A Bismut–Elworthy inequality for a Wasserstein diffusion on the circle

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Abstract

We introduce in this paper a strategy to prove gradient estimates for some infinite-dimensional diffusions on L_2 -Wasserstein spaces. For a specific example of a diffusion on the L_2 -Wasserstein space of the torus, we get a Bismut–Elworthy–Li formula up to a remainder term and deduce a gradient estimate with a rate of blow-up of order $\mathcal{O}(t^{-(2+\varepsilon)})$.

Keywords Bismut–Elworthy–Li formula · Wasserstein diffusion · Interacting particle system · Lions’ differential calculus

Mathematics Subject Classification Primary 60H10 · 60H07 ; Secondary 60H30 · 60K35 · 60J60

1 Introduction

In [42], von Renesse and Sturm introduced a diffusion process on the L_2 -Wasserstein space $\mathcal{P}_2(\mathbb{R})$, satisfying following properties: the large deviations in small time are given by the Wasserstein distance and the martingale term which arises when expanding any smooth function ϕ of the measure argument along the process has exactly the square norm of the Wasserstein gradient of ϕ as local quadratic variation. There are several examples of diffusions on Wasserstein spaces, either constructed via Dirichlet forms [2, 28, 42], as limits of particle systems [21, 22] or as a system of infinitely many particles [32, 34]. Some diffusive properties of those processes have already been

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proved, as e.g. a large deviation principle [29] or as a restoration of uniqueness for McKean–Vlasov equations [34].

We prove in this paper another well-known diffusive property. Indeed, we control the gradient of the semi-group associated to a diffusion process on the L_2 -Wasserstein space. For finite-dimensional diffusions, this gradient estimate can be obtained from a Bismut–Elworthy–Li integration by parts formula. Bismut, Elworthy and Li showed that the gradient of the semi-group $P_t\phi$ associated to the stochastic differential equation $dX_t = \sigma(X_t)dW_t + b(X_t)dt$ on $\mathbb{R}^n$ can be expressed as follows

$$\nabla(P_t\phi)_{x_0}(v_0) = \frac{1}{t}\mathbb{E}\left[\phi(X_t)\int_0^t\langle V_s, \sigma(X_s)dW_s\rangle\right],$$

where V_s is a certain stochastic process starting at v_0 (see [3, 18, 19]). In particular, that equality shows that $\|\nabla(P_t\phi)_{x_0}(v_0)\|$ is of order $t^{-1/2}$ for small times. Important domains of application of Bismut–Elworthy–Li formulae are among others geometry [1, 39, 40], non-linear PDEs [13, 43] or finance [20, 35]. Recent interest has emerged for similar results in infinite dimension. First, Bismut–Elworthy–Li formulae were proved for Kolmogorov equations on Hilbert spaces and for reaction-diffusion systems in bounded domains of $\mathbb{R}^n$, see [11, 15, 16]. Recently, Crisan and McMurray [12] and Baños [5] proved Bismut–Elworthy–Li formulae for McKean–Vlasov equations $dX_t = b(t, X_t, \mu_t)dt + \sigma(t, X_t, \mu_t)dW_t$, with $\mu_t = \mathcal{L}(X_t)$. For other recent smoothing results on McKean–Vlasov equations and mean-field games, see also [4, 7, 9, 10].

1.1 A gradient estimate for a Wasserstein diffusion on the torus

In this paper, we construct a system of infinitely many particles moving on the one-dimensional torus $\mathbb{T} = \mathbb{S}^1$, identified with the interval $[0, 2\pi]$. Considering for each time the empirical measure associated to that system, we get a diffusion process on $\mathcal{P}(\mathbb{T})$, space of probability measures on $\mathbb{T}$. Then we average out that process over the realizations of an additive noise β . This averaging increases the regularity of the process and leads to a gradient estimate of the associated semi-group.

To state more precisely the main result of the paper, let us introduce the following equation

$$x_t^g(u) = g(u) + \sum_{k \in \mathbb{Z}} f_k \int_0^t \Re(e^{-ikx_s^g(u)} dW_s^k) + \beta_t, \quad t \geq 0, \quad u \in [0, 1]. \quad (1)$$

Hereabove, β , $(W^{\Re, k})_{k \in \mathbb{Z}}$ and $(W^{\Im, k})_{k \in \mathbb{Z}}$ are independent standard real-valued Brownian motions, the notation $\Re$ denotes the real part of a complex number and $W^k := W^{\Re, k} + iW^{\Im, k}$. The sequence $(f_k)_{k \in \mathbb{Z}}$ is fixed and real-valued, typically of the form $f_k = \frac{C}{(1+k^2)^{\alpha/2}}$. Lastly, the initial condition $g : [0, 1] \rightarrow \mathbb{R}$ is a C^1 -function with positive derivative satisfying $g(1) = g(0) + 2\pi$, so that g is seen as the quantile function of a probability measure ν_0^g on $\mathbb{T}$.

For each $u \in [0, 1]$, $(x_t^g(u))_{t \in [0, T]}$ represents the trajectory of the stochastic particle starting at $g(u)$, which is driven both by a common noise $W = (W^k)_k$ and by an

idiosyncratic noise β , taking over the terminology usually used for McKean–Vlasov equation. This terminology is justified since Eq. (1) can be seen as the counterpart on the torus $\mathbb{T}$ of the equation on the real line studied in [34] and defined by

$$y_t^g(u) = g(u) + \int_{\mathbb{R}} f(k) \int_0^t \Re(e^{-iky_s^g(u)} dw(k, s)) + \beta_t, \quad t \geq 0, u \in \mathbb{R},$$

where $(w(k, t))_{k \in \mathbb{R}, t \in [0, T]}$ is a complex-valued Brownian sheet on $\mathbb{R} \times [0, T]$.

Moreover, we will show that the cloud of all particles is spread over the whole torus. More precisely, for each $t \in [0, T]$, the probability measure $\nu_t^g = \text{Leb}_{[0, 1]} \circ (x_t^g)^{-1}$ has a density q_t^g w.r.t. Lebesgue measure on $\mathbb{T}$ such that $q_t^g(x) > 0$ for all $x \in \mathbb{T}$. Instead of studying the process $(\nu_t^g)_{t \in [0, T]}$, we consider a more regular process defined by averaging over the realizations of β :

$$\mu_t^g = (\text{Leb}_{[0, 1]} \otimes \mathbb{P}^\beta) \circ (x_t^g)^{-1}, \quad (2)$$

i.e. μ_t^g is the probability measure on $\mathbb{T}$ with density $p_t^g(x) := \mathbb{E}^\beta [q_t^g(x)]$, $x \in \mathbb{T}$, w.r.t. Lebesgue measure¹. In other words, since we assume that β and $(W^k)_{k \in \mathbb{Z}}$ are independent, μ_t^g is the conditional law of x_t^g given $(W^k)_{k \in \mathbb{Z}}$.

The semi-group associated to $(\mu_t^g)_{t \in [0, T]}$ is defined, for any bounded and continuous function $\phi : \mathcal{P}_2(\mathbb{T}) \rightarrow \mathbb{R}$, by $P_t \phi(\mu_0^g) := \mathbb{E} [\phi(\mu_t^g)]$. Alternatively, denoting the lift of ϕ by $\widehat{\phi}(X) := \phi((\text{Leb}_{[0, 1]} \otimes \mathbb{P}^\beta) \circ X^{-1})$ for any random variable $X \in L_2([0, 1] \times \Omega^\beta)$, we can also define the semi-group by $\widehat{P_t \phi}(g) = \mathbb{E} [\widehat{\phi}(x_t^g)]$.

The main theorem of this paper states the following upper bound for the Fréchet derivative of $g \mapsto \widehat{P_t \phi}(g)$ depending only on the L_∞ -norm of ϕ . Assume that $g \in \mathcal{C}^{3+\theta}$ with $\theta > 0$ and let h be a 1-periodic $\mathcal{C}^1$ -function. If ϕ is sufficiently regular and if $f_k = \frac{C}{(1+k^2)^{\alpha/2}}$, $k \in \mathbb{Z}$, with $\alpha \in (\frac{7}{2}, \frac{9}{2})$, then there is C_g independent of h such that

$$\left| \widehat{D P_t \phi}(g) \cdot h \right| = \left| \frac{d}{d\rho} \Big|_{\rho=0} P_t \phi(\mu_0^{g+\rho h}) \right| \leq C_g \frac{\|\phi\|_{L_\infty}}{t^{2+\theta}} \|h\|_{\mathcal{C}^1}, \quad (3)$$

for any $t \in (0, T]$. The precise assumptions over ϕ and the precise statement of this theorem are given below by Definition 11 and Theorem 15, respectively. Moreover, C_g depends polynomially on $\|g'''\|_{L_\infty}$, $\|g''\|_{L_\infty}$, $\|g'\|_{L_\infty}$ and $\|\frac{1}{g'}\|_{L_\infty}$.

1.2 Comments on the main result

The order α of the polynomial decrease of $(f_k)_{k \in \mathbb{Z}}$ has a key role in this paper. In Eq. (1), the diffusion coefficient in front of the noise W is written as a Fourier series, $(f_k)_{k \in \mathbb{Z}}$ being the sequence of Fourier coefficients. Therefore, it should not be surprising that the larger α is, the more regular the solution to (1) is². Nevertheless,

¹ We assume that β and W are defined on probability spaces $(\Omega^\beta, \mathcal{G}^\beta, \mathbb{P}^\beta)$ and $(\Omega^W, \mathcal{G}^W, \mathbb{P}^W)$, respectively.

