equation: $A(u) = -i(\Delta u + |u|^2 u)$; [10].
   (c) The incompressible Navier–Stokes equation $A(u) = -\Delta u + (u \cdot \nabla)u$; [5, 6, 16, 35].

(d) The Landau-Lifshitz equation: $A(u) = u \times (u \times \Delta u) - u \times \Delta u$; [1].

(iv) Very few results are available for the numerical approximation of stochastic versions of degenerate parabolic equations, such as the stochastic porous-medium equation [17].

For the first two cases, semigroup techniques are often used to construct mild solutions of (1); a comprehensive exposition of this theory may be found in the monograph [9]. Variational approaches were developed in [26, 34] to accommodate nonlinear equations where the concept of a mild solution is not available. The more general notion of a "weak martingale solution" is required to obtain the existence of solutions for the last two equations in (iii), and (iv).

The collective effort of this work is a unification of techniques from stochastic analysis and numerical analysis of PDE's, resulting in a general convergence theory for implementable discretizations of a wide class of nonlinear SPDE's. This theory provides the technical tools needed to realize the Lax Richtmeyer meta–theorem:

A numerical scheme converges if (and only if) it is stable and consistent.

For this purpose, we distill and adapt ideas from [1, 5, 20, 21, 31] to develop a general convergence theory for numerical schemes comprising of the following steps:

1. **Estimates:** Structural properties of the particular SPDE inherited by the discrete schemes are used to bound the numerical approximations uniformly with respect to discretization parameters. While the specific structure and bounds are problem dependent, standard tools from stochastic analysis (independence, filtrations, adaptedness) are utilized to accommodate the stochastic term.
2. **Compactness:** Compactness properties of Banach spaces with both weak and strong topologies are used in an essential fashion when the operator A is nonlinear. For deterministic PDE's ($g \equiv 0$ in (1)) the Banach–Alaoglu and Lions–Aubin theorems are used to identify limits of approximate solutions. In the stochastic setting the solutions are random variables taking values in Banach spaces and the deterministic arguments are augmented with the Prokhorov theorem on compactness in distribution to obtain convergence of laws.
3. **Convergence:** Concepts of weak and strong solutions are used in both, the deterministic and probabilistic setting to specify in what sense a function u is a solution of the equation. While the meaning of a weak solution is very different in each setting, it has the same purpose; it extends the concept of a solution to accommodate situations where strong (or classical) solutions may not exist. In this work, the concepts of a weak solution for the deterministic and stochastic setting are combined to construct weak martingale solutions as a limit of solutions to discrete approximations of (1).

Bounds upon the numerical approximations establish stability of the numerical schemes. For consistent (Galerkin) approximations of the parabolic problems under consideration we show

$$\text{stability} \quad \Rightarrow \quad \text{stability \& compactness} \quad \Rightarrow \quad \text{convergence},$$

so that the Lax Richtmeyer theorem is realized. The goal of this article is to present these ideas in a context accessible to audiences from either numerical PDE's or stochastic analysis. To achieve this, key results required from each area will be stated and their role explained prior to their use, and the following section reviews background material on the numerical approximation of PDE, stochastic analysis, and probability.

Following the introductory material (which may be skipped at first reading) Sect. 3 introduces the implicit Euler approximation of parabolic SPDE's, and a general convergence results is presented in Theorem 3.2. This theorem establishes convergence in law of the numerical approximations, and Sect. 3.1 considers convergence of the spatial terms under this mode of convergence. To illustrate an application of the abstracty theory, convergence of numerical approximations of SPDE's with $A : U \to U'$ linear is established in Sect. 4. The proof of Theorem 3.2 is taken up on Sect. 5. To reduce the technical overhead scalar valued Weiner noise is assumed, and at the end of this section the extension of the proof to include spatial noise is sketched. This extension is mainly technical in the sense that once the additional definitions, concepts, and properties are acquired, proofs in the simplified setting extend directly to the more general situation. Section 6 applies the general theory to establish convergence of numerical approximations for three prototypical nonlinear SPDE's.

2 Preliminaries from numerical and stochastic analysis

In this section, we give a terse review of the essential concepts from numerical PDE's and stochastic processes required for the development of weak martingale solutions to Eq. (1).

2.1 Numerical partial differential equations

This section reviews the abstract setting where tools from functional analysis can be applied to solve nonlinear PDE's. Solutions are sought in a Banach space U, and a pivot space construction is used to characterize the partial differential operator under consideration. Specifically, U is assumed to be densely embedded in a Hilbert space H, and when H is identified with its dual by the Riesz theorem we have $U \hookrightarrow H \hookrightarrow U'$. Then $u \in U$ is identified with the dual element $\iota(u) \in U'$ by

$$\iota(u)(v) = (u, v)_H, \qquad v \in U.$$

If $f \in U'$ we frequently write $(f, v) = f(v)$ so that $(f, v) = (f, v)_H$ when $f \in H$.

Solutions of time dependent problems are viewed as (strongly measurable) functions from the interval $[0, T]$ to various Banach spaces. The Bochner spaces are the natural Banach spaces that arise in this context; for example,

$$L^2[0, T; U] = \{u : [0, T] \to U \mid \int_0^T \|u(t)\|_U^2 \, dt < \infty\},$$
$$L^\infty[0, T; H] = \{u : [0, T] \to H \mid \operatorname{ess\,sup}_{0 \le t \le T} \|u(t)\|_H < \infty\}.$$

Similar notation is used for the continuous functions, $C[0, T; U]$, and Hölder continuous functions, $C^{0,\theta}[0, T; U]$, from $[0, T]$ to a Banach space U.

The space U is constructed so that the partial differential operator, A, in equation (1) maps U to U'. In this situation it is possible to define $a : U \times U \to \mathbb{R}$ by

$$a(u, v) = A(u)(v), \qquad u, v \in U.$$

The canonical example of this construction is the Laplacian $A(u) = -\Delta u$ on a bounded domain $D \subset \mathbb{R}^d$ with homogeneous boundary data. Letting $H = L^2(D)$ and U be the Sobolev space

$$U = H_0^1(D) \equiv \{u \in L^2(D) \mid \nabla u \in L^2(D)^d,\ u|_{\partial\Omega} = 0\},$$

then

$$A(u)(v) = (-\Delta u, v) \equiv \int_D \nabla u.\nabla v\, dx = a(u, v), \qquad u, v \in U.$$

In this setting, a weak solution of the (stationary) PDE $A(u) = f$ satisfies

$$u \in U \qquad a(u, v) = f(v), \qquad v \in U. \tag{2}$$

The solution of this second order PDE is "weak" in the sense that it is only required to have one square integrable derivative and the datum $f \in U'$ need not be regular. For linear problems the following theorem establishes existence of weak solutions in many situations.

Theorem 2.1 (Lax Milgram 1954) *Let U be a Hilbert space and $a : U \times U \to \mathbb{R}$ be bilinear. Suppose that there exist constants C_a, $c_a > 0$ such that*

$$|a(u, v)| \leq C_a \|u\|_U \|v\|_U, \quad \textit{and} \quad a(u, u) \geq c_a \|u\|_U^2, \qquad u, v \in U.$$

Then for each $f \in U'$ there exists a unique $u \in U$ such that

$$a(u, v) = f(v), \qquad v \in U.$$

Moreover, $\|u\|_U \leq \|f\|_{U'}/c_a$.

Given $f : (0, T) \to U'$ and $u^0 \in H$, a weak solution of the evolution equation $\partial_t u + A(u) = f$ on $(0, T)$ with $u(0) = u^0$ is a function $u : [0, T] \to U$ satisfying

$$(u(t), v)_H + \int_0^t a(u, v)\, ds = (u^0, v)_H + \int_0^t (f, v)\, ds, \qquad v \in U,\ t \in [0, T]. \tag{3}$$

The pivot space construction is used to characterize $\partial_t u(t) \in U'$ for almost every $t \in (0, T)$ through

$$(\partial_t u(t), v) = \lim_{h\to 0} \frac{(u(t+h) - u(t), v)_H}{h}, \qquad v \in U.$$

Now we turn to the discretization of problems (2) and (3). If $U_h \subset U$ is a finite dimensional subspace, a natural numerical scheme to approximate weak solutions of the stationary problem (2) is obtained by seeking a function $u_h \in U_h$ which satisfies the weak statement for each "test function" $v_h \in U_h$. To obtain a fully discrete scheme for the evolution equation (3) it is necessary to also approximate the time derivative. If $N \in \mathbb{N}$ and $\tau = T/N$ is a time step, the implicit Euler scheme computes approximations $\{u^n_{h\tau}\}_{n=1}^N \subset U_h$ of $\{u(t^n)\}_{n=1}^N$ on a uniform partition $\{t^n\}_{n=0}^N$ of $[0, T]$ as solutions of

$$(u^n_{h\tau} - u^{n-1}_{h\tau}, v_h)_H + \tau a(u^n_{h\tau}, v_h) = \tau(f^n_{h\tau}, v_h), \qquad v_h \in U_h, \quad n = 1, 2, \dots N, \tag{4}$$

with $u^0_{h\tau}$, and $f^n_{h\tau}$ approximations of u^0 and $f(t^n)$. The finite element methodology [3] provides a systematic method to construct finite dimensional subspaces of the function space U. These subspaces consist of piecewise polynomial functions on a partition of the domain $D \subset \mathbb{R}^d$; the index $h > 0$ denotes the maximal diameter of a partition (the mesh size). For linear PDEs, if the bilinear function $a : U \times U \to \mathbb{R}$ satisfies the hypotheses of the Lax Milgram theorem, then so too does

$$a_\tau(u_h, v_h) \equiv (u_h, v_h)_H + \tau a(u_h, v_h), \qquad u_h, v_h \in U_h,$$

which ensures the existence of a unique solution to the implicit Euler scheme (4).

For nonlinear operators $A : U \to U'$ compactness properties of U and H are required to obtain and identify limits of numerical solutions. For the parabolic problems under consideration, the space U will always be compactly embedded into the pivot space H; we write $U \hookrightarrow\!\!\!\rightarrow H$. For the evolution problem with $U \hookrightarrow\!\!\!\rightarrow H \hookrightarrow\!\!\!\rightarrow U'$ a typical compactness result for the associated Bochner spaces is the following [38, Theorems 5 and 7].

Theorem 2.2 *Let U be a Banach space.*

- *Let $U \hookrightarrow\!\!\!\rightarrow B \hookrightarrow U'$ be embeddings of Banach spaces, and $1 \le p \le \infty$. Then $L^p[0, T; U] \cap C^{0,\theta}[0, T; U'] \hookrightarrow\!\!\!\rightarrow L^p[0, T; B]$ (and in $C[0, T; B]$ if $p = \infty$).*
- *Let $U \hookrightarrow\!\!\!\rightarrow H \hookrightarrow U$ be embeddings with H a Hilbert space, then $C^{0,\theta}[0, T; U'] \cap L^1[0, T; U] \hookrightarrow\!\!\!\rightarrow L^2[0, T; H]$.*

2.1.1 Skorokhod space

The implicit Euler scheme (4) gives a sequence $\{u^n\}_{n=0}^N \subset U_h$ which can be interpolated to give either a piecewise affine function $\hat{u}_{h\tau}$ or a piecewise constant function

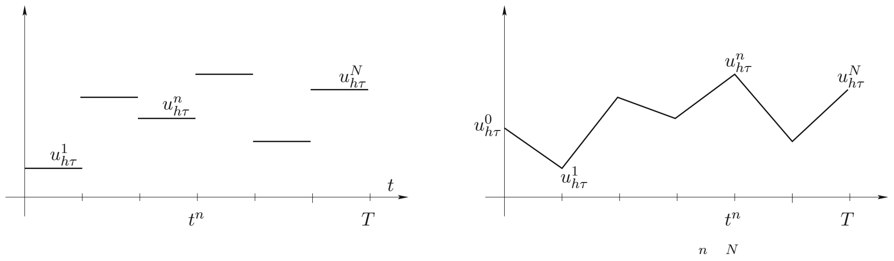

Fig. 1 Piecewise constant $u_{h\tau}$ and piecewise affine interpolation $\hat{u}_{h\tau}$ of $\{u_{h\tau}^n\}_{n=0}^N$

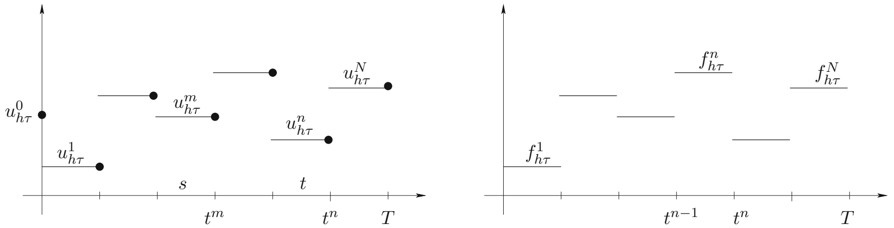

Fig. 2 Indexing of piecewise constant càglàd functions $u_{h\tau}$ and Bochner functions $f_{h\tau}$

$u_{h\tau}$ (see Fig. 1), which satisfy the equation

$$\frac{d\hat{u}_{h\tau}}{dt} + A(u_{h\tau}) = f_{h\tau}, \qquad \text{in } U'$$

on $[0, T]\backslash\{t^n\}_{n=0}^N$. Typically bounds are obtained by multiplying this equation by $u_{h\tau}$, so discontinuous trial and test functions should be admissible; however, it is desirable to retain some of the continuity properties of $\hat{u}_{h\tau}$. For this reason it is convenient to pose the problem in the Skorokhod–type space[1] (see Fig. 2),

$$\begin{aligned} G[0, T; U'] = \Big\{u : [0, T] \to U' \mid u(t) \\ = \lim_{s\to t^-} u(s) \text{ and } \lim_{s\to t^+} u(s) \text{ exist}\Big\}, \quad (i.e., \text{càglàd functions}). \end{aligned}$$

Developing a general theory in this context is extremely useful for applications since stochastic solutions are not very regular in time. Consistency errors of the form $A(u_{h\tau}) - A(\hat{u}_{h\tau})$ would arise if test functions were required to be continuous in time, and frequently these may not vanish as $(h, \tau) \to (0, 0)$.

The construction of the Skorokhod metric is technical, and for completeness we present it here; however, the explicit formula will not be needed. Let Λ be the set of strictly increasing functions $\lambda : [0, T] \to [0, T]$ satisfying $\lambda(0) = 0$ and $\lambda(T) = T$,

[1] Functions in the Skorokhod space $D[0, T; U']$ are continuous from the right with left limits (continue à droite, limite à gauche). For parabolic problems the initial datum is less regular than the solution at later times, so it is natural to consider functions $G[0, T; U']$ continuous from the left with right limits (continue à gauche, limite à droite).

and set

$$\gamma(\lambda) = \sup_{0 \le s < t \le T} \left| \ln \left(\frac{\lambda(t) - \lambda(s)}{t - s} \right) \right| .$$

The Skorokhod metric is

$$d_G(u, v) = \inf_{\lambda \in \Lambda} \max \left(\gamma(\lambda), \| u - v \circ \lambda \|_{L^\infty[0,T;U']} \right) .$$

The following lemma contains the properties of $G[0, T; U']$ required in the sequel [2].

Lemma 2.3 *Let U be a Banach space and $G[0, T; U']$ denote the space of càglàd functions from $[0, T]$ to U' endowed with the Skorokhod metric, $d_G(.,.)$.*

1. *$G[0, T; U']$ is complete and is separable when U' is separable.*
2. *The following embeddings are continuous,*

$$C[0, T; U'] \hookrightarrow G[0, T; U'] \hookrightarrow L^s[0, T; U'], \qquad 1 \le s < \infty.$$

   In addition, $\|u\|_{L^\infty[0,T;U']} \le d_G(0, u)$ so $G[0, T; U'] \subset L^\infty[0, T; U']$; however, the inclusion is not an embedding since convergence in $G[0, T; U']$ does not imply uniform convergence.
3. *If $d_G(u, u_n) \to 0$, then $u_n(t) \to u(t)$ for $t = 0$, $t = T$, and at every time $t \in (0, T)$ where u is continuous. In particular,*
   - *$u_n(t) \to u(t)$ for almost every $t \in [0, T]$ since there is at most a countable set of times $t \in [0, T]$ at which a function in $G[0, T; U']$ is discontinuous.*
   - *If the limit u is continuous, then $u_n(t) \to u(t)$ for every $t \in [0, T]$.*
4. *If $0 = t^0 < t^1 < \cdots < t^N = T$, then the linear function $\phi : C^{0,\theta}[0, T; U'] \to G[0, T; U']$ for which $\phi(u)$ is the piecewise constant càglàd interpolant of $\{u(t_i)\}_{i=0}^N$ is continuous, and*

$$d_G(\phi(u), \phi(v)) \le \|u - v\|_{C[0,T;U']} \quad \text{and}$$
$$d_G(\phi(u), u) \le \|u\|_{C^{0,\theta}[0,T;U']} \left(\max_{1 \le n \le N} (t^n - t^{n-1}) \right)^\theta .$$

2.2 Stochastic processes

All of the random variables we consider will be measurable mappings from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to a topological space $\mathbb{X}$ equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{X})$, and we adopt the terminology that a *(stochastic) process* is a function from a time interval $[0, T]$ to a set of random variables. Implicit in the statement of equation (1) is the presence of a filtration $\{\mathcal{F}(t)\}_{0 \le t \le T}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. In order to apply standard results from probability all filtrations are assumed to satisfy the "usual conditions" [22], namely,

1. $\mathcal{F}(0)$ contains all the null sets.
2. $\mathcal{F}(t) = \cap_{s>t}\mathcal{F}(s)$.

An analogous terminology is utilized for discrete filtrations, $\mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^N$.

The probability of a measurable set $B \in \mathcal{F}$ is denoted by $\mathbb{P}[B]$, and the expected value of a random variable X by $\mathbb{E}[X]$. The conditional expectation of a random variable X with respect to a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ is denoted by $\mathbb{E}[X|\mathcal{G}]$.

In order to exploit standard results from probability theory it is convenient to view a solution of Eq. (1) as both a random variable with values in a Bochner space (for example, $u \in L^2(\Omega, L^\infty[0,T;H]))$, and as a stochastic process (for example, $u \in L^2[0,T;L^p(\Omega,U')])$. While both may be viewed as Bochner spaces, a much richer theory is available for the subspace of stochastic processes *adapted* to a filtration; that is, when $u(t)$ is $\mathcal{F}(t)$-measurable for all $0 \le t \le T$.

To construct the stochastic integral of a random variable with values in equivalence classes of functions, such as $G : \Omega \to L^2[0,T;H]$, a jointly measurable adapted representation $g : [0,T]\times\Omega \to H$ is required for which $g(\cdot,\omega) \in G(\omega)$ for every $\omega \in \Omega$. Specifically, the stochastic integral is correctly defined only for jointly measurable adapted processes with paths in $L^2[0,T;H]$ almost surely. Such g exists if and only if $\Omega \to L^2[0,T;H] : \omega \mapsto \mathbf{1}_{[0,t]}G(\omega)$ is $\mathcal{F}(t)$-measurable for every $t \in [0,T]$, in which case an appropriate selection is the "precise representative" [13] given by

$$g(t,\omega) = \lim_{n\to\infty} n\int_{(t-1/n)^+}^{t} G(\omega)(s)\,ds, \qquad \text{if the limit exists,}$$

and $g(t,\omega) = 0$ otherwise. This representative is actually predictable; that is, measurable with respect to the σ-algebra generated by left continuous adapted processes. When identifying a random variable taking values in a Bochner space with a process we will tacitly assume that a jointly measurable element of the equivalence class is taken so that the stochastic calculus is available.

2.2.1 Martingales

An important class of adapted processes is the class of martingales. Given a filtration $\{\mathcal{F}(t)\}_{t\ge0}$ on a probability space, an adapted process $\{u(t)\}_{t\ge0}$ with values in a Banach space U is an $\{\mathcal{F}(t)\}_{t\ge0}$–martingale if at each time it is integrable, $\mathbb{E}[\|u(t)\|_U] < \infty$, and if it has conditionally independent increments, $\mathbb{E}[u(t)-u(s)|\mathcal{F}(s)] = 0$ when $s \le t$. In particular, $\mathbb{E}[u(t)|\mathcal{F}(s)] = \mathbb{E}[u(s)|\mathcal{F}(s)] = u(s)$; the second equality following since u is adapted. For $T > 0$ and H a Hilbert space, we denote by $\mathcal{M}_T^2(H)$ the set of all H-valued, square integrable martingales with continuous paths. If indistinguishable processes are considered as one process, this is a Banach space when endowed with the norm

$$\|X\|_{\mathcal{M}_T^2(H)} = \|X\|_{L^2(\Omega,L^\infty[0,T;H])} \equiv \Big(\mathbb{E}\big[\sup_{t\in[0,T]} \|X(t)\|_H^2\big]\Big)^{1/2}.$$

Note that the time at which the supremum is taken depends upon $\omega \in \Omega$, and two processes X and Y are *indistinguishable* if there exists a set $A \subset \Omega$ with $\mathbb{P}[A] = 1$ for which $X(t, \omega) = Y(t, \omega)$ for all $\omega \in A$ and $t \in [0, T]$.

The quadratic variation $\langle X \rangle$ of a process $X \in \mathcal{M}_T^2(H)$, defined next, plays a central role in the subsequent theory.

Definition 2.4 Let $T > 0$, H be a separable Hilbert space, and $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space. The quadratic variation $\langle X \rangle$ of $X \in \mathcal{M}_T^2(H)$ is a symmetric, non–negative bilinear process $\langle X(t) \rangle : H \times H \to \mathbb{R}$ satisfying:

1. (Adaptedness) For each $t \in [0, T]$ the real-valued random variable $\langle X(t) \rangle(u, v)$ is $\mathcal{F}(t)$-measurable.
2. (Continuity) $t \mapsto \langle X(t) \rangle(u, v)$ is continuous for every $\omega \in \Omega$ and $u, v \in H$.
3. (Normalization) $\langle X(0) \rangle = 0$.
4. (Monotonicity) $\langle X(t) \rangle(u, u) \geq \langle X(s) \rangle(u, u)$ for every $u \in H$ and $0 \leq s \leq t \leq T$.
5. (Variation) The function $t \mapsto (X(t), u)_H (X(t), v)_H - \langle X(t) \rangle(u, v)$ is a continuous real–valued martingale for each pair $u, v \in H$.

Note that for each time, $\langle X(t) \rangle$ is a semi–inner product so is characterized by $\{\langle X(t) \rangle(u, u) \mid u \in H\}$, or by the Riesz maps $L(t) : H \to H$ for which $(L(t)(u), v)_H = \langle X(t) \rangle(u, v)$. A *standard Wiener process* (or Brownian motion) is a real-valued martingale $W \in \mathcal{M}_T^2(\mathbb{R})$ satisfying $W(0) = 0$, with $\mathbb{E}[W(t)] = 0$, and quadratic variation $\langle W(t) \rangle = t$ for all $0 \leq t \leq T$.

The quadratic variation process appears in the isometry for Ito integrals, and the statement of the Burkholder–Davis–Gundy (BDG) inequalities. The construction of the Ito integral, and a proof of the BDG inequalities involve significant technical developments; however, numerical schemes typically involve processes taking values at discrete times which eliminates much of the technical overhead. Let $\{t^n\}_{n=0}^N$ be a uniform partition of $[0, T]$ with time step $\tau = T/N$, and $\{\mathcal{F}^n\}_{n=0}^N$ be a (discrete) filtration of $(\Omega, \mathcal{F}, \mathbb{P})$. In this context discrete Ito integrals take the form

$$X_\tau^n = \sum_{m=1}^{n} g_\tau^{m-1} \xi_\tau^m, \qquad n = 1, 2, \ldots, N \quad \text{and} \quad X_\tau^0 \equiv 0, \tag{5}$$

where g_τ^{m-1} is an $\mathcal{F}^{m-1}$–measurable random variable with values in a Hilbert space H, and for each $m = 1, 2, \ldots, N$ the increments $\{\xi_\tau^m\}_{n=1}^N$ are real–valued random variables which satisfy the following standing assumptions.

Assumption 2.5 (*with parameter* $p \geq 2$) For each $N \in \mathbb{N}$ let $\{t^n\}_{n=0}^N$ be the uniform partition of $[0, T]$ with time step $\tau = T/N$. Then $\left(\Omega, \mathcal{F}, \{\mathcal{F}^n\}_{n=0}^N, \mathbb{P}\right)$ is a (discretely) filtered probability space and the real-valued random variables $\{\xi_\tau^n\}_{n=1}^N$ satisfy

1. (Zero average) $\mathbb{E}[\xi_\tau^n] = 0$.
2. (Variance) $\mathbb{E}[|\xi_\tau^n|^2] = \tau \equiv T/N$.
3. (Bounds) $\xi_\tau^n \in L^p(\Omega)$, and there exists a constant $C > 0$ such that $\mathbb{E}[|\xi_\tau^n|^p] \leq C\tau^{p/2}$.
4. (Independence) ξ_τ^n is $\mathcal{F}^n$-measurable and independent of $\{\mathcal{F}^m \mid 0 \leq m \leq n-1\}$.

Increments of the form $\xi_\tau^n = W(t^n) - W(t^{n-1})$ with W a standard Wiener process on a filtration of $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ satisfy the above assumptions but lack computational realization. In a numerical context, discrete random variables $\{\xi_\tau^n\}_{n=1}^N$ taking values $\pm\sqrt{\tau}$ with the same probability of $1/2$, and $\mathcal{F}^n$ the σ-algebra generated by $\{\xi_\tau^m\}_{m=1}^n$ are a practical, convenient, and admissible choice satisfying Assumption 2.5. Setting

$$W_\tau^0 = 0 \qquad \text{and} \qquad W_\tau^n = \sum_{m=1}^n \xi_\tau^m, \qquad n = 1, 2, \dots, N, \tag{6}$$

the piecewise linear interpolant of $\{W_\tau^n\}_{n=0}^N$ is the discrete Ito integral with $g_\tau^{m-1} \equiv 1$ and plays the role of a standard Wiener process in the discrete setting.

Under Assumption 2.5, the process $\{X_\tau^n\}_{n=0}^N$ of Eq. (5) is adapted to $\{\mathcal{F}^n\}_{n=0}^N$, and

$$\mathbb{E}[X_\tau^n - X_\tau^{n-1} | \mathcal{F}^{n-1}] = \mathbb{E}[g_\tau^{n-1}\xi_\tau^n | \mathcal{F}^{n-1}] = g_\tau^{n-1}\mathbb{E}[\xi_\tau^n | \mathcal{F}^{n-1}] = 0.$$

This shows that $\{X_\tau^n\}_{n=0}^N$ is a (discrete) martingale; the discrete Ito isometry is then immediate,

$$\begin{aligned}
\mathbb{E}[\|X_\tau^n\|_H^2] &= \mathbb{E}\Big[\sum_{k,m=1}^n (g_\tau^{k-1}, g_\tau^{m-1})_H \xi^k \xi_\tau^m\Big] \\
&= \sum_{m=1}^n \mathbb{E}\big[\|g_\tau^{m-1}\|_H^2 |\xi_\tau^m|^2\big] + 2\sum_{k<m} \mathbb{E}\big[(g_\tau^{k-1}, g_\tau^{m-1})_H \xi_\tau^k \xi_\tau^m\big] \\
&= \sum_{m=1}^n \mathbb{E}[\|g_\tau^{m-1}\|_H^2]\tau.
\end{aligned}$$

The last line follows from Assumption 2.5_4,

$$\mathbb{E}[\|g_\tau^{m-1}\|_H^2 |\xi_\tau^m|^2] = \mathbb{E}[\|g_\tau^{m-1}\|_H^2]\,\mathbb{E}[|\xi_\tau^m|^2] = \mathbb{E}[\|g_\tau^{m-1}\|_H^2]\tau,$$

and when $k < m$ the cross terms vanish,

$$\mathbb{E}\left[(g_\tau^{k-1}, g_\tau^{m-1})_H \xi_\tau^k \xi_\tau^m\right] = \mathbb{E}\left[(g_\tau^{k-1}, g_\tau^{m-1})_H \xi_\tau^k\right] \mathbb{E}[\xi_\tau^m] = \mathbb{E}\left[(g_\tau^{k-1}, g_\tau^{m-1})_H \xi_\tau^k\right] \cdot 0.$$

A similar calculation shows that its discrete quadratic variation is

$$\langle X^n \rangle(u, v) = \sum_{m=1}^n \tau (g^{m-1}, u)_H (g^{m-1}, v)_H, \qquad n \geq 1.$$

Note that in the discrete setting $\langle X^n \rangle(u, v)$ must be $\mathcal{F}^{n-1}$–measurable (predictable). The following theorem shows that the quadratic and cross variations characterize the Ito integral.

Theorem 2.6 *Let U be a separable Banach space and H a Hilbert space with $U \hookrightarrow H \hookrightarrow U'$ dense inclusions. Let $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{0\le t\le T}, \mathbb{P})$ be a filtered probability space and X, g, and W be U'-, H- and real–valued process respectively, with X and W continuous. Suppose that for each $v \in U$ the processes*

$$
\begin{aligned}
&W(t), \quad W^2(t) - t, \quad (X(t), v),\\
&\quad (X(t), v)^2 - \int_0^t (g(s), v)_H^2\, ds, \; (X(t), v)W(t) - \int_0^t (g(s), v)_H\, ds,
\end{aligned}
$$

are all real-valued martingales. Then W is a standard Wiener process and

$$
\big(X(t), v\big) = \int_0^t \big(g(s), v\big)_H\, dW(s), \qquad v \in U.
$$

Proof (sketch) We show that the quadratic variation of $(X(t) - \int_0^t g\, dW, v)$ vanishes using the following calculus for the quadratic variations of real-valued martingales X and Y:

- $\langle X + Y\rangle = \langle X\rangle + 2\langle X, Y\rangle + \langle Y\rangle$, where the cross variation $\langle X, Y\rangle$ is determined from the parallelogram law, $4\langle X, Y\rangle = \langle X + Y\rangle - \langle X - Y\rangle$.
- If $Y(t) = \int_0^t g(s)\, dW(s)$ then $\langle X, Y\rangle(t) = \int_0^t g(s)\, d\langle X, W\rangle(s)$.

Using this calculus for the adapted process $(X(t) - \int_0^t g\, dW, v)$ gives the result.

$$
\begin{aligned}
&\left\langle (X, v) - \int_0^{\cdot} (g, v)_H\, dW \right\rangle(t)\\
&\quad = \big\langle (X, v)\big\rangle(t) - 2\left\langle (X, v), \int_0^{\cdot} (g, v)_H\, dW \right\rangle(t) + \left\langle \int_0^{\cdot} (g, v)_H\, dW \right\rangle(t)\\
&\quad = \big\langle (X, v)\big\rangle(t) - 2\int_0^t (g, v)_H\, d\big\langle (X, v), W\big\rangle + \left\langle \int_0^{\cdot} (g, v)_H\, dW \right\rangle(t)\\
&\quad = \int_0^t \big(g(s), v\big)_H^2\, ds - 2\int_0^t \big(g(s), v\big)_H^2\, ds + \int_0^t \big(g(s), v\big)_H^2\, ds = 0.
\end{aligned}
$$

The middle term takes the form shown since

$$
\begin{aligned}
&\big\langle (X, v) + W\big\rangle(t) = \int_0^t (g, v)_H^2\, ds + 2\int_0^t (g, v)_H\, ds + t, \qquad \text{and}\\
&\big\langle (X, v) - W\big\rangle(t) = \int_0^t (g, v)_H^2\, ds - 2\int_0^t (g, v)_H\, ds + t.
\end{aligned}
$$

Then $\big\langle (X, v), W\big\rangle(t) = \int_0^t \big(g(s), v\big)_H\, ds$, so that $d\big\langle (X, v), W\big\rangle(t) = \big(g(t), v\big)_H\, dt$. $\square$

The (discrete) BDG inequality, stated next, shows that moments of a discrete $\{\mathcal{F}^n\}_{n=0}^N$-martingale taking values in a Hilbert space may be bounded by their quadratic variations, [32, Remark 3.3].

Theorem 2.7 (Burkholder–Davis–Gundy (BDG)) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with (discrete) filtration $\{\mathcal{F}^n\}_{n=0}^N$ and let $\{X_\tau^n\}_{n=0}^N$ with $X_\tau^0 \equiv 0$ be a (discrete) $\{\mathcal{F}^n\}_{n=0}^N$-martingale taking values in a separable Hilbert space H. Then for each $p \geq 1$ there exist constants $0 < c_p < C_p$ such that*

$$
\begin{aligned}
c_p \mathbb{E}\left[\left(\sum_{n=1}^N \|X_\tau^n - X_\tau^{n-1}\|_H^2\right)^{p/2}\right] &\leq \mathbb{E}\left[\max_{0 \leq n \leq N} \|X_\tau^n\|_H^p\right] \\
&\leq C_p \mathbb{E}\left[\left(\sum_{n=1}^N \|X_\tau^n - X_\tau^{n-1}\|_H^2\right)^{p/2}\right].
\end{aligned}
$$

When the martingale is the discrete Ito integral (5), the moments of the quadratic variation can be bounded by the Bochner norms of $\{g_\tau^n\}_{n=0}^{N-1}$, which is the content of the following lemma.

Lemma 2.8 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with (discrete) filtration $\{\mathcal{F}^n\}_{n=0}^N$ and H be a Hilbert space. Let $\{g_\tau^n\}_{n=0}^{N-1} \subset L^p(\Omega, H)$ with g_τ^n $\mathcal{F}^n$–measurable, and $\{\xi_\tau^n\}_{n=1}^N \subset L^p(\Omega, \mathbb{R})$ satisfy Assumption 2.5, with $2 \leq p$. Then*

$$
\mathbb{E}\left[\left(\sum_{m=1}^n \|g_\tau^{m-1}\xi_\tau^m\|_H^2\right)^{p/2}\right] \leq C_p (n\tau)^{p/2-1} \sum_{m=1}^n \tau \|g_\tau^{m-1}\|_{L^p(\Omega,H)}^p \qquad n = 1, \ldots, N,
$$

where $C_p > 0$ is a constant depending upon p and the constant in Assumption 2.5$_3$.

Proof (sketch) The discrete process with $X_\tau^0 = 0$ and $X_\tau^n = \sum_{k=1}^n g_\tau^{k-1}\xi_\tau^k$ for $n = 1, 2, \ldots$ is a (discrete) martingale, and the Burkholder–Rosenthal inequality [33, Theorem 5.50] bounds the middle term in the BDG inequality as

$$
\begin{aligned}
\mathbb{E}\left[\max_{1 \leq k \leq n} \|\sum_{m=1}^k g_\tau^{m-1}\xi_\tau^m\|_H^p\right] \leq \beta_p \mathbb{E}&\left[\sum_{k=1}^n \mathbb{E}\left[\|g^{k-1}\tau\xi_\tau^k\|_H^2 \mid \mathcal{F}^{k-1}\right]\right]^{p/2} \\
&+\beta_p \mathbb{E}\left[\max_{1 \leq k \leq n} \|g_\tau^{k-1}\xi_\tau^k\|_H^p\right],
\end{aligned}
$$

where β_p is a constant depending only upon $p \geq 2$. Since g_τ^{k-1} is $\mathcal{F}^{k-1}$-measurable and ξ_τ^k is independent of $\mathcal{F}^{k-1}$ it follows that

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k=1}^n \mathbb{E}\left[\|g_\tau^{k-1}\xi_\tau^k\|_H^2 \mid \mathcal{F}^{k-1}\right]\right]^{p/2} \\
&= \mathbb{E}\left[\sum_{k=1}^n \|g_\tau^{k-1}\|_H^2 \mathbb{E}\left[(\xi_\tau^k)^2\right]\right]^{p/2}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\tau^{p/2}\mathbb{E}\left[\sum_{k=1}^{n}\|g_\tau^{k-1}\|_H^2\right]^{p/2}\\
&\le C\tau^{p/2}n^{p/2-1}\mathbb{E}\left[\sum_{k=1}^{n}\|g_\tau^{k-1}\|_H^p\right],
\end{aligned}
$$

where C is the constant in Assumption 2.5$_3$. The bound on the second term is direct,

$$
\begin{aligned}
\mathbb{E}\left[\max_{1\le k\le n}\|g_\tau^{k-1}\xi_\tau^k\|_H^p\right] &\le \mathbb{E}\left[\sum_{k=1}^{n}\|g_\tau^{k-1}\xi_\tau^k\|_H^p\right] = \sum_{k=1}^{n}\mathbb{E}\left[\|g_\tau^{k-1}\|_H^p\right]\mathbb{E}\left[(\xi_\tau^k)^p\right]\\
&\le C\tau^{p/2}\sum_{k=1}^{n}\mathbb{E}\left[\|g_\tau^{k-1}\|_H^p\right].
\end{aligned}
$$

□

2.2.2 Convergence in law

Below we construct numerical schemes whose solutions converge in law to a limit. In order to identify the limit as a solution of a stochastic differential equation, it is necessary to show that solutions of a discrete approximation of the equation (1) will pass to solutions of the SPDE (1) with this mode of convergence. In the deterministic setting the following two properties of Banach spaces are used ubiquitously to identify limits:

- Norm bounded subsets of reflexive Banach spaces are weakly sequentially compact. That is, if $A \subset U$ is a norm bounded set of a reflexive Banach space U, then there exist a sequence $\{u_n\}_{n=1}^\infty \subset A$ and u in the closed convex hull of A such that $u_n \rightharpoonup u$.
- Continuous convex functions $\psi : U \to \mathbb{R}$ are sequentially weakly lower semi–continuous. That is, if $\{u_n\}_{n=1}^\infty \subset U$ and $u_n \rightharpoonup u$ then $\psi(u) \le \liminf_{n\to\infty}\psi(u_n)$.
- If $u_n \rightharpoonup u$ in a uniformly convex Banach space and $\|u_n\| \to \|u\|$ then $u_n \to u$.

We present analogous results for random variables with convergence in law in place of weak convergence.

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $X : (\Omega, \mathcal{F}) \to (\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is a random variable with values in the topological space $\mathbb{X}$ with its Borel σ-algebra $\mathcal{B}(\mathbb{X})$, then the law of X on $\mathbb{X}$ is the measure

$$
\mathcal{L}(X)[B] = \mathbb{P}[\omega \in \Omega \mid X(\omega) \in B], \qquad B \in \mathcal{B}(\mathbb{X}).
$$

If $\{X_n\}_{n=1}^\infty$ is a sequence of such random variables, the laws converge (weakly) to the measure $\tilde{\mathbb{P}}$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, and we write $\mathcal{L}(X_n) \Rightarrow \tilde{\mathbb{P}}$, iff

$$
\tilde{\mathbb{E}}[\psi] \equiv \int_{\mathbb{X}} \psi(x)\, d\tilde{\mathbb{P}}(x) = \lim_{n\to\infty}\mathbb{E}[\psi \circ X_n] \qquad \psi \in C_b(\mathbb{X}),
$$

where $C_b(\mathbb{X})$ denotes the set of bounded continuous real-valued functions on $\mathbb{X}$. In the current context $\mathbb{X}$ will typically be a product of spaces; for example,

$$\begin{aligned} & \mathbb{X} = G[0,T;U'] \cap L^r[0,T;U]_{weak} \times L^{q'}[0,T;U'], \\ \text{or} \qquad & \mathbb{X} = G[0,T;U'] \cap L^r[0,T;U]_{weak} \times L^{q'}[0,T;U']_{weak}, \end{aligned}$$

where $L^r[0,T;U]_{weak}$ and $L^{q'}[0,T;U']_{weak}$ denote the spaces $L^r[0,T;U]$ and $L^{q'}[0,T;U']$ endowed with the weak topology. Frequently $C_b(\mathbb{X})$ does not contain all the functions needed to characterize the limits when the factor spaces have the weak topology; however, the lemma below shows that in many situations a larger class of test functions is available when the sequence of laws are tight.

Definition 2.9 Let $\mathbb{X}$ be a topological space and $\mathcal{B}(\mathbb{X})$ denote its Borel σ-algebra.

- A sequence of probability measures $\{\mathbb{P}_n\}_{n=1}^{\infty}$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is tight if for every $\epsilon > 0$ there exists a compact set $K_\epsilon \subset \mathbb{X}$ for which $\mathbb{P}_n[K_\epsilon] \geq 1-\epsilon$ for all $n = 1, 2, \ldots$.
- A sequence of random variables $\{X_n\}_{n=1}^{\infty}$ taking values in $\mathbb{X}$ is tight if their laws $\{\mathcal{L}(X_n)\}_{n=1}^{\infty}$ are tight.

Tight subsets of probability measures on separable metric spaces play a similar role to norm bounded sequences in reflexive Banach spaces in the sense that they are both weakly sequentially compact.

Lemma 2.10 *Let $\mathbb{X}$ be a topological space with a countable sequence of continuous functions separating points and $\{\mathbb{P}_k\}_{k=1}^{\infty}$ be tight on $\mathbb{X}$ and $\mathbb{P}_k \Rightarrow \mathbb{P}$.*

1. *Let $\zeta_k, \zeta : \mathbb{X} \to \mathbb{R}$ be Borel measurable for $k \in \mathbb{N}$. Define*

   $$N = \{x \in \mathbb{X} \mid \exists \{x_k\},\ x_k \to x \text{ in } \mathbb{X} \text{ such that } \{\zeta_k(x_k)\} \text{ does not converge to } \zeta(x)\}$$

   and assume that $\mathbb{P}^*[N] = 0$, i.e., $\inf\{\mathbb{P}[B] : N \subseteq B \in \mathcal{B}(\mathbb{X})\} = 0$.
   Then $\mathbb{P}_k[\zeta_k \in \cdot] \Rightarrow \mathbb{P}[\zeta \in \cdot]$ and if

   $$\lim_{R\to\infty}\left[\sup_k \int_{[|\zeta_k|>R]} |\zeta_k|\, d\mathbb{P}_k\right] = 0 \qquad \textit{then} \qquad \lim_{k\to\infty}\int_{\mathbb{X}} \zeta_k\, d\mathbb{P}_k = \int_{\mathbb{X}} \zeta\, d\mathbb{P}.$$

   In particular, if $\epsilon > 0$ and

   $$\sup_k \int_{\mathbb{X}} |\zeta_k|^{1+\varepsilon}\, d\mathbb{P}_k < \infty \qquad \textit{then} \qquad \lim_{k\to\infty}\int_{\mathbb{X}} \zeta_k\, d\mathbb{P}_k = \int_{\mathbb{X}} \zeta\, d\mathbb{P}.$$

2. *Let $\zeta : \mathbb{X} \to [0,\infty]$ be such that $[\zeta \leq t] \equiv \{x \in \mathbb{X} \mid \zeta(x) \leq t\}$ is sequentially closed for every $t \geq 0$. Then ζ is $\mathbb{P}$-measurable as well as $\mathbb{P}_k$-measurable for every $k \geq 1$ and*

   $$\int_{\mathbb{X}} \zeta\, d\mathbb{P} \leq \liminf_{k\to\infty}\int_{\mathbb{X}} \zeta\, d\mathbb{P}_k.$$

This lemma may be viewed as an extension of the classical Portmanteau theorem and is similar to the mapping theorem in [2, Theorem 2.7]. We provide a proof of this result in the Appendix. The following corollary uses this lemma to show that sequentially continuous test functions are available in the current setting. The class of weakly sequentially continuous functions is substantially larger than the weakly continuous functions since weakly convergent sequences are norm bounded while neighborhoods in the weak topology are not.

Corollary 2.11 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $1 < p < \infty$, and $\mathcal{U}$ be a separable reflexive Banach space, and let $\mathcal{U}_{weak}$ denote $\mathcal{U}$ endowed with the weak topology.*

- *Let $\psi : \mathcal{U} \to \mathbb{R}$ be weakly sequentially continuous. If the laws of $\{u_n\}_{n=1}^{\infty}$ converge on $\mathbb{X} = \mathcal{U}_{weak}$ to a probability measure $\tilde{\mathbb{P}}$ and $\{\psi(u_n)\}_{n=1}^{\infty}$ is bounded in $L^p(\Omega)$, then $\psi(u)$ is integrable on $(\mathbb{X}, \mathcal{B}(\mathbb{X}), \tilde{\mathbb{P}})$ and*
$$\tilde{\mathbb{E}}\left[\psi(u)\right] = \lim_{n\to\infty} \mathbb{E}\left[\psi(u_n)\right].$$
- *Let $\psi : \mathcal{U} \to \mathbb{R}$ be continuous, convex, and bounded below. If the laws of $\{u_n\}_{n=1}^{\infty}$ converge on $\mathbb{X} = \mathcal{U}_{weak}$ to a probability measure $\tilde{\mathbb{P}}$, then $\psi(u)$ is measurable on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ and*
$$\tilde{\mathbb{E}}\left[\psi(u)\right] \le \liminf_{n\to\infty} \mathbb{E}\left[\psi(u_n)\right].$$

Proof (sketch) The first result will follow from the first statement of the lemma. Since the Borel σ-algebras for $\mathcal{U}$ and $\mathcal{U}_{weak}$ coincide, the map ψ is Borel measurable. In addition, since ψ is weakly sequentially continuous it follows that the set N in the lemma is empty.

The final result follows from the second statement of the lemma and Mazur's theorem which states that continuous convex functions on a Banach space are weakly lower semi–continuous. □

The following example illustrates the use of these results to identify and bound initial and final values for the evolution problems under consideration.

Example 2.12 Let U be a separable Banach space, H a Hilbert space, and $U \hookrightarrow H \hookrightarrow U'$ be dense embeddings. Suppose that $\{u_n\}_{n=1}^{\infty}$ are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $G[0, T; U']$, and $\mathcal{L}(u_n) \Rightarrow \tilde{\mathbb{P}}$.

For $t \in [0, T]$ fixed, the mapping $G[0, T; U'] \ni u \mapsto u(t) \in U'$ is Borel, and if $p \ge 1$ the function $\zeta : U' \mapsto [0, \infty]$ given by
$$\zeta(u) = \begin{cases} \|u\|_H^p & u \in H, \\ \infty & \text{otherwise}, \end{cases}$$
is convex and lower semi–continuous. It follows from the second statement of Lemma 2.10 that
$$\tilde{\mathbb{E}}\left[\|u(t)\|_H^p\right] \le \liminf_{n\to\infty} \mathbb{E}\left[\|u_n(t)\|_H^p\right].$$

Next, suppose that $u_n(0)$ converges to a limit in $L^p(\Omega, H)$. Then the laws of $(u_n(0), u_n)$ are tight on $H \times G[0, T; U']$, so passing to a subsequence we may assume their laws converge to a limit, $\mathcal{L}(u_n(0), u_n) \Rightarrow \mathbb{Q}$, on $H \times G[0, T; U']$. If $f \in C_b(G[0, T; U'])$ then

$$\int_{H\times G[0,T;U']} f(u)\, d\mathbb{Q}(u^0, u) = \mathbb{E}^{\mathbb{Q}}[f(u)] = \lim_{n\to\infty} \mathbb{E}[f(u_n)] = \int_{G[0,T;U']} f(u)\, d\tilde{\mathbb{P}}(u),$$

shows that $\tilde{\mathbb{P}}$ is the second marginale of $\mathbb{Q}$.

Assume that $\|u_n^0\|_H$ and $\|u_n\|_{U'}$, and hence $\|(u_n^0, u_n)\|_{H\times G[0,T;U']}$, have bounded moments of order $p > 1$, and fix $v \in U$. Then the mapping $(u^0, u) \mapsto |(u^0 - u(0), v)|$ is continuous on $H \times G[0, T; U']$, and it follows from the first statement of the lemma that

$$\mathbb{E}^{\mathbb{Q}}\left[\left|(u^0 - u(0), v)\right|\right] = \lim_{n\to\infty} \mathbb{E}\left[\left|(u_n(0) - u_n(0), v)\right|\right] = 0,$$

whence $u(0) = u^0$ $\mathbb{Q}$-almost surely. From the Tonelli theorem we then conclude

$$\begin{aligned}
\tilde{\mathbb{E}}\,[\|u(0)\|_H] &= \int_{G[0,T;U']} \|u(0)\|_H\, d\tilde{\mathbb{P}}(u)\\
&= \int_{H\times G[0,T;U']} \|u(0)\|_H\, d\mathbb{Q}(u^0, u)\\
&= \int_{H\times G[0,T;U']} \|u^0\|_H\, d\mathbb{Q}(u^0, u)\\
&= \lim_{n\to\infty} \mathbb{E}\left[\|u_n^0\|_H\right],
\end{aligned}$$

the last line following since $(u^0, u) \mapsto \|u^0\|_H$ is continuous on $H \times G[0, T; U']$. Similarly, if $\|u_n^0\|_H$ has moments of order $p > 1$ then

$$\tilde{\mathbb{E}}\left[\|u(0)\|_H^s\right] = \lim_{n\to\infty} \mathbb{E}\left[\|u_n^0\|_H^s\right], \qquad 1 \le s < p.$$

2.3 Stochastic partial differential equations

Combining the ideas from the previous section provides a formulation of the stochastic evolution equation (1) amenable to analysis by results from functional analysis and probability theory. Letting $U \hookrightarrow H \hookrightarrow U'$ be dense embeddings and writing $a(u, v) = (A(u), v)$, a solution of (1) may be viewed as a process taking values in U which at each time $t \in [0, T]$ satisfies

$$(u(t), v)_H + \int_0^t a(u, v)\, ds = (u^0, v)_H + \int_0^t (f, v)\, ds + \int_0^t (g, v)_H\, dW, \qquad v \in U. \tag{7}$$

The last integral in this equation is the Ito integral corresponding to a Wiener process W defined on the filtered probability space. The distinction between a (stochastically) weak and strong solution of (7) is as follows:

- For a stochastically strong solution of (1), a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{0\le t\le T}, \mathbb{P})$ and random variables f, g, W, and u^0 *are specified*, and the solution $u : [0, T] \to U$ is a process adapted to $\{\mathcal{F}(t)\}_{0\le t\le T}$ which satisfies (7).
- For a stochastically weak solution of (1), *laws* $\mathbb{P}_f$, $\mathbb{P}_g$ and $\mathbb{P}_0$ of the data are specified, and a solution consists of a probability space$(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}(t)\}_{0\le t\le T}, \tilde{\mathbb{P}})$ and adapted processes u, f, g, and W, which satisfy

  - $\mathcal{L}(f) = \mathbb{P}_f$,
  - $\mathcal{L}(g) = \mathbb{P}_g$,
  - $\mathcal{L}(W)$ is an instance of the standard Wiener measure,
  - $\mathcal{L}(u(0)) = \mathbb{P}_0$,

  and (u, f, g, W) satisfy (7) $\tilde{\mathbb{P}}$ almost surely.

Clearly a strong solution is also a weak solution, the major distinction between the two concepts is that the construction of a filtered probability space is a part of the solution process for weak solutions. Since filtered probability spaces and Wiener processes are not available in a computational context, only weak solutions are computable in practice.

Definition 2.13 Let $T > 0$ and $U \hookrightarrow H$ be a dense embedding of the Banach space U into a Hilbert space H so that $U \hookrightarrow H \hookrightarrow U'$. Then $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}(t)\}_{0\le t\le T}, \tilde{\mathbb{P}})$ and random variables u, f, g, and W on this space are a weak martingale solution of (7) if

(i) $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}(t)\}_{0\le t\le T}, \tilde{\mathbb{P}})$ is a filtered probability space satisfying the usual conditions, f and g are adapted, and u^0 is $\mathcal{F}(0)$-measurable.
(ii) $W = \{W(t) \,|\, 0 \le t \le T\}$ is a standard real-valued Wiener process on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}(t)\}_{0\le t\le T}, \tilde{\mathbb{P}})$.
(iii) $u : [0, T] \times \Omega \to U$ is adapted to $\{\tilde{\mathcal{F}}(t)\}_{0\le t\le T}$, and

  (a) $u \in C[0, T; U']$ $\tilde{\mathbb{P}}$–a.s.,
  (b) Equation (7) holds $\tilde{\mathbb{P}}$-a.s., for every $v \in U$ and every $0 \le t \le T$.

The remainder of this manuscript considers the numerical approximation of weak martingale solutions using (pseudo-) random number generators to simulate the role of noise in (7). For simplicity of presentation we will consider a real-valued Wiener process; extensions to infinite-dimensional and cylindrical noise are outlined in Sect. 5.3.1.

2.3.1 Ito's formula

A version of Ito's formula is available for weak martingale solutions of stochastic PDE's taking values in a Banach space [25, 26, 34]. The Ito formula stated next considers weak martingale solutions of the equation $du = F\,dt + g\,dW$ with F

taking values in U' and g taking values in the pivot space H. Writing Eq. (1) as

$$du = (f - A(u))\,dt + g\,dW \equiv F\,dt + g\,dW,$$

shows that it takes the form assumed in the theorem.[2]

Theorem 2.14 *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^T, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, U be a separable Banach space, H a Hilbert space, and $U \hookrightarrow H \hookrightarrow U'$ be dense embeddings. With $1 < q < \infty$, let $F \in L^{q'}(\Omega, L^{q'}[0,T;U'])$ and $g \in L^2(\Omega, L^2[0,T;H])$ be jointly measurable (as functions of (t,ω)) adapted processes, and W be a standard Wiener process. If $u^0 \in L^2(\Omega, H)$, and a process $u \in L^q(\Omega, L^q[0,T;U])$ with $(u,g)_H \in L^2(\Omega \times (0,T))$ satisfies*

$$(u(t), v) = (u^0, v) + \int_0^t (F(s), v)\,ds + \int_0^t (g(s), v)_H\,dW(s), \qquad v \in U,$$

then there is an adapted version of u with values in $C[0,T;H]$ for which

$$\mathbb{E}\left[\sup_{0\le t\le T} \|u(t)\|_H^2\right] < \infty,$$

and

$$\mathbb{E}\left[(1/2)\|u(t)\|_H^2\right] = \mathbb{E}\left[(1/2)\|u^0\|_H^2 + \int_0^t (F(s), u(s)) + (1/2)\|g(s)\|_H^2\,ds\right].$$

2.3.2 Uniqueness of solutions

This section shows that if the solution of the deterministic equation is unique then the laws of weak martingale solutions of the corresponding SPDE with additive noise will also be unique. Writing equation (1) as

$$du = (f\,dt + g\,dW) - A(u)\,dt \equiv dV - A(u)\,dt,$$

then (the law of) V depends upon (laws of) the data (f, g, W). Theorem 2.17 below states that the law of a solution u to an equation of this form will depend only upon the law of V when $A(.)$ satisfies the following assumption.

Assumption 2.15 If $\lambda > 0$ and $u_1, u_2 \in C[0,T;U] \cap L^r[0,T;U']$ satisfy $A(u_1), A(u_2) \in L^1[0,T;U']$ and

$$(u_2(t) - u_1(t), w)_H + \int_0^t \lambda\left(A(u_2(s)) - A(u_1(s)), v\right)\,ds = 0, \quad t \in [0,T], \quad v \in U,$$

then $u_1 = u_2$. (Note that if this holds for some $T > 0$ then it holds for all $T > 0$.)

[2] If $1 \le q \le \infty$ then q' denotes the conjugate exponent of q, $1/q + 1/q' = 1$.

This assumption will always be considered in the context where U is a separable Banach space, H is a Hilbert space, the embeddings $U \hookrightarrow H \hookrightarrow U'$ are dense, and $A : U \to U'$.

Definition 2.16 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathbb{X}_1 = C[0,T;U'] \cap L^r[0,T;U']_{weak}$ with $1 < r < \infty$, and $A : U \to U'$. Then a pair of random variables (u, V) taking values in $\mathbb{X}_1 \times C[0,T;U']$ satisfy

$$du = dV - A(u)\,dt, \tag{8}$$

if $\mathbb{P}[u \in S] = 1$ for some σ-compact set S in $\mathbb{X}_1$, $A(u) \in L^1[0,T;U']$ almost surely, and

$$\mathbb{P}\left[(u(t), v)_H = (V(t), v) - \int_0^t (A(u(s)), v)\,ds\right] = 1, \qquad t \in [0,T], \quad v \in U.$$

The following theorems establish uniqueness when the partial differential operators satisfying Assumption 2.15, and may be viewed as extensions of the classical Yamada-Watanabe theory to the situation where the data f and g are random.

Theorem 2.17 (Joint Uniqueness in Law) *Let Assumption 2.15 hold. If (u^i, V^i) satisfy (8) on a probability space $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i)$ and $\mathcal{L}(V^0) = \mathcal{L}(V^1)$, then $\mathcal{L}(u^0, V^0) = \mathcal{L}(u^1, V^1)$.*

Theorem 2.18 (Strong Existence) *Let Assumption 2.15 hold and let there exist a solution $(\tilde{u}, \tilde{V})$ of (8) on some probability space. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, V is a $C[0,T;U']$-valued random variable with $\mathcal{L}(V) = \mathcal{L}(\tilde{V})$ then there exists a unique $\mathbb{X}_1$-valued random variable u with a σ-compact range such that (u, V) is a solution of (8). Moreover, u is $(\mathcal{F}_t^{V,0})$-adapted where $(\mathcal{F}_t^{V,0})$ denotes the $\mathbb{P}$-augmentation of the filtration generated by V.*

The proofs of these two theorems are presented in the Appendix.

3 Numerical approximation of SPDE's

To construct numerical approximations of the weak statement (7) let $U_h \subset U$ be a (finite-dimensional) subspace, and $\{t^n\}_{n=0}^N$ be a uniform partition of $[0, T]$ with time step $\tau = T/N > 0$. A (pseudo-) random number generator is used to generate sampled random variables $\xi_\tau^n(\omega) \in \mathbb{R}$ satisfying Assumptions 2.5. Then $u_{h\tau}^n \equiv u_{h\tau}^n(\omega) \in U_h$ is a solution of

$$\begin{aligned}
&(u_{h\tau}^n - u_{h\tau}^{n-1}, v_h)_H + \tau a(u_{h\tau}^n, v_h) \\
&= \tau(f_{h\tau}^n, v_h) + (g_{h\tau}^{n-1}, v_h)_H \xi_\tau^n, \quad v_h \in U_h, \quad 1 \le n \le N.
\end{aligned} \tag{9}$$

In this equation, $f_{h\tau}^n$ is a U'-valued approximation of $f(t^n)$, $g_{h\tau}^n$ an H-valued approximation of $g(t_n)$, and $u_{h\tau}^0$ is a U_h-valued approximation of u^0; for example,

$$f_{h\tau}^n = \frac{1}{\tau}\int_{t^{n-1}}^{t^n} f(s)\,ds \quad \text{and} \quad g_{h\tau}^n = \frac{1}{\tau}\int_{t^{n-1}}^{t^n} g(s)\,ds, \tag{10}$$

and $u_{h\tau}^0$ the orthogonal projection of u^0 onto $U_h \subset H$. In general, f and g may depend upon u ($u_{h\tau}$ in the discrete case), so both h and τ are included in the notation $f_{h\tau}^n$ and $g_{h\tau}^n$.

The specific bounds available for solutions of a particular equation (1) depend in an essential fashion upon the structure of the nonlinear operator A. For this reason a passage to the limit in this term in a numerical scheme is problem dependent. In contrast, there is a commonality of the structure in the temporal terms which facilitates a convergence theory for implicit Euler approximations of this class of problems provided bounds upon the solution are available.

Writing $F(t) = f(t) - A(u(t))$, the spatial dependence of the equation is characterized by a single process taking values in U'. With this notation the implicit Euler scheme (9) becomes: Find $u_{h\tau}^n(\omega) \in U_h$ such that

$$(u_{h\tau}^n, v_h)_H = (u_{h\tau}^{n-1}, v_h)_H + \tau(F_{h\tau}^n, v_h) + (g_{h\tau}^{n-1}, v_h)_H \xi_\tau^n, \qquad v_h \in U_h,\ 1 \le n \le N, \tag{11}$$

with the U'-valued $F_{h\tau}^n$ defined by $(F_{h\tau}^n, v) = (f_{h\tau}^n, v) - a(u_{h\tau}^n, v)$.

Theorem 3.2 below establishes conditions under which solutions of this abstract difference scheme will converge to a weak martingale solution. Assumption 2.5 on the stochastic increments $\{\xi_\tau^n\}_{n=1}^N$, and the following assumptions on the data and discrete spaces will be assumed throughout.

Assumption 3.1 $U \hookrightarrow H$ is a dense embedding of a Banach space U into a Hilbert space H. The discrete subspace $U_h \subset U$, and data of the numerical scheme (11) with time step $\tau = T/N$ with $N \in \mathbb{N}$ and $t^n \equiv n\tau$ satisfy:

1. $(\Omega, \mathcal{F}, \{\mathcal{F}^n\}_{n=0}^N, \mathbb{P})$ is a (discretely) filtered probability space satisfying the usual assumptions.
2. $\{F_{h\tau}^n\}_{n=1}^N$ is adapted to $\{\mathcal{F}^n\}_{n=1}^N$ with values in U'.
3. $\{g_{h\tau}^n\}_{n=0}^{N-1}$ is adapted to $\{\mathcal{F}^n\}_{n=0}^{N-1}$ with values in H.
4. The initial datum $u_{h\tau}^0$ is an H-valued random variable that is $\mathcal{F}^0$-measurable.
5. For each $v \in U$, there exists a sequence $\{v_h\}_{h>0} \subset U_h$ such that $\lim_{h\to 0} v_h = v$.
6. The restrictions of the orthogonal projections $P_h : H \to U_h$ to U are stable in the sense that there exists a constant $C > 0$ independent of $h > 0$ such that $\|P_h(v)\|_U \le C\|v\|_U$.

The last two conditions are density and stability conditions on the spatial discretizations and, in a finite element context, are satisfied under mild restrictions on the triangulations of the domain [7].

We make frequent use of the following notation. Piecewise constant temporal interpolants of $\{F_{h\tau}^n\}_{n=1}^N$, and $\{g_{h\tau}^{n-1}\}_{n=1}^N$ are denoted by $F_{h\tau}$, and $g_{h\tau}$ respectively. With

$\{u_{h\tau}^n\}_{n=0}^N$ taking values in U_h and $\{W_\tau^n\}_{n=0}^N$ as in (6), $\hat{u}_{h\tau}$ and $\hat{W}_\tau$ denote the piecewise linear interpolants respectively, and $u_{h\tau}$ will denote the piecewise constant càglàd interpolant; see Fig. 2. In Sect. 5 we establish the following theorem which is the main result of this manuscript.

Theorem 3.2 *Let $T > 0$, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, U be a separable reflexive Banach space, H a Hilbert space, and $U \hookrightarrow\!\!\!\rightarrow H \hookrightarrow\!\!\!\rightarrow U'$ be compact, dense embeddings. For every pair of numerical parameters (τ, h) with $\tau = T/N \in \mathbb{N}$ let Assumptions 3.1 and 2.5 hold with parameter $p > 2$, and let $\{u_{h\tau}^n\}_{n=0}^N$ be a solution of (11) with data $(u_{h\tau}^0, F_{h\tau}, g_{h\tau})$. Assume for some $1 < q, r < \infty$ that*

1. *$\{\|u_{h\tau}\|_{L^p(\Omega, L^r[0,T;U])}\}_{h,\tau>0}$ is bounded.*
2. *$\{\|F_{h\tau}\|_{L^p(\Omega, L^{q'}[0,T;U'])}\}_{h,\tau>0}$ is bounded.*
3. *$\{F_{h\tau}(u_{h\tau})\}_{h,\tau>0}$ is bounded in $L^{p/2}(\Omega, L^1(0, T))$.*
4. *$\{\|g_{h\tau}\|_{L^p(\Omega, L^p[0,T;H])}\}_{h,\tau>0}$ is bounded.*
5. *The initial data $\{u_{h\tau}^0\}_{h,\tau>0}$ are bounded in $L^p(\Omega, H)$ and converge in $L^2(\Omega, H)$ to u^0 as $(h, \tau) \to (0, 0)$.*

Then the following properties hold.

1. $\{\|u_{h\tau}\|_{L^p(\Omega, L^\infty[0,T;H])}\}_{h,\tau>0}$ and $\{\|\hat{u}_{h\tau}\|_{L^p(\Omega, C^{0,\theta}[0,T;U'])}\}_{h,\tau>0}$ with $0 < \theta < \min(1/2 - 1/p, 1/q)$ are bounded.

2. There exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a random variable (u, F, g, W) on $\tilde{\Omega}$ with values in

$$\mathbb{X} \equiv G[0, T; U'] \cap L^r[0, T; U]_{weak} \times L^{q'}[0, T; U']_{weak} \times L^2[0, T; H]_{weak} \times C[0, T],$$

and a subsequence $(h_k, \tau_k) \to (0, 0)$ for which the laws of $\big\{(u_{h_k\tau_k}, F_{h_k\tau_k}, g_{h_k\tau_k}, \hat{W}_{\tau_k})\big\}_{k=1}^\infty$ converge to the law of (u, F, g, W),

$$\mathcal{L}(u_{h_k\tau_k}, F_{h_k\tau_k}, g_{h_k\tau_k}, \hat{W}_{\tau_k}) \Rightarrow \mathcal{L}(u, F, g, W),$$

with $\tilde{\mathbb{P}}\big[u \in C[0, T; U'] \cap L^\infty[0, T; H]\big] = 1$. Here $L^r[0, T; U]_{weak}$ and $L^{q'}[0, T; U']_{weak}$ denote the spaces $L^r[0, T; U]$ and $L^{q'}[0, T; U']$ endowed with the weak topology.

3. If, in addition, the laws of $\{g_{h\tau}\}_{h,\tau>0}$ are tight on $L^2[0, T; H]$ (for example, if $g_{h\tau}$ converges in $L^2(\Omega, L^2[0, T; H])$) then the laws converge along a subsequence $(h_k, \tau_k) \to (0, 0)$ on

$$\mathbb{X} \equiv G[0, T; U'] \cap L^r[0, T; U]_{weak} \times L^{q'}[0, T; U']_{weak} \times L^2[0, T; H] \times C[0, T],$$

and there exists a filtration $\{\tilde{\mathcal{F}}(t)\}_{0\le t\le T}$ satisfying the usual conditions for which F is adapted, g has a predictable representative in $L^2((0, T) \times \tilde{\Omega}; U')$, and W is a real-valued Wiener process, such that for all $0 \le t \le T$

$$(u(t), v)_H = (u^0, v)_H + \int_0^t (F, v)\, ds + \int_0^t (g, v)\, dW, \qquad v \in U. \tag{12}$$

4. *If $U_0 \subset U$ is a subspace and if Assumption 3.1$_5$ is weakened to:*

   (5') For each $v \in U_0$, there exists a sequence $\{v_h\}_{h>0} \subset U_h$ such that $v_h \to v$ for $h \to 0$.

   the above still hold except that

$$(u(t), v)_H = (u^0, v)_H + \int_0^t (F, v)\, ds + \int_0^t (g, v)\, dW, \qquad v \in U_0.$$

5. *If additionally $V \hookrightarrow U'$ is a separable reflexive Banach space and $\{u_{h\tau}\}_{h,\tau>0}$ is bounded in $L^p(\Omega, L^s[0,T;V])$ for some $1 < s < \infty$, then the laws converge along some subsequence $(h_k, \tau_k) \to (0,0)$ on*

$$\begin{aligned}\mathbb{X} \equiv\ & G[0,T;U'] \cap L^r[0,T;U]_{weak} \cap L^s[0,T;V]_{weak}\\ & \times L^{q'}[0,T;U']_{weak} \times L^2[0,T;H] \times C[0,T].\end{aligned}$$

   If $U \hookrightarrow\!\!\!\rightarrow V$ is compact and $1 \le \hat{s} < s$, then the laws converge in

$$\begin{aligned}\mathbb{X} \equiv\ & G[0,T;U'] \cap L^r[0,T;U]_{weak} \cap L^{\hat{s}}[0,T;V]\\ & \times L^{q'}[0,T;U']_{weak} \times L^2[0,T;H] \times C[0,T].\end{aligned}$$

6. *If $F_{h\tau} = \sum_{\ell=1}^L F_{h\tau}^{(\ell)}$ and each summand is bounded as in Hypothesis 2, then the above holds mutatis mutandis with*

$$\mathbb{X} \equiv G[0,T;U'] \cap L^r[0,T;U]_{weak} \times L^{q'}[0,T;U']_{weak}^L \times L^2[0,T;H] \times C[0,T],$$

   and

$$(u(t), v)_H = (u^0, v)_H + \int_0^t \Big(\sum_{\ell=1}^L F^{(\ell)}, v\Big)\, ds + \int_0^t (g, v)\, dW, \qquad v \in U.$$

   If $F_{h\tau}^{(\ell)}$ converges strongly in $L^p(\Omega, L^{q'}[0,T;U'])$ for an index $1 \le \ell \le L$, then the laws converge when the corresponding factor space of $L^{q'}[0,T;U']^L$ has the strong topology.