² see Proposition 4 below.

when we apply Girsanov's Theorem with respect to W , which is part of a standard method introduced by Thalmaier and Wang [39,40], we need α to be sufficiently small, in order to be able to invert the Fourier series³. So there is a balance regarding the choice of α , which explains why we assume in our main result α to be bounded from above and from below.

Moreover, the question of the order α is highly related to the rate $t^{-(2+\theta)}$ appearing in (3). Usually, we expect a rate of $t^{-1/2}$ for diffusions. As we have already mentioned, this rate follows directly from a Bismut-Elworthy-Li integration by parts formula. However, adapting the usual strategy based on Kunita's expansion as in [39,40], we do not get an *exact* integration by parts formula here. Indeed, the failure of the latter strategy in our case comes from the fact that it is impossible to choose α which is simultaneously large enough to ensure a sufficient regularity of the solution and small enough to be able to invert the Fourier series. We refer to remark 22 below for a justification of that claim.

Therefore, the main new strategy introduced in this paper is to regularize the derivative of the solution⁴. By doing this, we get an *approximate* integration by parts formula, in the sense that there is an additional remainder term appearing in the formula:

$$\frac{d}{d\rho}\Big|_{\rho=0} P_t \phi(\mu_0^{g+\rho g'h}) = \frac{1}{t} \mathbb{E}^W \mathbb{E}^\beta \left[\phi(\mu_t^g) \sum_{k \in \mathbb{Z}} \int_0^t \Re(\overline{\lambda_s^{k,\varepsilon}} dW_s^k) \right] + \mathcal{O}(\varepsilon),$$

where $\lambda^{k,\varepsilon}$ is a stochastic process⁵. Controlling the remainder term leads us, by a final bootstrap argument, to the desired upper bound on $|\widehat{DP_t \phi}(g) \cdot h|$, at the prize of worsening the rate of blow-up. We are not claiming that the rate of $t^{-(2+\theta)}$ is sharp, but we expect that a rate of $t^{-1/2}$ is unachievable for this process. Let us mention that the author improves this rate of blow-up to $t^{-(1+\theta)}$, at the prize of assuming $\mathcal{C}^{4+\theta}$ -regularity of g and h . Since the proof is long and technical, it is not included in this paper but we refer to [33, Chapter IV] for all the details and for an application to a gradient estimate for an inhomogeneous SPDE with Hölder continuous source term.

Furthermore, the idiosyncratic noise β is important as well. Of course, the addition of β does not change dramatically the dynamics of the process, since it acts as a rotation on the circle of the whole system. Nevertheless, as it has already been pointed out, the diffusion process $(\mu_t^g)_{t \in [0, T]}$ is defined by an average over the realizations of β . The importance of that averaging is consistent with SPDE theory. Indeed, $(\mu_t^g)_{t \in [0, T]}$ solves the following equation:

$$d\mu_t^g - \frac{1 + \sum_{k \in \mathbb{Z}} f_k^2}{2} \Delta(\mu_t^g) dt + \partial_x \left(\sum_{k \in \mathbb{Z}} f_k \Re(e^{-ik \cdot} dW_t^k) \mu_t^g \right) = 0,$$

³ see Lemma 21 below.

⁴ More exactly, we regularize the function A^g defined by (14) into a convolution $A^{g,\varepsilon}$ defined by (15).

⁵ defined in Lemma 21.

with initial condition $\mu_t^g|_{t=0} = \mu_0^g$. The noise β manifests in the additional term $\frac{1}{2}$ in front of $\Delta(\mu_t^g)$. On the level of the densities, $(p_t^g)_{t \in [0, T]}$ solves the following equation

$$dp_t^g(v) = -\partial_v \left(p_t^g(v) \sum_{k \in \mathbb{Z}} f_k \Re(e^{-ikv} dW_t^k) \right) + \lambda (p_t^g)''(v) dt,$$

with $\lambda = \frac{1 + \sum_{k \in \mathbb{Z}} f_k^2}{2}$. Denis and Stoica showed in [14, 17] that the above equation is well-posed—and they also gave energy estimates—if λ is strictly larger than the critical threshold $\lambda_{\text{crit}} = \frac{\sum_{k \in \mathbb{Z}} f_k^2}{2}$. If we considered Eq. (1) *without* β , we would exactly obtain the above equation with $\lambda = \lambda_{\text{crit}}$. Therefore, it seems that adding a level of randomness is crucial to get our estimate. Precisely, in our above-described strategy, the Brownian motion β plays a key role in controlling the remainder term.

In addition, let us note that we study a process on the one-dimensional circle $\mathbb{T}$, and not on the real line as e.g. in [22, 34]. We made this choice rather for technical reasons, in order to deal with processes of compactly supported measures having a positive density on the whole space. The main result is restricted to functions g which have a strictly positive derivative, meaning that the associated measure has a density with respect to Lebesgue measure on the torus. The constant C_g tends to infinity when $\min_{u \in [0, 1]} g'(u)$ gets closer to zero. The assumptions on the regularity of g and h seem reasonable since our model is close to the process $(\mu_t^{\text{MMAF}})_t$ called *modified massive Arratia flow* introduced in [22], which has highly singular coefficients. Indeed, Konarovskiy and von Renesse showed in [29] that $(\mu_t^{\text{MMAF}})_t$, which is almost surely of finite support for all $t > 0$, solves the following SPDE:

$$d\mu_t^{\text{MMAF}} = \Gamma(\mu_t^{\text{MMAF}})dt + \text{div}(\sqrt{\mu_t^{\text{MMAF}}} dW_t),$$

where Γ is defined by $\langle f, \Gamma(\nu) \rangle = \frac{1}{2} \sum_{x \in \text{Supp}(\nu)} f''(x)$ for any $f \in C_b^2(\mathbb{R})$.

Organization of the paper

The goal of Sect. 2 is to define properly Eq. (1) and to state the main result of the paper. The proof of the theorem is then divided into four main steps. We start in the short Sect. 3 by splitting the gradient of the semi-group into two parts, one regularized term and one remainder term, which are separately studied in Sects. 4 and 5, respectively. Finally in Sect 6, we complete the proof by a bootstrap argument.

2 Statement of the main result

The main result of the paper is stated in Paragraph 2.4. Before, we define precisely the diffusion on the torus (Paragraph 2.1), its associated semi-group (Paragraph 2.2) and the assumptions on the test functions (Paragraph 2.3).

2.1 A diffusion on the torus

In this paper, we study the following stochastic differential equation on a fixed time interval $[0, T]$

$$dx_t^g(u) = \sum_{k \in \mathbb{Z}} f_k \Re \left(e^{-ikx_t^g(u)} dW_t^k \right) + d\beta_t, \quad t \in [0, T], \quad u \in \mathbb{R}, \quad (4)$$

with initial condition $x_0^g = g$. In this paragraph, we first define the assumptions made on W^k , β , f_k and g , where we emphasise the interpretation of $(x_t^g)_{t \in [0, T]}$ as a diffusion on the torus. Then we state existence, uniqueness and some important properties of solutions to Eq. (4).

Let β , $(W_t^{\Re, k})_{k \in \mathbb{Z}}$, $(W_t^{\Im, k})_{k \in \mathbb{Z}}$ be a collection of independent standard real-valued Brownian motions. Thus $W^k = W_t^{\Re, k} + iW_t^{\Im, k}$ denotes a $\mathbb{C}$ -valued Brownian motion. The notation $\Re$ denotes the real part of a complex number, so that Eq. (4) can alternatively be written as follows

$$dx_t^g(u) = \sum_{k \in \mathbb{Z}} f_k \cos(kx_t^g(u)) dW_t^{\Re, k} + \sum_{k \in \mathbb{Z}} f_k \sin(kx_t^g(u)) dW_t^{\Im, k} + d\beta_t.$$

Definition 1 We say that $f := (f_k)_{k \in \mathbb{Z}}$ is of order $\alpha > 0$ if there are $c > 0$ and $C > 0$ such that $\frac{c}{\langle k \rangle^\alpha} \leq |f_k| \leq \frac{C}{\langle k \rangle^\alpha}$ for every $k \in \mathbb{Z}$, where $\langle k \rangle := (1 + |k|^2)^{1/2}$.

Note that if f is of order $\alpha > \frac{1}{2}$, then for each $u \in \mathbb{R}$, the particle $(x_t^g(u))_{t \in [0, T]}$ has a finite quadratic variation equal to $\langle x^g(u), x^g(u) \rangle_t = \left(\sum_{k \in \mathbb{Z}} f_k^2 + 1 \right) t$.

Let $\mathbb{T}$ be the one-dimensional torus, that we identify with the interval $[0, 2\pi]$. $\mathcal{P}(\mathbb{T})$ denotes the space of probability measures on the torus. We consider the L_2 -Wasserstein metric $W_2^{\mathbb{T}}$ on $\mathcal{P}(\mathbb{T})$, defined by $W_2^{\mathbb{T}}(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left(\int_{\mathbb{T}^2} d^{\mathbb{T}}(x, y)^2 d\pi(x, y) \right)^{1/2}$, where $\Pi(\mu, \nu)$ is the set of probability measures on $\mathbb{T}^2$ with first marginal μ and second marginal ν and where $d^{\mathbb{T}}$ is the distance on the torus defined by $d^{\mathbb{T}}(x, y) := \inf_{k \in \mathbb{Z}} |x - y - 2k\pi|$, where $x, y \in \mathbb{R}$.

Definition 2 Let $\mathcal{G}^1$ be the set of $\mathcal{C}^1$ -functions $g : \mathbb{R} \rightarrow \mathbb{R}$ such that for every $u \in \mathbb{R}$, $g'(u) > 0$ and $g(u + 1) = g(u) + 2\pi$. Let $\sim$ be the following equivalence relation on $\mathcal{G}^1$: $g_1 \sim g_2$ if and only if there exists $c \in \mathbb{R}$ such that $g_2(\cdot) = g_1(\cdot + c)$. We denote by $\mathbf{G}^1$ the set of equivalence classes $\mathcal{G}^1 / \sim$.

An interpretation for Definition 2 is that the initial condition g is seen as the quantile function (or inverse c.d.f. function) associated with the measure $\nu_0^g \in \mathcal{P}(\mathbb{T})$ with density $p(x) = \frac{1}{g'(g^{-1}(x))}$, $x \in [0, 2\pi]$, with respect to Lebesgue measure on $\mathbb{T}$. There is a one-to-one correspondence between $\mathbf{G}^1$ and the set of positive densities on the torus, see Paragraph A.1 in “Appendix” for more details.

Existence, uniqueness, continuity and differentiability of solutions to (4) depend on the order α of f and on the regularity of g , as shown by the following two propositions. The proofs, which are classical, are left to “Appendix”.

Proposition 3 *Let $g \in \mathcal{G}^1$ and f be of order $\alpha > \frac{3}{2}$. Then for each $u \in \mathbb{R}$, strong existence and pathwise uniqueness in $\mathcal{C}(\mathbb{R} \times [0, T])$ hold for Eq. (4). Moreover almost surely, for every $u \in [0, 1]$, $(x_t^g(u))_{t \in [0, T]}$ satisfies Eq. (4) and for every $t \in [0, T]$, $u \mapsto x_t^g(u)$ is strictly increasing.*

Proof See paragraph A.2 in “Appendix”. $\square$

For every $j \in \mathbb{N}$ and $\theta \in [0, 1)$, let $\mathcal{C}^{j+\theta}$ denote the set of $\mathcal{C}^j$ -functions whose j^{th} derivative is θ -Hölder continuous. By extension, $\mathcal{G}^{j+\theta} \subseteq \mathcal{G}^1$ and $\mathbf{G}^{j+\theta} \subseteq \mathbf{G}^1$, are the subsets of $\mathcal{G}^1$ and of $\mathbf{G}^1$ consisting of all $\mathcal{C}^{j+\theta}$ -functions and $\mathcal{C}^{j+\theta}$ -equivalence classes, respectively.

Proposition 4 *Let $j \geq 1$, $\theta \in (0, 1)$, $g \in \mathcal{G}^{j+\theta}$ and f be of order $\alpha > j + \frac{1}{2} + \theta$. Then almost surely, for every $t \in [0, T]$, the map $u \mapsto x_t^g(u)$ is a $\mathcal{C}^{j+\theta'}$ -function for every $\theta' < \theta$. Moreover, its first derivative satisfies almost surely*

$$\partial_u x_t^g(u) = g'(u) \exp \left(\sum_{k \in \mathbb{Z}} f_k \int_0^t \Re \left(-i k e^{-i k x_s^g(u)} dW_s^k \right) - \frac{t}{2} \sum_{k \in \mathbb{Z}} f_k^2 k^2 \right),$$

$$u \in \mathbb{R}, t \in [0, T]. \quad (5)$$

Proof See paragraph A.2 in “Appendix”. $\square$

The next proposition states that the flow of the SDE preserves the equivalence classes of quantile functions that we introduced in Definition 2.

Proposition 5 *Let $\theta \in (0, 1)$, $g \in \mathcal{G}^{1+\theta}$ and f be of order $\alpha > \frac{3}{2} + \theta$. Then almost surely, for every $t \in [0, T]$, the map $u \mapsto x_t^g(u)$ belongs to $\mathcal{G}^1$. Moreover, if $g_1 \sim g_2$, then almost surely, $x_t^{g_1} \sim x_t^{g_2}$ for every $t \in [0, T]$.*

Proof By Propositions 3 and 4, it is clear that $u \mapsto x_t^g(u)$ belongs to $\mathcal{C}^1$ and that $\partial_u x_t^g(u) > 0$ for every $u \in \mathbb{R}$. Furthermore, let $(y_t^g)_{t \in [0, T]}$ be the process defined by $y_t^g(u) := x_t^g(u + 1) - 2\pi$. By definition of $\mathcal{G}^{1+\theta}$, $g(u + 1) - 2\pi = g(u)$, thus for every $t \in [0, T]$ and $u \in \mathbb{R}$, $y_t^g(u) = g(u) + \sum_{k \in \mathbb{Z}} f_k \int_0^t \Re \left(e^{-i k y_s^g(u)} dW_s^k \right) + \beta_t$. Therefore $(y_t^g(u))_{t \in [0, T], u \in \mathbb{R}}$ and $(x_t^g(u))_{t \in [0, T], u \in \mathbb{R}}$ satisfy the same equation and belong to $\mathcal{C}(\mathbb{R} \times [0, T])$. By Proposition 3, there is a unique solution in this space. Thus for every $t \in [0, T]$ and every $u \in \mathbb{R}$, $x_t^g(u + 1) - 2\pi = x_t^g(u)$. We deduce that x_t^g belongs to $\mathcal{G}^1$ for every $t \in [0, T]$.

The proof of the second statement is similar; if there is $c \in \mathbb{R}$ such that $g_2(u) = g_1(u + c)$ for every $u \in \mathbb{R}$, then the processes $(x_t^{g_2}(u))_{t \in [0, T], u \in \mathbb{R}}$ and $(x_t^{g_1}(u + c))_{t \in [0, T], u \in \mathbb{R}}$ satisfy the same equation and are equal. $\square$

By Proposition 5, we are able to give a meaning to Eq. (4) with initial value g in $\mathbf{G}^{1+\theta}$. Indeed, for each $t \in [0, T]$, the solution x_t^g will take its values in $\mathbf{G}^{1+\theta'}$ for every $\theta' < \theta$. More generally, by Proposition 4, if the initial condition g belongs to $\mathbf{G}^{j+\theta}$ for $j \geq 1$ and $\theta \in (0, 1)$, then for each $t \in [0, T]$, x_t^g belongs to $\mathbf{G}^{j+\theta'}$ for every $\theta' < \theta$.

Furthermore, the L_p -norms (in the space variable) of the derivatives $\partial_u^{(j)} x_t^g$, $j \geq 1$, and of $\frac{1}{\partial_u x_t^g}$ can be easily controlled with respect to the initial conditions. All the inequalities that will be needed later in this paper were listed and proved in Lemma 39 in “Appendix”.

To conclude this paragraph, let us mention that the solution to Eq. (4) can be equivalently seen as a solution to the following parametric SDE

$$Z_t^x = x + \sum_{k \in \mathbb{Z}} f_k \int_0^t \Re \left(e^{-ikZ_s^x} dW_s^k \right) + \beta_t, \quad x \in \mathbb{R}. \quad (6)$$

Under the same assumptions over f , well-posedness and regularity of solutions to Eq. (6) can be shown, see Proposition 40 in “Appendix”. Moreover, Z_t is closely related to x_t^g by the following identities.

Proposition 6 *Let $g \in \mathbf{G}^{1+\theta}$ and f be of order $\alpha > \frac{3}{2} + \theta$ for some $\theta \in (0, 1)$. Then almost surely, for every $t \in [0, T]$ and $u \in [0, 1]$,*

$$Z_t^{g(u)} = x_t^g(u), \quad (7)$$

$$\partial_x Z_t^{g(u)} = \frac{\partial_u x_t^g(u)}{g'(u)}. \quad (8)$$

Proof See end of paragraph A.2 in “Appendix”. $\square$

2.2 Semi-group averaged out by idiosyncratic noise

According to Proposition 5, $u \in [0, 1] \mapsto x_t^g(u)$ is for each fixed t a quantile function of the measure $\nu_t^g \in \mathcal{P}(\mathbb{T})$ defined by $\nu_t^g := \text{Leb}_{[0,1]} \circ (x_t^g)^{-1}$. However, the stochastic process $(\nu_t^g)_{t \in [0, T]}$ is not regular enough to obtain a gradient estimate for the associated semi-group. Therefore, we average out the realization of the noise β by defining $\mu_t^g := (\text{Leb}_{[0,1]} \otimes \mathbb{P}^\beta) \circ (x_t^g)^{-1}$. In terms of densities, if q_t^g is the density of ν_t^g , then $p_t^g(x) := \mathbb{E}^\beta [q_t^g(x)]$, $x \in \mathbb{T}$, is the density of μ_t^g .