This theorem can be viewed as an instance of the Lax–Richtmeyer equivalence theorem or an infinite dimensional version of Donsker's theorem with random walk in U'. The stability hypothesis of the Lax–Richtmeyer theorem is identified with the bounds assumed upon $\{(u_{h\tau}, F_{h\tau}, g_{h\tau})\}_{h,\tau>0}$, and the convergence is as stated. The analysis of numerical schemes for each of the examples introduced at the beginning of Section 1 all included the following steps.

1. Bounds upon the approximate solution were first derived which always contained the hypotheses of Theorem 3.2 as a subset. The implicit Euler scheme has been used ubiquitously in both the deterministic (PDE) and probabilistic (SODE) setting

and bounds for stochastic PDE's follow upon integrating the ideas from these two disciplines.

2. The ideas introduced in Sect. 5 below for the proof of Theorem 3.2 were utilized to establish convergence to a weak martingale solution. In addition to those introduced in the previous two sections, these include appropriate versions of the Kolmogorov–Centsov theorem to establish pathwise continuity, and the theorems of Prokhorov and Lions–Aubin to establish compactness.
3. Compactness properties were developed in order to show that the limit F took the form $F = f - A(u)$ (and $g \equiv \gamma(u)$ if $g_{h\tau} \equiv \gamma_{h\tau}(u_{h\tau})$). This involves an interchange of limits; the numerical scheme will be "consistent" if $F_{h\tau} \equiv F(u_{h\tau}) \Rightarrow F(u)$.

Frequently the last step, which involves the spatial terms, was not well-delineated from the previous step which establishes convergence of the time stepping scheme. In the deterministic setting consistency is usually direct once the compactness is established; however, in the stochastic setting additional arguments are required. In the next section we illustrate how convergence in law is used to establish consistency. Note that if additional bounds are available for a specific problem (as in Statement 3.2 of the theorem) more test functions are available when the solutions converge in law, and these can be used to show consistency.

3.1 Consistency of the spatial terms

Theorem 3.2 shows that the implicit Euler scheme (11) is consistent in the sense that (along a subsequence) the laws of the discrete solution $(u_{h\tau}, F_{h\tau}, g_{h\tau}, \hat{W}_\tau)$ converge to the laws of a limit $\tilde{\mathbb{P}} = \mathcal{L}(u, F, g, W)$ satisfying (12). In order to recover a solution of (7) it is necessary to show that $F = f - A(u)$ on the support of $\tilde{\mathbb{P}}$, and, if the diffusion term depends upon the solution, $g_{h\tau} = \gamma(u_{h\tau})$, that $g = \gamma(u)$. Convergence in law will be used to show this; recall that this mode of convergence guarantees that

$$\mathbb{E}[\phi(u_{h\tau}, F_{h\tau}, g_{h\tau}, \hat{W}_\tau)] \to \int_{\mathbb{X}} \phi(u, F, g, W)\, d\tilde{\mathbb{P}}(u, F, g, W), \quad \text{for all} \quad \phi \in C_b(\mathbb{X}).$$

A judicious selection of test functions in Lemma 2.10 and Corollary 2.11 is made to establish consistency.

In the examples $F = f - A(u) = F^{(1)} + F^{(2)}$ is a sum, and Statement 3.2 in Theorem 3.2 shows that it is sufficient to consider consistency of each term separately. Specifically, with

$$\mathbb{X} \equiv G[0, T; U'] \cap L^r[0, T; U]_{weak} \times L^{q'}[0, T; U']^2_{weak} \times L^2[0, T; H] \times C[0, T],$$

we have

$$\mathcal{L}(u_{h_k\tau_k}, (f_{h_k\tau_k}, A(u_{h\tau})), g_{h_k\tau_k}, \hat{W}_{\tau_k}) \Rightarrow \mathcal{L}(u, (f, a), g, W) \equiv \tilde{\mathbb{P}}, \qquad \text{on } \mathbb{X}.$$

Typically, the data $\{f_{h\tau}\}_{h,\tau>0}$ are an approximation of a specified random variable with law $\mathbb{P}_f$, and the discrete approximations are constructed so that $\mathcal{L}(f_{h\tau}) \Rightarrow \mathbb{P}_f$

on $L^{q'}[0, T; U']$. This will be the case if, for example, $f_{h\tau}$ converges to a limit in $L^p(\Omega, L^{q'}[0, T; U'])$. It is then immediate that $\mathcal{L}(f) = \mathbb{P}_f$.

When $\mathcal{L}(u_{h\tau}) \Rightarrow \mathcal{L}(u)$ and $\mathcal{L}(A(u_{h\tau})) \Rightarrow \mathcal{L}(a)$ it is necessary to show $a = A(u)$ on the support of $\tilde{\mathbb{P}}$. The next example shows that this is easily verified when A is linear, and the following example uses Corollary 2.11 to establish this for a nonlinear problem.

Example 3.3 (linear equations) Let $A : U \to U'$ be linear and continuous, $\|A(u)\|_{U'} \le C_a \|u\|_U$. For $v \in L^2[0, T; U]$ fixed, the mapping $u \mapsto A(u)(v)$ is linear and continuous on $L^2[0, T; U']$, hence weakly continuous, so

$$\phi(u, (f, a), g, W) = \left| \int_0^T (a - A(u), v)\, ds \right| \wedge 1$$

is continuous on $\mathbb{X} = L^2[0, T; U]_{weak} \times L^2[0, T; U']^2_{weak} \times L^2[0, T; H] \times C[0, T]$ and bounded. Consistency is then immediate,

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\left| \int_0^T (a - A(u), v)\, ds \right| \wedge 1 \right] \\
&\quad = \lim_{(h_k, \tau_k) \to (0,0)} \mathbb{E}\left[\left| \int_0^T (A(u_{h_k\tau_k}) - A(u_{h_k\tau_k}), v)\, ds \right| \wedge 1 \right] = 0.
\end{aligned}$$

The next example considers the common situation where the spatial operator is a compact perturbation of a linear operator.

Example 3.4 (stochastic Navier Stokes equation) Solutions of the stochastic Navier–Stokes equation take values in the divergence free Sobolev space $U_0 = \{u \in H_0^1(D)^3 \mid div(u) = 0\}$, with $D \subset \mathbb{R}^3$ bounded and Lipschitz. However, numerical solutions are computed in the larger space $U = H_0^1(D)^3$, and the spatial operator $A : U \to U'$ is

$$\begin{aligned}
(A(u), v) &= (1/2)\left((u.\nabla)u, v\right) - (1/2)\left(u, (u.\nabla)v\right) + \left(2\mu D(u), \nabla v\right) \\
&\equiv \sum_{ij=1}^{3} \int_D (1/2)\left(u_j \frac{\partial u_i}{\partial x_j} v_i - u_i u_j \frac{\partial v_i}{\partial x_j}\right) + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \frac{\partial v_i}{\partial x_j}, \qquad v \in U,
\end{aligned} \tag{13}$$

where $D(u) = 1/2(\nabla u + \nabla u^T)$. The last term on the right is bilinear and continuous and is accommodated as in the prior example. Appropriate exponents for this example are $r = 2$, $q = 8$, $q' = 8/7$.

Let $\hat{A} : U \to U'$ denote the operator

$$(\hat{A}(u), v) = (1/2)\left((u.\nabla)u, v\right) - (1/2)\left(u, (u.\nabla)v\right).$$

For $v \in L^4[0, T; U]$ fixed, we show that

$$\phi(u, \hat{F}) = \left| \int_0^T (\hat{F} - \hat{A}(u), v)\, ds \right|$$

is sequentially continuous on $\tilde{\mathbb{X}} \equiv G[0, T; U'] \cap L^2[0, T; U]_{weak} \times L^{8/7}[0, T; U']_{weak}$ and has a finite moment of order p when the solution has moments of order $2p$. Thus if $\mathcal{L}(u_{h\tau}, \hat{F}_{h\tau}) \Rightarrow \hat{\mathbb{P}}$ on $\hat{\mathbb{X}}$ with $\hat{F}_{h\tau} \equiv \hat{A}(u_{h\tau})$, then $\phi(u_{h\tau}, \hat{F}_{h\tau}) \equiv 0$ and from Corollary 2.11 we conclude that

$$\hat{\mathbb{E}}\left[\left| \int_0^T (\hat{F} - \hat{A}(u), v)\, ds \right| \right] = \lim_{(h,\tau)\to(0,0)} \mathbb{E}\left[\left| \int_0^T (\hat{F}_{h\tau} - \hat{A}(u_{h\tau}), v)\, ds \right| \right] = 0,$$

whence $\hat{\mathbb{P}}[\hat{F} = \hat{A}(u)] = 1$.

Since the mapping $L^{8/7}[0, T; U']_{weak} \ni \hat{F} \mapsto \int_0^T (\hat{F}, v) ds$ is continuous it suffices to show that

$$G[0, T; U'] \cap L^2[0, T; U]_{weak} \ni u \mapsto \int_0^T (\hat{A}(u), v)\, ds$$

is sequentially continuous. We sketch a proof of this; a detailed discussion of this operator is available in every text on the Navier–Stokes equations [14, 15, 39].

A calculation using Hölder's inequality and the Sobolev embedding theorem, $U \hookrightarrow L^6(D)$ in three dimensions, shows

$$|(\hat{A}(u_2) - \hat{A}(u_1), v)| \le C \|u_2 - u_1\|_{L^3(D)} \Big(\|u_1\|_U + \|u_2\|_U \Big) \|v\|_U.$$

Integration by parts for functions with homogeneous boundary data is used to obtain a bound without any derivatives on the difference $u_2 - u_1$. Using the interpolation estimate $\|u\|_{L^2(D)} \le \|u\|_U^{1/2} \|u\|_{U'}^{1/2}$ it follows that

$$\|u\|_{L^3(D)} \le \|u\|_{L^2(D)}^{1/2} \|u\|_{L^6(D)}^{1/2} \le C \|u\|_{U'}^{1/4} \|u\|_U^{3/4},$$

so

$$|(\hat{A}(u_2) - \hat{A}(u_1), v)| \le C \|u_2 - u_1\|_{U'}^{1/4} \Big(\|u_1\|_U^{7/4} + \|u_2\|_U^{7/4} \Big) \|v\|_U.$$

In particular, setting $u_1 = u$ and $u_2 = 0$ and integrating in time, it follows that

$$\left| \int_0^T (\hat{A}(u), v)\, ds \right| \le \|u\|_{L^\infty[0,T;U']}^{1/4} \|u\|_{L^2[0,T;U]}^{7/4} \|v\|_{L^8[0,T;U]},$$

so $\hat{A}$ maps bounded sets in $G[0,T;U']\cap L^2[0,T;U]$ to bounded sets in $L^{8/7}[0,T;U']$, and Hölder's inequality (with $s=8$, $s'=8/7$) shows

$$\mathbb{E}\left[\left|\int_0^T (\hat{A}(u),v)\,ds\right|^p\right] \le \mathbb{E}\left[\|u\|^{2p}_{L^\infty[0,T;U']}\right]^{1/8} \mathbb{E}\left[\|u\|^{2p}_{L^2[0,T;U]}\right]^{7/8} \|v\|_{L^8[0,T;U]},$$

so has a moment of order $p>1$ if the solution has a moment greater than 2.

If $u_n \to u$ in $G[0,T;U']\cap L^2[0,T;U]_{weak}$, then $\{u_n\}_{n=1}^\infty$ converges in $L^2[0,T;U']$ and is bounded in $L^2[0,T;U]$. An application of Hölder's inequality then shows

$$\begin{aligned}&\int_0^T \left|\left(\hat{A}(u_n)-\hat{A}(u),v\right)\right| ds\\ &\quad\le C\|u_n-u\|^{1/4}_{L^2[0,T;U']}\left(\|u_n\|^{7/4}_{L^2[0,T;U]}+\|u\|^{7/4}_{L^2[0,T;U]}\right)\|v\|_{C[0,T;U]}\to 0.\end{aligned}$$

Since the embedding $C[0,T;U']\hookrightarrow L^8[0,T,U]$ is dense it follows that $\hat{A}(u_n)\rightharpoonup \hat{A}(u)$ in $L^{8/7}[0,T,U']$.

The fully implicit approximation of the nonlinear term has $\hat{F}_{h\tau}=\hat{A}(u_{h\tau})$; semi–implicit schemes approximate the convective term with the operator

$$(\hat{F}^n_{h\tau},v)=(1/2)\left((u^{n-1}_{h\tau}.\nabla)u^n_{h\tau},v\right)-(1/2)\left(u^n_{h\tau},(u^{n-1}_{h\tau}.\nabla)v\right),$$

so that each time step only requires the solution of a linear system. The choice preserves skew symmetry, $(\hat{F}^n_{h\tau},u^n_{h\tau})=0$, and using the embedding theorems as above shows

$$|(\hat{F}^n_{h\tau}-\hat{A}(u^n_{h\tau}),v)|\le C\|\hat{u}_{h\tau}\|^{1/4}_{C^{0,\theta}[0,T;U']}\left(\|u^{n-1}_{h\tau}\|^{7/4}_U+\|u^n_{h\tau}\|^{7/4}_U\right)\|v\|_U\tau^{\theta/4},$$

and

$$\begin{aligned}&\mathbb{E}\left[\int_0^T |(\hat{F}_{h\tau}-\hat{A}(u_{h\tau}),v)|^p\,ds\right]\\ &\quad\le C\mathbb{E}\left[\|\hat{u}_{h\tau}\|^{2p}_{C^{0,\theta}[0,T;U']}\right]^{1/8}\mathbb{E}\left[\|u_{h\tau}\|^{2p}_{L^2[0,T;U]}\right]^{7/8}\|v\|_{L^8[0,T;U]}\tau^{\theta/4}.\end{aligned}$$

Theorem 3.2 bounds the Hölder norm $L^p(\Omega,C^{0,\theta}[0,T;U'])$, so this term vanishes as $\tau\to 0$, and consistency of this approximation of the nonlinear term follows.

The stochastic Eq. (1) is said to have *additive noise* if the law of the function g in Eq. (1) is specified a priori. In this case $\{g_{h\tau}\}_{h,\tau>0}$ is an approximation of a specified random variable with law $\mathbb{P}_g$, and the discrete approximations are constructed so that $\mathcal{L}(g_{h\tau})\Rightarrow\mathbb{P}_g$ on $L^{q'}[0,T;U']$. When the stochastic term depends upon the solution, $g=\gamma(u)$, the equation is said to have *multiplicative noise*, and it is necessary to verify that this equation holds in the limit. The following elementary lemma is useful in this context.

Lemma 3.5 *Let U be a separable Banach space, H be a Hilbert space and $U \hookrightarrow H$ be continuous embeddings. If $\gamma : L^s[0,T;H] \cap L^r[0,T;U]_{weak} \to L^2[0,T;H]$ is sequentially continuous then γ maps tight sequences to tight sequences.*

Proof Compact subsets of $L^2[0,T;H] \cap L^r[0,T;U]_{weak}$ are metrizable, so γ maps compact subsets to compact subsets. Thus if $\epsilon > 0$, $K_\epsilon \subset L^s[0,T;H] \cap L^r[0,T;U]_{weak}$ is compact, and $\mathbb{P}[u_k \in K_\epsilon] \geq 1-\epsilon$, then

$$\mathbb{P}[\gamma(u_k) \in \gamma(K_\epsilon)] = \mathbb{P}[u_k \in \gamma^{-1}(\gamma(K_\epsilon))] \geq \mathbb{P}[u_k \in K_\epsilon] \geq 1-\epsilon.$$

□

Example 3.6 Let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $U \subset H^1(D)$ and $H = L^2(D)$. Suppose that $\gamma : (0,T) \times D \times \mathbb{R} \to \mathbb{R}$ is Caratheodory [37]; that is, $\gamma(t,x,u)$ is measurable in (t,x) with u fixed, and continuous in u with (t,x) fixed, and suppose that

$$|\gamma(t,x,u)| \leq C|u|^{3/2} + k(t,x), \quad a.e.\, x \in (0,T) \times D, \quad u \in \mathbb{R},$$

and $k \in L^2[0,T;L^2(D)]$. Let $g(t,x,u) = \gamma(t,x,u(t,x))$ also denote the realization of γ on the Lebesgue spaces, then $\gamma : L^3[0,T;L^3(D)] \to L^2[0,T;L^2(D)]$ is continuous [37].

We show $\gamma : L^6[0,T;H] \cap L^2[0,T;U]_{weak} \to L^2[0,T;H]$ is sequentially continuous. For this purpose, recall that Statement 5 of Theorem 3.2 shows that $\{u_{h\tau}\}_{h,\tau \geq 0}$ is tight in $L^s[0,T;H]$ for all $s > 1$.

The first step is to note that the Sobolev embedding theorem shows $U \hookrightarrow L^6(D)$, and since $1/3 = \theta/2 + (1-\theta)/6$ when $\theta = 1/2$ it follows that

$$\|u\|_{L^3(D)} \leq \|u\|_{L^2(D)}^{1/2} \|u\|_{L^6(D)}^{1/2} \leq C\|u\|_H^{1/2} \|u\|_U^{1/2}.$$

Integrating in time and Hölder's inequality (with $s = 4$ and $s' = 4/3$) shows

$$\|u\|_{L^3[0,T;L^3(D)]} \leq C\|u\|_{L^6[0,T;H]}^{1/2} \|u\|_{L^2[0,T;U]}^{1/2}.$$

Since weakly convergent sequences in $L^2[0,T;U]$ are bounded, and γ is continuous from $L^3[0,T;L^3(D)]$ to $L^2[0,T;H]$, sequential continuity of $\gamma : L^6[0,T;H] \cap L^2[0,T;U]_{weak} \to L^2[0,T;H]$ follows.

Finally, note that

$$\begin{aligned}
\mathbb{E}[\|\gamma(u)\|_{L^2[0,T;H]}^p] &\leq C\mathbb{E}\left[\|u\|_{L^6[0,T;H]}^{p/2} \|u\|_{L^2[0,T;U]}^{p/2} + \|k\|_{L^2[0,T;L^2(D)]}\right] \\
&\leq C\left(\mathbb{E}[\|u\|_{L^6[0,T;H]}^p]^{1/2}\, \mathbb{E}[\|u\|_{L^2[0,T;U]}^p]^{1/2} + 1\right),
\end{aligned}$$

so $\gamma(u)$ inherits moment bounds from u. From Corollary 2.11 it follows that if $\mathcal{L}(u_{h\tau}) \Rightarrow \mathcal{L}(u)$ in $L^6[0,T;H] \cap L^2[0,T;U]_{weak}$ then $\mathcal{L}(\gamma(u_{h\tau})) \Rightarrow \mathcal{L}(\gamma(u))$ on $L^2[0,T;H]$.

3.2 Computational model

Strong solutions are never realized in a computational context since this would require a filtered probability space to be input as part of the problem specification. Instead, a random number generator is seeded and then iterated to generate a sequence $\{b_p(\omega)\}_{p=1}^{\infty}$ which exhibits the statistics of a sequence of real-valued i.i.d. variables $\{b_p\}_{p=1}^{\infty}$ sampled at a point $\omega \in \Omega$ determined by the seed. Typically their law is the uniform (Lebesgue) measure on $(0, 1)$. Given laws of the data, $\mathcal{L}(f, g, W)$, the random numbers are then used to engineer samples $(f_{h\tau}^n(\omega), g_{h\tau}^n(\omega), \xi_\tau^n(\omega))$ of random variables with laws $\mathcal{L}(f_{h\tau}, g_{h\tau}, \hat{W}_{h\tau}) \Rightarrow \mathcal{L}(f, g, W)$.

Example 3.7 If $\mathcal{L}(b_n)$ is the Lebesgue measure on $(0, 1)$ and $\xi_\tau^n(\omega) = \sqrt{12\tau}(b_n(\omega) - 1/2)$ then

$$\mathbb{E}[\xi_\tau^n] = 0, \qquad \mathbb{E}[(\xi_\tau^n)^2] = \tau, \qquad \text{and} \qquad \mathbb{E}[|\xi_\tau^n|^p] = \tfrac{(3\tau)^{p/2}}{(p+1)}.$$

It follows that $\{\xi^n\}_{n=1}^N$ will satisfy Assumption 2.5. In addition, if

$$f_{h\tau}^n(x, \omega) = \Phi_{h\tau}^n(x, b_1(\omega), \ldots, b_n(\omega)) \qquad \text{with} \qquad \Phi_{h\tau}^n \in C(D \times \mathbb{R}^n; U_h),$$

then $f_{h\tau}$ will be adapted to $\mathcal{F}_{h\tau}^n \equiv \sigma(b_1, \ldots, b_n)$.

If the law $\mathcal{L}(u_{h\tau})$ of a solution of the implicit Euler scheme (9) depends only upon the laws of the data $\mathcal{L}(f_{h\tau}, g_{h\tau}, W_{h\tau})$ (and the law of the initial data if not deterministic), then for (h, τ) fixed, solutions $\{(f_{h\tau}^{(p)}(\omega), g_{h\tau}^{(p)}(\omega), W_{h\tau}^{(p)}(\omega))\}_{p=1}^{\infty}$ of the implicit Euler scheme computed using distinct subsets of the random numbers will be i.i.d. In this context Monte-Carlo quadrature can be used to compute the statistics of a solution guaranteed by Theorem 3.2. If $\tilde{\mathbb{P}}$ is the measure and $\{(h_k, \tau_k)\}_{k=1}^{\infty}$ is the subsequence whose existence is guaranteed by Theorem 3.2, then

$$\tilde{\mathbb{E}}[\phi(u)] = \lim_{h_k, \tau_k \to 0} \mathbb{E}[\phi(u_{h\tau})] = \lim_{h_k, \tau_k \to 0} \left(\lim_{P \to \infty} \frac{1}{P} \sum_{p=1}^{P} \phi(u_{h_k \tau_k}^{(p)}(\omega)) \right), \text{ almost surely,}$$

for any function $\phi : G[0, T; U'] \times L^r[0, T; U]_{weak} \to \mathbb{R}$ satisfying the hypotheses of Lemma 2.10.

When the law $\mathcal{L}(u)$ of the solution to (1) is uniquely determined by the law $\mathcal{L}(f, g, W)$ of the data, it is unnecessary to pass to a subsequence provided $\mathcal{L}(f_{h\tau}, g_{h\tau}, \hat{W}_{h\tau}) \Rightarrow \mathcal{L}(f, g, W)$. This is typically achieved by constructing $(f_{h\tau}^n(\omega), g_{h\tau}^n(\omega))$ to be projections or interpolants of specified functions onto the discrete spaces (*e.g.* as in equation (10)) to give a Cauchy sequence in $L^p(\Omega; L^{q'}[0, T; U']] \times L^p(\Omega; L^p[0, T; H])$. In the examples below it is assumed that $\{(f_{h\tau}, g_{h\tau})\}_{h\tau > 0}$ converges in $L^p(\Omega; L^{q'}[0, T; U']) \times L^p(\Omega; L^p[0, T; H])$ whenever we wish to assert uniqueness.

4 The stochastic heat equation

In this section, we construct a weak martingale solution of the stochastic heat equation. While (stochastically) strong solutions exist for this problem [9, 34], we choose this simplified framework to eliminate many technical issues that would otherwise obfuscate the essential structure; more general nonlinear SPDE's are presented in Sect. 6.

Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and $[0, T]$ be a time interval. Adopting the notation commonly used in stochastic analysis, the heat equation with a stochastic source takes the form: find a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{0\le t\le T}, \mathbb{P})$ satisfying the usual conditions, an adapted process $u : [0, T] \times D \times \Omega \to \mathbb{R}$, and a standard Wiener process $W : [0, T] \times \Omega \to \mathbb{R}$ such that

$$du - \Delta u\, dt = f\, dt + g\, dW \qquad u|_{t=0} = u^0, \qquad u|_{\partial D} = 0, \tag{14}$$

with data $f, g : [0, T] \times D \times \Omega \to \mathbb{R}$ that are adapted to $\{\mathcal{F}(t)\}_{0\le t\le T}$ and u^0 measurable on $\mathcal{F}(0)$. Multiplying the heat equation by a test function v vanishing on the boundary and integrating by parts shows

$$\begin{aligned}\int_D u(t) v\, dx + \int_0^t \int_D \nabla u.\nabla v\, dxds &= \int_D u^0 v\, dx + \int_0^t f v\, dxds \\ &+ \int_0^t \left(\int_D g v\, dx \right) dW, \quad 0 \le t \le T.\end{aligned} \tag{15}$$

Setting $H = L^2(D)$, $U = H_0^1(D)$ and defining $a : U \times U \to \mathbb{R}$ by

$$a(u, v) = \int_D \nabla u.\nabla v\, dx,$$

it follows that a solution of the heat equation with stochastic source is an instance of the stochastic evolution equation exhibited in Eq. (7). Convergence of the discrete scheme (9) with these operators will be established under the following hypotheses.

Assumption 4.1 Let $U \hookrightarrow H \hookrightarrow U'$ be a dense embedding of separable Hilbert spaces, and the operators and data for Eq. (7) satisfy

1. $a : U \times U \to \mathbb{R}$ is bilinear, continuous, and coercive. Specifically, there exist constants $c_a, C_a > 0$ such that

$$|a(u, v)| \le C_a \|u\|_U \|v\|_U, \quad \text{and} \quad a(u, u) \ge c_a \|u\|_U^2, \qquad u, v \in U.$$

2. For every $h > 0$, U_h is a finite dimensional subspace of U, and $\{t^n\}_{n=0}^N$ is a uniform partition of $[0, T]$ with time-step $\tau = T/N$.
3. For each pair of parameters (h, τ), $\mathcal{F}^0$ is generated by u^0 and $\{\mathcal{F}^n\}_{n=1}^N$ is the discrete filtration with $\mathcal{F}^n = \sigma\left(\{(u_{h\tau}^m, f_{h\tau}^m, g_{h\tau}^m, \xi_\tau^m)\}_{m=0}^n\right)$.

Granted Assumptions 4.1 and 2.5 with $p \geq 2$, the existence to the discrete scheme (9) is direct; fix $\omega \in \Omega$ and write Eq. (9) as $u_{h\tau}^n(\omega) \in U_h$,

$$\left(u_{h\tau}^n(\omega), v_h\right)_H + \tau a\left(u_{h\tau}^n(\omega), v_h\right) = \left(u_{h\tau}^{n-1}(\omega), v_h\right)_H$$
$$+\tau\left(f_{h\tau}^n(\omega), v_h\right) + \left(g_{h\tau}^{n-1}(\omega), v_h\right)\xi_\tau^n(\omega), \quad v_h \in U_h.$$

Upon selecting a basis for U_h this becomes a system of linear equations, $\mathbb{A}\mathbf{u}(\omega) = \mathbf{b}(\omega)$, with as many equations as unknowns; moreover,

$$\mathbf{v}^\top \mathbb{A}\mathbf{v} = \big(v_h, v_h\big)_H + \tau a\big(v_h, v_h\big) \geq \|v_h\|_H^2 + \tau c_a \|v_h\|_U^2, \qquad v_h \in U_h,$$

so $\mathbb{A}$ is nonsingular and $u_{h\tau}^n$ is a continuous function of the data $(u_{h\tau}^{n-1}, f_{h\tau}^n, g_{h\tau}^{n-1}, \xi_\tau^n)$. Since measurability of random variables is always with respect to the Borel σ-algebra on the target space, continuity of the solution operator guarantees that $u_{h\tau}^n$ is $\mathcal{F}^n$–measurable whence the sequence $\{u_{h\tau}^n\}_{n=0}^N$ is adapted to $\{\mathcal{F}^n\}_{n=0}^N$.

4.1 Bounds

We begin by recalling bounds satisfied by the deterministic equation

$$u \in U, \qquad (\partial_t u, v)_H + a(u, v) = (f, v), \qquad v \in U,$$

with the bilinear function satisfying Assumption 4.1. The fundamental estimate is found upon selecting $v = u$ to get

$$(1/2)\frac{d}{dt}\|u\|_H^2 + c_a\|u\|_U^2 \leq (f, u) \leq \|f\|_{U'}\|u\|_U.$$

Integration in time then shows

$$\|u\|_{L^\infty[0,T;H]}^2 + c_a\|u\|_{L^2[0,T;U]}^2 \leq \|u(0)\|_U^2 + (1/2c_a)\|f\|_{L^2[0,T;U']}^2.$$

The analogous statement for the discrete scheme (4) is obtained upon selecting the test function $v_h = u_{h\tau}^n$, and the corresponding estimate is

$$\max_{1\leq n\leq N}\|u_{h\tau}^n\|_H^2 + \sum_{m=1}^N \|u_{h\tau}^m - u_{h\tau}^{m-1}\|_H^2 + c_a\sum_{m=1}^N \tau\|u_{h\tau}^m\|_U^2$$
$$\leq C\left(\|u_{h\tau}^0\|_H^2 + \sum_{m=1}^N \tau\|f_\tau^m\|_{U'}^2\right).$$

The second term on the left is an additional dissipative term inherent to the implicit Euler scheme which arises when completing the square of the approximate time derivative,

$$(u - v, u)_H = (1/2)\|u\|_H^2 + (1/2)\|u - v\|_H^2 - (1/2)\|v\|_H^2. \tag{16}$$

Consider next the discrete scheme (9) with bilinear form satisfying Assumption 4.1. To bound its solution, independence of the increments and the dissipative term in the Euler scheme are used in an essential fashion. With $\omega \in \Omega$ fixed, selecting the test function in equation (9) to be $v = u_{h\tau}^n(\omega)$ gives

$$\begin{aligned} &\|u_{h\tau}^n\|_H^2 + \|u_{h\tau}^n - u_{h\tau}^{n-1}\|_H^2 + 2c_a\tau\|u_{h\tau}^n\|_U^2 \\ &\quad \le \|u_{h\tau}^{n-1}\|_H^2 + 2\tau(f_{h\tau}^n, u_{h\tau}^n) + 2(g_{h\tau}^{n-1}, u_{h\tau}^n)\xi_\tau^n. \end{aligned} \tag{17}$$

To bound the last term properties of the stochastic increments from Assumption 2.5 are utilized. Writing this term as

$$(g_{h\tau}^{n-1}, u_{h\tau}^n)\xi_\tau^n = (g_{h\tau}^{n-1}, u_{h\tau}^n - u_{h\tau}^{n-1})\xi_\tau^n + (g_{h\tau}^{n-1}, u_{h\tau}^{n-1})\xi_\tau^n,$$

and taking the expected value we have

- $(g_{h\tau}^{n-1}, u_{h\tau}^{n-1})_H$ is $\mathcal{F}^{n-1}$-measurable, so is independent of ξ_τ^n, and since the average of ξ_τ^n vanishes it follows that

$$\mathbb{E}\left[(g_{h\tau}^{n-1}, u_{h\tau}^{n-1})\xi_\tau^n\right] = \mathbb{E}[(g_{h\tau}^{n-1}, u_{h\tau}^{n-1})]\,\mathbb{E}[\xi_\tau^n] = 0.$$

- $\|g_{h\tau}^{n-1}\|_H$ and $|\xi_\tau^n|$ are also independent, so an application of the Cauchy-Schwarz inequality gives

$$\begin{aligned} \mathbb{E}\left[(g_{h\tau}^{n-1}, u_{h\tau}^n - u_{h\tau}^{n-1})\xi_\tau^n\right] &\le \left(\mathbb{E}[\|u_{h\tau}^n - u_{h\tau}^{n-1}\|_H^2]\right)^{1/2} \left(\mathbb{E}[\|g_{h\tau}^{n-1}\|_H^2 |\xi_\tau^n|^2]\right)^{1/2} \\ &= \left(\mathbb{E}[\|u_{h\tau}^n - u_{h\tau}^{n-1}\|_H^2]\right)^{1/2} \left(\mathbb{E}[\|g_{h\tau}^{n-1}\|_H^2]\mathbb{E}[|\xi_\tau^n|^2]\right)^{1/2} \\ &= \left(\mathbb{E}[\|u_{h\tau}^n - u_{h\tau}^{n-1}\|_H^2]\right)^{1/2} \left(\tau\mathbb{E}[\|g_{h\tau}^{n-1}\|_H^2]\right)^{1/2}. \end{aligned}$$

Taking the expected value of both sides of equation (17), this bound is used to estimate the stochastic term,

$$\begin{aligned} &\|u_{h\tau}\|^2_{L^\infty[0,T;L^2(\Omega,H)]} + \|u_{h\tau}\|^2_{L^2[0,T;L^2(\Omega,U)]} \\ &\quad \le C\left(\|u_{h\tau}^0\|^2_{L^2(\Omega,H)} + \|f_{h\tau}\|^2_{L^2[0,T;L^2(\Omega,U')]} + \|g_{h\tau}\|^2_{L^2[0,T;L^2(\Omega,H)]}\right). \end{aligned} \tag{18}$$

This estimate bounds $u_{h\tau}$ in the Bochner space $L^\infty[0, T; L^2(\Omega, H)]$; however, we also wish to identify $u_{h\tau}$ as a random variable taking values in the Bochner space $L^\infty[0, T; H]$. For any Banach space U, the canonical correspondences

$$L^2[0, T; L^2(\Omega, U)] \simeq L^2((0, T) \times \Omega, U) \simeq L^2(\Omega, L^2[0, T; U]),$$

allow functions in these spaces to be identified as a random variable with values in $L^2[0,T;U]$. In general it is not possible to identify $L^\infty[0,T;L^2(\Omega,H)]$ with $L^2(\Omega,L^\infty[0,T;U])$; however, the BDG inequality shows that the norms on these two spaces are equivalent on the subspace of martingales. The following lemma uses the property that the stochastic term in (9) is a martingale to bound the solution in $L^2(\Omega,L^\infty[0,T;H])$.

Lemma 4.2 *Let Assumptions 2.5 and 4.1 with $p \geq 2$ hold and $u_{h\tau}$ be a solution of the implicit Euler scheme (9) with initial condition $u^0_{h\tau} \in L^p(\Omega,H)$, and data $f_{h\tau} \in L^p(\Omega,L^2[0,T;U'])$, and $g_{h\tau} \in L^p(\Omega,L^p[0,T;H])$. Then there exists a constant $C = C(p) > 0$ such that*

$$\|u_{h\tau}\|_{L^p(\Omega,L^\infty[0,T;H])} + \sqrt{c_a}\|u_{h\tau}\|_{L^p(\Omega,L^2[0,T;U])} + \mathbb{E}\left[\left(\sum_{m=1}^N \|u^m_{h\tau} - u^{m-1}_{h\tau}\|^2_H\right)^{p/2}\right]^{1/p}$$
$$\leq C\left(\|u^0_{h\tau}\|_{L^p(\Omega,H)} + \|f_\tau\|_{L^p(\Omega,L^2[0,T;U'])} + T^{1/2-1/p}\|g_\tau\|_{L^p(\Omega,L^p[0,T;H])}\right). \tag{19}$$

Proof Sum each side of inequality (17) to obtain

$$\begin{aligned}
&\|u^n_{h\tau}\|^2_H + \sum_{m=1}^n \|u^m_{h\tau} - u^{m-1}_{h\tau}\|^2_H + 2c_a\sum_{m=1}^n \tau\|u^m_{h\tau}\|^2_U \\
&\leq \|u^0_{h\tau}\|^2_H + 2\sum_{m=1}^n \tau(f^m_{h\tau},u^m_{h\tau}) + 2\sum_{m=1}^n (g^{m-1}_{h\tau},u^m_{h\tau})_H\xi^m_\tau \\
&\leq \|u^0_{h\tau}\|^2_H + 2\sum_{m=1}^n \tau\|f^m_{h\tau}\|_{U'}\|u^m_{h\tau}\|_U + 2\sum_{m=1}^n \|g^{m-1}_{h\tau}\|_H\|u^m_{h\tau} - u^{m-1}_{h\tau}\|_H|\xi^m_\tau| \\
&\quad +2\sum_{m=1}^n (g^{m-1}_{h\tau},u^{m-1}_{h\tau})_H\xi^m_\tau,
\end{aligned}$$

and use the Cauchy–Schwarz and Young inequalities to get

$$\begin{aligned}
&\|u^n_{h\tau}\|^2_H + \sum_{m=1}^n \|u^m_{h\tau} - u^{m-1}_{h\tau}\|^2_H + c_a\sum_{m=1}^n \tau\|u^m_{h\tau}\|^2_U \\
&\leq C\Big(\|u^0_{h\tau}\|^2_H + \sum_{m=1}^n \tau\|f^m_{h\tau}\|^2_{U'} + \sum_{m=1}^n \|g^{m-1}_{h\tau}\|^2_H|\xi^m_\tau|^2 \\
&\quad +\Big|\sum_{m=1}^n (g^{m-1}_{h\tau},u^{m-1}_{h\tau})_H\xi^m_\tau\Big|\Big).
\end{aligned}$$

Raising each side to the power $p/2$ and using Assumption 2.5_3 shows

$$
\begin{aligned}
&\|u_{h\tau}^n\|_H^p + \left(\sum_{m=1}^n \|u_{h\tau}^m - u_{h\tau}^{m-1}\|_H^2\right)^{p/2} + \left(c_a \sum_{m=1}^n \tau \|u_{h\tau}^m\|_U^2\right)^{p/2} \\
&\quad \le C \left(\|u_{h\tau}^0\|_H^p + \|f_{h\tau}\|_{L^2[0,T;U']}^p + \left(\sum_{m=1}^n \|g_{h\tau}^{m-1}\|_H^2 |\xi_\tau^m|^2\right)^{p/2} \right. \\
&\qquad \left. + \left|\sum_{m=1}^n (g_{h\tau}^{m-1}, u_{h\tau}^{m-1})_H \xi_\tau^m\right|^{p/2} \right) \\
&\quad \le C \left(\|u_{h\tau}^0\|_H^p + \|f_{h\tau}\|_{L^2[0,T;U']}^p + n^{p/2-1} \sum_{m=1}^n \|g_\tau^{m-1}\|_H^p |\xi_\tau^m|^p \right. \\
&\qquad \left. + \left|\sum_{m=1}^n (g_{h\tau}^{m-1}, u_{h\tau}^{m-1})_H \xi_\tau^m\right|^{p/2} \right).
\end{aligned}
$$

Taking the maximum over $1 \le n = n(\omega) \le N$ and using the property that

$$
\begin{aligned}
\mathbb{E}\left[n^{p/2-1} \|g_{h\tau}^{m-1}\|_H^p |\xi_\tau^m|^p\right] &\le C(p) N^{p/2-1} \mathbb{E}\left[\|g_{h\tau}^{m-1}\|_H^p\right] \tau^{p/2} \\
&\le C(p) T^{p/2-1} \mathbb{E}\left[\tau \|g_{h\tau}^{m-1}\|_H^p\right],
\end{aligned}
$$

shows

$$
\begin{aligned}
&\mathbb{E}\left[\max_{1\le n\le N} \|u_{h\tau}^n\|_H^p + \left(\sum_{m=1}^N \|u_{h\tau}^m - u_{h\tau}^{m-1}\|_H^2\right)^{p/2} + c_a^{p/2} \|u_{h\tau}\|_{L^2[0,T;U]}^p\right] \\
&\le C\Big(\|u_{h\tau}^0\|_{L^p(\Omega,H)}^p + \|f_{h\tau}\|_{L^p(\Omega,L^2[0,T;U'])}^p + T^{p/2-1} \|g_{h\tau}\|_{L^p(\Omega,L^p[0,T;H])}^p \\
&+ \mathbb{E}\left[\max_{1\le n\le N} |\sum_{m=1}^n (g_{h\tau}^{m-1}, u_{h\tau}^{m-1})_H \xi_\tau^m|^{p/2}\right]\Big).
\end{aligned} \tag{20}
$$

The last term is a discrete Ito integral (c.f. Eq. 5),

$$
X_\tau^n = \sum_{m=1}^n (g_{h\tau}^{m-1}, u_{h\tau}^{m-1})_H \xi_\tau^m,
$$

and is bounded using the discrete BDG inequality (Theorem 2.7) and Lemma 2.8. With $\epsilon > 0$ to be selected below,

$$
\mathbb{E}\left[\max_{0\le n\le N} |X_\tau^n|^{p/2}\right] \le C \sum_{m=1}^N \tau \|(g_{h\tau}^{m-1}, u_{h\tau}^{m-1})_H\|_{L^{p/2}(\Omega)}^{p/2} T^{p/4-1}
$$

$$
\begin{aligned}
&\leq C \sum_{m=1}^{N} \tau \|g_{h\tau}^{m-1}\|_{L^p(\Omega,H)}^{p/2} \|u_{h\tau}^{m-1}\|_{L^p(\Omega,H)}^{p/2} T^{p/4-1} \\
&\leq C \left(\max_{0\leq m\leq N-1} \|u_{h\tau}^{m}\|_{L^p(\Omega,H)}^{p/2} \right) \sum_{m=1}^{N} \tau \|g_{h\tau}^{m-1}\|_{L^p(\Omega,H)}^{p/2} T^{p/4-1} \\
&\leq \epsilon \left(\max_{0\leq m\leq N-1} \|u_{h\tau}^{m}\|_{L^p(\Omega,H)}^{p} \right) \\
&\quad + (2C^2/\epsilon) \left(\sum_{m=1}^{N} \tau \|g_{h\tau}^{m-1}\|_{L^p(\Omega,H)}^{p/2} \right)^2 T^{p/2-2} \\
&\leq \epsilon \|u_{h\tau}^{0}\|_{L^p(\Omega,H)}^{p} + \epsilon \max_{1\leq m\leq N} \|u_{h\tau}^{m}\|_{L^p(\Omega,H)}^{p} \\
&\quad + (2C^2/\epsilon) T^{p/2-1} \|g_{h\tau}\|_{L^p(\Omega,L^p[0,T;H])}^{p}.
\end{aligned}
$$

The proof now follows since the middle term, with an appropriate choice of $\epsilon > 0$, can be absorbed into the left-hand side of Eq. (20). □

4.2 Passage to the limit

Setting $(F, v) = (f, v) - a(u, v)$, the weak statement of the stochastic heat equation (15) is an instance of the abstract problem (12), and its discretization is of the form (11). The bounds in Lemma 4.2 are sufficient to verify the hypotheses of Theorem 3.2, and convergence of the discrete scheme to a weak martingale solution of (14) will follow.

Theorem 4.3 *Let U be a separable reflexive Banach space, H a Hilbert space, $U \hookrightarrow\!\!\!\!\rightarrow H$ be a compact, dense embedding, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let the operators of the abstract difference scheme (9), and data satisfy Assumptions 4.1 and 3.1 respectively, and let the stochastic increments satisfy Assumptions 2.5 with $p \in (2, \infty)$. Denote the discrete Wiener process with increments $\{\xi_\tau^m\}_{m=1}^N$ by $\hat{W}_\tau^n$, and let $\{u_{h\tau}\}_{h,\tau>0}$ be a sequence of solutions of the corresponding implicit Euler scheme (9) with data satisfying:*

1. *$\{u_{h\tau}^0\}$ is bounded in $L^p(\Omega, H)$ and converges to a limit u^0 in $L^2(\Omega, H)$ as $h \to 0$.*
2. *$\{f_{h\tau}\}$ is bounded in $L^p(\Omega, L^2[0, T; U'])$ and converges in $L^2(\Omega, L^2[0, T; U'])$ as $\tau, h \to 0$.*
3. *$\{g_{h\tau}\}$ is bounded in $L^p(\Omega, L^p[0, T; H])$ and converges in $L^2(\Omega, L^2[0, T; H])$ as $\tau, h \to 0$.*

Let

$$
\mathbb{X} = G[0, T; U'] \cap L^2[0, T; U]_{weak} \times L^2[0, T; U']_{weak} \times L^2[0, T; H] \times C[0, T].
$$

Then there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a random variable (u, F, g, W) on $\tilde{\Omega}$ with values in $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ for which the laws of $\big\{(u_{h\tau}, (f_{h\tau}, A(u_{h\tau})), g_{h\tau}, \hat{W}_\tau)\big\}_{h,\tau>0}$

converge to the law of $(u, (f, A(u)), g, W)$*,*

$$\mathcal{L}(u_{h\tau}, (f_{h\tau}, A(u_{h\tau})), g_{h\tau}, \hat{W}_\tau) \Rightarrow \mathcal{L}(u, (f, A(u)), g, W).$$

In addition, there exists a filtration $\{\tilde{\mathcal{F}}(t)\}_{0\le t\le T}$ *satisfying the usual conditions for which* (u, f, g, W) *is adapted and* W *is a real-valued Wiener process for which*

$$(u(t), v)_H + \int_0^t a(u, v)\, ds = (u^0, v)_H + \int_0^t (f, v)\, ds + \int_0^t (g, v)\, dW, \qquad v \in U.$$

Proof Under the assumptions of the theorem solutions of the implicit Euler scheme satisfy the bounds stated in Lemma 4.2; in particular, $\{u_{h\tau}\}_{h,\tau>0}$ is bounded in $L^p(\Omega, L^2[0, T; U])$. With

$$F^n_{h\tau}(v_h) = (f^n_{h\tau}, v_h) - a(u^n_{h\tau}, v_h),$$

it is immediate that $F^n_{h\tau}$ is $\mathcal{F}^n$-measurable, and since $a : U \times U \to \mathbb{R}$ is bilinear and continuous, $\{F_{h\tau}\}_{h,\tau>0}$ is bounded in $L^p(\Omega, L^2[0, T; U'])$, and it follows from the Cauchy-Schwarz inequality that $\{F_{h\tau}(u_{h\tau})\}_{h\tau>0}$ is bounded in $L^{p/2}(\Omega, L^1(0, T))$. This establishes the hypotheses of Theorem 3.2 (with $r = q = 2$) which guarantees the existence of a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}(t)\}_{0\le t\le T}, \tilde{\mathbb{P}})$, a subsequence $(h_k, \tau_k) \to (0, 0)$ and a limit (u, f, g, W) for which the laws of $(u_{h_k\tau_k}, F_{h_k\tau_k}, g_{h_k\tau_k}, \hat{W}_{\tau_k})$ convergence as asserted in the theorem and

$$(u(t), v)_H = (u^0, v)_H + \int_0^t (F, v)\, ds + \int_0^t (g, v)_H\, dW, \quad v \in U.$$

To verify that $(F, v) = (f, v) - a(u, v)$ for $v \in U$ note that the mapping $u \mapsto f - a(u, .)$ is affine so is continuous from $L^2[0, T; U]$ to $L^2[0, T; U']$ with both the weak and strong topologies. This is the setting of Example 3.3 where it was shown that F takes the required form. Finally, A satisfies Assumption 2.15 since solutions of the deterministic heat equation are unique so Theorem 2.17 is applicable. It follows that $\mathcal{L}(u)$ is uniquely determined by $\mathcal{L}(f, g, W)$; in particular, passing to a subsequence was unnecessary. □

5 Construction of a Martingale solution

This section is devoted to the proof of Theorem 3.2. Throughout U will denote a Banach space densely embedded in a Hilbert space H so that $U \hookrightarrow H \hookrightarrow U'$, and U_h will denote a (finite dimensional) subspace of U, and $\tau = T/N$ the time step for the implicit Euler scheme (11).

5.1 Bounds and pathwise continuity

The following lemma is essentially a restatement of Lemma 4.2 adapted to the current setting where bounds upon the solution are assumed.

Lemma 5.1 *Let $1 \le q \le \infty$, and $\{u_{h\tau}^n\}_{n=1}^N$ be a U_h-valued solution of the implicit Euler scheme (11) with increments and data satisfying Assumptions 2.5 with $p \ge 2$ and 3.1 respectively. If $u_{h\tau}^0 \in L^p(\Omega, U_h)$, $F_{h\tau} \in L^p(\Omega, L^{q'}[0,T; U_h'])$, and $g_{h\tau} \in L^p(\Omega, L^p[0,T; H])$, then there exists a constant $C = C(p) > 0$ such that the piecewise constant interpolant $u_{h\tau}$ satisfies*

$$\|u_{h\tau}\|_{L^p(\Omega, L^\infty[0,T;H])} + \mathbb{E}\left[\left(\sum_{m=1}^N \|u_{h\tau}^m - u_{h\tau}^{m-1}\|_H^2\right)^{p/2}\right]^{1/p}$$
$$\le C\left(\|u_{h\tau}^0\|_{L^p(\Omega,H)} + \|F_{h\tau}(u_{h\tau})\|_{L^{p/2}(\Omega,L^1(0,T))}^{1/2} + T^{1/2-1/p}\|g_{h\tau}\|_{L^p(\Omega,L^p[0,T;H])}\right).$$

Proof (sketch) Setting $v_h = u_{h\tau}^n$ in equation (11), and completing the square shows

$$\|u_{h\tau}^n\|_H^2 + \|u_{h\tau}^n - u_{h\tau}^{n-1}\|_H^2 \le \|u_{h\tau}^{n-1}\|_H^2 + 2\tau(F_{h\tau}^n, u_{h\tau}^n) + 2(g^{n-1}, u_{h\tau}^n)_H \xi_\tau^n.$$

The proof then is identical to that of Lemma 4.2 with $c_a = 0$ and $F_{h\tau}^n$ in place of $f_{h\tau}^n$. □

Pathwise continuity is an essential property of martingale solutions; that is, for almost every $\omega \in \Omega$ the map $t \mapsto u(\omega, t)$ is continuous. Solutions of nonlinear PDE's may not be pathwise continuous into the pivot space H; however, continuity into the dual space U' follows from standard arguments. Specifically, Hölder continuity into U' is established by showing that solutions of the numerical scheme (11) satisfy the hypothesis of the following theorem [8, Theorem 3.3].

Theorem 5.2 (Kolmogorov–Centsov) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{X}$ be a Banach space, and $u \in L^1(\Omega, L^p[0,T;\mathcal{X}])$. If for some $0 < \theta \le 1$ there exists $\hat{C} > 0$ such that for all $0 \le \delta < T$*

$$\mathbb{E}\left[\int_\delta^T \|u(t) - u(t-\delta)\|_{\mathcal{X}}^p \, dt\right] \le \hat{C}^p \delta^{1+\theta p},$$

then there exists a modification of u on a null set of $(0,T)$ such that $u(\omega) \in C^{0,\theta'}[0,T;\mathcal{X}]$ for almost every $\omega \in \Omega$ and all $0 < \theta' < \theta$; in particular $\mathbb{E}\left[\|u\|_{C^{0,\theta'}[0,T;\mathcal{X}]}^p\right] \le C$.

Piecewise linear interpolants $\hat{u}_{h\tau}$ of numerical schemes are Lipschitz (in the time variable), so no modification is required; the bound on the Hölder norm is the essential content.

The following theorem bounds translates of solutions of the difference scheme (11) appearing in the Kolmogorov–Centsov theorem. The spatial discretization plays no role in this lemma; U is an arbitrary Banach space. Setting $U = U_h$ establishes Hölder continuity of the discrete solution for almost all paths in the dual space U_h' which has norm

$$\|u\|_{U_h'} = \sup_{v_h \in U_h} \frac{(u, v_h)_H}{\|v_h\|_U}.$$

This is a norm on U_h and a semi–norm on U with $\|u\|_{U_h'} \le C\|u\|_{U'}$. If $P_h : H \to U_h$ denotes the orthogonal projection and $u_h \in U_h$ then

$$\|u_h\|_{U'} = \sup_{v \in U} \frac{(u_h, v)_H}{\|v\|_U} = \sup_{v \in U} \frac{(u_h, P_h(v))_H}{\|P_h(v)\|_U} \frac{\|P_h(v)\|_U}{\|v\|_U} \le \|u_h\|_{U_h'} \left(\sup_{v \in U} \frac{\|P_h(v)\|_U}{\|v\|_U} \right).$$

In a finite element context the supremum on the right is bounded independently of h under mild conditions on the underlying mesh [7]. In this situation a function taking values in U_h is bounded in U' when it is bounded in U_h', and this is where Assumption 3.1$_{(6)}$ is used.

Theorem 5.3 *Let $1 \le q \le \infty$, and $U \hookrightarrow H$ be an embedding of a Banach space into the Hilbert space H so that $U \hookrightarrow H \hookrightarrow U'$. Let $0 = t^0 < t^1 < \ldots < t^N = T$ be a uniform partition of $[0, T]$ with time step τ, and $(\Omega, \mathcal{F}, \{\mathcal{F}^n\}_{n=0}^N, \mathbb{P})$ be a (discretely) filtered probability space. Let $\{u_\tau^n\}_{n=0}^N$ be an adapted process taking values in U, satisfying the difference scheme*

$$(u_\tau^n - u_\tau^{n-1}, v)_H = \tau(F_\tau^n, v) + (g_\tau^{n-1}, v)_H \xi_\tau^n, \qquad v \in U,$$

with

- *$\{\xi_\tau^n\}_{n=1}^N$ satisfying Assumption 2.5 with $p > 2$.*
- *$u_\tau^0 \in L^p(\Omega, U')$.*
- *$F_\tau \in L^p(\Omega, L^{q'}[0, T; U'])$, and F_τ^n is $\mathcal{F}^n$-measurable for $1 \le n \le N$,*
- *$g_\tau \in L^p(\Omega, L^p[0, T; H])$, and g_τ^{n-1} is $\mathcal{F}^{n-1}$-measurable for $1 \le n \le N$,*

where $F_\tau(t) = F_\tau^n$ and $g_\tau(t) = g_\tau^{n-1}$ on (t^{n-1}, t^n) denote the piecewise constant functions. Then the piecewise linear interpolant $\hat{u}_\tau$ of $\{u_\tau^n\}_{n=0}^N$ satisfies

$$\begin{aligned} &\mathbb{E}\Big[\int_\delta^T \|\hat{u}_\tau(t) - \hat{u}_\tau(t-\delta)\|_{U'}^p \, dt\Big] \\ &\quad \le C\left(\|F_\tau\|_{L^p(\Omega, L^{q'}[0,T;U'])}^p + \|g_\tau\|_{L^p(\Omega, L^p[0,T;U'])}^p\right) \delta^{1+\theta p}, \quad 0 < \delta < T, \end{aligned}$$

with $\theta = \min(1/2 - 1/p, 1/q)$. In particular, $\hat{u}_\tau$ is bounded in $L^p(\Omega, C^{0,\theta'}[0, T; U'])$ for all $0 < \theta' < \theta$, and the difference between the piecewise constant interpolant u_τ and $\hat{u}_\tau$ is bounded by

$$\|u_\tau - \hat{u}_\tau\|_{L^p(\Omega, L^\infty[0,T;U'])} \le C\tau^{\theta'} \left(\|F_\tau\|_{L^p(\Omega, L^{q'}[0,T;U'])} + \|g_\tau\|_{L^p(\Omega, L^p[0,T;U'])}\right).$$

As in the deterministic setting [40], we first consider the situation where $\delta = m\tau$ is an integer multiple of the time step, and present this as a separate lemma. The proof of the theorem is an extension this result to arbitrary $0 < \delta < T$.

Lemma 5.4 *Under the hypotheses of Theorem 5.3,*

$$\mathbb{E}\Big[\sum_{n=m}^{N} \tau \|u_\tau^n - u_\tau^{n-m}\|_{U'}^p\Big] \le C\left(\|F_\tau\|^p_{L^p(\Omega, L^{q'}[0,T;U'])}(m\tau)^{1+p/q} + \|g_\tau\|^p_{L^p(\Omega, L^p[0,T;U'])}(m\tau)^{p/2}\right),$$

for all $1 \le m \le N$ *when* $1/p + 1/q \le 1$*; otherwise,* $p \le q'$ *and*

$$\mathbb{E}\Big[\sum_{n=m}^{N} \tau \|u_\tau^n - u_\tau^{n-m}\|_{U'}^p\Big] \le C\left(\|F_\tau\|^p_{L^p(\Omega, L^p[0,T;U'])}(m\tau)^{p} + \|g_\tau\|^p_{L^p(\Omega, L^p[0,T;U'])}(m\tau)^{p/2}\right).$$

Proof Let $v \in U$ and sum the difference scheme from $n - m + 1$ to n to obtain

$$\begin{aligned}(u_\tau^n - u_\tau^{n-m}, v)_H &= \sum_{k=n-m+1}^{n} \tau (F_\tau^k, v) + \sum_{k=n-m+1}^{n} (g_\tau^{k-1}\xi_\tau^k, v)_H \\ &\le \left(\sum_{k=n-m+1}^{n} \tau \|F_\tau^k\|_{U'} + \Big\|\sum_{k=n-m+1}^{n} g_\tau^{k-1}\xi_\tau^k\Big\|_{U'}\right)\|v\|_U.\end{aligned}$$

Taking the supremum on the left over $v \in U$ with $\|v\|_U = 1$, raising both sides to the power p, and summing shows

$$\mathbb{E}\Big[\sum_{n=m}^{N} \tau \|u_\tau^n - u_\tau^{n-m}\|_{U'}^p\Big] \le C\mathbb{E}\Big[\sum_{n=m}^{N} \tau \left\{\left(\sum_{k=n-m+1}^{n} \tau \|F_\tau^k\|_{U'}\right)^p + \Big\|\sum_{k=n-m+1}^{n} g_\tau^{k-1}\xi_\tau^k\Big\|_{U'}^p\right\}\Big].$$

The first term on the right is bounded using Hölder's inequality. If $q' \le p$ then

$$\begin{aligned}\mathbb{E}\Big[\sum_{n=m}^{N} \tau \left(\sum_{k=n-m+1}^{n} \tau \|F_\tau^k\|_{U'}\right)^p\Big] &\le \mathbb{E}\Big[\sum_{n=m}^{N} \tau \left(\sum_{k=n-m+1}^{n} \tau \|F_\tau^k\|_{U'}^{q'}\right)^{p/q'} (m\tau)^{p/q}\Big] \\ &\le \mathbb{E}\Big[\sum_{n=m}^{N} \tau \left(\sum_{k=n-m+1}^{n} \tau \|F_\tau^k\|_{U'}^{q'}\right) \|F_\tau\|^{p-q'}_{L^{q'}[0,T;U']}(m\tau)^{p/q}\Big] \\ &\le \mathbb{E}\left[\|F_\tau\|^p_{L^{q'}[0,T;U']}\right](m\tau)^{1+p/q}.\end{aligned}$$

When $q' = p$ the exponent in the last term is $1 + p/q = 1 + p(1 - 1/q') = p$.

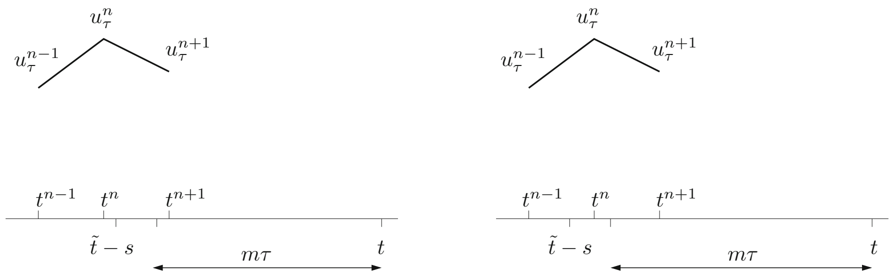

Fig. 3 Translates in Theorem 5.3, $(\tilde{t}-s,\tilde{t}) \subset (t^n, t^{n+1})$(left) and $t^n \in (\tilde{t}-s,\tilde{t})$(right)

The second term is a (discrete) Ito integral and is bounded using the discrete BDG inequality, Theorem 2.7, and Lemma 2.8,

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{n=m}^{N} \tau \left(\| \sum_{k=n-m+1}^{n} g_\tau^{k-1}\xi_\tau^k \|_{U'}\right)^p \Big] \\
&\leq C \sum_{n=m}^{N} \tau \left(\sum_{k=n-m+1}^{n} \tau \|g_\tau^{k-1}\|^p_{L^p(\Omega,U')} \right) (m\tau)^{p/2-1} \\
&\leq C \sum_{k=1}^{N} \tau \|g_\tau^{k-1}\|^p_{L^p(\Omega,U')} (m\tau)^{p/2}.
\end{aligned}$$

$\square$

Proof (of Theorem 5.3) Write $\delta = m\tau + s$ with $0 \leq m < N$ and $0 \leq s \leq \tau$. From the triangle inequality

$$\begin{aligned}
\|\hat{u}_\tau(t) - \hat{u}_\tau(t-\delta)\|_{U'} \leq &\|\hat{u}_\tau(t) - \hat{u}_\tau(t-m\tau)\|_{U'} \\
&+\|\hat{u}_\tau(\tilde{t}) - \hat{u}_\tau(\tilde{t}-s)\|_{U'}, \quad \text{where} \quad \tilde{t} = t - m\tau.
\end{aligned}$$

The previous lemma shows that for all $1 \leq m \leq N$

$$\begin{aligned}
\mathbb{E}\Big[\int_{m\tau}^{T} \|\hat{u}_\tau(t) - \hat{u}_\tau(t-m\tau)\|^p_{U'}\,dt\Big] &\leq C(p)\mathbb{E}\Big[\sum_{n=m}^{N} \tau \|u_\tau^n - u_\tau^{n-m}\|^p_{U'}\Big] \\
&\leq C(p,f,g)(m\tau)^{1+\theta p}, \qquad (21)
\end{aligned}$$

so it suffices to show that translates of size $0 < s \leq \tau$ can be bounded by $s^{1+\theta p}$. For piecewise linear functions there are two cases, see Fig. 3.

- If $\tilde{t} \in (t^n + s, t^{n+1})$ then

$$\|\hat{u}_\tau(\tilde{t}) - \hat{u}_\tau(\tilde{t}-s)\|_{U'} = (s/\tau)\|u_\tau^{n+1} - u_\tau^n\|_{U'}.$$

- If $\tilde{t} \in (t^n, t^n + s)$ use the triangle inequality to write

$$\|\hat{u}_\tau(\tilde{t}) - \hat{u}_\tau(\tilde{t} - s)\|_{U'} \leq \|\hat{u}_\tau(\tilde{t}) - u_\tau^n\|_{U'} + \|u_\tau^n - \hat{u}_\tau(\tilde{t} - s)\|_{U'}.$$

Explicit formulas for the piecewise linear interpolants on each interval show

$$\begin{aligned}&\|\hat{u}_\tau(\tilde{t}) - u_\tau^n\|_{U'} + \|u_\tau^n - \hat{u}_\tau(\tilde{t} - s)\|_{U'} \\ &\quad \leq \big(\|u_\tau^{n-1} - u_\tau^n\|_{U'} + \|u_\tau^{n+1} - u_\tau^n\|_{U'}\big)(s/\tau) \qquad t^n \leq t \leq t^n + s.\end{aligned}$$

Inequality in (21) with $m = 1$ then gives

$$\begin{aligned}\mathbb{E}\Big[\int_\tau^T \|\hat{u}(t) - \hat{u}(t - s)\|_{U'}^p \, dt\Big] &\leq C(p)(s/\tau)^p \mathbb{E}\Big[\sum_{n=1}^N \tau \|u_\tau^n - u_\tau^{n-1}\|_{U'}^p\Big] \\ &\leq C(p, f, g)\tau^{1+\theta p}(s/\tau)^p \leq C(p, f, g)s^{1+\theta p},\end{aligned}$$

where the last inequality holds since $1 + \theta p \leq p$ when $p \geq 2$, and $s/\tau \leq 1$. □

5.2 Compactness

The Prokhorov theorem, stated next, will be used to establish convergence of the laws of the solutions to the implicit Euler equation (11). The key hypothesis of this theorem requires a sequence of probability measures $\{\mathbb{P}_n\}_{n=1}^\infty$ on a topological space $\mathbb{X}$ endowed with its Borel σ-algebra $\mathcal{B}(\mathbb{X})$ to be tight (see Definition 2.9).

Theorem 5.5 (Prokhorov) *Let $\mathbb{X}$ be a topological space with the property that there exists a countable family of real-valued continuous functions which separates points of $\mathbb{X}$. Let $\{\tilde{\mathbb{P}}_n\}_{n=1}^\infty$ be a tight sequence of probability measures on $\mathbb{X}$ with its Borel σ-algebra $\mathcal{B}(\mathbb{X})$. Then there exist a subsequence $\{\tilde{\mathbb{P}}_{n_k}\}_{k\in\mathbb{N}}$ and a probability measure $\tilde{\mathbb{P}}$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ for which $\tilde{\mathbb{P}}_{n_k} \Rightarrow \tilde{\mathbb{P}}$.*

If the probability measures in this theorem are the laws of random variables $\tilde{\mathbb{P}}_n = \mathcal{L}(X_n)$ taking values in $\mathbb{X}$, and $\mathcal{L}(X_{n_k}) \Rightarrow \tilde{\mathbb{P}}$ we can write $\tilde{\mathbb{P}} = \mathcal{L}(X)$ where $X : \mathbb{X} \to \mathbb{X}$ is the identity function identified as a random variable on $\tilde{\Omega} \equiv (\mathbb{X}, \mathcal{B}(\mathbb{X}), \tilde{\mathbb{P}})$.

Below we will set

$$\mathbb{X} = G[0, T; U'] \cap L^r[0, T; U]_{weak} \times L^{q'}[0, T; U']_{weak} \times L^2[0, T; U'] \times C[0, T],$$

and the probabilities in the Prokhorov theorem to be the laws of $X_{h\tau} = (u_{h\tau}, F_{h\tau}, g_{h\tau}, \hat{W}_\tau)$. Recall that $\hat{W}_\tau$ denotes the piecewise linear interpolant of (6), and $L^r[0, T; U]_{weak}$ and $L^{q'}[0, T; U']_{weak}$ denote the indicated spaces endowed with the weak topology. When U is separable and reflexive and $1 < r, q' < \infty$, classical results from functional analysis can be used to exhibit a countable family of real-valued functions on $\mathbb{X}$ which separate points.

A convenient way to establish compactness of piecewise constant functions in $G[0, T; U']$ is to use the Arzela–Ascoli theorem to show that their corresponding piecewise linear interpolants are compact in $C[0,T;U']$. The following lemma makes this precise, and also shows that the laws concentrate on $C[0,T;U']$.

Lemma 5.6 *Let U be a separable reflexive Banach space, H a Hilbert space, $U \hookrightarrow H \hookrightarrow U'$ be dense embeddings, and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For $n = 1, 2, \ldots$, let $\{u_n^i\}_{i=0}^n$ be U-valued processes, and define their càglàd and piecewise linear interpolants on $[0,T]$ by*

$$u_n = u_n^0 \mathbf{1}_{\{0\}} + \sum_{i=0}^{n-1} u_n^i \mathbf{1}_{(t_n^i, t_n^{i+1}]} \quad \textit{and}$$

$$\hat{u}_n(t) = \frac{t_n^{i+1} - t}{t_n^{i+1} - t_n^i} u_n^i + \frac{t - t_n^i}{t_n^{i+1} - t_n^i} u_n^{i+1} \quad \textit{for} \quad t \in [t_n^i, t_n^{i+1}],$$

where $t_n^i = iT/n$.

If $\{\mathcal{L}(\hat{u}_n)\}_{n=1}^\infty$ is tight on $C[0,T;U']$ and $\{\mathcal{L}(u_n)\}_{n=1}^\infty$ is tight on $L^r_{weak}[0,T;U]$, then $\{\mathcal{L}(u_n)\}_{n=1}^\infty$ is tight on $G[0,T;U'] \cap L^r_{weak}[0,T;U]$, and if μ is any accumulation point of $\{\mathcal{L}(u_n)\}_{n=1}^\infty$ on $G[0,T;U'] \cap L^r_{weak}[0,T;U]$ then

$$\mu(C[0,T;U'] \cap L^r[0,T;U]) = 1.$$

Note that the Borel subsets of $L^r[0,T;U]_{weak}$ and $L^r[0,T;U]$ coincide since U is separable. This lemma, which we prove in the Appendix, is used to establish tightness of solutions to the numerical scheme.