To be more precise, we define three sources of randomness, for the noises W^k , β and the initial condition g , respectively. Let $(\Omega^W, \mathcal{G}^W, (\mathcal{G}_t^W)_{t \in [0, T]}, \mathbb{P}^W)$ and $(\Omega^\beta, \mathcal{G}^\beta, (\mathcal{G}_t^\beta)_{t \in [0, T]}, \mathbb{P}^\beta)$ be filtered probability spaces satisfying usual conditions, on which we define a $(\mathcal{G}_t^W)_{t \in [0, T]}$ -adapted collection $((W_t^k)_{t \in [0, T]})_{k \in \mathbb{Z}}$ of independent $\mathbb{C}$ -valued Brownian motions and a $(\mathcal{G}_t^\beta)_{t \in [0, T]}$ -adapted standard Brownian motion $(\beta_t)_{t \in [0, T]}$, respectively. Let $(\Omega^0, \mathcal{G}^0, \mathbb{P}^0)$ be another probability space rich enough to support $\mathbf{G}^1$ -valued random variables with any possible distribution. We denote by $\mathbb{E}^\beta$, $\mathbb{E}^W$ and $\mathbb{E}^0$ the expectations associated to $\mathbb{P}^\beta$, $\mathbb{P}^W$ and $\mathbb{P}^0$, respectively. Let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in [0, T]}, \mathbb{P})$ be the filtered probability space defined by $\Omega := \Omega^W \times \Omega^\beta \times \Omega^0$, $\mathcal{G} := \mathcal{G}^W \otimes \mathcal{G}^\beta \otimes \mathcal{G}^0$, $\mathcal{G}_t := \sigma((\mathcal{G}_s^W)_{s \leq t}, (\mathcal{G}_s^\beta)_{s \leq t}, \mathcal{G}^0)$ and $\mathbb{P} := \mathbb{P}^W \otimes \mathbb{P}^\beta \otimes \mathbb{P}^0$. Without loss of generality, we assume the filtration $(\mathcal{G}_t)_{t \in [0, T]}$ to be complete and, up to adding negligible subsets to $\mathcal{G}^0$, we assume that $\mathcal{G}_0 = \mathcal{G}^0$.

Definition 7 Fix $t \in [0, T]$ and $\omega \in \Omega^W \times \Omega^0$. Then $x_t^g(\omega)$ is a random variable from $[0, 1] \times \Omega^\beta$ to $\mathbb{R}$. We denote by $\mu_t^g(\omega)$ its law, that is:

$$\mu_t^g(\omega) := (\text{Leb}_{[0,1]} \otimes \mathbb{P}^\beta) \circ (x_t^g(\omega))^{-1}.$$

In particular, μ_t^g is a random variable defined on $\Omega^W \times \Omega^0$ with values in $\mathcal{P}_2(\mathbb{R})$.

Define now the semi-group $(P_t)_{t \in [0, T]}$ associated to $(\mu_t^g)_{t \in [0, T]}$. Let $\phi : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ be a bounded and continuous function. Let $\widehat{\phi} : L_2([0, 1] \times \Omega^\beta) \rightarrow \mathbb{R}$ be the lifted function of ϕ , defined by $\widehat{\phi}(X) := \phi((\text{Leb}_{[0,1]} \otimes \mathbb{P}^\beta) \circ X^{-1})$. In other words, $\widehat{\phi}(X) = \phi(\mathcal{L}_{[0,1] \times \Omega^\beta}(X))$, where $\mathcal{L}_{[0,1] \times \Omega^\beta}(X)$ denotes the law of the random variable $X : [0, 1] \times \Omega^\beta \rightarrow \mathbb{R}$.

Definition 8 For every $t \in \mathbb{R}$ and $\mu \in \mathcal{P}_2(\mathbb{R})$,

$$P_t \phi(\mu) := \mathbb{E}^W [\widehat{\phi}(Z_t^X)],$$

where (Z_t^X) is the solution to SDE (6) and $\mu = \text{Leb}_{[0,1]} \circ X^{-1}$.

Proposition 9 $P_t \phi$ is well-defined and for every $t \in [0, T]$ and $g \in \mathbf{G}^1$,

$$P_t \phi(\mu_0^g) = \mathbb{E}^W [\widehat{\phi}(x_t^g)] = \mathbb{E}^W [\phi(\mu_t^g)]. \quad (9)$$

Proof By Proposition 40, the parametric SDE (6) is strongly well-posed. Thus, if $X, X' \in L_2[0, 1]$ have same law, i.e. $\mathcal{L}_{[0,1]}(X) = \mathcal{L}_{[0,1]}(X')$, then $\mathbb{P}^W$ -almost surely for every $t \in [0, T]$, $\mathcal{L}_{[0,1] \times \Omega^\beta}(Z_t^X) = \mathcal{L}_{[0,1] \times \Omega^\beta}(Z_t^{X'})$. It follows that $\mathbb{E}^W [\widehat{\phi}(Z_t^X)] = \mathbb{E}^W [\widehat{\phi}(Z_t^{X'})]$ for all $t \in [0, T]$, so $\widehat{P_t \phi}(X) := \mathbb{E}^W [\widehat{\phi}(Z_t^X)]$ does not depend on the representative X of the law μ .

Moreover $\mathbb{P}^W$ -almost surely, $\widehat{\phi}(x_t^g) = \phi(\mu_t^g)$. Furthermore by Proposition 6, $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -almost surely, for every $t \in [0, T]$ and for every $u \in [0, 1]$, $Z_t^{g(u)} = x_t^g(u)$. In particular, $\mathbb{P}^W$ -almost surely and for every $t \in [0, T]$, $Z_t^{g(u)} = x_t^g(u)$ holds true $\text{Leb}_{[0,1]} \otimes \mathbb{P}^\beta$ -almost surely. Therefore,

$$P_t \phi(\mu_0^g) = \widehat{P_t \phi}(g) = \mathbb{E}^W [\widehat{\phi}(Z_t^g)] = \mathbb{E}^W [\widehat{\phi}(x_t^g)] = \mathbb{E}^W [\phi(\mu_t^g)],$$

which proves (9). □

2.3 Assumptions on the test functions

The semi-group $(P_t)_{t \in [0, T]}$ acts on bounded and continuous functions $\phi : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$. We will assume the assumptions on ϕ defined hereafter.

Definition 10 We define an equivalence class on $\mathcal{P}_2(\mathbb{R})$ by: $\mu \sim \nu$ if and only if $\mu(A + 2\pi\mathbb{Z}) = \nu(A + 2\pi\mathbb{Z})$ for any $A \in \mathcal{B}[0, 2\pi]$. A function $\phi : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ is said to be $\mathbb{T}$ -stable if $\phi(\mu) = \phi(\nu)$ for any $\mu \sim \nu$. In particular, ϕ induces a map from $\mathcal{P}(\mathbb{T})$ to $\mathbb{R}$.

In particular, for a $\mathbb{T}$ -stable function ϕ and $X \in L_2(\Omega)$, $\widehat{\phi}(X) = \widehat{\phi}(\{X\})$, where $\{x\}$ is the unique number in $[0, 2\pi)$ such that $x - \{x\} \in 2\pi\mathbb{Z}$. Let us mention two important classes of examples of $\mathbb{T}$ -stable functions:

- if $h : \mathbb{R} \rightarrow \mathbb{R}$ is a 2π -periodic function, the map $\phi : \mu \in \mathcal{P}_2(\mathbb{R}) \mapsto \int_{\mathbb{R}} h(x) d\mu(x)$ is $\mathbb{T}$ -stable. The 2π -periodicity condition ensures that $\widehat{\phi}(X) = \mathbb{E}[h(X)] = \mathbb{E}[h(\{X\})] = \widehat{\phi}(\{X\})$.
- if $h : \mathbb{R} \rightarrow \mathbb{R}$ is a 2π -periodic function, the map $\phi : \mu \in \mathcal{P}_2(\mathbb{R}) \mapsto \int_{\mathbb{R}^2} h(x - y) d(\mu \otimes \mu)(x, y)$ is also $\mathbb{T}$ -stable.

If the reader is not familiar with the L-derivative $\partial_\mu \phi$, we refer to Paragraph A.3 in “Appendix” for a short introduction.

Definition 11 A function $\phi : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ is said to satisfy the ϕ -assumptions if the following three conditions hold:

- ($\phi 1$) ϕ is $\mathbb{T}$ -stable, bounded and continuous on $\mathcal{P}_2(\mathbb{R})$.
- ($\phi 2$) ϕ is L-differentiable and $\sup_{\mu \in \mathcal{P}_2(\mathbb{R})} \int_{\mathbb{R}} |\partial_\mu \phi(\mu)(x)|^2 d\mu(x) < +\infty$.
- ($\phi 3$) The Fréchet derivative $D\widehat{\phi}$ is Lipschitz-continuous: there is $C > 0$ such that

$$\mathbb{E} \left[|D\widehat{\phi}(X) - D\widehat{\phi}(Y)|^2 \right] \leq C \mathbb{E} \left[|X - Y|^2 \right], \quad X, Y \in L_2(\Omega).$$

Remark 12 Assumption ($\phi 2$) implies that $\widehat{\phi}$ is Lipschitz-continuous. Therefore $\widehat{\phi}$ belongs to $\mathcal{C}_b^{1,1}(L_2(\Omega))$, the space of bounded and Lipschitz continuous functions on $L_2(\Omega)$ whose Fréchet derivative is also bounded and Lipschitz continuous on $L_2(\Omega)$. Let us mention that the inf-sup convolution method introduced by Lasry and Lions allows to construct, for each bounded uniformly continuous function φ defined on $L_2(\Omega)$, a sequence $(\varphi_n)_n$ of $\mathcal{C}_b^{1,1}(L_2(\Omega))$ -functions converging uniformly to φ on $L_2(\Omega)$, see [31].

The following statement shows that the class of functions satisfying the ϕ -assumptions is stable under the action of $(P_t)_{t \in [0, T]}$.