Theorem 5.7 *Let U be a separable reflexive Banach space, H a Hilbert space, and $U \hookrightarrow\!\!\!\!\rightarrow H$ be a compact, dense embedding, and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Assume that the spaces, data, and increments of the scheme (11) satisfy Assumptions 3.1 and 2.5 with $p \in (2, \infty)$, and that the initial data $\{u^0_{h\tau}\}$ are bounded in $L^p(\Omega, H)$ and converge in $L^2(\Omega, H)$ to a limit u^0 as $(h, \tau) \to (0, 0)$.*

Let $u_{h\tau}$ denote the piecewise constant càglàd interpolant of $\{u^n_{h\tau}\}_{n=0}^N$ in time and assume for some $1 < q, r < \infty$ that

1. *$\{\|u_{h\tau}\|_{L^p(\Omega, L^r[0,T;U])}\}_{h,\tau>0}$ is bounded.*
2. *$\{\|F_{h\tau}\|_{L^p(\Omega, L^{q'}[0,T;U'])}\}_{h,\tau>0}$ is bounded.*
3. *$\{\|g_{h\tau}\|_{L^p(\Omega, L^p[0,T;H])}\}_{h,\tau>0}$ is bounded.*

Then the laws of $\{(u_{h\tau}, F_{h\tau}, g_{h\tau}, \hat{W}_\tau)\}_{h,\tau>0}$ are tight on

$$\mathbb{X} = G[0,T;U'] \cap L^r[0,T;U]_{weak} \times L^{q'}[0,T;U']_{weak} \times L^2[0,T;H]_{weak} \times C[0,T].$$

In addition,

- *If $\{g_{h\tau}\}_{h,\tau>0}$ is Cauchy in $L^p(\Omega, L^2[0,T;H])$ then the laws $\{\mathcal{L}(g_{h\tau})\}_{h,\tau>0}$ are tight on $L^2[0,T;H]$.*
- *The piecewise linear interpolants $\{\hat{u}_{h\tau}\}_{h,\tau>0}$ are tight in $C[0,T;U'] \cap L^r[0,T;U]_{weak}$.*

- *If additionally $V \hookrightarrow U'$ is a separable reflexive Banach space and $\{u_{h\tau}\}_{h,\tau>0}$ is bounded in $L^p(\Omega, L^s[0,T;V])$ for some $1 < s < \infty$, then the laws $\{\mathcal{L}(u_{h\tau}\}_{h,\tau>0}$ are tight on*

$$G[0,T;U'] \cap L^r[0,T;U]_{weak} \cap L^s[0,T;V]_{weak}.$$

If $U \hookrightarrow\!\!\!\rightarrow V$ is compact and $1 \le \hat{s} < s$, then the laws are tight on

$$G[0,T;U'] \cap L^r[0,T;U]_{weak} \cap L^{\hat{s}}[0,T;V].$$

Proof To establish tightness for the laws we exhibit large compact sets in each of the factor spaces of $\mathbb{X}$.

- If $U \hookrightarrow\!\!\!\rightarrow H$ then $U \hookrightarrow\!\!\!\rightarrow H \hookrightarrow\!\!\!\rightarrow U'$, and for $\theta > 0$

$$C^{0,\theta}[0,T;U'] \cap L^r[0,T;U] \hookrightarrow\!\!\!\rightarrow C[0,T;U'].$$

Fix $0 < \theta < \min(1/2 - 1/p, 1/q)$ and let

$$K_\epsilon = \{\hat{u} \in C^{0,\theta}[0,T;U'] \mid \|\hat{u}\|^p_{L^r[0,T;U]} \le 1/\epsilon \;\text{ and }\; \|\hat{u}\|^p_{C^{0,\theta}[0,T;U']} \le 1/\epsilon\}.$$

Then

$$\begin{aligned}\mathcal{L}(\hat{u}_{h\tau})[C[0,T;U'] \setminus K_\epsilon] &= \mathbb{P}\left[\{\omega \in \Omega \mid \hat{u}_{h\tau} \notin K_\epsilon\}\right] \\ &\le \mathbb{P}\left[\{\omega \in \Omega \mid \|\hat{u}_{h\tau}\|^p_{L^r[0,T;U]} > 1/\epsilon \;\text{ or }\; \|\hat{u}_{h\tau}\|^p_{C^{0,\theta}[0,T;U']} > 1/\epsilon\}\right] \\ &\le C\left(\|\hat{u}_{h\tau}\|^p_{L^p(\Omega,L^r[0,T;U])} + \|\hat{u}_{h\tau}\|^p_{L^p(\Omega,C^{0,\theta}[0,T;U'])}\right)\epsilon,\end{aligned}$$

where the last line follows from Chebyshev's inequality. The hypotheses assumed upon the data and Theorem 5.3 bound the two norms in the last expression independently of h and τ which shows $\mathcal{L}(\hat{u}_{h\tau})[K_\epsilon] \ge 1 - C\epsilon$, and tightness on $C[0,T;U']$ follows.
If $U \hookrightarrow\!\!\!\rightarrow V$ then $L^r[0,T;U] \cap C^{0,\theta}[0,T;U'] \hookrightarrow\!\!\!\rightarrow L^r[0,T;V]$ and the same argument shows that $\{\mathcal{L}(\hat{u}_{h\tau})\}_{h,\tau>0}$ are tight in $L^r[0,T;V]$. The mapping $\hat{u}_{h\tau} \mapsto u_{h\tau}$ is continuous on $L^r[0,T;V]$, so maps compact sets to compact sets, so $\{\mathcal{L}(u_{h\tau})\}_{h,\tau>0}$ is also tight on $L^r[0,T;V]$.
If $1 \le r < s$ and $K \subset L^r[0,T;V]$ is compact, then $K \cap L^s[0,T;V]$ is compact in $L^{\hat{s}}[0,T;V]$ for $1 \le \hat{s} < s$. Thus if $\{u_{h\tau}\}_{h,\tau>0}$ is also bounded in $L^p(\Omega, L^s[0,T;V])$ the laws are also tight in $L^{\hat{s}}[0,T;V]$.
- Since U is reflexive and $1 < q < \infty$ the Banach–Alaoglu theorem shows

$$L^{q'}[0,T;U']_{strong} \hookrightarrow\!\!\!\rightarrow L^{q'}[0,T;U']_{weak}.$$

Letting K_ϵ be the closed ball in $L^{q'}[0,T;U']$ centered at the origin with radius $1/\epsilon$, Chebyshev's inequality shows

$$\begin{aligned}\mathcal{L}(F_{h\tau})[L^{q'}[0,T;U'] \setminus K_\epsilon] &= \mathbb{P}\Big[\{\omega \in \Omega \mid \|F_{h\tau}\|_{L^{q'}[0,T;U']} > 1/\epsilon\}\Big] \\ &\le \|F_{h\tau}\|_{L^p(\Omega, L^{q'}[0,T;U'])}\epsilon.\end{aligned}$$

Since closed balls in $L^{q'}[0,T;U']$ are weakly compact, and $\|F_{h\tau}\|_{L^p(\Omega,L^{q'}[0,T;U'])}$ is bounded independently of k, it follows that $\{\mathcal{L}(F_{h\tau})\}_{h,\tau>0}$ is tight in $L^{q'}[0,T;U']_{weak}$.
The same argument shows $\{\mathcal{L}(\hat{u}_{h\tau})\}_{h,\tau>0}$ is tight in $L^r[0,T;U]_{weak}$, and is also tight in $L^s[0,T;V]_{weak}$ when $\{u_{h\tau}\}_{h,\tau>0}$ is bounded in $L^p(\Omega, L^s[0,T;V])$. The previous lemma then shows that the laws of $\{u_{h\tau}\}_{h,\tau>0}$ are tight on $G[0,T;U'] \cap L^r[0,T;U]_{weak}$.

- The laws of a strongly convergent sequence in $L^p(\Omega;X)$ are always tight and converge weakly to the limit. In particular, if $\{g_{h\tau}\}_{h,\tau>0}$ is Cauchy in $L^2(\Omega, L^2[0,T;U'])$, then $\{\mathcal{L}(g_{h\tau})\}_{h,\tau>0}$ are tight and converge weakly to $\mathcal{L}(g)$.
- The discrete Wiener process $\hat{W}_\tau$ interpolating $\{W_\tau^n\}_{n=0}^N$ is Hölder continuous. Briefly, from Lemma 2.8 (with $H=\mathbb{R}$ and $g^m=1$) it follows that

$$\begin{aligned}\mathbb{E}\Big[\sum_{n=m}^N \tau |W_\tau^n - W_\tau^{n-m}|^p\Big] &= \mathbb{E}\Big[\sum_{n=m}^N \tau \left|\sum_{k=n-m+1}^n \xi_\tau^k\right|^p\Big] \\ &\le C\sum_{n=m}^N \tau (m\tau)^{p/2}.\end{aligned}$$

Since $p>2$ the Kolmogorov-Centsov Theorem 5.2 bounds the expected value of the Hölder norm of $\hat{W}_\tau$ with exponent $\theta < \min(1/2 - 1/p, 1)$. Tightness then follows from the Arzella-Ascolli theorem since $C^{0,\theta}[0,T]$ is compactly embedded in $C[0,T]$. □

5.3 Convergence: Proof of Theorem 3.2

This section establishes convergence along subsequences of solutions of the numerical scheme (11) to a weak martingale solution of the problem (7) which we write as $du = F\,dt + g\,dW$. Convergence is established using the Prokhorov theorem to construct a measure $\tilde{\mathbb{P}}$ on the product space

$$\tilde{\Omega} = G[0,T;U'] \cap L^r[0,T;U] \times L^{q'}[0,T;U'] \times L^2[0,T;H] \times C[0,T], \tag{22}$$

and a filtration for which the projections

$$
\begin{aligned}
u : \tilde{\Omega} \to C[0,T;U'] \cap L^r[0,T;U], \quad F : \tilde{\Omega} \to L^{q'}[0,T;U'], \\
g : \tilde{\Omega} \to L^2[0,T;U'], \quad W : \tilde{\Omega} \to C[0,T],
\end{aligned}
$$

defined by

$$
u(\tilde{\omega}) = \tilde{\omega}_1, \quad F(\tilde{\omega}) = \tilde{\omega}_2, \quad g(\tilde{\omega}) = \tilde{\omega}_3, \quad W(\tilde{\omega}) = \tilde{\omega}_4, \quad \text{with} \quad \tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_4), \tag{23}
$$

are random variables satisfying (7). To verify that these variables are a solution, independence properties of the approximating scheme are used to show that

$$
X(t) \equiv u(t) - u^0 - \int_0^t F\,ds,
$$

is a martingale with respect to the filtration generated by (u, F, g, W). The final step is to verify that W is a Wiener process and $X(t) = \int_0^t g\,dW$.

The following lemma, which characterizes when one process is independent of the filtration generated by another, is useful in this context.

Lemma 5.8 *Let $\{\mathbb{X}_t\}_{t=0}^T$ be topological spaces, $\{Y(t)\}_{t=0}^T$ be $\mathbb{X}_t$-valued Borel measurable random variables, and let $\{\mathcal{F}_t\}_{t=0}^T$ be the filtration given by $\mathcal{F}_t = \sigma(Y(s) : 0 \le s \le t)$. An integrable process $\{X(t)\}_{t=0}^T$ adapted to this filtration taking values in a separable Banach space $\mathbb{X}$ is a martingale with respect to the filtration if and only if*

$$
\mathbb{E}\left[\Big(X(t) - X(s)\Big) \prod_{i=1}^m \prod_{j=1}^n \phi_{ij}\Big(\psi_{ij}(Y(s_j))\Big)\right] = 0
$$

holds for all times $0 \le s_1 < \ldots < s_m \le s < t \le T$, all $\phi_{ij} \in C_b(\mathbb{R})$, and for all $\psi_{1j}, \ldots, \psi_{nj} \in \mathcal{A}_{s_j}$, where $\mathcal{A}_s$ is a subset of real-valued functions on $\mathbb{X}_s$ for which $\sigma(\mathcal{A}_s) = \mathcal{B}(\mathbb{X}_s)$ (the Borel σ-algebra on $\mathbb{X}_s$).

The Dynkin lemma shows that the criteria in this lemma are equivalent to $\mathbb{E}[X(t) - X(s) \mid \mathcal{F}_s] = 0$.

Example 5.9 In the proof below

$$
\mathbb{X}_t = G[0,t;U'] \cap L^r[0,t;U]_{weak} \times L^{q'}[0,t;U']_{weak} \times L^2[0,t;H] \times C[0,t].
$$

A countable set of continuous functions, $\mathcal{A}_t$, generating $\mathcal{B}(\mathbb{X}_t)$ is

$$
(u, F, g, W) \mapsto z_1 \int_a^b \big(u(s), v\big)\,ds + z_2 \int_a^b \big(F(s), v\big)\,ds
$$

$$+z_3 \int_a^b \big(g(s), v\big)\, ds + z_4 \int_a^b W(s)\, ds,$$

for $a < b$ in $[0, t] \cap \mathbb{Q}$, v in a dense subset of U, and $z_1, z_2, z_3, z_4 \in \{0, 1\}$.

Proof (of Theorem 3.2)

1. Lemma 5.1 and Theorem 5.3 show that $\{\|u_{h\tau}\|_{L^p(\Omega, L^\infty[0,T;H])}\}_{h,\tau>0}$ and $\{\|\hat{u}_{h\tau}\|_{L^p(\Omega, C^{0,\theta}[0,T;U'])}\}_{h,\tau>0}$ with $0 < \theta < \min(1/2 - 1/p, 1/q)$ are bounded.
2. Let

$$\mathbb{X} = G[0,T;U'] \cap L^r[0,T;U]_{weak} \times L^{q'}[0,T;U']_{weak} \times L^2[0,T;H]_{weak} \times C[0,T],$$

and $(\tilde{\Omega}, \tilde{\mathcal{F}}) = (\mathbb{X}, \mathcal{B}(\mathbb{X}))$ be the corresponding measurable space endowed with the Borel σ-algebra. Let $\tilde{\mathbb{P}}_{hk}$ denote the law of $(u_{hk}, F_{hk}, g_{hk}, \hat{W}_\tau)$; that is

$$\tilde{\mathbb{P}}_{h\tau}[B_1 \times B_2 \times B_3 \times B_4] := \mathbb{P}\big[(u_{h\tau} \in B_1) \wedge (F_{h\tau} \in B_2) \wedge (g_{h\tau} \in B_3) \wedge (\hat{W}_\tau \in B_4)\big]$$

for

$$\begin{aligned} &B_1 \in \mathcal{B}(G[0,T;U'] \cap L^r[0,T;U]),\ B_2 \in \mathcal{B}(L^{q'}[0,T;U']),\\ &B_3 \in \mathcal{B}(L^2[0,T;H]),\ B_4 \in \mathcal{B}(C[0,T]). \end{aligned}$$

Theorem 5.7 shows that the measures $\{\tilde{\mathbb{P}}_{h\tau}\}_{h,\tau>0}$ form a tight family, so by the Prokhorov theorem we may pass to a subsequence $(h_k, \tau_k) \to (0,0)$ for which $\tilde{\mathbb{P}}_{h_k\tau_k} \Rightarrow \tilde{\mathbb{P}}$, and Lemma 5.6 shows that

$$\tilde{\mathbb{P}}\Big[\big\{(u, F, g, W) \mid u \in C[0,T;U'] \cap L^r[0,T;U]\big\}\Big] = 1.$$

Below we write $\tilde{\mathbb{P}}_k = \tilde{\mathbb{P}}_{h_k\tau_k}$, $u_k = u_{h_k\tau_k}$ etc.
3. If $\{g_{h\tau}\}_{h,\tau>0}$ is tight in $L^2[0,T;H]$, set

$$\mathbb{X} = G[0,T;U'] \cap L^r[0,T;U]_{weak} \times L^{q'}[0,T;U']_{weak} \times L^2[0,T;H] \times C[0,T],$$

and, since the Borel subsets of $L^r[0,T;U]$ and $L^{q'}[0,T;U']$ with the weak and strong topologies coincide, repeat the argument in the previous step to assert $\mathbb{P}_k \Rightarrow \tilde{\mathbb{P}}$ with the stronger topology. We show that the equation (12) holds on the support of $\tilde{\mathbb{P}}$.
For each $0 < t \le T$, let $X(t) : \mathbb{X} \to U'$ be the function $X(t)(\omega) = u(t) - u(0) - \int_0^t F\, ds$ where $\omega \equiv (u, F, g, W)$. We construct a filtration of $(\tilde{\Omega}, \tilde{\mathcal{F}})$ for which $X(t)$ is a square integrable martingale.
For $v \in U$ and $0 \le t \le T$ fixed we first verify that $(X(t), v)$ is square integrable on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and show

$$\int_{\mathbb{X}} (X(t), v)\, d\tilde{\mathbb{P}} = \lim_{k\to\infty} \mathbb{E}\left[\left(u_k^{n_k} - u_k^0, v\right)_H - \int_0^t (F_k, v)\, ds\right],$$

when $0 \leq n_k\tau_k - t < \tau$ with $n_k \in \mathbb{N}$, *i.e.*, $u_k(t) = u_k^{n_k}$ as in Fig. 2.
To do this, define $\zeta : \mathbb{X} \to \mathbb{R}$ by $\zeta(u, F, g, W) = (X(t), v)$. Since the mapping $u \mapsto u(t)$ is Borel on $G[0, T; U']$, and the coordinate projections ($(u, F, g, W) \mapsto u$, $(u, F, g, W) \mapsto F$, etc.) are continuous, it follows that $X(t)$, and hence ζ is Borel measurable. Set

$$N = \Big\{(u, F, g, W) \in \mathbb{X} \mid \exists(\bar{u}_k, \bar{F}_k, \bar{g}_k, \bar{W}_k) \to (u, F, g, W) \\ \text{such that } \zeta(\bar{u}_k, \bar{F}_k, \bar{g}_k, \bar{W}_k) \nrightarrow \zeta(u, F, g, W)\Big\}.$$

Claim N has null outer $\tilde{\mathbb{P}}$ measure, $\tilde{\mathbb{P}}^*[N] = 0$.

Proof • If $\bar{F}_k \to F$ in $L^{q'}[0, T; U']_{weak}$ it is immediate that $\int_0^t (\bar{F}_k, v)\, ds \to \int_0^t (F, v)\, ds$.
• If $\bar{u}_k \to u$ in $G[0, T; U'] \cap L^r[0, T; U]_{weak}$ then

$$(\bar{u}_k(t), v)_H \nrightarrow (u(t), v)_H \qquad \Rightarrow \qquad u \notin C[0, T; U'].$$

From Lemma 5.6 we conclude $\tilde{\mathbb{P}}^*[N] \leq \tilde{\mathbb{P}}^*\big[\big\{(u, F, g, W) \mid u \notin C[0, T; U'] \cap L^r[0, T; U]\big\}\big] = 0$. □

Since

$$|\zeta(u_k, F_k, g_k, \hat{W}_k)| \leq \Big(2\|u_k\|_{L^\infty[0,T;U']} + \|F_k\|_{L^{q'}[0,T,U']}\, t^{1/q}\Big)\, \|v\|_U,$$

and since Lemma 5.1 bounds the p^{th} moment of the right-hand side with $p > 2$, it follows from Lemma 2.10 (with $\zeta_k = \zeta$) that $(X(t), v)$ is square integrable and

$$\int_{\tilde{\Omega}} (X(t), v)\, d\tilde{\mathbb{P}} = \int_{\mathbb{X}} \zeta\, d\tilde{\mathbb{P}} \\ = \lim_{k\to\infty} \int_{\mathbb{X}} \zeta\, d\tilde{\mathbb{P}}_k \equiv \lim_{k\to\infty} \mathbb{E}\left[(u_k^{n_k} - u_k^0, v)_H - \int_0^t (F_k, v)\, ds\right].$$

Next, let $\{\tilde{\mathcal{F}}(t)\}_{t=0}^T$ be the coarsest filtration on $\tilde{\Omega}$ for which each of the mappings

$$\tilde{\Omega} \mapsto G[0, t; U'] \cap L^q[0, T; U]_{weak} \times L^{q'}[0, t; U']_{weak} \times L^2[0, t; H] \times C[0, t] \equiv \mathbb{X}_t,$$

given by

$$(u, F, g, W) \mapsto (u|_{[0,t]}, F_{[0,t]}, g_{[0,t]}, W|_{[0,t]}), \qquad 0 \leq t \leq T,$$

is a measurable map $(\tilde{\Omega}, \tilde{\mathcal{F}}) \to (\mathbb{X}_t, \mathcal{B}(\mathbb{X}_t))$.

Claim For $v \in U$ fixed, the real-valued random variable $(X(t), v) = (u(t) - u(0), v)_H - \int_0^t (F, v)\,ds$ is a martingale on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}(t)\}_{0\le t\le T}, \tilde{\mathbb{P}})$.

Proof Fix $0 \le s_1 < \ldots < s_m \le s < t \le T$, $\phi_{ij} \in C_b(\mathbb{R})$ and let $\psi_{1j}, \ldots, \psi_{nj}$ be functions in the generating set of $\mathcal{B}(\mathbb{X}_{s_j})$ given in Example 5.9. Then let $\phi \in C_b(\mathbb{X}, \mathbb{R})$ be the function

$$\phi(u, F, g, W) = \prod_{i=1}^{m} \prod_{j=1}^{n} \phi_{ij}\big(\psi_{ij}(u, F, g, W)\big),$$

and let $v_k \to v$ with $v_k \in U_{h_k}$. Define $\zeta, \zeta_k : \mathbb{X} \to \mathbb{R}$ to be the functions

$$\zeta(u, F, g, W) = \big(X(t) - X(s), v\big)\phi \text{ and } \zeta_k(u, F, g, W) = \big(X(t) - X(s), v_k\big)\phi.$$

If

$$N = \Big\{(u, F, g, W) \in \mathbb{X} \mid \exists(\bar{u}_k, \bar{F}_k, \bar{g}_k, \bar{W}_k) \to (u, F, g, W) \\ \text{such that } \zeta_k(\bar{u}_k, \bar{F}_k, \bar{g}_k, \bar{W}_k) \nrightarrow \zeta(u, F, g, W)\Big\},$$

then, as above, $\tilde{\mathbb{P}}^*[N] = 0$ and ξ and ξ_k have moments of order $p > 2$. Lemma 2.10 then gives

$$\begin{aligned}\int_{\tilde{\Omega}} (X(t) - X(s), v)\phi\, d\tilde{\mathbb{P}} &= \int_{\tilde{\Omega}} \zeta\, d\tilde{\mathbb{P}} = \lim_{k\to\infty} \int_{\tilde{\Omega}} \zeta_k\, d\tilde{\mathbb{P}}_k \\ &\equiv \lim_{k\to\infty} \mathbb{E}\left[(u_k^{n_k} - u_k^{m_k} - \int_s^t F_k\, dr, v_k)\phi\right] \\ &= \lim_{k\to\infty} \mathbb{E}\left[(u_k^{n_k} - u_k^{m_k} - \int_{t^{m_k}}^{t^{n_k}} F_k\, dr, v_k)\phi\right] \\ &\quad - \lim_{k\to\infty} \mathbb{E}\left[\Big(\int_s^{t^{m_k}} F_k\, dr - \int_t^{t^{n_k}} F_k\, dr, v_k\Big)\phi\right],\end{aligned} \tag{24}$$

where $0 \le n_k\tau_k - t < \tau_k$ and $0 \le m_k\tau_k - s < \tau_k$ since $u_k(t) = u_k(n_k\tau_k)$ and $u_k(s) = u_k(m_k\tau_k)$; see Fig. 2(left).

We verify that each term on the right-hand side vanishes to conclude from Lemma 5.8 that increments of X are independent and X is a martingale.

Summing each side of the Euler scheme (11) shows

$$(u_k^{n_k}, v_k) = (u_k^{m_k}, v_k) + \tau \sum_{j=m_k+1}^{n_k} (F_k^j, v_k) + \sum_{j=m_k+1}^{n_k} (g_k^{j-1}, v_k)_H \xi_k^j.$$

Multiplying this equation by ϕ and rearranging gives

$$\mathbb{E}\left[(u_k^{n_k} - u_k^{m_k} - \int_{t^{m_k}}^{t^{n_k}} F_k\, dr, v_k)\phi\right] = \sum_{j=m_k+1}^{n_k} \mathbb{E}\left[(g_k^{j-1}, v_k)_H \phi \xi_k^j\right]$$
$$= \sum_{j=m_k+1}^{n_k} \mathbb{E}\left[(g_k^{j-1}, v_k)_H \phi\right] \mathbb{E}[\xi_k^j] = 0,$$

where the last two steps follow since ϕ is $\mathcal{F}(t^{m_k})$ measurable, $\mathcal{F}(t^{m_k}) \subset \mathcal{F}(t^{j-1})$ when $j \geq m_k + 1$, and ξ_k^j is independent of $\mathcal{F}(t^{j-1})$ with zero average.

The last term in (24) vanishes since $F_k \in L^1(\Omega, L^{q'}[0, T; U'])$ with $1 < q < \infty$,

$$\mathbb{E}\left[\int_s^{t^{m_k}} \|F_k\|_{U'}\, dr + \int_t^{t^{n_k}} \|F_k\|_{U'}\, dr\right]$$
$$\leq \mathbb{E}\left[\|F_k\|_{L^{q'}[0,T;U']}\right] \left(|t^{m_k} - s|^{1/q} + |t^{n_k} - t|^{1/q}\right) \to 0.$$

□

The arguments used here can be repeated to show that for each $v \in U$ the processes

$$W(t), \quad W^2(t) - t, \quad (X(t), v)^2 - \int_0^t (g(s), v)^2\, ds, \quad \text{and}$$
$$(X(t), v)W(t) - \int_0^t (g(s), v)\, ds,$$

are also real-valued martingales on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}(t)\}_{t=0}^T)$. The Martingale Representation Theorem 2.6 then shows that W is a real-valued Wiener process and

$$(u(t), v) = (u^0, v) + \int_0^t (F(s), v)\, ds + \int_0^t (g, v)\, dW, \qquad 0 \leq t \leq T,$$

holds $\tilde{\mathbb{P}}$–a.s. for every $v \in U$. Moreover, since paths of W are continuous, W is also a Wiener process for the augmentation of $\{\tilde{\mathcal{F}}(t)\}_{0\leq t\leq T}$ satisfying the usual conditions, so (u, F, g, W) is also a weak martingale solution with respect to the augmented filtration.

Finally, since $u_k^0 = u_k(0)$, the map $u \mapsto u(0)$ is continuous on $G[0, T; U']$, and the initial data is assumed to converge and has moments of order $p > 2$, it follows that $\mathcal{L}(u_k^0) \Rightarrow \mathcal{L}(u(0))$ on U'.

4. Assumption 3.1_5 was required in the previous step to assert that for each $v \in U$ there existed a sequence $v_k \in U_{h_k}$

$$\begin{aligned}\xi(u, F, g, W) &\equiv (X(t) - X(s), v)\phi \\ &= \lim_{k\to\infty} (X(t) - X(s), v_k)\phi \equiv \lim_{k\to\infty} \xi_k(u, F, g, W).\end{aligned}$$

This construction is still possible when the assumption is relaxed to (5') provided $v \in U_0$.

5. Theorem 5.7 shows that if $\{u_{h\tau}\}_{h,\tau>0}$ is bounded in $L^s[0,T;V]$ then it is also tight in $L^s[0,T;V]_{weak}$, and if $U \hookrightarrow\!\!\!\rightarrow V$ then it is also tight in $L^{\hat{s}}[0,T;V]$ for $1 \le \hat{s} < s$, in which case it is possible to pass to a subsequence for which $\mathcal{L}(u_{h_k\tau_k}) \Rightarrow \mathcal{L}(u)$ with the stronger topologies.
6. If $F_{h\tau} = \sum_{\ell=1}^{L} F_{h\tau}^{(\ell)}$ and each summand satisfies the second hypothesis of the theorem then so too does that $F_{h\tau}$. In addition, Theorem 5.7 shows that each summand is tight in $L^{q'}[0,T;U']_{weak}$ (and $L^{q'}[0,T;U']$ if it is convergent). Passing to a subsequence for which $\mathcal{L}(F_{h\tau}^{(\ell)}) \Rightarrow \mathcal{L}(F^{(\ell)})$ we have $\mathcal{L}(F_{h\tau}) \Rightarrow \mathcal{L}(\sum_{\ell=1}^{L} F^{(\ell)})$ since addition is continuous under weak convergence. □

5.3.1 Infinite–dimensional Wiener process

The existence theory for parabolic SPDE's of the form (7) extends to the situation where the noise is a Wiener (resp. cylindrical Wiener) process W in a separable Hilbert space K. In this situation g takes values in $\mathcal{L}(K,H)$, the continuous linear functions from K to the pivot space H. Upon introducing an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ for K, SPDE's with this noise satisfy

$$\begin{aligned}\big(u(t),v\big)_H &+ \int_0^t a(u,v)\,ds = (u^0,v)_H + \int_0^t (f,v)\,ds\\ &+ \sum_{j=1}^{\infty}\int_0^t (g_j,v)_H\,dW_j, \quad v \in U,\end{aligned} \tag{25}$$

where where $g_j = g(e_j)$, and $W_j = (W,e_j)_K$ (resp. $W_j = W(e_j)$) are standard real-valued independent Wiener processes. The space $\mathbb{X}$ in Theorem 3.2 then has the form

$$\begin{aligned}\mathbb{X} &\equiv G[0,T;U'] \cap L^q[0,T;U]_{weak} \times L^{q'}[0,T;U']_{weak} \times L^2[0,T;H]_{weak}^{\mathbb{N}}\\ &\quad \times C[0,T]^{\mathbb{N}},\end{aligned}$$

and it is necessary to assume that the limits g_j of the approximating sequences $g_{j,h\tau}$ as $(h,\tau) \to (0,0)$ satisfy

$$\sum_{j=1}^{\infty}\int_0^T |(g_j,v)|^2\,ds < \infty \quad \text{a.s. for every } v \in U,$$

so that the series in (25) converges.

6 Examples

In this chapter we present three examples that illustrate the applicability of the convergence theory for parabolic systems that exhibit distinctly different structural properties. In the first instance we consider the incompressible Navier–Stokes equation driven by multiplicative noise which has the structure of a diffusion equation. The stability estimate for this class of problems follows upon multiplying the equation by the solution itself. The second example is a gradient flow for which the spatial operator is the gradient of a (typically non–convex) stored energy function, $I(u)$. In the deterministic setting stability follows upon multiplying the equation by the time derivative of the solution. However, in the stochastic setting this is not possible, so it is necessary to multiply the equation by $A(u)$ instead. The final example considers the situation where A is a maximal monotone operator.

6.1 Structural properties

In this section we review how structural properties of the spatial operators give rise to specific bounds upon the solution. Following this, we recall a convenient statement of the Brouwer fixed point theorem which is used ubiquitously in the deterministic setting to establish existence of solutions to the discrete problems. Since solutions of the nonlinear problems may not be unique, in the stochastic setting it is necessary to establish the existence of a measurable selection.

6.1.1 Bounding solutions

Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $T > 0$, and $f \in L^2[0,T; L^2(D)]$ be given. The classical heat equation with Neumann boundary data,

$$\partial_t u - \Delta u = f, \quad \text{in } (0,T) \times D, \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial D} = 0, \tag{26}$$

has the structure of both, a classical diffusion equation and a gradient flow. Multiplying by u and integrating shows

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(D)} + \|\nabla u\|^2_{L^2(D)} = (f, u),$$

while multiplying by $\partial_t u$ gives

$$\|\partial_t u\|^2_{L^2(D)} + \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2(D)} = (f, \partial_t u).$$

When a stochastic term is included on the right-hand side of (26), there is a loss of temporal regularity and the scalar product of $\partial_t u$ and the stochastic term can not be bounded. Since the spatial regularity is not degraded to the same extent, it is frequently possible to multiply the equation by the variational derivative of the energy. The energy for the heat equation is $I(u) = (1/2)\|\nabla u\|^2_{L^2(D)}$, and multiplying equation (26) by $\delta I(u)/\delta u = -\Delta u$ gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2(D)} + \|\Delta u\|^2_{L^2(D)} = -(f, \Delta u).$$

The second problem that we present in Section 6.3 has this structure as well and, in addition, the solution takes values in a manifold. In this instance the PDE can be viewed as an equation on the tangent space so the stochastic term needs to be restricted appropriately; this results in Stratonovich noise.

The numerical schemes will satisfy an estimate of the form

$$I(u^n_{h\tau}) + \frac{1}{2}\|u^n_{h\tau} - u^{n-1}_{h\tau}\|^2_H + \tau\|a^n_{h\tau}\|^q_U \le I(u^{n-1}_{h\tau}) + \tau(f^n_{h\tau}, a^n_{h\tau}) + (g^{n-1}_\tau, u^n_{h\tau})\xi^n_\tau, \tag{27}$$

where the energy $I(u)$ is non–negative, $a^n_{h\tau}$ is (a discrete approximation of) the variational derivative, and $g^{n-1}_{h\tau}$ may depend upon $u^{n-1}_{h\tau}$. The following mild generalization of Lemma 4.2 establishes bounds upon the solution. To accommodate examples of gradient flows, such as the heat equation where $\|u\|_H = \|\nabla u\|_{L^2(D)}$, the pairing $(.,.)_H$ is only assumed to be a semi–inner product.

Lemma 6.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $U \hookrightarrow H$ be an embedding of a normed linear space into a semi–inner product space H. Suppose that $I : U \to \mathbb{R}$ is continuous and satisfies $\|u\|^2_H \le I(u)$ for $u \in U$. Let Assumptions 2.5 and 4.1 hold, and inequality (27) be satisfied with random variables for which:*

- *$\{f^n_{h\tau}\}_{n=1}^N$ takes values in U' and $\{a^n_{h\tau}\}_{n=1}^N$ takes values in U.*
- *$\{u^n_{h\tau}\}_{n=0}^N$ takes values in U_h and is adapted to the filtration $\{\mathcal{F}^n\}_{n=0}^N$.*
- *$\{g^n_{h\tau}\}_{n=0}^{N-1}$ takes values in H and is adapted to the filtration $\{\mathcal{F}^n\}_{n=0}^N$, and there exists a constant $C > 0$ such that $\|g^{n-1}_{h\tau}\|_H \le CI(u^{n-1}_{h\tau})^{1/2} + k^{n-1}_{h\tau}$ where $k^{n-1}_{h\tau} \in L^p(\Omega)$ for some $p \ge 2$.*

Then

$$\begin{aligned}
&\|\max_{1\le n\le N} I(u^n_{h\tau})^{1/2}\|_{L^p(\Omega)} + \|a_{h\tau}\|^{q/2}_{L^{pq/2}(\Omega, L^q[0,T;U])} \\
&+\mathbb{E}\left[\left(\sum_{n=1}^N \|u^n_{h\tau} - u^{n-1}_{h\tau}\|^2_H\right)^{p/2}\right]^{1/p} \\
&\le C(p,T)\Big(1 + CT/N\Big)^{N/p}\left(\|I(u^0_{h\tau})^{1/2}\|_{L^p(\Omega)} + \|f_{h\tau}\|^{q'/2}_{L^{pq'/2}(\Omega, L^{q'}[0,T;U'])}\right. \\
&\left.+\|k_{h\tau}\|_{L^p((0,T)\times\Omega)}\right).
\end{aligned}$$

Proof (Sketch) Starting from (27), then, upon neglecting the dependence of g upon u, the estimate

$$\|\max_{1\le n\le N} I(u_{h\tau}^n)^{1/2}\|_{L^p(\Omega)} + \|a_{h\tau}\|_{L^{pq/2}(\Omega,L^q[0,T;U])}^{q/2} + \mathbb{E}\left[\left(\sum_{n=1}^N \|u_{h\tau}^n - u_{h\tau}^{n-1}\|_H^2\right)^{p/2}\right]^{1/p}$$
$$\le C(p)\left(\|I(u_{h\tau}^0\|)_{L^p(\Omega)}^{1/2} + \|f_\tau\|_{L^{pq'/2}(\Omega,L^{q'}[0,T;U'])}^{q'/2} + T^{1/2-1/p}\|g_\tau\|_{L^p(\Omega,L^p[0,T;H])}\right)$$

follows *mutatis mutandis* as in the proof of Lemma 4.2. Bounding the last term as

$$\begin{aligned}\|g_{h\tau}\|_{L^p(\Omega,L^p[0,T;H])} &\le \|k_{h\tau}\|_{L^p((0,T)\times\Omega)} + C\left(\sum_{n=1}^N \tau\|I(u_{h\tau}^{n-1})^{1/2}\|_{L^p(\Omega)}^p\right)^{1/p}\\ &\le \|k_{h\tau}\|_{L^p((0,T)\times\Omega)} + C\left(\sum_{n=0}^{N-1} \tau\|\max_{0\le m\le n} I(u_{h\tau}^m)^{1/2}\|_{L^p(\Omega)}^p\right)^{1/p},\end{aligned}$$

and noting that the upper bound N was arbitrary shows

$$M^n + \|a_{h\tau}\|_{L^{pq/2}(\Omega,L^q[0,t^n;U])}^{pq/2} + \mathbb{E}\left[\left(\sum_{m=1}^n \|u_{h\tau}^m - u_{h\tau}^{m-1}\|_H^2\right)^{p/2}\right]$$
$$\le C(p,T)\left(M^0 + \|f_{h\tau}\|_{L^{pq'/2}(\Omega,L^{q'}[0,t^n;Up])}^{pq'/2} + \|k_{h\tau}\|_{L^p((0,t^n)\times\Omega)}^p + \sum_{m=0}^{n-1}\tau M^m\right),$$

where $M^n \equiv \|\max_{0\le m\le n} I(u_{h\tau}^m)^{1/2}\|_{L^p(\Omega)}^p$. The lemma now follows from the discrete Gronwall inequality. □

6.1.2 Existence and measurability of solutions

Given $\omega \in \Omega$, solutions of the discrete problems will be established using the following formulation of Brouwer's fixed point theorem [37, Proposition 2.1].

Theorem 6.2 *Let $\psi : \mathbb{R}^M \to \mathbb{R}^M$ be continuous and suppose that there exists $R > 0$ such that $\psi(\mathbf{u}).\mathbf{u} \ge 0$ whenever $|\mathbf{u}| = R$. Then there exists $\mathbf{u} \in \mathbb{R}^M$ with $|\mathbf{u}| \le R$ for which $\psi(\mathbf{u}) = 0$.*

In the numerical context $\mathbf{u}$ is the vector of coefficients representing the solution $u_{h\tau}^n(\omega) \in U_h$ for a given basis of U_h, and at each time step ψ will depend on the sample point $\omega \in \Omega$ implicitly through the stochastic increment, data, and the solution at the prior time step, *i.e.*,

$$\psi(\omega,\mathbf{u}) \equiv \psi\left(\mathbf{u};\, u_{h\tau}^{n-1}(\omega),\, f_{h\tau}^n(\omega),\, g_{h\tau}^{n-1}(\omega),\, \xi_{h\tau}^n(\omega)\right).$$

In all instances the dependence of ψ upon ω will be $\mathcal{F}^n$-measurable, and in this situation the following lemma shows that it is possible to select an $\mathcal{F}^n$-measurable solution of $\psi(\omega, \mathbf{u}) = 0$ for every $\omega \in \Omega$.

Lemma 6.3 *Let $(\Omega, \mathcal{F})$ be a measurable space, $\psi : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ be a mapping for which*

- *$\omega \mapsto \psi(\omega, \mathbf{u})$ is $\mathcal{F}$-measurable for every $\mathbf{u} \in \mathbb{R}^d$.*
- *$\mathbf{u} \mapsto \psi(\omega, \mathbf{u})$ is continuous for every $\omega \in \Omega$.*
- *For every $\omega \in \Omega$, there exists $\mathbf{u} \in \mathbb{R}^d$ such that $\psi(\omega, \mathbf{u}) = 0$.*

Then there exists an $\mathcal{F}$-measurable mapping $\mathbf{u} : \Omega \to \mathbb{R}^d$ such that $\psi(\omega, \mathbf{u}(\omega)) = 0$ holds for every $\omega \in \Omega$.

Results of this form appear in [11, 16] and are obtained using the following lemma from [27].

Lemma 6.4 (Kuratowski and Ryll-Nardzewski [27]) *Let $(\Omega, \mathcal{F})$ be a measurable space, Y a complete, separable metric space, and for every $\omega \in \Omega$ let $F(\omega)$ be a non-empty closed set in Y such that*

$$\{\omega \in \Omega : F(\omega) \cap G \neq \emptyset\} \in \mathcal{F} \tag{28}$$

holds for every open set G in Y. Then there exists an $\mathcal{F}$-measurable mapping $\zeta : \Omega \to Y$ such that $\zeta(\omega) \in F(\omega)$ holds for every $\omega \in \Omega$.

Remark 6.5 The hypothesis (28) holds if $\{\omega \in \Omega : F(\omega) \cap B \neq \emptyset\} \in \mathcal{F}$ for every closed ball B in Y since every open set G in a separable metric space is a countable union of closed balls.

Proof (of Lemma 6.3) Define $F(\omega) = \{\mathbf{u} \in \mathbb{R}^d : |\psi(\omega, \mathbf{u})| = 0\}$ for $\omega \in \Omega$. Then $F(\omega)$ is non-empty and closed and

$$\{\omega \in \Omega : F(\omega) \cap B \neq \emptyset\} = \{\omega \in \Omega : \inf_{\mathbf{u} \in B} |\psi(\omega, \mathbf{u})| = 0\} \in \mathcal{F}$$

holds for every closed ball B in $\mathbb{R}^d$ as $\omega \mapsto \inf_{\mathbf{u} \in B} |\psi(\omega, \mathbf{u})|$ is $\mathcal{F}$-measurable. The existence of a measurable solution of $\psi(\omega, \mathbf{u}(\omega))$ then follows from the Kuratowski Ryll-Nardzewski Lemma 6.4. $\square$

The proof of the Kuratowski Ryll-Nardzewski Lemma is not constructive so it is not clear that the computed solutions are measurable. If the time step τ is sufficiently small, solutions of the nonlinear problem can often be established using the Banach fixed point theorem. In this situation solutions depend continuously upon the data, and hence are measurable; however, usually the bound on the time step is prohibitively small and fixed point iterations converge slowly, so a (quasi) Newton method is employed. If, for every $\omega \in \Omega$, convergence is achieved for a bounded number of iterations, the solution would depend continuously upon the data, and measurability would follow.

6.2 Stochastic Navier–Stokes equation

The strong form of the incompressible stochastic Navier–Stokes equations on a bounded Lipschitz domain $D \subset \mathbb{R}^3$ takes the form

$$\begin{aligned} du + \Big((u.\nabla)u - D(u) + \nabla p\Big)\, dt &= f\, dt + g(u)\, dW, \\ div(u) &= 0, \end{aligned} \tag{29}$$

with initial and boundary conditions

$$u\big|_{t=0} = u^0, \qquad u\big|_{\partial D} = 0,$$

and W an $\mathbb{R}$-valued Wiener process. Here u is the vector-valued velocity of the fluid, p the pressure, and f and g are vector-valued and $D(u)$ is the symmetric part of the gradient[3] as in (13). In the above

$$g(u)(t, x, \omega) = \gamma\Big(t, x, u(t, x, \omega)\Big) \tag{30}$$

where $\gamma : (0, T) \times D \times \mathbb{R} \to \mathbb{R}^d$ is Caratheodory with linear growth. That is, for $u \in \mathbb{R}$ fixed $(t, x) \mapsto \gamma(t, x, u)$ is measurable, and for $(t, x) \in (0, T) \times D$ fixed $u \mapsto \gamma(t, x, u)$ is continuous, and $|\gamma(t, x, u)| \le C|u| + k(t, x)$ where $k \in L^p[0, T; L^2(D)]$ with $p > 4$.

To pose these equations in the abstract setting introduced in Sect. 3 let

$$U = H^1_0(D)^3, \quad H = L^2(D)^3, \quad U_0 = \{u \in U : div(u) = 0\ a.e.\ in\ D\}, \tag{31}$$

and consider the weak statement of (29) for which u takes values in $L^2[0, T; U_0]$ and satisfies

$$\begin{aligned} (u(t), v)_H + \int_0^t \Big\{((u.\nabla)u, v) + (D(u), \nabla v)\Big\}\, ds \\ = (u^0, v)_H + \int_0^t (f, v)\, ds + \int_0^t (g(u), v)_H\, dW, \quad v \in U_0, \end{aligned}$$

where $(.,.)$ denotes an L^2 pairing on D. Restricting the test functions to be in the space of divergence-free functions eliminates the pressure which is necessary since even in the deterministic setting the temporal regularity of p is very low [28].

To motivate the numerical scheme, recall that in the deterministic setting the natural stability estimate is found upon taking the dot product of the equation (29) with the solution and integrating by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_H + \Big((u.\nabla)u, u\Big) + \|u\|^2_U = (f, u),$$

[3] The symmetric part of the velocity gradient is denoted as $D(u)$ in fluid mechanics. The distinction between $D(u)$ and the domain $D \subset \mathbb{R}^d$ will always be clear by both notation and context.

where $\|u\|_U \equiv \|D(u)\|_{L^2(D)}$ is equivalent to the usual norm on U. The key step is to observe that the cubic term (which for large data could not be dominated by the quadratic terms) is skew symmetric; specifically, integration by parts shows

$$\Big((u.\nabla)u, v\Big) = -\Big(u, (u.\nabla)v\Big) + \Big(div(u)\, u, v\Big), \qquad u, v \in U. \tag{32}$$

It follows that $((u.\nabla)u, u) = 0$ when $u \in U_0$, so bounds upon the solution in $L^\infty[0, T; H] \cap L^2[0, T; U]$ follow as for the heat equation.

In general, it is difficult to construct subspaces of the divergence-free space U_0 with good approximation properties, so in a numerical context a velocity and pressure pair are constructed; $(u_h, p_h) \in V_h \times P_h$ with $V_h \subset U$ and $P_h \subset L^2(D)/\mathbb{R}$. The divergence-free condition is then approximated by requiring u_h to take values in the "discretely divergence-free subspace" $U_h \subset V_h$ defined by

$$U_h = \{u_h \in V_h \mid (div(u_h), q_h) = 0,\ q_h \in P_h\}. \tag{33}$$

Note that $U_h \not\subset U_0$, and in order to guarantee that functions $u \in U_0$ can be well-approximated by functions $u_h \in U_h$ the pair (V_h, P_h) is required to satisfy the discrete inf–sup (Ladyzhenskaya–Babuska–Brezzi) condition [4]: there exists a constant $c > 0$ independent of h such that

$$\sup_{v_h \in V_h} \frac{\big(p_h, div(v_h)\big)}{\|\nabla v_h\|_{L^2(D)}} \geq c\|p_h\|_{L^2(D)/\mathbb{R}}, \qquad p_h \in P_h. \tag{34}$$

We now come back to (29), and a corresponding discretization. Letting $\tau = T/N$ be a time step and $\{\xi_\tau^n\}_{n=1}^N$ be stochastic increments, for each $n = 1, 2, \ldots, N$ and all $\omega \in \Omega$ we let $\big(u_{h\tau}^n(\omega), p_{h\tau}^n(\omega)\big) \in V_h \times P_h$ satisfy

$$\begin{aligned}(u_{h\tau}^n - u_{h\tau}^{n-1}, v_h) &+ (\tau/2)\left((u_{h\tau}^n.\nabla)u_{h\tau}^n, v_h\right) - (\tau/2)\left(u_{h\tau}^n, (u_{h\tau}^n.\nabla), v_h\right)\\ &+\tau\left(D(u_{h\tau}^n), \nabla v_h\right) - \tau(p_{h\tau}^n, div\, v_h) = \tau(f_{h\tau}^n, v_h) + \big(g_{h\tau}^{n-1}, v_h\big)\xi_\tau^n, \quad v_h \in V_h,\end{aligned} \tag{35}$$

$$(div(u_{h\tau}^n), q_h) = 0, \qquad q_h \in P_h,$$

where

$$g_{h\tau}^{n-1}(x, \omega) = \frac{1}{\tau}\int_{t^{n-1}}^{t^n} \gamma\left(t, x, u_{h\tau}^{n-1}(x, \omega)\right) dt. \tag{36}$$

The second equation is simply the requirement that $u_{h\tau}^n(\omega) \in U_h$, and it is immediate that the term involving the pressure vanishes when $v_h \in U_h$. Equation (32) was used to formulate an approximation of the convective derivative that is skew symmetric when $div(u_{h\tau}^n)$ may not vanish. Note too that the convective derivative $(u_{h\tau}^n.\nabla)$ could be lagged to $(u_{h\tau}^{n-1}.\nabla)$ to give a linearly implicit scheme.

The methodology introduced in Sect. 6.1.2 is used to establish a (measurable) solution of the discrete scheme (35). Given a basis for U_h, an element $u_h(\omega) \in U_h$ is identified with an element $\mathbf{u}(\omega)$ of $\mathbb{R}^M$ where $M = dim(U_h)$. With $\omega \in \Omega$ fixed, and identifying an element $v_h \in U_h$ with a vector of coefficients $\mathbf{v} \in \mathbb{R}^M$, the Riesz theorem is used to construct $\psi : \mathbb{R}^M \to \mathbb{R}^M$ satisfying

$$\psi(\mathbf{u}).\mathbf{v} := (u_h - u_{h\tau}^{n-1}, v_h) + (\tau/2)\Big((u_h.\nabla)u_h, v_h\Big) - (\tau/2)\Big(u_h, (u_h.\nabla), v_h\Big) \\ +\tau\Big(D(\nabla u_h), \nabla v_h\Big) - \tau(f_{h\tau}^n, v_h) - \big(g_{h\tau}^{n-1}, v_h\big)\xi_\tau^n, \qquad \mathbf{v} \in \mathbb{R}^M.$$

Fixing $\omega \in \Omega$, and setting $v_h = u_h(\omega)$ then leads to

$$\begin{aligned}\psi(\mathbf{u}).\mathbf{u} &= \frac{1}{2}\Big(\|u_h\|_H^2 + \|u_h - u_{h\tau}^{n-1}\|_H^2 - \|u_{h\tau}^{n-1}\|_H^2\Big) \\ &\quad +\tau\|u_h\|_U^2 - \tau(f_{h\tau}^n, u_h) - \big(g_{h\tau}^{n-1}, u_h\big)\xi_\tau^n \\ &\geq \frac{1}{2}\Big(\|u_h\|_H^2 - \|u_{h\tau}^{n-1}\|_H^2\Big) + \tau\|u_h\|_U^2 \\ &\quad -\Big(\tau\|f_{h\tau}^n\|_{U'} + \|g_{h\tau}^{n-1}\|_{U'}|\xi_{h\tau}^n|\Big)\|u_h\|_U,\end{aligned}$$

and it is clear that this is non–negative whenever

$$\min\Big(\|u_h\|_H, \|u_h\|_U\Big) \geq \max\Big(\|u_{h\tau}^{n-1}\|_H, \|f_{h\tau}^n\|_{U'} + \|g_{h\tau}^{n-1}\|_{U'}|\xi_{h\tau}^n|/\tau\Big).$$

Existence of a pressure then follows from the inf–sup condition.

To bound the solutions, set $v_h = u_{h\tau}^n(\omega)$ in the discrete weak statement (35) and complete the square (as in (16)) to get

$$\begin{aligned}&(1/2)\|u_{h\tau}^n\|_H^2 + (1/2)\|u_{h\tau}^n - u_{h\tau}^{n-1}\|_H^2 + \tau\|u_{h\tau}^n\|_U^2 \\ &\quad = (1/2)\|u_{h\tau}^{n-1}\|_H^2 + \tau(f_{h\tau}^n, u_{h\tau}^n) + (g_{h\tau}^{n-1}, u_{h\tau}^n)\xi_\tau^n,\end{aligned} \tag{37}$$

which, due to the skew symmetry of the nonlinear term, is identical in form to the corresponding equation (17) for the heat equation.

The following theorem establishes convergence of solutions to the numerical scheme (35) to a weak martingale solution of the stochastic Navier–Stokes equation (29).

Theorem 6.6 *Fix $T > 0$, and let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let $U = H_0^1(D)^3$, $H = L^2(D)^3$, $U_0 \subset U$ be the divergence-free subspace, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let Assumptions 3.1, and 2.5 hold with parameter $2p > 4$. Let $\tau = T/N$ with $N \in \mathbb{N}$ denote a time step, and let $\{(V_h, P_h)\}_{h>0} \subset U \times L^2(D)/\mathbb{R}$ be finite-dimensional subspaces satisfying:*

- *For each* $(v, q) \in U \times L^2(D)/\mathbb{R}$ *there exists a sequence* $\{(v_h, q_h)\}_{h>0}$ *with* $(v_h, q_h) \in V_h \times P_h$ *such that* $(v_h, q_h) \to (v, q)$ *as* $h \to 0$.
- *The restriction of the orthogonal projection* $Q_h : H \to V_h$ *to* U *is stable. That is, there exists* $C > 0$ *independent of* h *such that* $\|Q_h u\|_U \leq C\|u\|_U$.
- *The discrete inf–sup condition* (34) *holds with a constant* $c > 0$ *independent of* $h > 0$. *Denote the discretely divergence-free subspace by* $U_h = \{u_h \in V_h \mid (div(u_h), q_h) = 0,\ q_h \in P_h\}$.

Let $\{u_{h\tau}\}_{h,\tau>0}$ *be a sequence of solutions of* (35) *with data satisfying*

1. $\{u^0_{h\tau}\}_{h,\tau>0}$ *is bounded in* $L^{2p}(\Omega, H)$ *and converges in* $L^2(\Omega, H)$.
2. $\{f_{h\tau}\}_{h,\tau>0}$ *is bounded in* $L^{2p}(\Omega, L^2[0, T; U'])$ *and converges in* $L^2(\Omega, L^2[0, T; U'])$.
3. $g_{h\tau}$ *is given by Eq.* (36) *with* $\gamma : (0, T) \times D \times \mathbb{R} \to \mathbb{R}^d$ *Caratheodory with linear growth,* i.e., $|\gamma(t, x, u)| \leq C|u| + k(t, x)$ *with* $k \in L^{2p}[0, T; L^2(D)]$.

Denote the discrete Wiener process with increments $\{\xi^n_\tau\}_{n=1}^N$ *by* $\hat{W}_\tau$, *and write*

$$(A(u), v) = \Big(D(u), \nabla v\Big) + (1/2)\Big((u.\nabla)u, v\Big) - (1/2)\Big(u, (u.\nabla), v\Big).$$

Then there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, *a random variable* $(u, (f, a), g, W)$ *on* $\tilde{\Omega}$ *with values in* $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ *with*

$$\begin{aligned}\mathbb{X} = G[0, T; U'] \cap L^2[0, T; U]_{weak} \times (L^{4/3}[0, T; U']\\ \times L^{4/3}[0, T; U']_{weak}) \times L^2[0, T; H] \times C[0, T],\end{aligned}$$

and a subsequence $(\tau_k, h_k) \to (0, 0)$ *for which the*

$$\mathcal{L}(u_{h_k\tau_k}, (f_{h_k\tau_k}, A(u^n_{h\tau})), g_{h_k\tau_k}, \hat{W}_{\tau_k}) \Rightarrow \mathcal{L}(u, (f, a), g, W) \equiv \tilde{\mathbb{P}}.$$

In addition, $\tilde{\mathbb{P}}[div(u) = 0] = 1$, $\tilde{\mathbb{P}}[a = A(u)] = 1$, *and there exists a filtration* $\{\tilde{\mathcal{F}}(t)\}_{0\leq t\leq T}$ *satisfying the usual conditions for which* (u, f, g, W) *is adapted and* W *is a real-valued Wiener process for which*

$$\begin{aligned}&(u(t), v)_H + \int_0^t \Big((u.\nabla)u, v) + (D(u), \nabla v)\Big)\, ds\\ &\quad = (u^0, v)_H + \int_0^t (f, v)\, ds + \int_0^t (g(u), v)\, dW, \quad v \in U_0,\end{aligned}$$

where $g(u)(t, x, \omega) = \gamma(t, x, u(t, x, \omega))$.

Proof Lemma 6.1 is first used to bound the solutions of the numerical scheme. Equation (37) establishes the bounds needed at each time step, and using the structural properties of γ give

$$\|g^{n-1}_{h\tau}\|_{L^2(D)} \leq C\left(\|u^{n-1}_{h\tau}\|_{L^2(D)} + k^n_\tau\right) \quad \text{where} \quad k^n_\tau = (1/\sqrt{\tau})\|k\|_{L^2[t^{n-1}, t^n; L^2(D)]}. \tag{38}$$

Lemma 6.1 with parameters $2p$ and $q = q' = 2$ then shows

$$\|u_{h\tau}\|_{L^{2p}(\Omega,L^\infty[0,T;H])} + \|u_{h\tau}\|_{L^{2p}(\Omega,L^2[0,T;U])}$$
$$\leq C\left(\|u^0_{h\tau}\|_{L^{2p}(\Omega,H)} + \|f_{h\tau}\|_{L^{2p}(\Omega,L^2[0,T;U]')} + \|k\|_{L^{2p}[0,T;L^2(D)]}\right).$$

We now verify that $\{(u_{h\tau}, (f_{h\tau}, A(u_{h\tau})), g_{h\tau}, \hat{W}_\tau)\}_{h,\tau>0}$ satisfies the hypotheses of Theorem 3.2 with parameters $r = 2$, $q = 8$, and $q' = 8/7$.

1. The embedding $H^1(D) \hookrightarrow L^6(D)$ is first used to verify $\|u\|_{L^3(D)} \leq C\|u\|_H^{1/2}\|u\|_U^{1/2}$. Then

$$|(A(u), v)| \leq (\|u\|_U)\,\|v\|_U + (1/2)\|u\|_{L^3(D)}\|u\|_U\|v\|_{L^6(D)}$$
$$+(1/2)\|u\|_{L^3(D)}\|u\|_{L^6(D)}\|v\|_U,$$

so that

$$\|A(u)\|_{U'} \leq \|u\|_U + C\|u\|_H^{1/2}\|u\|_U^{3/2}.$$

Repeated application of Hölder's inequality then shows

$$\|A(u_{h\tau})\|_{L^p(\Omega,L^{4/3}[0,T;U'])} \leq \|u_{h\tau}\|_{L^p(\Omega,L^{4/3}[0,T;U])}$$
$$+C\|u_{h\tau}\|^{1/2}_{L^{2p}(\Omega,L^\infty[0,T;H])}\|u_{h\tau}\|^{3/2}_{L^{2p}(\Omega,L^2[0,T;U])}.$$

The bounds upon $u_{h\tau}$ and embedding $L^{4/3}[0, T; U'] \hookrightarrow L^{8/7}[0, T; U']$ then show $\|A(u_{h\tau})\|_{L^p(\Omega,L^{8/7}[0,T;U'])}$ is also bounded.

2. Writing $F^n_{h\tau} = f^n_{h\tau} - A(u^n_{h\tau})$ we have $(F^n_{h\tau}, u^n_{h\tau}) = (f^n_{h\tau}, u^n_{h\tau}) - \|u^n_{h\tau}\|^2_U$, and the Cauchy-Schwarz inequality gives

$$\|(F^n_{h\tau}, u^n_{h\tau})\|_{L^{p/2}(\Omega,L^1(0,T))} \leq \|f_{h\tau}\|_{L^p(\Omega,L^2[0,T;U]')}\|u_{h\tau}\|_{L^p(\Omega,L^2[0,T;U])}$$
$$+\|u_{h\tau}\|^2_{L^p(\Omega,L^2[0,T;U])}.$$

3. Equation (38), and the bounds upon $\{u_{h\tau}\}_{h,\tau>0}$ show that $\{g_{h\tau}\}_{h,\tau>0}$ will be bounded in $L^{2p}(\Omega, L^{2p}[0, T; H])$ provided $\{k_\tau\}_{\tau>0}$ is bounded in $L^{2p}(0, T)$ (note that k is deterministic). This follows from repeated applications of Hölder's inequality,

$$\|k_\tau\|^{2p}_{L^{2p}(0,T)} = \sum_{n=1}^N \tau\left(\frac{1}{\tau}\int_{t^{n-1}}^{t^n}\|k(t)\|^2_H\right)^p \leq \sum_{n=1}^N\int_{t^{n-1}}^{t^n}\|k(t)\|^{2p}_H\,dt$$
$$= \|k\|^{2p}_{L^{2p}[0,T;H]}.$$

4. The initial data u^0 satisfies the properties assumed in Theorem 3.2 by hypothesis.

It follows that upon passing to a sub–sequence $(h_k, \tau_k) \to 0$ there exist a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}(t)\}_{t=0}^T, \tilde{\mathbb{P}})$, a random variable (u, F, g, W) with values in $\mathbb{X}$ for which W is a standard Wiener process, such that

$$\mathcal{L}(u_{h_k\tau_k}, (f_{h_k\tau_k}, A(u_{h_k\tau_k})), g_{h_k\tau_k}, \hat{W}_{h_k\tau_k}) \Rightarrow \mathcal{L}(u, (f, a), g, W) \equiv \tilde{\mathbb{P}}$$

and

$$(u(t), v)_H = (u^0, v)_H + \int_0^t (f - a, v)\, ds + \int_0^t (g, v)_H\, dW, \qquad v \in U_0, 0 \le t \le T.$$

For $q \in L^2[0, T; L^2(D)]$ fixed, the function

$$(u, (f, a), g, W) \mapsto \left| \int_0^T (div(u), q)\, ds \right| \wedge 1$$

is continuous and bounded on $\mathbb{X}$. Letting $q_k \in L^2[0, T; P_{h_k}]$ be chosen so that $q_k \to q$ it follows that

$$\begin{aligned} \tilde{\mathbb{E}}\left[\left| \int_0^T (div(u), q) \right| \wedge 1 \right] &= \lim_{k\to\infty} \mathbb{E}\left[\left| \int_0^T (div(u_{h_k\tau_k}), q) \right| \wedge 1 \right] \\ &= \lim_{k\to\infty} \mathbb{E}\left[\left| \int_0^T (div(u_{h_k\tau_k}), q_k) \right| \wedge 1 \right] = 0, \end{aligned}$$

whence $\tilde{\mathbb{P}}[div(u) = 0] = 1$. Finally, Example 3.4 shows that $(a, v) = (D(u), \nabla v) + (1/2)((u.\nabla)u, v) - (1/2)(u, (u.\nabla)v)$ almost surely on the support of $\tilde{\mathbb{P}}$.

To verify that $g(t, x, \omega) = \gamma(t, x, u(t, x, \omega)$ (we write $g = \gamma(u)$) on the support of $\tilde{\mathbb{P}}$, note that the map $u(t, x) \mapsto \gamma(t, x, u(t, x))$ is continuous from $L^2[0, T; H]$ to itself, so if $v \in L^2[0, T; H]$ is fixed

$$\tilde{\mathbb{E}}\left[\left| (\gamma(u) - g, v)_{L^2[0,T;H]} \right| \wedge 1 \right] = \lim_{k\to\infty} \mathbb{E}\left[\left| (\gamma(u_{h_k\tau_k}) - g_{h_k,\tau_k}, v)_{L^2[0,T;H]} \right| \wedge 1 \right].$$

For the numerical scheme $g_{h\tau}(\omega)$ is the orthogonal projection of $\gamma(u_{h\tau}(\omega))$ onto the subspace of functions in $L^2[0, T; H]$ which are piecewise constant in time. Thus if $v_k \in L^2[0, T; H]$ is piecewise constant in time and $v_k \to v$ we have

$$\begin{aligned} \tilde{\mathbb{E}}\left[\left| (\gamma(u) - g, v)_{L^2[0,T;H]} \right| \wedge 1 \right] &= \lim_{k\to\infty} \mathbb{E}\left[\left| (\gamma(u_{h_k\tau_k}) - g_{h_k,\tau_k}, v_k)_{L^2[0,T;H]} \right| \wedge 1 \right] \\ &= 0. \end{aligned}$$

□

6.3 Harmonic heat flow

The stochastic harmonic heat flow equation on a domain $D \subset \mathbb{R}^3$ is the vector-valued equation

$$du + (-\Delta u + \lambda u)\,dt = f\,dt + (u \times \gamma) \circ dW, \qquad \text{with constraint} \quad u \in \mathbb{S}^2,$$

and initial and boundary data $u|_{t=0} = u^0$ and $\partial u/\partial n|_{\partial D} = 0$. Here λ is a Lagrange multiplier dual to the constraint $|u| = 1$, and

$$(u \times \gamma) \circ dW \equiv (1/2)(u \times \gamma) \times \gamma\,dt + (u \times \gamma)\,dW$$

denotes the Stratonovich integral. In order to preserve the constraint the noise term is selected to be tangent to $u \in \mathbb{S}^2$, and in order to eliminate a significant amount of technical overhead we will assume that the datum $\gamma \in \mathbb{R}^3$ is independent of $x \in D$. The numerical analysis of the spatially dependent data (and operator-valued colored noise) is undertaken in [1] for the stochastic Landau–Lifshitz–Gilbert equation.

The analysis of the harmonic heat flow equation is complicated by the fact that solutions may exhibit singularities. In this situation essentially nothing is known about the structure of the Lagrange multiplier, and this gap in the theory plagues both the construction and analysis of numerical schemes. For this reason the constraint is usually approximated using a penalty scheme and this is the approach considered here. Specifically, we consider numerical approximations of the equation

$$du + \Big(-\Delta u + D\phi(u)\Big)\,dt = f\,dt + (u \times \gamma) \circ dW, \tag{39}$$

where $\phi(u) = (1/2\epsilon)(|u|^2 - 1)^2$ with $\epsilon > 0$. The drift term on the left is the variational derivative of the energy

$$I(u) = \int_D \frac{1}{2}|\nabla u|^2\,dx + \phi(u),$$

and in the deterministic case bounds upon the solution independent of the penalty constant ϵ follow upon taking the product of the equation with either u_t or $-\Delta u + D\phi(u)$ to obtain

$$\|u_t\|^2_{L^2(D)} + \frac{d}{dt}I(u) = (f, u_t), \quad \text{or}$$
$$\frac{d}{dt}I(u) + \|-\Delta u + D\phi(u)\|^2_{L^2(D)} = \Big(f, -\Delta u + D\phi(u)\Big).$$

When the stochastic term is present, we derive an analog of the second estimate. However, in a numerical context where $u_h(\omega) \in U_h \subset U \equiv H^1(D)^3$, the function $-\Delta u_h(\omega) + \phi\big(u_h(\omega)\big) \notin U_h$ is not available as a test function. For this reason we will use a mixed method where $a \equiv -\Delta u + D\phi(u)$ is introduced as an additional variable.

Letting $\tau = T/N$ with $N \in \mathbb{N}$ be a time step and $f^n_{h\tau} \simeq f(n\tau)$, we approximate solutions of (39) by $\left(u^n_{h\tau}(\omega), a^n_{h\tau}(\omega)\right) \in U_h \times U_h$,

$$(u^n_{h\tau} - u^{n-1}_{h\tau}, v_h) + \tau(a^n_{h\tau}, v_h) = \tau(f^n_{h\tau}, v_h) + \left(u^{n-1/2}_{h\tau} \times \gamma, v_h\right)\xi^n_\tau \tag{40}$$

$$(a^n_{h\tau}, b_h) = (\nabla u^n_{h\tau}, \nabla b_h) + (1/\epsilon)\left((|u^n_{h\tau}|^2 + |u^{n-1}_{h\tau}|^2 - 2)u^{n-1/2}_{h\tau}, b_h\right), \tag{41}$$

for all $(v_h, b_h) \in U_h \times U_h$, where $u^{n-1/2}_{h\tau} \equiv (1/2)(u^n_{h\tau} + u^{n-1}_{h\tau})$ and ξ^n_τ are stochastic increments satisfying Assumption 2.5. This scheme was constructed so that:

- The approximation of $D\phi(u) = (2/\epsilon)(|u|^2 - 1)u$ in (41) inherits a discrete version of the identity $(D\phi(u), u_t) = d\phi/dt$,

$$(1/\epsilon)\left((|u^n_{h\tau}|^2 + |u^{n-1}_{h\tau}|^2 - 2)u^{n-1/2}_{h\tau}, u^n_{h\tau} - u^{n-1}_{h\tau}\right) = \phi(u^n_{h\tau}) - \phi(u^{n-1}_{h\tau}).$$

  This is essential in order to obtain bounds independent of ϵ.
- Since $D\phi(u)$ is parallel to u it follows that $(D\phi(u), u \times \gamma) = 0$. The discrete approximation of $D\phi(u)$ is parallel to $u^{n-1/2}_{h\tau}$ and is perpendicular to the coefficient $u^{n-1/2}_{h\tau} \times \gamma$ of ξ^n_τ.
  Note too that $u^{n-1/2}_{h\tau}(\omega) \times \gamma \in U_h$, so is admissible as test function, and $\left(\nabla u, \nabla(u \times \gamma)\right) = 0$ when $u \in U$; both following since $g(x) = \gamma \in \mathbb{R}^3$ was taken to be independent of x.