Proposition 13 Assume that f is of order $\alpha > \frac{5}{2}$. Let $\phi : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ be a function satisfying the ϕ -assumptions. Then for every $t \in [0, T]$, $P_t \phi : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ also satisfies the ϕ -assumptions. Moreover, for any fixed $t \in [0, T]$, the Fréchet derivative of $g \mapsto P_t \phi(\mu_0^g)$ is given by

$$\begin{aligned} \frac{d}{d\rho} \Big|_{\rho=0} P_t \phi(\mu_0^{g+\rho h}) &= D\widehat{P_t \phi}(g) \cdot h = \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 D\widehat{\phi}(x_t^g)_u \frac{\partial_u x_t^g(u)}{g'(u)} h(u) du \right] \\ &= \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \partial_\mu \phi(\mu_t^g)(x_t^g(u)) \frac{\partial_u x_t^g(u)}{g'(u)} h(u) du \right]. \end{aligned} \quad (10)$$

Note that $D\widehat{\phi}(x_t^g)$ is an element of the dual of $L_2([0, 1] \times \Omega^\beta)$, identified here with an element of $L_2([0, 1] \times \Omega^\beta)$. The proof of Proposition 13 is given in Paragraph A.4 in “Appendix”.

2.4 Statement of the main theorem

The main result of this paper is a gradient estimate for the semi-group $(P_t)_{t \in [0, T]}$ associated to $(\mu_t^g)_{t \in [0, T]}$, which is given at points $g \in \mathbf{G}^{3+\theta}$ and for directions of perturbations h defined as follows.

Definition 14 We denote by Δ^1 the set of 1-periodic C^1 -functions $h : \mathbb{R} \rightarrow \mathbb{R}$. We define the following norm on Δ^1 : $\|h\|_{C^1} := \sup_{u \in [0, 1]} |h(u)| + \sup_{u \in [0, 1]} |h'(u)|$.

A simple computation shows that for $|\rho| \ll 1$, $g + \rho h$ still belongs to $\mathbf{G}^1$. Let us state the main theorem.

Theorem 15 Let $\phi : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ satisfy the ϕ -assumptions. Let $\theta \in (0, 1)$ and f be of order $\alpha = \frac{7}{2} + \theta$. Let $g \in \mathbf{G}^{3+\theta}$ and $h \in \Delta^1$ be two deterministic functions. Then there is C_g independent of h such that for every $t \in (0, T]$

$$\left| \frac{d}{d\rho} \Big|_{\rho=0} P_t \phi(\mu_0^{g+\rho h}) \right| \leq C_g \frac{\|\phi\|_{L^\infty}}{t^{2+\theta}} \|h\|_{C^1}, \quad (11)$$

where C_g is bounded when $\|g'''\|_{L^\infty} + \|g''\|_{L^\infty} + \|g'\|_{L^\infty} + \|\frac{1}{g'}\|_{L^\infty}$ is bounded.

In the following section, we split the l.h.s. of (11) into two terms, I_1 and I_2 , which will be studied separately in Sects. 4 and 5, respectively.

3 Preparation for the proof

To start this paragraph, we rewrite the gradient of the semi-group in terms of the L-derivative of $P_t \phi$ and of the linear functional derivative of $P_t \phi$. We refer to paragraph A.3 in “Appendix” for a short remainder about definitions and relationships between the different types of derivatives.

For convenience, the following lemma is written for a perturbation $g'h$ instead of h (the corresponding result for h can be naturally obtained by applying the following formula to $\frac{h}{g'}$ instead of h). For later purposes, the lemma is stated for random functions g and h , with a $\mathcal{G}_0$ -measurable randomness. Recall that within this framework, g and h are independent of $((W^k)_{k \in \mathbb{Z}}, \beta)$.

Lemma 16 Let ϕ , θ and f be as in Theorem 15. Let g and h be $\mathcal{G}_0$ -measurable random variables with values respectively in $\mathbf{G}^{3+\theta}$ and Δ^1 . Then for every $t \in [0, T]$,

$$\begin{aligned} \frac{d}{d\rho} \Big|_{\rho=0} P_t \phi(\mu_0^{g+\rho g'h}) &= \int_0^1 \partial_\mu(P_t \phi)(\mu_0^g)(g(u)) g'(u) h(u) \, du \\ &= \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \partial_\mu \phi(\mu_t^g)(x_t^g(u)) \partial_u x_t^g(u) h(u) \, du \right] \\ &= - \int_0^1 \frac{\delta P_t \phi}{\delta m}(\mu_0^g)(g(u)) h'(u) \, du. \end{aligned} \quad (12)$$

Proof Fix ω^0 in an almost-sure event of Ω^0 such that $g = g(\omega^0)$ belongs to $\mathbf{G}^{3+\theta}$ and $h = h(\omega^0)$ belongs to Δ^1 . Since g' is 1-periodic and positive, $\rho_0 := \frac{1}{\|h'\|_{L^\infty}} \inf_{u \in \mathbb{R}} g'(u)$ is positive and $g + \rho h$ belongs to $\mathbf{G}^1$ for every $\rho \in (-\rho_0, \rho_0)$. The first equality in (12) follows from the definition of the L-derivative:

$$\begin{aligned} \frac{d}{d\rho} \Big|_{\rho=0} P_t \phi(\mu_0^{g+\rho g' h}) &= \frac{d}{d\rho} \Big|_{\rho=0} \widehat{P_t \phi}(g + \rho g' h) = D \widehat{P_t \phi}(g) \cdot (g' h) \\ &= \int_0^1 D \widehat{P_t \phi}(g)_u g'(u) h(u) du \\ &= \int_0^1 \partial_\mu(P_t \phi)(\mu_0^g)(g(u)) g'(u) h(u) du. \end{aligned} \quad (13)$$

The second equality in (12) was already stated in Proposition 13. For the third equality in (12), we use the relationship between the L-derivative and the functional linear derivative (see Proposition 41)

$$\begin{aligned} \int_0^1 \partial_\mu(P_t \phi)(\mu_0^g)(g(u)) g'(u) h(u) du &= \int_0^1 \partial_v \left\{ \frac{\delta P_t \phi}{\delta m}(\mu_0^g) \right\} (g(u)) g'(u) h(u) du \\ &= \int_0^1 \partial_u \left\{ \frac{\delta P_t \phi}{\delta m}(\mu_0^g)(g(\cdot)) \right\} (u) h(u) du \\ &= \left[\frac{\delta P_t \phi}{\delta m}(\mu_0^g)(g(u)) h(u) \right]_0^1 \\ &\quad - \int_0^1 \frac{\delta P_t \phi}{\delta m}(\mu_0^g)(g(u)) h'(u) du, \end{aligned}$$

by an integration by parts formula. Furthermore, $v \mapsto \frac{\delta P_t \phi}{\delta m}(\mu_0^g)(v)$ is 2π -periodic. This follows from Proposition 45 in “Appendix”, because on the one hand $P_t \phi$ satisfies the ϕ -assumptions by Proposition 13 and on the other hand the probability measure μ_0^g has density g' on the torus, which is strictly positive everywhere. It follows that $\frac{\delta P_t \phi}{\delta m}(\mu_0^g)(g(1)) = \frac{\delta P_t \phi}{\delta m}(\mu_0^g)(g(0) + 2\pi) = \frac{\delta P_t \phi}{\delta m}(\mu_0^g)(g(0))$. Since h is 1-periodic, we conclude that $\left[\frac{\delta P_t \phi}{\delta m}(\mu_0^g)(g(u)) h(u) \right]_0^1 = 0$. $\square$

By Proposition 5, x_t^g belongs to $\mathbf{G}^1$ for every $t \in [0, T]$. In particular, $u \mapsto x_t^g(u)$ is invertible and its inverse $F_t^g := (x_t^g)^{-1}$ satisfies $F_t^g(x + 2\pi) = F_t^g(x) + 1$. We define $(A_t^g)_{t \in [0, T]}$ by

$$A_t^g := (\partial_u x_t^g)(F_t^g(\cdot)) h(F_t^g(\cdot)). \quad (14)$$

and $(A_t^{g, \varepsilon})_{t \in [0, T]}$ by

$$A_t^{g, \varepsilon} := A_t^g * \varphi_\varepsilon = \int_{\mathbb{R}} A_t^g(\cdot - y) \varphi_\varepsilon(y) dy, \quad (15)$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and $\varphi_\varepsilon(x) = \frac{1}{\varepsilon} \varphi(\frac{x}{\varepsilon})$.

Lemma 17 For any $t \in [0, T]$ and $\varepsilon > 0$, A_t^g is a 2π -periodic C^1 -function and $A_t^{g,\varepsilon}$ is a 2π -periodic C^∞ -function.

Proof The periodicity property follows from the fact that h and $\partial_u x_t^g$ are 1-periodic and from $F_t^g(x + 2\pi) = F_t^g(x) + 1$. Furthermore, by Proposition 4, $u \mapsto x_t^g(u)$ belongs to $C^{3+\theta'}$ for some $\theta' < \theta$, thus $\partial_u x_t^g \in C^{2+\theta'}$. Therefore, A_t^g is a C^1 -function. The properties of $A_t^{g,\varepsilon}$ follows. $\square$

We conclude this paragraph by splitting the derivative of $P_t\phi$ into two terms. I_1 involves the regularized function $A_t^{g,\varepsilon}$, whereas I_2 is a remainder term for which we have to show that it is small with respect to $|\varepsilon|$.