Fixing $\omega \in \Omega$, selecting the test functions in (40)–(41) to be

$$(v_h, b_h) = \left(a^n_{h\tau}, u^n_{h\tau} - u^{n-1}_{n\tau} + (u^{n-1/2}_{n\tau} \times \gamma)\xi^n_\tau\right)(\omega)$$

and using these structural properties shows

$$\begin{aligned}
&\frac{1}{2}\|\nabla u^n_{h\tau}\|^2_{L^2(D)} + \|\phi(u^n_{h\tau})\|_{L^1(D)} + \frac{1}{2}\|\nabla(u^n_{h\tau} - u^{n-1}_{h\tau})\|^2_{L^2(D)} + \tau\|a^n_{h\tau}\|^2_{L^2(D)} \\
&= \frac{1}{2}\|\nabla u^{n-1}_{h\tau}\|^2_{L^2(D)} + \|\phi(u^{n-1}_{h\tau})\|_{L^1(D)} + \tau\left(f^n_{h\tau}, a^n_{h\tau}\right) \\
&\quad + \left(\nabla u^n_{h\tau}, \nabla(u^{n-1/2}_{h\tau} \times \gamma)\right)\xi^n_\tau.
\end{aligned} \tag{42}$$

Lemma 6.1 with $U = H = L^2(D)^3$ then establishes bounds upon the gradient of the solution independent of ϵ; an additional calculation then establishes a bound upon the (spatial) average of the solution, and the Poincare inequality then bounds the solution itself.

Lemma 6.7 *Let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain, set $U = H^1(D)^3$ and $H = L^2(D)^3$, and let $I : U \to \mathbb{R}$ be the function $I(u) = (1/2)(\|\nabla u\|^2_{L^2(D)} + \|\phi(u)\|_{L^1(D)})$ where $\phi(u) = (1/(2\epsilon))(|u|^2 - 1)^2$, with $\epsilon > 0$ fixed. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.*

Suppose that the Assumptions 3.1, and 2.5 with parameter $p > 2$ hold, and that $\{u^0_{h\tau}\}$ is bounded in $L^p(\Omega, U)$, and $\{f_{h\tau}\}$ is bounded in $L^p(\Omega, L^2[0,T;H])$. Then there exists a sequence $\{(u^n_{h\tau}, a^n_{h\tau})\}_{n\geq 1}$ of $U_h \times U_h$-valued random variables adapted to $\{\mathcal{F}^n\}_{n=0}^N$ which satisfy (40)–(41) and

$$\text{(i)}\quad \mathbb{E}\left[\max_{1\leq n\leq N}\left(\|\nabla u^n_{h\tau}\|^p_H + \|\phi(u^n_{h\tau})\|^{p/2}_{L^1(D)}\right) + \left(\sum_{n=1}^N \|\nabla(u^n_{h\tau} - u^{n-1}_{h\tau})\|^2_H\right)^{p/2} + \left(\sum_{n=1}^N \tau\|a^n_{h\tau}\|^2_H\right)^{p/2}\right]^{1/p}$$
$$\leq C\left(\|\nabla u^0_{h\tau}\|_{L^p(\Omega,H)} + \|\phi(u^0_{h\tau})\|^{1/2}_{L^{p/2}(\Omega,L^1(D))} + \|f_{h\tau}\|_{L^p(\Omega,L^2[0,T;H])}\right).$$

$$\text{(ii)}\quad \mathbb{E}\left[\max_{1\leq n\leq N}\|u^n_{h\tau}\|^p_H + \left(\sum_{n=1}^N \|u^n_{h\tau} - u^{n-1}_{h\tau}\|^2_H\right)^{p/2}\right]^{1/p}$$
$$\leq C\left(\|u^0_{h\tau}\|_{L^p(\Omega,U)} + \|\phi(u^0_{h\tau})\|^{1/2}_{L^{p/2}(\Omega,L^1(D))} + \|f_{h\tau}\|_{L^p(\Omega,L^2[0,T;H])}\right).$$

Proof Theorem 6.2 will be used to establish the existence of a solution to the scheme by solving for the variable $\delta u(\omega) = u^n_{h\tau}(\omega) - u^{n-1}_{h\tau}(\omega)$. Inductively assume that the U_h-valued random variable $u^{n-1}_{h\tau}(\omega)$ is given and use the Riesz theorem to construct the solution operator $a_h : U_h \to U_h$ of equation (41) with $u^n_{h\tau} = u^{n-1}_{h\tau} + \delta u$. Upon introducing a basis for U_h the Riesz theorem on $\mathbb{R}^M$ with $M = dim(U_h)$ guarantees the existence of a continuous function $\psi : \mathbb{R}^M \to \mathbb{R}^M$ which, for each $\omega \in \Omega$, satisfies

$$\psi(\delta\mathbf{u}).\mathbf{v} = \left(\delta u + \tau a_h(\delta u) - \tau f^n_{h\tau} - \left((\delta u/2 + u^{n-1}_{h\tau}) \times \gamma\right)\xi^n_\tau, v_h\right)_H, \qquad \mathbf{v} \in \mathbb{R}^M,$$

where $\delta\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^M$ denote the vectors of coefficients of U_h-valued functions $\delta u(\omega)$ and v_h. Using equation (41) we find

$$\begin{aligned}\left(a_h(\delta u), \delta u\right)_H &= \left(\nabla(\delta u + u^{n-1}_{h\tau}), \nabla\delta u\right)_H + \|\phi(\delta u + u^{n-1}_{h\tau})\|_{L^1(D)} \\ &\quad -\|\phi(u^{n-1}_{h\tau})\|_{L^1(D)} \\ &= \frac{1}{2}\left(\|\nabla\delta u\|^2_H + \|\nabla(\delta u + u^{n-1}_{h\tau})\|^2_H - \|\nabla u^{n-1}_{h\tau}\|^2_H\right) \\ &\quad +\|\phi(\delta u + u^{n-1}_{h\tau})\|_{L^1(D)} - \|\phi(u^{n-1}_{h\tau})\|_{L^1(D)}.\end{aligned}$$

Zeros of $\psi(.)$ then exist since $(\delta u \times \gamma, \delta u) = 0$, so

$$\begin{aligned}\psi(\delta\mathbf{u}).\delta\mathbf{u} &= \|\delta u\|^2_H + \frac{\tau}{2}\|\nabla\delta u\|^2_H + \frac{\tau}{2}\|\nabla(\delta u + u^{n-1}_{h\tau})\|^2_H + \tau\|\phi(\delta u + u^{n-1}_{h\tau})\|_{L^1(D)} \\ &\quad - \left(u^{n-1}_{h\tau} \times \gamma, \delta u\right)_H \xi^n_\tau - \tau(f^n_{h\tau}, \delta u) - \frac{\tau}{2}\|\nabla u^{n-1}_{h\tau}\|^2_H - \tau\|\phi(u^{n-1}_{h\tau})\|_{L^1(D)}\end{aligned}$$

is non–negative whenever $\|\delta u\|^2_{L^2(D)} + \tau\|\nabla\delta u\|^2_{L^2(D)}$ is sufficiently large. Equation (42) and the measurable selection theorem Lemma 6.3 then establish the hypothesis of Lemma 6.1 from which estimate (i) in the lemma follows.

To establish estimate (ii), let $\bar{u}^n_{h\tau} = (1/|D|)\int_\Omega u^n_{h\tau}\, dx$ denote the spatial average. For $\omega \in \Omega$ fixed, on selecting the test function in (40) to be $v_h = \bar{u}^{n-1/2}_{h\tau}(\omega)$ and summing gives

$$
\begin{aligned}
|\bar{u}^n_{h\tau}|^2 &= |\bar{u}^0_{h\tau}|^2 + (1/|D|)\sum_{m=1}^{n} \tau(f^m_{h\tau} - a^m_{h\tau}, \bar{u}^{m-1/2}_{h\tau}) \\
&\le |\bar{u}^0_{h\tau}|^2 + (1/|D|)\left(\sum_{m=1}^{n} \tau\|f^m_{h\tau} - a^m_{h\tau}\|^2_H\right)^{1/2}\left(\sum_{m=1}^{n} \tau|D|\,(\bar{u}^{m-1/2}_{h\tau})^2\right)^{1/2} \\
&\le |\bar{u}^0_{h\tau}|^2 + \|f_{h\tau} - a_{h\tau}\|_{L^2[0,T;H]}(T|D|)^{1/2}\max_{0\le m\le n}|\bar{u}^m_{h\tau}|, \qquad 1 \le n \le N.
\end{aligned}
$$

It readily follows that

$$
\|\max_{1\le m\le N}|\bar{u}^m_{h\tau}|\|_{L^p(\Omega)} \le C(p, T/|D|)\left(\|\bar{u}^0_{h\tau}\|_{L^p(\Omega)} + \|f_{h\tau} - a_{h\tau}\|_{L^p(\Omega, L^2[0,T;H])}\right).
$$

Next, select the test function in (40) to be $v_h = \bar{u}^n_{h\tau}(\omega)$ to obtain

$$
\begin{aligned}
\frac{1}{2}|\bar{u}^n_{h\tau}|^2 + \frac{1}{2}|\bar{u}^n_{h\tau} - \bar{u}^{n-1}_{h\tau}|^2 + \frac{\tau}{|D|}|\bar{u}^n_{h\tau}|^2 &= \frac{1}{2}|\bar{u}^{n-1}_{h\tau}|^2 \\
+\frac{\tau}{|D|}\left(f^m_{h\tau} - a^m_{h\tau} - \bar{u}^n_{h\tau}, \bar{u}^n_{h\tau}\right) &+ \frac{1}{2|D|}\left(\bar{u}^n_{h\tau}, \bar{u}^{n-1}_{h\tau} \times \gamma\right)\xi^n_\tau.
\end{aligned}
$$

Lemma 6.1 with $H = U = \mathbb{R}^3$ then shows

$$
\begin{aligned}
&\|\max_{1\le n\le N}|\bar{u}^n_{h\tau}|\|_{L^p(\Omega,H)} + \|\bar{u}_{h\tau}\|_{L^p(\Omega,L^2(0,T))} + \mathbb{E}\left[\left(\sum_{n=1}^{N}|\bar{u}^n_{h\tau} - \bar{u}^{n-1}_{h\tau}|^2\right)^{p/2}\right]^{1/p} \\
&\le C(p,T)\left(\|\bar{u}^0_{h\tau}\|_{L^p(\Omega,H)} + \|f_{h\tau} - a_{h\tau} - \bar{u}_{h\tau}\|_{L^p(\Omega,L^2[0,T;H])}\right).
\end{aligned}
$$

Estimate (ii) in the lemma now follows from the bounds upon $a_{h\tau}$ and $\bar{u}_{h\tau}$ obtained above and the Poincare inequality. □

To cast the above scheme into the setting of Theorem 3.2, set

$$
F^n_{h\tau} = f^n_{h\tau} - a^n_{h\tau} + (1/2\tau)(u^n_{h\tau} - u^{n-1}_{h\tau}) \times \gamma\,\xi^n_\tau \qquad \text{and } g^{n-1}_{h\tau} = u^{n-1}_{h\tau} \times \gamma, \tag{43}
$$

so that the equation (40) becomes

$$
(u^n_{h\tau} - u^{n-1}_{h\tau}, v_h)_H = \tau(F^n_{h\tau}, v_h)_H + (g^{n-1}_{h\tau}, v_h)_H\xi^n_\tau, \qquad v_h \in U_h.
$$

The following lemma bounds the last term of $F_{h\tau}$, which is the discrete analog of the Stratonovich correction.

Lemma 6.8 *Under the hypothesis of Lemma 6.7 with parameter* $p \geq 4$

$$\|F_{h\tau}^{(2)}\|_{L^{8/3}(\Omega, L^{4/3}[0,T;H])} \leq \mathbb{E}\left[\left(\sum_{n=1}^{N} \|u_{h\tau}^n - u_{h\tau}^{n-1}\|_H^2\right)^4\right]^{1/8},$$

where $F_{h\tau}^{(2)}$ *denotes the piecewise constant function in time taking values* $(1/2\tau)(u_{h\tau}^n - u_{h\tau}^{n-1}) \times \gamma\, \xi_\tau^n$ *on* (t^{n-1}, t^n).

Proof First compute

$$\begin{aligned}\|F_{h\tau}^{(2)}\|_{L^{4/3}[0,T;H]}^{4/3} &\leq (|\gamma|/2\tau)^{4/3} \sum_{n=1}^{N} \tau \|u_{h\tau}^n - u_{h\tau}^{n-1}\|_H^{4/3} |\xi_\tau^n|^{4/3} \\ &\leq C\tau^{-1/3} \left(\sum_{n=1}^{N} \|u_{h\tau}^n - u_{h\tau}^{n-1}\|_H^2\right)^{2/3} \left(\sum_{n=1}^{N} |\xi_\tau^n|^4\right)^{1/3}.\end{aligned}$$

The stochastic increments satisfy $\mathbb{E}\left[|\xi_\tau^n|^4\right] \leq C\tau^2$ when $p \geq 4$, and will cancel the factor of $\tau^{-1/3}$;

$$\begin{aligned}\|F_{h\tau}^{(2)}\|_{L^{8/3}(\Omega, L^{4/3}[0,T;H])}^{8/3} &= \mathbb{E}\left[\|F_{h\tau}^{(2)}\|_{L^{4/3}[0,T;H]}^{8/3}\right] \\ &\leq C\tau^{-2/3}\mathbb{E}\left[\left(\sum_{n=1}^{N} \|u_{h\tau}^n - u_{h\tau}^{n-1}\|_H^2\right)^{4/3} \left(\sum_{n=1}^{N} |\xi_\tau^n|^4\right)^{2/3}\right] \\ &\leq C\tau^{-2/3}\mathbb{E}\left[\left(\sum_{n=1}^{N} \|u_{h\tau}^n - u_{h\tau}^{n-1}\|_H^2\right)^4\right]^{1/3} \mathbb{E}\left[\sum_{n=1}^{N} |\xi_\tau^n|^4\right]^{2/3} \\ &\leq CT^{2/3}\mathbb{E}\left[\left(\sum_{n=1}^{N} \|u_{h\tau}^n - u_{h\tau}^{n-1}\|_H^2\right)^4\right]^{1/3},\end{aligned}$$

which completes the proof. □

Theorem 6.9 *Fix* $T > 0$ *and let* $D \subset \mathbb{R}^3$ *be a bounded Lipschitz domain,* $U = H^1(D)^3$, $H = L^2(D)^3$, *and let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. Let the Assumptions 3.1, and 2.5 hold with* $p = 8$ *moments. Let* $\tau = T/N$ *with* $N \in \mathbb{N}$ *denote a time step, and let* $\{U_h\}_{h>0} \subset U$ *be finite dimensional subspaces satisfying:*

- *For each* $u \in U$ *there exists a sequence* $\{(u_h)\}_{h>0}$ *with* $u_h \in U_h$ *such that* $u_h \to u$ *as* $h \to 0$.

- *The restriction of the orthogonal projection $P_h : H \to U_h$ to U is stable. That is, there exists $C > 0$ independent of h such that $\|P_h u\|_U \le C\|u\|_U$.*

Let $\{(u_{h\tau}, a_{h\tau})\}_{h,\tau>0}$ denote the solution of (40)–(41) with data satisfying

1. *$\{u^0_{h\tau}\}_{h,\tau>0}$ is bounded in $L^8(\Omega, U)$ and converges to a limit u^0 in $L^2(\Omega, U)$ as $(h, \tau) \to 0$.*
2. *$\{f_{h\tau}\}_{h,\tau>0}$ is bounded in $L^8(\Omega, L^2[0, T; H])$ and converges to a limit f in $L^{8/3}(\Omega, L^{4/3}[0, T; H])$ as $(h, \tau) \to 0$.*

Denote the discrete Wiener process with increments $\{\xi^n_\tau\}_{n=1}^N$ by $\hat{W}_{h\tau}$, and let

$$F^n_{h\tau} = f^n_{h\tau} - a^n_{h\tau} + (1/2\tau)(u^n_{h\tau} - u^{n-1}_{h\tau}) \times \gamma\, \xi^n_\tau \qquad \text{and} \qquad g^{n-1}_{h\tau} = u^{n-1}_{h\tau} \times \gamma.$$

Then there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a random variable (u, F, g, W) on $\tilde{\Omega}$ with values in $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ with

$$\begin{aligned}\mathbb{X} = G[0, T; U'] \cap L^4[0, T; U]_{weak} \cap L^4[0, T; L^4(D)^3]\\ \times L^{4/3}[0, T; U']_{weak} \times L^2[0, T; H] \times C[0, T],\end{aligned}$$

and a subsequence $(\tau_k, h_k) \to (0, 0)$ for which the laws of $\big\{(u_{h_k\tau_k}, F_{h_k\tau_k}, g_{h_k\tau_k}, \hat{W}_{\tau_k})\big\}_{k=1}^\infty$ converge to the law of (u, F, g, W),

$$\mathcal{L}(u_{h_k\tau_k}, F_{h_k\tau_k}, g_{h_k\tau_k}, \hat{W}_{\tau_k}) \Rightarrow \mathcal{L}(u, F, g, W).$$

In addition, there exists a filtration $\{\tilde{\mathcal{F}}(t)\}_{0\le t\le T}$ satisfying the usual conditions for which (u, f, g, W) is adapted and W is a real-valued Wiener process for which

$$(u(t), v) = (u^0, v) + \int_0^t (F, v)\, ds + \int_0^t (u \times \gamma, v)_H\, dW, \qquad v \in H^1(D), \quad (44)$$

where

$$(F, v) = f - (\nabla u, \nabla v) - (D\phi(u), v) - (1/2)\Big(u \times \gamma) \times \gamma, v\Big). \qquad (45)$$

Proof We verify that $\big\{(u_{h\tau}, F_{h\tau}, g_{h\tau}, \hat{W}_\tau)\big\}_{h,\tau>0}$ satisfy the hypothesis of Theorem 3.2 with parameters $r = q = 4$, $q' = 4/3$, $p = 8/3$, and $L^s[0, T; V] = L^4[0, T; L^4(D)^3]$ in Statement 3.2 of the theorem.

Note first that $\|\phi(u)\|_{L^1(D)} \le C\|u\|^4_{L^4(D)} \le C\|u\|^4_U$ since $H^1(D) \hookrightarrow L^4(D)$. Then under the hypotheses assumed upon the data

$$\|\phi(u^0_{h\tau})\|_{L^{4/3}(\Omega,L^1(D))} \le C\|u^0_{h\tau}\|^4_{L^{16/3}(\Omega,U)} \le C\|u^0_{h\tau}\|^4_{L^8(\Omega,U)} < \infty.$$

1. Lemma 6.7 bounds $\{u_{h\tau}\}_{h,\tau>0}$ in $L^8(\Omega, L^\infty[0, T; U]) \hookrightarrow L^4(\Omega, L^4[0, T; U])$.

2. Lemma 6.7 bounds $\{a_{h\tau}\}_{h,\tau>0}$ in $L^8(\Omega, L^2[0,T;H]) \hookrightarrow L^{8/3}(\Omega, L^{4/3}[0,T;U'])$. Combining this with the bound in Lemma 6.8 shows $\{F_{h\tau}\}_{h,\tau>0}$ is bounded in $L^{8/3}(\Omega, L^{4/3}[0,T;U'])$.
3. Since $L^4[0,T;U]' = L^{4/3}[0,T;U']$ it is immediate that $(F_{h\tau}, u_{h\tau})$ is bounded in $L^{4/3}(\Omega, L^1(0,T))$.
4. The embedding $U = H^1(D)^3 \hookrightarrow\!\!\!\!\rightarrow L^4(D)^3$ is compact, and $\{u_{h\tau}\}_{h,\tau>0}$ is bounded in $L^8(\Omega, L^\infty[0,T;U]) \hookrightarrow L^4[0,T;L^4(D)^3]$, so from Statement 5 of Theorem 3.2 it follows that upon passing to a subsequence $\mathcal{L}(u_{h\tau}) \Rightarrow \mathcal{L}(u)$ on $L^4[0,T;L^4(D)^3]$.
5. Bounds upon $\{u_{h\tau}\}_{h,\tau>0}$ immediately bound $g_{h\tau} = u_{h\tau} \times \gamma$ in $L^{8/3}(\Omega, L^{8/3}[0,T;H])$. In addition, it is immediate that $\mathcal{L}(g_{h\tau}) \Rightarrow \mathcal{L}(g)$ in $L^2[0,T;H]$ when $\mathcal{L}(u_{h\tau}) \Rightarrow \mathcal{L}(u)$ on $L^4[0,T;L^4(D)^3] \hookrightarrow L^2[0,T;H]$.

It follows that upon passing to a sub–sequence $(h_k, \tau_k) \to (0,0)$ there exist a filtered probability space, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\mathcal{F}(t)\}_{0\le t\le T}, \tilde{\mathbb{P}})$, and a random variable (u, F, g, W) taking values in $\mathbb{X}$ for which

$$\mathcal{L}(u_{h_k\tau_k}, F_{h_k\tau_k}, g_{h_k\tau_k}, \hat{W}_{h_k\tau_k}) \Rightarrow \mathcal{L}(u, F, g, W),$$

where W is a standard Wiener process, and equation (44) is satisfied.

To show that F takes the form shown in (45), write $F_{h\tau} = f_{h\tau} + F^{(1)}_{h\tau} + F^{(2)}_{h\tau}$ with

$$(F^{(1)}_{h\tau}, v) = \sum_{n=1}^{T/\tau} -\tau(a^n_{h\tau}, v^n_\tau) \text{ and } (F^{(2)}_{h\tau}, v) = (1/2)\sum_{n=1}^{T/\tau}(u^n_{h\tau} - u^{n-1}_{h\tau}) \times \gamma, v^n_\tau)\, \xi^n_\tau,$$

where v^n_τ is the average of $v \in L^4[0,T;U]$ on $\big((n-1)\tau, n\tau\big)$. Since each summand is bounded in $L^{8/3}(\Omega, L^{4/3}[0,T;U'])$ we may assume $(f_{h\tau}, F^{(1)}_{h\tau}, F^{(2)}_{h\tau}) \Rightarrow (f, F^{(1)}, F^{(2)})$ on $L^{(4/3)}[0,T;U']^3_{weak}$ so that $F = f + F^{(1)} + F^{(2)}$.

Let $A^{(1)} : L^4[0,T;U]_{weak} \cap L^4[0,T;L^4[0,T;L^4(D)] \to L^{4/3}[0,T;U']$ be characterized by

$$\begin{aligned}\big(A^{(1)}(u), v\big) &= \int_0^T (\nabla u, \nabla v) + \big(D\phi(u), v\big)\, ds \\ &= \int_0^T (\nabla u, \nabla v) + (2/\epsilon)\big((u^2-1)u, v\big)\, ds.\end{aligned}$$

The map is continuous, and if $v \in L^4[0,T;U]$ then $\{(A^{(1)}(u_{h\tau}), v)\}_{h,\tau>0}$ is bounded in $L^{4/3}(\Omega)$ and the extended Portmanteau Lemma 2.10 shows

$$\tilde{\mathbb{E}}\Big[|(A^{(1)}(u), v)|\Big] = \lim_{k\to\infty} \mathbb{E}\Big[|(A^{(1)}(u_k), v)|\Big] = \lim_{k\to\infty} \mathbb{E}\left[\sum_{n=1}^{T/\tau_k} \tau_k \left|\Big(A^{(1)}(u^n_k), v^n_{\tau_k}\Big)\right|\right],$$

where we write $u_k \equiv u_{h_k \tau_k}$, and v_{τ_k} is the piecewise constant interpolant of $\{v^n_{\tau_k}\}_{n=1}^{T/\tau_k}$. We then compute

$$
\begin{aligned}
\tilde{\mathbb{E}}\big[\big|\big(A^{(1)}(u) - F^{(1)}, v\big)\big|\big] &= \lim_{k\to\infty} \mathbb{E}\left[\tau_k \left|\sum_{n=1}^{T/\tau_k} \big(a^n_k - D\phi(u^n_k), v^n_{\tau_k}\big) - \big(\nabla u^n_k, \nabla v^n_{\tau_k}\big)\right|\right] \\
&= \lim_{k\to\infty} \mathbb{E}\left[(\tau_k/\epsilon)\left|\sum_{n=1}^{T/\tau_k} \Big((|u^n_k|^2 + |u^{n-1}_k|^2 - 2)u^{n-1/2}_k - 2(|u^n_k|^2 - 1)u^n_k, v^n_{\tau_k}\Big)\right|\right] \\
&= \lim_{k\to\infty} \mathbb{E}\left[(\tau_k/\epsilon)\left|\sum_{n=1}^{T/\tau_k} (1/2)\Big((|u^n_k|^2 + |u^{n-1}_k|^2 - 2)(u^{n-1}_k - u^n_k), v^n_{\tau_k}\Big)\right.\right. \\
&\quad \left.\left. + \Big((|u^{n-1}_k|^2 - |u^n_k|^2)u^n_k, v^n_{\tau_k}\Big)\right|\right].
\end{aligned}
$$

Bounding the right-hand side using Hölder's inequality, and the embedding $U \hookrightarrow L^6(D)$ give

$$
\begin{aligned}
&\tilde{\mathbb{E}}\big[\big|\big(A^{(1)}(u) - F^{(1)}, v\big)\big|\big] \\
&\le (C/\epsilon) \lim_{k\to\infty} \mathbb{E}\left[\sum_{n=1}^{T/\tau_k} \tau_k \|u^n - u^{n-1}\|_{L^2(D)} (\|u^n\|^2_{L^6(D)} + \|u^{n-1}\|^2_{L^6(D)}) \|v^n_{\tau_k}\|_{L^6(D)}\right] \\
&\le (C/\epsilon) \lim_{k\to\infty} \mathbb{E}\left[\sum_{n=1}^{T/\tau_k} \tau_k \|u^n - u^{n-1}\|^2_{L^2(D)}\right]^{1/2} \|u_k\|^2_{L^4(\Omega, L^4[0,T;U])} \|v\|_{L^4[0,T;U]} \\
&= \lim_{k\to\infty} O(\sqrt{\tau_k}) = 0
\end{aligned}
$$

where the last line follows from the estimate in Lemma 6.7 on the norm of the differences.

To identify the Stratonovich term, define $A^{(2)} : L^4[0,T; L^4[0,T; L^4(D)] \to L^{4/3}[0,T; U']$ by

$$
\big(A^{(2)}(u), v\big) = (1/2) \int_0^T \Big((u \times \gamma) \times \gamma, v\Big)\, ds \qquad v \in L^4[0,T; U].
$$

Again this operator is continuous, and the extended Portmanteau Lemma 2.10 shows

$$
\begin{aligned}
\tilde{\mathbb{E}}\Big[\big|(F^{(2)} - A^{(2)}(u), v)\big|\Big] &= \lim_{k\to\infty} \mathbb{E}\Big[\big|(F^{(2)}_k - A^{(2)}(u_k), v)\big|\Big] \\
&= \lim_{k\to\infty} \mathbb{E}\left[(1/2)\left|\sum_{n=1}^{T/\tau_k} \Big((u^{n-1}_k - u^n_k) \times \gamma \xi^n_{\tau_k} - (u^n_k \times \gamma) \times \gamma \tau_k, v^n_\tau\Big)\right|\right].
\end{aligned}
$$

Using the discrete scheme (40) to rewrite the first term gives

$$\tilde{\mathbb{E}}\left[\left|(F^{(2)} - A^{(2)}(u), v)\right|\right] = \lim_{k\to\infty} \mathbb{E}\left[(1/2)\left|\sum_{n=1}^{T/\tau_k} \left(\left(f_k^n - a_k^n\right)\tau_k \xi_{\tau_k}^n\right.\right.\right.$$
$$\left.\left.\left.-\left(u_k^{n-1/2}(\xi_k^n)^2 - u_k^n \tau_k\right) \times \gamma\right), \gamma \times v_\tau^n\right)\right|\right]$$
$$= \lim_{k\to\infty} \mathbb{E}\left[(1/2)\left|\sum_{n=1}^{T/\tau_k}\left(\left(f_k^n - a_k^n\right)\tau_k \xi_{\tau_k}^n - (1/2)\left(u_k^{n-1} - u_k^n\right)\right.\right.\right.$$
$$\left.\left.\left.\times\gamma\,(\xi_k^n)^2 + u_k^n\left((\xi_k^n)^2 - \tau_k\right) \times \gamma, \gamma \times v_\tau^n\right)\right|\right].$$

Each of the three summands on the right vanishes in the limit. The first term is bounded using Hölder's inequality and the bounds assumed upon the moments of the stochastic increments,

$$\mathbb{E}\left[\left|\sum_{n=1}^{T/\tau_k}\left(\left(f_k^n - a_k^n\right)\tau_k \xi_{\tau_k}^n \gamma \times v_\tau^n\right)\right|\right] \le |\gamma|\, \|f_k - a_k\|_{L^2(\Omega, L^2[0,T;L^2(D)])}$$
$$\mathbb{E}\left[\sum_{n=1}^{T/\tau_k} \tau_k (\xi_k^n)^4\right]^{1/4} \|v\|_{L^4[0,T;L^4(D)]}$$
$$\le |\gamma|\, \|f_k - a_k\|_{L^2(\Omega, L^2[0,T;L^2(D)])} (T\tau_k^2)^{1/4}$$
$$\|v\|_{L^4[0,T;L^4(D)]}.$$

To show that the second term vanishes we use the bound on the differences $u^{n-1} - u^n$ from Lemma 6.7,

$$\mathbb{E}\Big[\Big|\sum_{n=1}^{T/\tau_k}\left(\left(u_k^{n-1} - u_k^n\right) \times \gamma\,(\xi_k^n)^2, \gamma \times v_\tau^n\right)\Big|\Big] \le |\gamma|^2$$
$$\mathbb{E}\Big[\sum_{n=1}^{T/\tau_k} \|u_k^{n-1} - u_k^n\|_{L^2(D)}^2\Big]^{\frac{1}{2}} \mathbb{E}\Big[\sum_{n=1}^{T/\tau_k} (\xi_k^n)^4\Big]^{\frac{1}{4}} \|v\|_{L^4[0,T;L^4(D)]}$$
$$\le |\gamma|^2 \mathbb{E}\left[\sum_{n=1}^{T/\tau_k} \|u_k^{n-1} - u_k^n\|_{L^2(D)}^2\right]^{1/2}$$
$$(T\tau_k)^{1/4} \|v\|_{L^4[0,T;L^4(D)]}.$$

The final term is bounded as

$$\mathbb{E}\Big[\Big|\sum_{n=1}^{T/\tau_k}\left(u_k^n\left((\xi_k^n)^2 - \tau_k\right) \times \gamma, \gamma \times v_\tau^n\right)\Big|\Big]$$

$$\leq |\gamma|^2 \, \|u_k\|_{L^4(\Omega, L^4[0,T;L^4(D)])} \mathbb{E}\Big[\sum_{n=1}^{T/\tau_k} \big((\xi_k^n)^2 - \tau_k\big)^2\Big]^{\frac{1}{2}} \|v\|_{L^4[0,T;L^4(D)]}$$
$$\leq |\gamma|^2 \, \|u_k\|_{L^4(\Omega, L^4[0,T;L^4(D)])} C(T\tau_k)^{1/2} \|v\|_{L^4[0,T;L^4(D)]},$$

where the final line follows from the properties the stochastic increments,

$$\mathbb{E}\left[\left((\xi_k^n)^2 - \tau_k\right)^2\right] = \mathbb{E}\left[(\xi_k^n)^4 - 2\tau_k(\xi_k^n)^2 + \tau_k^2\right] = \mathbb{E}\left[(\xi_k^n)^4\right] - \tau_k^2 \leq C\tau_k^2.$$

□

6.4 Monotone operators

The canonical example of a maximally monotone operator is the q Laplacian, $A : U \to U'$, characterized by

$$(A(u), v) = \int_D |\nabla u|^{q-2} \nabla u . \nabla v \, dx, \qquad u, v \in U,$$

defined on the Sobolev space

$$U = W_0^{1,q}(D) = \{u \in L^q(D) \mid \nabla u \in L^q(D)^d \text{ and } u|_{\partial D} = 0\},$$

with $D \subset \mathbb{R}^d$ a bounded domain with Lipschitz boundary. In this section we consider the stochastic version of evolution equations taking the form

$$du + A(u)\, dt = f\, dt + g\, dW, \qquad u(0) = u^0, \tag{46}$$

with $A : U \to U'$ satisfying the following assumptions.

Assumption 6.10 U is a separable reflexive Banach space and H is a Hilbert space with $U \hookrightarrow\!\!\!\rightarrow H \hookrightarrow\!\!\!\rightarrow U'$, and there exist constants $C, c > 0$ and $q \in (1, \infty)$ such that

1. Monotone: $(A(v) - A(u), v - u) \geq 0$ for all $u, v \in U$.
2. Demicontinuous: $A : U_{strong} \to U'_{weak}$ is continuous.
3. Bounded: $\|A(u)\|_{U'} \leq C(1 + \|u\|_U^{q-1})$ for all $u \in U$.
4. Coercive: $(A(u), u) \geq c\|u\|_U^q$ for all $u \in U$.