Proposition 18 Under the same assumptions as in Lemma 16 and for every $t \in [0, T]$

$$\frac{d}{d\rho} \Big|_{\rho=0} P_t\phi(\mu_0^{g+\rho g'h}) = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &:= \frac{1}{t} \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^t \partial_\mu \phi(\mu_t^g)(x_t^g(u)) \frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)} A_s^{g,\varepsilon}(x_s^g(u)) ds du \right]; \\ I_2 &:= \frac{1}{t} \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^t \partial_\mu \phi(\mu_t^g)(x_t^g(u)) \frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)} (A_s^g - A_s^{g,\varepsilon})(x_s^g(u)) ds du \right]. \end{aligned}$$

Proof By definition of $(A_t^g)_{t \in [0, T]}$,

$$\begin{aligned} h(u) &= \frac{1}{t} \int_0^t h(u) ds = \frac{1}{t} \int_0^t \frac{\partial_u x_s^g(F_s^g(x_s^g(u))) h(F_s^g(x_s^g(u)))}{\partial_u x_s^g(u)} ds \\ &= \frac{1}{t} \int_0^t \frac{A_s^g(x_s^g(u))}{\partial_u x_s^g(u)} ds. \end{aligned}$$

Therefore, Eq. (12) rewrites

$$\begin{aligned} \frac{d}{d\rho} \Big|_{\rho=0} P_t\phi(\mu_0^{g+\rho g'h}) \\ = \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \partial_\mu \phi(\mu_t^g)(x_t^g(u)) \partial_u x_t^g(u) \left(\frac{1}{t} \int_0^t \frac{A_s^g(x_s^g(u))}{\partial_u x_s^g(u)} ds \right) du \right]. \end{aligned}$$

The r.h.s. of the latter equality is clearly equal to $I_1 + I_2$. $\square$

The proof of Theorem 15 is now divided into three main steps. In the following two sections, we will study separately I_1 and I_2 . This will lead to the estimate stated in Corollary 32. In Sect 6, we conclude the proof by iterating the result of Corollary 32 over successive time intervals.

4 Analysis of I_1

In order to control I_1 , we adapt in this section a method of proof introduced in [39]. To follow that strategy, we take benefit from the fact that $A_t^{g,\varepsilon}$, in contrast to A_t^g , is as regular as needed. The drawback is that the control on I_1 blows up when ε goes to 0, which is the reason why the explosion rate is $t^{-2-\theta}$ in Theorem 15 and not $t^{-1/2}$ as in [39].

Let us define

$$a_t(u) = \int_0^t \frac{g'(u)}{\partial_u x_s^g(u)} A_s^{g,\varepsilon}(x_s^g(u)) ds.$$

Using that notation, I_1 rewrites as follows

$$I_1 = \frac{1}{t} \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \partial_\mu \phi(\mu_t^g)(x_t^g(u)) \frac{\partial_u x_t^g(u)}{g'(u)} a_t(u) du \right].$$

The goal of this section is to prove the following inequality:

Proposition 19 *Let ϕ , θ and f be as in Theorem 15. Let g and h be $\mathcal{G}_0$ -measurable random variables with values respectively in $\mathbf{G}^{3+\theta}$ and Δ^1 . Then there is $C > 0$ independent of g , h and θ such that for every $t \in [0, T]$, for every $\varepsilon \in (0, 1)$,*

$$|I_1| \leq C \frac{\|\phi\|_{L_\infty}}{\varepsilon^{3+2\theta} \sqrt{t}} C_1(g) \|h\|_{C^1}. \quad (16)$$

where $C_1(g) = 1 + \|g'''\|_{L_4}^2 + \|g''\|_{L_\infty}^6 + \|g'\|_{L_\infty}^8 + \left\| \frac{1}{g'} \right\|_{L_\infty}^8$.

The proof of the proposition is based on writing the SDE satisfied by $(Z_t^{g(u)+\rho a_t(u)})_{t \in [0, T]}$, where Z_t^x is the solution to (6). We recall this expansion, known as Kunita's theorem, in the following lemma:

Lemma 20 *Let f be of order $\alpha > \frac{3}{2} + \theta$ for some $\theta \in (0, 1)$. Let $(\zeta_t)_{t \in [0, T]}$ be a $(\mathcal{G}_t)_{t \in [0, T]}$ -adapted process such that $t \mapsto \zeta_t$ is absolutely continuous, $\zeta_0 = 0$ almost surely and $\mathbb{E} \left[\int_0^T |\zeta_t| dt \right]$ is finite. Then almost surely, for every $x \in \mathbb{R}$, $t \in [0, T]$ and $\rho \in \mathbb{R}$,*

$$Z_t^{x+\rho \zeta_t} = Z_0^x + \sum_{k \in \mathbb{Z}} f_k \int_0^t \Re \left(e^{-ik Z_s^{x+\rho \zeta_s}} dW_s^k \right) + \beta_t + \rho \int_0^t \partial_x Z_s^{x+\rho \zeta_s} \dot{\zeta}_s ds.$$

Proof This is an application of Theorem 3.3.1 in [27, Chapter III]. $\square$

4.1 Fourier inversion on the torus

A key ingredient in the study of I_1 is the following Fourier inversion of $A_t^{g,\varepsilon}$, which we will later use in order to apply Girsanov's theorem.

Lemma 21 Let $\theta \in (0, 1)$, $g \in \mathbf{G}^{3+\theta}$ and f be of order $\alpha = \frac{7}{2} + \theta$. Fix $\varepsilon \in (0, 1)$. Then there is a collection of $(\mathcal{G}_t)_{t \in [0, T]}$ -adapted $\mathbb{C}$ -valued processes $((\lambda_t^k)_{t \in [0, T]})_{k \in \mathbb{Z}}$ such that for every $t \in [0, T]$, the following equality holds

$$A_t^{g, \varepsilon}(y) = \sum_{k \in \mathbb{Z}} f_k e^{-iky} \lambda_t^k, \quad (17)$$

and such that there is a constant $C > 0$ independent of ε satisfying for each $t \in [0, T]$

$$\mathbb{E}^W \mathbb{E}^\beta \left[\sum_{k \in \mathbb{Z}} \int_0^t |\lambda_s^k|^2 ds \right] \leq C \frac{t}{\varepsilon^{6+4\theta}} \|h\|_{\mathcal{C}^1}^2 C_1(g)^2, \quad (18)$$

where $C_1(g) = 1 + \|g'''\|_{L^4}^2 + \|g''\|_{L^\infty}^6 + \|g'\|_{L^\infty}^6 + \left\| \frac{1}{g'} \right\|_{L^\infty}^8$.

Proof Fix $t \in (0, T]$. The map $y \mapsto A_t^{g, \varepsilon}(y)$ is a 2π -periodic $\mathcal{C}^1$ -function. Therefore, by Dirichlet's Theorem, that map is equal to the sum of its Fourier series

$$A_t^{g, \varepsilon}(y) = \sum_{k \in \mathbb{Z}} c_k(A_t^{g, \varepsilon}) e^{-iky},$$

where $c_k(A) := \frac{1}{2\pi} \int_0^{2\pi} A(y) e^{iky} dy$ for every 2π -periodic function A and for every $k \in \mathbb{Z}$.

Let us define $\lambda_t^k := \frac{c_k(A_t^{g, \varepsilon})}{f_k}$. Since $(A_t^{g, \varepsilon})_{t \in [0, T]}$ is $(\mathcal{G}_t)$ -adapted, it is clear that for each $k \in \mathbb{Z}$, $(\lambda_t^k)_{t \in [0, T]}$ is also $(\mathcal{G}_t)$ -adapted. Equality (17) clearly holds true. Moreover,

$$\sum_{k \in \mathbb{Z}} \int_0^t |\lambda_s^k|^2 ds = \sum_{k \in \mathbb{Z}} \int_0^t \left| \frac{c_k(A_s^{g, \varepsilon})}{f_k} \right|^2 ds.$$

Compute the Fourier coefficient $c_k(A_s^{g, \varepsilon})$:

$$\begin{aligned} c_k(A_s^{g, \varepsilon}) &= c_k(A_s^g * \varphi_\varepsilon) = \frac{1}{2\pi} \int_0^{2\pi} \left(\int_{\mathbb{R}} A_s^g(y-x) \varphi_\varepsilon(x) dx \right) e^{iky} dy \\ &= \int_{\mathbb{R}} \varphi_\varepsilon(x) e^{ikx} \left(\frac{1}{2\pi} \int_0^{2\pi} A_s^g(y-x) e^{ik(y-x)} dy \right) dx \\ &= c_k(A_s^g) \int_{\mathbb{R}} \varphi(x) e^{ik\varepsilon x} dx. \end{aligned} \quad (19)$$

Since $\int_{\mathbb{R}} \varphi(x) e^{ik\varepsilon x} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-x^2/2} e^{ik\varepsilon x} dx = e^{-k^2\varepsilon^2/2}$, there is in particular $C > 0$ such that for every $k \in \mathbb{Z} \setminus \{0\}$ and for every $\varepsilon > 0$, $|\int_{\mathbb{R}} \varphi(x) e^{ik\varepsilon x} dx| \leq \frac{C}{|k\varepsilon|^{3+2\theta}}$.

Moreover, A_s^g is a $\mathcal{C}^1$ -function. Thus there is C independent of k and s such that for every $k \in \mathbb{Z} \setminus \{0\}$, $|c_k(A_s^g)| \leq \frac{C}{|k|} \|\partial_x A_s^g\|_{L^\infty}$. Furthermore, $|c_0(A_s^{g, \varepsilon})| = |c_0(A_s^g)| \leq$

$\|A_s^g\|_{L_\infty}$. Since f is of order $\alpha = \frac{7}{2} + \theta$, there is C such that for every $k \in \mathbb{Z} \setminus \{0\}$, $\frac{1}{|f_k|} \leq C|k|^{\frac{7}{2}+\theta}$. Thus we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \int_0^t \frac{|c_k(A_s^{g,\varepsilon})|^2}{f_k^2} ds &= \int_0^t \frac{|c_0(A_s^{g,\varepsilon})|^2}{f_0^2} ds + \sum_{k \neq 0} \int_0^t \frac{|c_k(A_s^{g,\varepsilon})|^2}{f_k^2} ds \\ &\leq C \int_0^t \|A_s^g\|_{L_\infty}^2 ds + C \sum_{k \neq 0} \int_0^t |k|^{7+2\theta} \frac{1}{|k\varepsilon|^{6+4\theta}} \frac{1}{|k|^2} \|\partial_x A_s^g\|_{L_\infty}^2 ds \\ &\leq \frac{C}{\varepsilon^{6+4\theta}} \int_0^t \|A_s^g\|_{C^1}^2 ds, \end{aligned}$$

because $1 \leq \frac{1}{\varepsilon}$ and because the sum $\sum_{k \neq 0} \frac{1}{|k|^{1+2\theta}}$ converges. Thus there is a constant $C > 0$ independent of ε satisfying the $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -almost surely for every $t \in [0, T]$

$$\sum_{k \in \mathbb{Z}} \int_0^t |\lambda_s^k|^2 ds \leq \frac{C}{\varepsilon^{6+4\theta}} \int_0^t \|A_s^g\|_{C^1}^2 ds. \quad (20)$$

Let us compute $\mathbb{E}^W \mathbb{E}^\beta \left[\|A_s^g\|_{C^1}^2 \right]$. Recall that $\|A_s^g\|_{L_\infty} \leq \|\partial_u x_s^g\|_{L_\infty} \|h\|_{L_\infty}$. Thus for every $s \in [0, T]$,

$$\begin{aligned} \mathbb{E}^W \mathbb{E}^\beta \left[\|A_s^g\|_{L_\infty}^2 \right] &\leq \|h\|_{L_\infty}^2 \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \|\partial_u x_t^g\|_{L_\infty}^2 \right] \\ &\leq C \|h\|_{L_\infty}^2 (1 + \|g''\|_{L_2}^2 + \|g'\|_{L_\infty}^4), \end{aligned}$$

where we used inequality (71) given in “Appendix”. Moreover, the derivative of A_s^g is equal to:

$$\partial_x A_s^g(x) = \frac{(h' \partial_u x_s^g + h \partial_u^{(2)} x_s^g)(F_s^g(x))}{\partial_u x_s^g(F_s^g(x))} = h'(F_s^g(x)) + \frac{h(F_s^g(x)) \partial_u^{(2)} x_s^g(F_s^g(x))}{\partial_u x_s^g(F_s^g(x))}.$$

We deduce that

$$\|\partial_x A_s^g\|_{L_\infty} \leq C \|h\|_{C^1} \left(1 + \|\partial_u^{(2)} x_s^g\|_{L_\infty} \left\| \frac{1}{\partial_u x_s^g} \right\|_{L_\infty} \right). \quad (21)$$

Therefore, for every $s \leq T$,

$$\begin{aligned} \mathbb{E}^W \mathbb{E}^\beta \left[\|\partial_x A_s^g\|_{L_\infty}^2 \right] &\leq C \|h\|_{C^1}^2 \left(1 + \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \|\partial_u^{(2)} x_t^g\|_{L_\infty}^4 \right] + \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \left\| \frac{1}{\partial_u x_t^g} \right\|_{L_\infty}^4 \right] \right) \\ &\leq C \|h\|_{C^1}^2 \left(1 + \|g'''\|_{L_4}^4 + \|g''\|_{L_\infty}^{12} + \|g'\|_{L_\infty}^{16} + \left\| \frac{1}{g'} \right\|_{L_\infty}^{16} \right), \end{aligned} \quad (22)$$

by (71) and (72) and because g belongs to $\mathbf{G}^{3+\theta}$ and $\alpha = \frac{7}{2} + \theta$. We deduce that

$$\begin{aligned} & \mathbb{E}^W \mathbb{E}^\beta \left[\sum_{k \in \mathbb{Z}} \int_0^t |\lambda_s^k|^2 ds \right] \\ & \leq \frac{Ct}{\varepsilon^{6+4\theta}} \|h\|_{\mathcal{C}^1}^2 \left(1 + \|g'''\|_{L^4}^4 + \|g''\|_{L^\infty}^{12} + \|g'\|_{L^\infty}^{16} + \left\| \frac{1}{g'} \right\|_{L^\infty}^{16} \right), \end{aligned}$$

which is inequality (18). This completes the proof of the lemma. $\square$

Remark 22 After the proof of Lemma 21, we can now explain precisely why a regularization of A^g was needed. Imagine for a while that instead of looking for the Fourier inverse of $A^{g,\varepsilon}$ in (17), we were looking for the Fourier inverse of A^g . In order to prove an inequality like (18), we would have to show that

$$\sum_{k \in \mathbb{Z}} (1 + k^2)^\alpha \int_0^t |c_k(A_s^g)|^2 ds < \infty.$$

The latter sum converges if $\mathbb{E} \left[\int_0^t \|A_s^g\|_{\mathcal{C}^p}^2 ds \right]$ is bounded for a certain $p > 1 + 2\alpha$. In turn, if the latter expectation is bounded then for almost every s , $y \mapsto A_s^g(y)$ is of class $\mathcal{C}^p$. But we know, by definition (14) of A_s^g and by Proposition 4, that $A_s^g \in \mathcal{C}^p$ if $h \in \mathcal{C}^p$, $g \in \mathcal{C}^{p+\theta}$ and $\alpha > p + \frac{1}{2}$. The regularity of h is not a big problem, since we could simply assume higher regularity in the assumptions of Theorem 15. However, it is impossible to choose α so that both inequalities $p > 1 + 2\alpha$ and $\alpha > p + \frac{1}{2}$ hold simultaneously. Regularizing A^g allows to work with $A_s^{g,\varepsilon} \in \mathcal{C}^p$ without having to assume that $\alpha > p + \frac{1}{2}$.

4.2 A Bismut–Elworthy-like formula

Let us state and prove an integration by parts formula, close to Bismut–Elworthy formula.

Proposition 23 *Let $\theta \in (0, 1)$, $g \in \mathbf{G}^{3+\theta}$ and f be of order $\alpha = \frac{7}{2} + \theta$. For every $t \in (0, T]$,*

$$I_1 = \frac{1}{t} \mathbb{E}^W \mathbb{E}^\beta \left[\phi(\mu_t^g) \sum_{k \in \mathbb{Z}} \int_0^t \Re(\overline{\lambda_s^k} dW_s^k) \right], \quad (23)$$

where $\overline{\lambda_s^k}$ denotes the complex conjugate of λ_s^k .

In view of proving Proposition 23, let us introduce the following stopping times. Let M_0 be an integer large enough so that for every $u \in \mathbb{R}$, $\frac{1}{M_0} < g'(u) < M_0$. For every $M \geq M_0$, define:

$$\tau_M^1 := \inf\{t \geq 0 : \|\partial_u x_t^g\|_{L^\infty} \geq M\} \wedge T;$$

$$\begin{aligned}\tau_M^2 &:= \inf\{t \geq 0 : \left\| \frac{1}{\partial_u x_t^g(\cdot)} \right\|_{L_\infty} \geq M\} \wedge T; \\ \tau_M &:= \tau_M^1 \wedge \tau_M^2.\end{aligned}\quad (24)$$

Since $g \in \mathbf{G}^{3+\theta}$ and f is of order $\alpha > \frac{7}{2}$, inequalities (71) and (72) imply that

$$\mathbb{P}^W \otimes \mathbb{P}^\beta [\tau_M < T] \xrightarrow{M \rightarrow +\infty} 0. \quad (25)$$

Lemma 24 *Let $M \geq M_0$, $\theta \in (0, 1)$, $g \in \mathbf{G}^{3+\theta}$ and f be of order $\alpha = \frac{7}{2} + \theta$. Fix $\varepsilon \in (0, 1)$. Then there is a collection of $(\mathcal{G}_t)_{t \in [0, T]}$ -adapted $\mathbb{C}$ -valued processes $((\lambda_t^{k, M})_{t \in [0, T]})_{k \in \mathbb{Z}}$ such that for every $t \in [0, T]$, the following equality holds*

$$\mathbb{1}_{\{t \leq \tau_M\}} A_t^{g, \varepsilon}(y) = \sum_{k \in \mathbb{Z}} f_k e^{-iky} \lambda_t^{k, M}, \quad (26)$$

and there is a constant $C_{M, \varepsilon} > 0$ such that $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -almost surely, $\sum_{k \in \mathbb{Z}} \int_0^T |\lambda_t^{k, M}|^2 dt \leq C_{M, \varepsilon}$.