Theorem 6.11 *Let U be a separable reflexive Banach space, H a Hilbert space, $U \hookrightarrow\!\!\!\rightarrow H$ be a compact, dense embedding, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $1 < q < \infty$ and the operators of the abstract difference scheme (9) and data satisfy Assumptions 6.10 and 3.1 respectively and let the stochastic increments satisfy Assumption 2.5 with $p > 4$. Denote the discrete Wiener process with increments $\{\xi_\tau^m\}_{m=1}^N$ by $\hat{W}_\tau^n$, and let $\{u_{h\tau}\}_{h,\tau>0}$ be a sequence of solutions of the corresponding implicit Euler scheme (9) with data satisfying:*

1. $\{u^0_{h\tau}\}$ *is bounded in* $L^p(\Omega, H)$ *and converges in* $L^2(\Omega, H)$ *as* $h \to 0$.
2. $\{f_{h\tau}\}$ *is bounded in* $L^{pq'/2}(\Omega, L^{q'}[0, T; U'])$ *and converges as* $\tau, h \to 0$.
3. $\{g_{h\tau}\}$ *is bounded in* $L^p(\Omega, L^p[0, T; H])$ *and converges in* $L^2(\Omega, L^2[0, T; H])$ *as* $\tau, h \to 0$.

Let

$$\mathbb{X} \equiv G[0, T; U'] \cap L^q[0, T; U]_{weak} \times L^{q'}[0, T; U'] \\ \times L^{q'}[0, T; U']_{weak} \times L^2[0, T; H] \times C[0, T] .$$

Then there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *and a random variable* (u, f, a, g, W) *on* $\tilde{\Omega}$ *with values in* $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ *for which the laws of* $\{u_{h\tau}, f_{h\tau}, A(u_{h\tau}), g_{h\tau}, \hat{W}_\tau)\}_{k=1}^\infty$ *converge to the law of* (u, f, a, g, W)*,*

$$\mathcal{L}(\hat{u}_{h\tau}, f_{h\tau}, A(u_{h\tau}), g_{h\tau}, \hat{W}_\tau) \Rightarrow \mathcal{L}(u, f, a, g, W).$$

In addition, $\tilde{\mathbb{P}}[u \in C[0, T; U'] \cap L^\infty[0, T; H]] = \tilde{\mathbb{P}}[a = A(u)] = 1$*, and there exists a filtration* $\{\tilde{\mathcal{F}}(t)\}_{0 \le t \le T}$ *satisfying the usual conditions for which* (u, f, g, W) *is adapted and* W *is a real-valued Wiener process for which*

$$(u(t), v)_H + \int_0^t (A(u), v)\, ds = (u^0, v)_H + \int_0^t (f, v)\, ds + \int_0^t (g, v)\, dW, \qquad v \in U.$$

In the previous examples the proof of consistency used the property that the principle part of the operator $A : U \to U'$ was linear. For monotone operators this is no longer the case and the following lemmas provide the properties required to establish the assertion $\tilde{\mathbb{P}}[a = A(u)]$ in the proof of Theorem 6.11. The first result is used to establish consistency in the deterministic setting [37, Lemmas III.2.1 and III.4.2].

Lemma 6.12 *Let* $A : U \to U'$ *be monotone, demicontinuous, and bounded (*i.e.*, bounded sets map to bounded sets).*

- *If* $u_n \rightharpoonup u$ *in* U *and* $A(u_n) \rightharpoonup a$ *in* U' *and* $\limsup_{n\to\infty}(A(u_n), u_n) \le (a, u)$*, then* $a = A(u)$.
- *If* A *satisfies Assumptions 6.10, then so too does its realization* $\mathcal{A} : L^q[0, T; U] \to L^{q'}[0, T; U']$ *given by*

$$(\mathcal{A}(u), v) = \int_0^T (A(u(t), v(t))\, dt.$$

The following lemma is the analog of this lemma for random variables. In the proof of Theorem 6.11 this lemma will be used with Banach space $\mathcal{U} = L^q[0, T; U]$.

Lemma 6.13 (Identification) *Let* $\mathcal{U}$ *be a separable reflexive Banach space and* $\mathcal{A} : \mathcal{U} \to \mathcal{U}'$ *be monotone, demicontinuous and bounded. Let* $(\Omega, \mathbb{P}, \mathcal{F})$ *be a probability triple and* $\{u_n\}_{n=1}^\infty$ *be random variables with values in* $\mathcal{U}$ *satisfying:*

- $\mathcal{L}(u_n, \mathcal{A}(u_n)) \Rightarrow \mathcal{L}(u, a)$ *in* $\mathcal{U}_{weak} \times \mathcal{U}'_{weak}$.

- $\sup_n \mathbb{E}\left[\|u_n\|_{\mathcal{U}}^s + \|\mathcal{A}(u_n)\|_{\mathcal{U}'}^{s'}\right] < \infty$ *for some* $s > 1$.
- $\liminf_{n\to\infty} \mathbb{E}[(\mathcal{A}(u_n), u_n)] \le \mathbb{E}[(a, u)]$.

Then $\mathcal{L}(u, a)[a = \mathcal{A}(u)] = 1$.

We postpone the proof of this lemma until the end of this section.

Proof (of Theorem 6.11) Writing $a(u, v) = (A(u), v)$, we consider the numerical approximation of solutions to equation (46) using the scheme (9) with data (10) from Section 3. Selecting the test function $v_h = u^n_{h\tau}$ in the discrete scheme (9), the coercivity hypothesis gives the bound

$$\begin{aligned}(1/2)\|u^n\|_H^2 + (1/2)\|u^n - u^{n-1}\|_H^2 + c\tau\|u^n\|_U^q \le (1/2)\|u^{n-1}\|_H^2 \\ +\tau(f^n_{h\tau}, u^n_{h\tau}) + (g^{m-1}_{h\tau}, u^m_{h\tau})_H \xi^n_\tau.\end{aligned} \tag{47}$$

It follows from Lemma 6.1 that

$$\begin{aligned}&\|\max_{0\le t\le T} \hat{u}_{h\tau}\|_{L^p(\Omega,H)} + \|u_{h\tau}\|^{q/2}_{L^{pq/2}(\Omega,L^q[0,T;U])} \\ &\le C(T)\left(\|u^0_{h\tau}\|_{L^p(\Omega,H)} + \|f_{h\tau}\|^{q'/2}_{L^{pq'/2}(\Omega,L^{q'}[0,T;U'])} + \|g_{h\tau}\|_{L^p(\Omega,L^2[0,T;H])}\right).\end{aligned}$$

Granted bounds upon the data (u^0, f, g), this estimate establishes the hypotheses of Theorem 3.2 with $F_{h\tau} = f_{h\tau} - A(u_{h\tau}) \equiv F^{(1)}_{h\tau} + F^{(2)}_{h\tau}$ (and moment parameter $\min(pq/2, pq'/2) > 2$), so that, upon passing to a subsequence, there exist a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}(t)\}_{0\le t\le T}, \tilde{\mathbb{P}})$ and a random variable (u, f, a, g, W) with values in $\mathbb{X}$ for which $\mathcal{L}(u_{h\tau}, f_{h\tau}, A(u_{h\tau}), g_{h\tau}, \hat{W}_{h\tau}) \Rightarrow \mathcal{L}(u, f, a, g, W)$ and

$$\begin{aligned}(u(t), v) = (u^0, v) + \int_0^t (f(s) - a(s), v)\, ds \\ + \int_0^t (g(s), v)_H\, dW(s), \qquad 0 \le t \le T,\ v \in U.\end{aligned}$$

Since $A : U \to U'$ satisfies Assumptions 2.15, uniqueness in law holds for solutions of (46), so that upon showing $a = A(u)$ it will follow that it whole sequence converges as asserted in the statement of the theorem.

Lemma 6.13 with $s = pq/2$ is used to verify that $a = A(u)$. Since A has $(q-1)$ growth it follows that

$$\|A(u)\|^{pq'/2}_{L^{q'}[0,T;U']} \le C\left(1 + \|u\|^{pq/2}_{L^q[0,T;U]}\right).$$

Then $s > 1$ and $s' < pq'/2$ when $q > 1$ and $p > 2$, so the growth hypothesis of Lemma 6.13 is satisfied. The third hypothesis is established by showing that the continuous and discrete pairings satisfy

$$\tilde{\mathbb{E}}\left[\int_0^T (a, u)\, ds\right] = \tilde{\mathbb{E}}\left[(1/2)\left(\|u(0)\|_H^2 - \|u(T)\|_H^2\right)\right.$$

$$+\int_0^T \left((f,u)+(1/2)\|g\|_H^2\right)\,ds\Bigg], \tag{48}$$

$$\mathbb{E}\left[\int_0^T (A(u_{h\tau}),u_{h\tau})\,ds\right] \le \mathbb{E}\Bigg[(1/2)\left(\|u_{h\tau}^0\|_H^2-\|u_{h\tau}^N\|_H^2\right) + \int_0^T \left((f_{h\tau},u_{h\tau})+(1/2)\|g_{h\tau}\|_H^2\right)\,ds\Bigg] \tag{49}$$

and to then show that the limit on the right-hand side of the second equation is bounded by the right-hand side of the first.

To verify equation (48), recall that Ito's formula, Theorem 2.14, shows

$$\tilde{\mathbb{E}}\left[(1/2)\|u(T)\|_H^2\right]=\tilde{\mathbb{E}}\left[(1/2)\|u(0)\|_H^2+\int_0^T\left((f-a,u)+(1/2)\|g\|_H^2\right)\,ds\right],$$

which is precisely equation (48).

To verify equation (49), select the test function $v_h=u_{h\tau}^n$ in the discrete scheme (9) to get

$$\begin{aligned}(1/2)\|u_{h\tau}^n\|_H^2&+(1/2)\|u_{h\tau}^n-u_{h\tau}^{n-1}\|_H^2\phi_n+(A(u_{h\tau}^n),u_{h\tau}^n)\\ &=(1/2)\|u_{h\tau}^{n-1}\|_H^2+f(u_{h\tau}^n)\phi^n+(g_{h\tau}^{n-1},u_{h\tau}^n)\xi^n.\end{aligned}$$

Summing this identity and independence of the increments, $\mathbb{E}[(g_{h\tau}^{n-1},u_{h\tau}^{n-1})_H\xi^n]=0$, shows

$$\begin{aligned}&\mathbb{E}\left[\sum_{n=1}^N(1/2)\|u_{h\tau}^N\|_H^2+(1/2)\|u_{h\tau}^n-u_{h\tau}^{n-1}\|_H^2+\int_0^T(A(u_{h\tau}),u_{h\tau})\,ds\right]\\ &=\mathbb{E}\left[(1/2)\|u_{h\tau}^0\|_H^2+\sum_{n=1}^N(f_{h\tau}^n,u_{h\tau}^n)+\sum_{n=1}^N(g_{h\tau}^{n-1},u_{h\tau}^n-u^{n-1})_H\xi^n\right].\end{aligned}$$

Equation (49) follows upon bounding the last term as

$$\begin{aligned}\mathbb{E}\left[\sum_{n=1}^N(g_{h\tau}^{n-1},u_{h\tau}^n-u^{n-1})_H\xi^n\right]&\le\frac{1}{2}\mathbb{E}\left[\sum_{n=1}^N\|g_{h\tau}^{n-1}\|_H^2(\xi^n)^2\right]\\ &+\frac{1}{2}\mathbb{E}\left[\sum_{n=1}^N\|u_{h\tau}^n-u^{n-1}\|_H^2\right],\end{aligned}$$

and recalling that the variance of the increments is the time step, $\mathbb{E}\left[\|g_{h\tau}^{n-1}\|_H^2(\xi^n)^2\right]=\mathbb{E}\left[\|g_{h\tau}^{n-1}\|_H^2\tau\right]$.

To pass to the limit on the right of (49), recall that Example 2.12 shows that, under the hypotheses of the theorem,

$$\tilde{\mathbb{E}}\left[\|u(0)\|_H^2\right] = \lim_{h,\tau\to 0} \mathbb{E}\left[\|u_{h\tau}^0\|_H^2\right] \quad \text{and} \quad \tilde{\mathbb{E}}\left[\|u(T)\|_H^2\right] \le \lim_{h,\tau\to 0} \mathbb{E}\left[\|u_\tau^N\|_H^2\right],$$

where $N = T/\tau$. The function

$$(u, f, a, g, W) \mapsto \int_0^T (f, u) + (1/2)\|g\|_H^2 \, ds$$

is continuous on $\mathbb{X}$ and the numerical approximation of each term has moments with modulus strictly greater than one, so

$$\lim_{h,\tau\to 0} \int_0^T (f_{h\tau}, u_{h\tau}) + (1/2)\|g_{h\tau}\|_H^2 \, ds = \int_0^T (f, u) + (1/2)\|g\|_H^2 \, ds \,.$$

□

We finish this section with the proof of Lemma 6.13.

Proof (of Lemma 6.13) Since $\mathcal{U}$ is separable and reflexive it follows that $\mathcal{U}'$ is also separable, and if u is a Borel measurable random variable with values in $\mathcal{U}$ then $\mathcal{A}(u)$ is a Borel measurable random variable in $\mathcal{U}'$ since $\mathcal{A}$ is demi–continuous. The separability of $\mathcal{U}$ and $\mathcal{U}'$ also implies that

$$\mathcal{B}(\mathcal{U}_{weak} \times \mathcal{U}'_{weak}) = \mathcal{B}(\mathcal{U} \times \mathcal{U}') = \mathcal{B}(\mathcal{U}) \otimes \mathcal{B}(\mathcal{U}').$$

Define $\mathbb{X} = \mathcal{U}_{weak} \times \mathcal{U}'_{weak}$, denote by $\tilde{\mathbb{P}}$ the law of (u, a) on $\mathcal{B}(\mathbb{X})$, and let $B_1, \ldots, B_m$ be Borel sets in $\mathbb{X}$ such that

$$\tilde{\mathbb{P}}\left[\partial B_1 \cup \cdots \cup \partial B_k\right] = 0.$$

Fix $v_1, \ldots, v_k \in \mathcal{U}$ and define

$$f(z) = \sum_{j=1}^k \mathbf{1}_{B_j}(z) v_j.$$

Then $f : \mathbb{X} \to \mathcal{U}_{strong}$ and $\mathcal{A}(f) : \mathbb{X} \to \mathcal{U}'_{strong}$ are uniformly bounded on $\mathbb{X}$ and continuous with respect to sequences $z_n \to z$ where z belongs to $\mathbb{X} \setminus (\partial B_1 \cup \cdots \cup \partial B_k)$; a set of $\tilde{\mathbb{P}}$-measure one. In particular, by the extended Portmanteau Lemma 2.10,

$$\lim_{n\to\infty} \mathbb{E}\left[\Big(\mathcal{A}(u_n), f(u_n, \mathcal{A}(u_n))\Big)\right] = \widetilde{\mathbb{E}}\left[\Big(a, f(u, a)\Big)\right] \tag{50}$$

$$\lim_{n\to\infty} \mathbb{E}\left[\Big(\mathcal{A}(f(u_n, \mathcal{A}(u_n))), u_n\Big)\right] = \widetilde{\mathbb{E}}\left[\Big(\mathcal{A}(f(u, a)), u\Big)\right] \tag{51}$$

$$\lim_{n\to\infty} \mathbb{E}\left[\Big(\mathcal{A}(f(u_n,\mathcal{A}(u_n))),\, f(u_n,\mathcal{A}(u_n))\Big)\right] = \widetilde{\mathbb{E}}\left[\Big(\mathcal{A}(f(u,a)),\, f(u,a)\Big)\right] \quad (52)$$

despite $\mathcal{A}$ not being weakly continuous. By monotonicity,

$$\mathbb{E}\left[\Big(\mathcal{A}(u_n) - \mathcal{A}(f(u_n,\mathcal{A}(u_n))),\, u_n - f(u_n,\mathcal{A}(u_n))\Big)\right] \geq 0$$

so, by the upper semi-continuity assumption on $\{\mathbb{E}\left[\big(\mathcal{A}(u_n), u_n\big)\right]\}$ and (50)–(52),

$$\widetilde{\mathbb{E}}\left[\Big(a - \mathcal{A}(f(u,a)),\, u - f(u,a)\Big)\right] \geq 0. \quad (53)$$

Now $\mathcal{B}_0 = \{B \in \mathcal{B}(\mathbb{X}) : \tilde{\mathbb{P}}(\partial B) = 0\}$ is an algebra such that $\sigma(\mathcal{B}_0) = \mathcal{B}(\mathbb{X})$, thus, if $B_1, \ldots, B_k$ belong to $\mathcal{B}(\mathbb{X})$, then there exist $B_1^n, \ldots, B_k^n$ in $\mathcal{B}_0$ with $n \in \mathbb{N}$ such that

$$f_n(z) = \sum_{j=1}^{k} \mathbf{1}_{B_j^n}(z) v_j \to f(z) = \sum_{j=1}^{k} \mathbf{1}_{B_j}(z) v_j, \qquad \tilde{\mathbb{P}}\text{-almost surely.}$$

Consequently, (53) holds for every Borel simple function f. Demi-continuity of $\mathcal{A}$ then implies that (53) holds for every Borel measurable bounded function f, which then extends (53) to $f \in L^s[(\mathbb{X}, \mathcal{B}(\mathbb{X}), \tilde{\mathbb{P}}); \mathcal{U}]$ by a cut-off argument. In particular, if $\xi : \mathbb{X} \to U$ is Borel measurable and bounded, then applying $f = \pi_1 + t\xi$ to (53) and letting $t \to 0$, we get

$$\tilde{\mathbb{E}}\left[\Big(a - \mathcal{A}(u),\, \xi(u,a)\Big)\right] = 0$$

by demi-continuity of $\mathcal{A}$. In particular, $\tilde{\mathbb{P}}[a = \mathcal{A}(u)] = 1$. □

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A Laws and random variables

Classical probability is well developed for random variables taking values in Polish (complete separable metric) spaces; however, the weak topologies of Banach space that arise for problems involving partial differential operators are not metrizable. In this appendix extensions of the classical results to the current setting are presented.

A.1 Portmanteau theorem for non-metrizable spaces

The following proof is a generalization of the proof of the mapping theorem in [2, Theorem 2.7] which admits sequences of functions which may not be continuous but may, for example, be sequentially continuous or lower semi–continuous.

Proof (of Lemma 2.10) To prove the first assertion, define $\nu_k = \mathbb{P}_k(\zeta_k \in \cdot)$ and $\nu = \mathbb{P}(\zeta_k \in \cdot)$. For $\epsilon > 0$, let C_ε be a compact subset[4] of $\mathcal{X}$ such that $\mathbb{P}_k(C_\varepsilon) \geq 1 - \varepsilon$ and let V be a closed set in $\mathbb{R}$. Then

$$\limsup_{k\to\infty} \nu_k(V) \leq \varepsilon + \limsup_{k\to\infty} \mathbb{P}_k([\zeta_k \in V] \cap C_\varepsilon) \leq \varepsilon + \mathbb{P}\left(\overline{\bigcup_{k\geq n} [\zeta_k \in V] \cap C_\varepsilon}\right)$$

holds by the Portmanteau theorem for every $n \geq 1$, hence

$$\limsup_{k\to\infty} \nu_k(V) \leq \varepsilon + \mathbb{P}\left(\bigcap_{n\geq 1} \overline{\bigcup_{k\geq n} [\zeta_k \in V] \cap C_\varepsilon}\right) \leq \varepsilon + \mathbb{P}(\zeta \in V) + \mathbb{P}^*(N),$$

thus $\nu_k \Rightarrow \nu$ by the Portmanteau theorem.

For the second assertion, let C_ε be a compact set as above. Then

$$\limsup_{k\to\infty} \mathbb{P}_k([\zeta \leq t] \cap C_\varepsilon) \leq \mathbb{P}([\zeta \leq t] \cap C_\varepsilon)$$

by the Portmanteau theorem. So $\liminf_{k\to\infty} \mathbb{P}_k[\zeta > t] \geq \mathbb{P}[\zeta > t]$ and

$$\int_{\mathcal{X}} \zeta \, d\mathbb{P} = \int_0^\infty \mathbb{P}[\zeta > t]\, dt \leq \liminf_{k\to\infty} \int_0^\infty \mathbb{P}_k[\zeta > t]\, dt = \liminf_{k\to\infty} \int_{\mathcal{X}} \zeta \, d\mathbb{P}_k$$

by the Fatou lemma. □

A.2 Proof of Lemma 5.6

Lemma A.1 *Let $Z = C[0, T; U']$ or $G[0, T; U']$, $R \in (0, \infty)$ and define*

$$M_R = \{u \in Z \cap L^r[0, T; U]_{weak} : \|u\|_{L^r[0,T;U]} \leq R\}.$$

Then M_R is closed and metrizable. In particular

- *If $\mathcal{F}$ is a compact in Z then $\mathcal{F} \cap M_R$ is a compact in $Z \cap L^r[0, T; U]_{weak}$.*
- *If $\mathcal{F}$ is a compact in $Z \cap L^r[0, T; U]_{weak}$ then $\mathcal{F}$ is a compact in Z and there exists $R > 0$ such that $\mathcal{F} \subseteq M_R$.*

[4] Note that compacts subsets of $\mathcal{X}$ are metrizable.

Proof Closed balls of separable reflexive Banach spaces (here $L^r[0,T;U]$), equipped with the weak topology, are metrizable and intersections of metric spaces is also a metric space. $\square$

Proof (of Lemma 5.6) Let us consider the modulus of continuity (see (6.2) in [12, Section 3.6])

$$w(u,\delta)=\inf\left\{\sup\left\{\|u(t)-u(s)\|_{U'}\,:\,s,t\in(s_j,s_{j+1}],\,0\le j\le m\right\}\,\mid\,\min_j(s_{j+1}-s_j)>\delta\right\},$$

and observe that $w(u_n,\delta)=0$ if $\delta<T/n$ and

$$w(u_n,\delta)\le 2m(\hat u_n,T/n)+m(\hat u_n,2\delta)\le 3m(\hat u_n,2\delta)\quad\text{if}\quad\delta\ge T/n,$$

where m is the standard modulus of continuity in $C[0,T;U']$. In particular,

$$w(u_n,\delta)\le 3m(\hat u_n,2\delta)\quad\delta\in(0,T).$$

Also, $Rg(u_n)\subseteq Rg(\hat u_n)$. Hence, tightness of $\mathcal{L}(\hat u_n)$ in $C[0,T;U']$ implies tightness of $\mathcal{L}(u_n)$ in $G[0,T;U']$. If μ is the accumulation probability measure then there exists a subsequence n_k such that

- $\mathcal{L}(u_{n_k})\Rightarrow\mu$ in $G[0,T;U']\cap L^r[0,T;U]_{weak}$,
- $\mathcal{L}(u_{n_k},\hat u_{n_k})\Rightarrow\theta$ in $G[0,T;U']\cap L^r[0,T;U]_{weak}\times C[0,T;U']$.

Then μ is the first marginal of θ, and

$$d_G(u_n,\hat u_n)\le\|u_n-\hat u_n\|_{L^\infty[0,T;U']}\le m(\hat u^n,T/n),$$

so

$$1=\lim_{k\to\infty}\mathcal{L}\left(u_{n_k},\hat u_{n_k}\right)\{(x,y):d(x,y)\le\varepsilon\}\le\theta\left\{(x,y):d(x,y)\le\varepsilon\right\},\qquad\varepsilon>0,$$

by the Portmanteau theorem. Hence $\theta(V)=1$ where $V=\{(x,y)\,:\,x=y\}$ and

$$\begin{aligned}&\mu(C[0,T;U']\cap L^r[0,T;U]_{weak})\\&\quad=\theta(C[0,T;U']\cap L^r[0,T;U]_{weak}\times C[0,T;U'])\\&\quad=\theta(C[0,T;U']\cap L^r[0,T;U]_{weak}\times C[0,T;U']\cap V)\\&\quad=\theta(G[0,T;U']\cap L^r[0,T;U]_{weak}\times C[0,T;U']\cap V)=1.\end{aligned}$$ $\square$

A.3 Proof of Theorems 2.17 and 2.18

We adopt the context of Section 2.3.2; specifically, U is a separable Banach space, H is a Hilbert space, and $U\hookrightarrow H\hookrightarrow U'$ are dense embeddings, and write $\mathbb{X}_1=C[0,T;U']\cap L^r[0,T;U]_{weak}$.

The proof Theorems will 2.17 and 2.18 follow from the following two results for random variables taking values in topological spaces. We start with a lemma on existence of a regular version of a random probability measure.

Lemma A.2 *Let X be a topological space such that there exist real continuous functions $h_n : X \to \mathbb{R}$ and, for every $x_0, x_1 \in X$ distinct, there exists $n \in \mathbb{N}$ satisfying $h_n(x_0) \neq h_n(x_1)$. Let $(H, \mathcal{H}, \mu)$ be a probability space,*

1. *$r_B : H \to [0, 1]$ be $\mathcal{H}$-measurable for every $B \in \mathcal{B}(X)$,*
2. *$\mu(r_\emptyset = 0) = 1$, $\mu(r_X = 1) = 1$,*
3. *$\mu(r_{B_0} + r_{B_1} + r_{B_2} + \cdots = r_B) = 1$ whenever $B_0, B_1, B_2, \ldots$ are pair-wise disjoint Borel sets in X and B denotes their union,*
4. *$\mu(r_S = 1) = 1$ for some σ-compact set S in X.*

Then there exists

1. *$R_B : H \to [0, 1]$ which is $\mathcal{H}$-measurable for every $B \in \mathcal{B}(X)$,*
2. *$B \mapsto R_B(h)$ is a Borel probability measure supported in S, for every $h \in H$,*
3. *$\mu(r_B = R_B) = 1$ for every $B \in \mathcal{B}(X)$.*

Proof Existence of regular versions of random probability measures is well know for Polish spaces. Use the functions $\{h_n\}$ to construct an injective mapping $F : X \to Z$ for a suitable Polish space Z. If C is a compact set in X then $F|C : C \to F[C]$ is a homeomorphism. Hence $F|S : S \to F[S]$ and $(F|S)_{-1} : F[S] \to S$ are Borel measurable. Denote by K a regular version of the random probability measure $r_{F^{-1}[A]}(h)$ for $A \in \mathcal{B}(Z)$ and $h \in H$, *i.e.*,

1. $K_A : H \to [0, 1]$ is $\mathcal{H}$-measurable for every $A \in \mathcal{B}(Z)$,
2. $A \mapsto K_A(h)$ is a Borel probability measure for every $h \in H$,
3. $\mu(K_A = r_{F^{-1}[A]}) = 1$ for every $A \in \mathcal{B}(Z)$,

and define $U_B(h) = K_{F[B\cap S]}(h)$ for $B \in \mathcal{B}(X)$ and $h \in H$. Then

1. $U_B : H \to [0, 1]$ is $\mathcal{H}$-measurable for every $B \in \mathcal{B}(X)$,
2. $B \mapsto U_B(h)$ is a Borel measure for every $h \in H$,
3. $\mu(U_B = r_B) = 1$ for every $B \in \mathcal{B}(X)$.

Now we define $K_B(h) = U_B(h)$ for $h \in [U_S = 1]$ and $K_B(h) = \delta_s(B)$ for $h \notin [U_S = 1]$. □

Proposition A.3 *Let Assumption 2.15 hold and θ be a Borel probability measure on $C[0, T; U']$. Then there exists a Borel measurable mapping*

$$k_\theta : C[0, T; U'] \to \mathbb{X}_1$$

with a range in a σ-compact set, and with the following property: If (u, V) is a solution of (8) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{L}(V) = \theta$ then

$$\mathbb{P}\,[u = k_\theta(V)] = 1.$$

Proof The proof follows the argument of the Yamada-Watanabe theorem. Let $\mathbb{Y} = C[0, T; U']$ and assume that (u^i, V^i) is a solution of (8) on a probability space $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i)$ with $\mathcal{L}(V^i) = \theta$. Then

$$\mathcal{B}(\mathbb{X}_1) \otimes \mathcal{B}(\mathbb{X}_1) \otimes \mathcal{B}(\mathbb{Y}) = \mathcal{B}(\mathbb{X}_1 \times \mathbb{X}_1 \times \mathbb{Y}),$$
$$\mathcal{B}(\mathbb{X}_1) \otimes \mathcal{B}(\mathbb{Y}) = \mathcal{B}(\mathbb{X}_1 \times \mathbb{Y}), \quad \mathcal{B}(\mathbb{X}_1) \otimes \mathcal{B}(\mathbb{X}_1) = \mathcal{B}(\mathbb{X}_1 \times \mathbb{X}_1)$$

because

$$\{u \in \mathbb{X}_1 : \|u\|_{L^r[0,T;U]} \leq n\}$$

is separable and metrizable for every $n \in \mathbb{N}$. In particular, $\mathcal{L}(u^i, V^i)$, $i = 0, 1$ are Borel probability measures on $\mathbb{X}_1 \times \mathbb{Y}$. If Q is a Borel set in $\mathbb{X}_1$ then $\mathcal{L}(u^i, V^i)(Q \times \cdot)$ is absolutely continuous with respect to θ. So, by Lemma A.2, there exists $R^i : \mathbb{Y} \times \mathcal{B}(\mathbb{X}_1) \to [0, 1]$ such that

1. $R^i(\cdot, Q) : \mathbb{Y} \to [0, 1]$ is Borel measurable for every $Q \in \mathcal{B}(\mathbb{X}_1)$,
2. $Q \mapsto R^i(y, Q)$ is a Borel probability measure supported in S^i for every $y \in \mathbb{Y}$

and

$$\mathcal{L}(u^i, V^i)(Q \times J) = \int_J R^i(y, Q)\, d\theta(y), \qquad Q \in \mathcal{B}(\mathbb{X}_1), \quad J \in \mathcal{B}(\mathbb{Y}), \quad i = 0, 1.$$

Define a Borel probability measure

$$\mathbb{P}^*(L) = \int_{\mathbb{Y}} (R^0_y \otimes R^1_y)(L^y)\, d\theta(y), \qquad L \in \mathcal{B}(\mathbb{X}_1 \times \mathbb{X}_1 \times \mathbb{Y}).$$

and random variables $U^1(a, b, c) = a$, $U^2(a, b, c) = b$ and $V(a, b, c) = c$ on $\mathbb{X}_1 \times \mathbb{X}_1 \times \mathbb{Y}$. Then

$$\mathcal{L}(U^0, V) = \mathcal{L}(u^0, V^0), \qquad \mathcal{L}(U^1, V) = \mathcal{L}(u^1, V^1)$$

so

$$\mathbb{P}^*\left[U^i(t) = V(t) - \int_0^t A(U^i(s))\, ds\right] = 1, \qquad t \in [0, T], \quad i = 0, 1$$

and, by the uniqueness of the deterministic equation, we obtain that

$$\mathbb{P}^*\left[U^0 = U^1\right] = 1.$$

Hence, if we denote by D the diagonal in $\mathbb{X}_1 \times \mathbb{X}_1$, we get

$$1 = \mathbb{P}^*(D \times \mathbb{Y}) = \int_{\mathbb{Y}} (R^0_y \otimes R^1_y)(D)\, d\theta(y).$$

In particular $(R^0_y \otimes R^1_y)(D) = 1$ for every $y \in M \in \mathcal{B}(\mathbb{Y})$ where $\theta(M) = 1$. So there exists a unique $k(y) \in \mathbb{X}_1$ such that $R^0_y = R^1_y = \delta_{k(y)}$ for every $y \in M$. Set $k(y) = x$ for $y \notin M$ where $x \in \mathbb{X}_1$ is arbitrary. Now $k : \mathbb{Y} \to \mathbb{X}_1$ is Borel measurable with the range in a σ-compact set in $\mathbb{X}_1$ since

$$\{y \in \mathbb{Y} : k(y) \in B\} \cap M = \{y \in \mathbb{Y} : R^0(y, B) = 1\} \cap M,$$

and

$$\mathcal{L}(u^i, V^i)(N) = \theta(\{y \in \mathbb{Y} : (k(y), y) \in N\}), \qquad N \in \mathcal{B}(\mathbb{X}_1 \times \mathbb{Y}).$$

In particular,

$$\mathbb{P}^i\,[u^i = k(V^i)] = 1, \qquad i = 0, 1.$$

□

Proof (of Theorem 2.17) Proposition A.3 yields that

$$\mathcal{L}(u^0, V^0) = \mathcal{L}(k_\theta(V^0), V^0) = \mathcal{L}(k_\theta(V^1), V^1) = \mathcal{L}(u^1, V^1)$$

where $\theta := \mathcal{L}(V^0) = \mathcal{L}(V^1)$. □

Proof (of Theorem 2.18) We apply Proposition A.3 with $\theta = \mathcal{L}(V)$ and $u = k_\theta(V)$. To prove that u is $(\mathcal{F}_t^{V,0})$-adapted let $\tau \in (0, T]$ and define $\lambda = \tau/T \in (0, 1]$, $\tilde{u}_\lambda(t) = \tilde{u}(\lambda t)$, $\tilde{V}_\lambda(t) = \tilde{V}(\lambda t)$ and $V_\lambda(t) = V(\lambda t)$ for $t \in [0, T]$, and $\theta_\tau := \mathcal{L}(\tilde{V}_\lambda) = \mathcal{L}(V_\lambda)$. Then $(\tilde{u}_\lambda, \tilde{V}_\lambda)$ solve

$$du = dV - \lambda A(u)\,dt$$

since $\{w(\lambda\cdot) : w \in S\}$ is σ-compact in $\mathbb{X}_1$ when $w \mapsto w(\lambda\cdot)$ is continuous from $\mathbb{X}_1$ to $\mathbb{X}_1$. If we define $u_\lambda := k_{\theta_\tau}(V_\lambda)$ then

$$du_\lambda = dV_\lambda - \lambda A(u_\lambda)\,dt \quad \text{a.s.}$$

But we also have that

$$du(\lambda\cdot) = dV_\lambda - \lambda A(u(\lambda\cdot))\,dt \quad \text{a.s.}$$

so Assumption 2.15 yields that $u_\lambda(T) = u(\lambda T) = u(\tau)$ a.s. Now $u_\lambda = k_{\theta_\tau}(V_\lambda)$ is $\mathcal{F}_T^{V_\lambda,0}$-measurable and $\mathcal{F}_T^{V_\lambda,0} = \mathcal{F}_\tau^{V,0}$. So $u(\tau)$ is $\mathcal{F}_\tau^{V,0}$-measurable. □

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