Proof Define for every $t \in [0, T]$, $\lambda_t^{k, M} := \mathbb{1}_{\{t \leq \tau_M\}} \frac{c_k(A_t^{g, \varepsilon})}{f_k} = \mathbb{1}_{\{t \leq \tau_M\}} \lambda_t^k$. Similarly as in the proof of Lemma 21, there is a constant $C > 0$ such that for every $k \in \mathbb{Z} \setminus \{0\}$ and for every $\varepsilon > 0$, $|\int_{\mathbb{R}} \varphi(x) e^{ik\varepsilon x} dx| \leq \frac{C}{|k\varepsilon|^{4+2\theta}}$. Furthermore, for every $k \in \mathbb{Z}$, $|c_k(A_s^g)| \leq \|A_s^g\|_{L_2(\mathbb{T})}$. Thus we have

$$\begin{aligned}\sum_{k \in \mathbb{Z}} \int_0^T |\lambda_t^{k, M}|^2 dt &\leq C \int_0^T \mathbb{1}_{\{t \leq \tau_M\}} |c_0(A_t^g)|^2 dt \\ &\quad + \sum_{k \neq 0} \int_0^T \mathbb{1}_{\{t \leq \tau_M\}} |k|^{7+2\theta} \frac{1}{|k\varepsilon|^{8+4\theta}} |c_k(A_t^g)|^2 dt \\ &\leq \frac{C}{\varepsilon^{8+4\theta}} \int_0^T \mathbb{1}_{\{t \leq \tau_M\}} \|A_t^g\|_{L_2(\mathbb{T})}^2 dt.\end{aligned}$$

By Definition (14), for every $s \in [0, T]$,

$$\begin{aligned}\mathbb{1}_{\{t \leq \tau_M\}} \|A_t^g\|_{L_2(\mathbb{T})} &\leq C \mathbb{1}_{\{t \leq \tau_M\}} \|A_t^g\|_{L_\infty(\mathbb{T})} \leq C \mathbb{1}_{\{t \leq \tau_M^1\}} \|h\|_{L_\infty} \|\partial_u x_t^g\|_{L_\infty} \\ &\leq CM \|h\|_{L_\infty}.\end{aligned}$$

Since the constant does not depend on t , we deduce the statement of the lemma. $\square$

Define the $(\mathcal{G}_t)$ -adapted process $(a_t^M)_{t \in [0, T]}$ by $a_t^M = a_{t \wedge \tau_M}$, in other words:

$$a_t^M(u) := \int_0^t \mathbb{1}_{\{s \leq \tau_M\}} \frac{g'(u)}{\partial_u x_s^g(u)} A_s^{g, \varepsilon}(x_s^g(u)) ds. \quad (27)$$

We easily check that for every $u \in \mathbb{R}$, $a_0^M(u) = 0$ and that $\dot{a}_t^M(u) = \frac{g'(u)}{\partial_u x_t^g(u)} \mathbb{1}_{\{t \leq \tau_M\}} A_t^{g,\varepsilon}(x_t^g(u))$ is a 1-periodic and continuous function of $u \in \mathbb{R}$.

Lemma 25 *Let $\varepsilon \in (0, 1)$. For every $M \geq M_0$, there are two constants C_M^a (depending on T, M, g' and h) and $C_{M,\varepsilon}$ (depending on T, M, ε and h) such that for every $t \in [0, T]$*

$$\|a_t^M\|_{L_\infty} \leq C_M^a; \quad (28)$$

$$\int_0^{t \wedge \tau_M} \|A_s^{g,\varepsilon}\|_{C^1}^2 ds \leq C_{M,\varepsilon}. \quad (29)$$

Proof By definition of τ_M , $|a_t^M(u)| \leq T \|g'\|_{L_\infty} M \sup_{s \leq \tau_M} \|A_s^{g,\varepsilon}\|_{L_\infty}$ for every $t \in [0, T]$ and $u \in \mathbb{R}$. Since $A_s^{g,\varepsilon}$ is 2π -periodic and $A_s^{g,\varepsilon} = A_s^g * \varphi_\varepsilon$, with $\|\varphi_\varepsilon\|_{L_1(\mathbb{R})} = 1$, we have $\|A_s^{g,\varepsilon}\|_{L_\infty} \leq \|A_s^g\|_{L_\infty(\mathbb{T})}$. Recall that by definition (14), $\|A_s^g\|_{L_\infty(\mathbb{T})} \leq \|h\|_{L_\infty} \|\partial_u x_s^g\|_{L_\infty}$. We deduce that

$$\mathbb{1}_{\{s \leq \tau_M\}} \|A_s^{g,\varepsilon}\|_{L_\infty(\mathbb{T})} \leq \|h\|_{L_\infty} \mathbb{1}_{\{s \leq \tau_M^1\}} \|\partial_u x_s^g\|_{L_\infty} \leq M \|h\|_{L_\infty}. \quad (30)$$

Therefore, inequality (28) holds with $C_M^a := T \|g'\|_{L_\infty} M^2 \|h\|_{L_\infty}$.

For every $t \in [0, T]$, $\partial_x A_t^{g,\varepsilon} = A_t^g * \partial_x \varphi_\varepsilon$. Since $\|\partial_x \varphi_\varepsilon\|_{L_1(\mathbb{R})} \leq \frac{C}{\varepsilon}$, we obtain

$$\mathbb{1}_{\{t \leq \tau_M\}} \|\partial_x A_t^{g,\varepsilon}\|_{L_\infty(\mathbb{T})} \leq \frac{C}{\varepsilon} \|h\|_{L_\infty} \mathbb{1}_{\{t \leq \tau_M^1\}} \|\partial_u x_t^g\|_{L_\infty} \leq \frac{CM}{\varepsilon} \|h\|_{L_\infty}. \quad (31)$$

It follows from (30) and (31) that $\int_0^{t \wedge \tau_M} \|A_s^{g,\varepsilon}\|_{C^1}^2 ds \leq T \frac{C}{\varepsilon^2} M^2 \|h\|_{L_\infty}^2$ for every $t \in [0, T]$, whence we obtain (29). $\square$

Using the constant C_M^a appearing in (28), we define $\rho_0 := \rho_0(M) = \frac{1}{2C_M^a}$. The following lemma makes use of Kunita's expansion.

Lemma 26 *Let f be of order $\alpha = \frac{7}{2} + \theta$. Define the auxiliary process $(Y_t^{\rho,M})_{t \in [0,T]}$ as the solution to:*

$$\begin{aligned} Y_t^{\rho,M}(u) &= g(u) + \sum_{k \in \mathbb{Z}} f_k \int_0^t \Re \left(e^{-ikY_s^{\rho,M}(u)} dW_s^k \right) \\ &\quad + \beta_t + \rho \int_0^t \mathbb{1}_{\{s \leq \tau_M\}} A_s^{g,\varepsilon}(Y_s^{\rho,M}(u)) ds. \end{aligned} \quad (32)$$

Then there exists C depending on M, f, g', h, T and ε such that for every $\rho \in (-\rho_0, \rho_0)$ and for every $t \in [0, T]$,

$$\mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |Z_t^{g(u) + \rho a_t^M(u)} - Y_t^{\rho,M}(u)|^2 du \right]^{1/2} \leq C |\rho|^{5/4}. \quad (33)$$

Proof Fix $u \in \mathbb{R}$ and write the equation satisfied by $(Z_t^{g(u)+\rho a_t^M(u)})_{t \in [0, T]}$. We apply Kunita's expansion (Lemma 20) with $(\zeta_t)_{t \in [0, T]} := (a_t^M(u))_{t \in [0, T]}$, $x = g(u)$ and $\zeta_t = a_t^M(u)$. By inequality (28), we have $\mathbb{E} \left[\int_0^T |\zeta_t| dt \right] \leq C_M$. Thus $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -almost surely,

$$\begin{aligned} Z_t^{g(u)+\rho a_t^M(u)} &= Z_0^{g(u)} + \sum_{k \in \mathbb{Z}} f_k \int_0^t \Re \left(e^{-ikZ_s^{g(u)+\rho a_s^M(u)}} dW_s^k \right) + \beta_t + \rho \int_0^t \partial_x Z_s^{g(u)+\rho a_s^M(u)} \dot{a}_s(u) ds \\ &= g(u) + \sum_{k \in \mathbb{Z}} f_k \int_0^t \Re \left(e^{-ikZ_s^{g(u)+\rho a_s^M(u)}} dW_s^k \right) + \beta_t + \rho \int_0^t \mathbb{1}_{\{s \leq \tau_M\}} A_s^{g, \varepsilon}(Z_s^{g(u)}) ds \\ &\quad + \rho \int_0^t (\partial_x Z_s^{g(u)+\rho a_s^M(u)} - \partial_x Z_s^{g(u)}) \frac{g'(u)}{\partial_u x_s^g(u)} \mathbb{1}_{\{s \leq \tau_M\}} A_s^{g, \varepsilon}(x_s^g(u)) ds, \end{aligned} \quad (34)$$

where we used the identities (7) and (8).

Comparing Eq. (34) with Eq. (32) satisfied by $(Y_t^{\rho, M}(u))_{t \in [0, T]}$, we have for every $t \in [0, T]$,

$$\mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |Z_t^{g(u)+\rho a_t^M(u)} - Y_t^{\rho, M}(u)|^2 du \right] \leq 3(E_1 + \rho^2 E_2 + \rho^2 E_3), \quad (35)$$

where

$$\begin{aligned} E_1 &:= \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left| \sum_{k \in \mathbb{Z}} \int_0^t f_k \Re \left((e^{-ikZ_s^{g(u)+\rho a_s^M(u)}} - e^{-ikY_s^{\rho, M}(u)}) dW_s^k \right) \right|^2 du \right]; \\ E_2 &:= \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left| \int_0^t \mathbb{1}_{\{s \leq \tau_M\}} (A_s^{g, \varepsilon}(Z_s^{g(u)}) - A_s^{g, \varepsilon}(Y_s^{\rho, M}(u))) ds \right|^2 du \right]; \\ E_3 &:= \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left| \int_0^t (\partial_x Z_s^{g(u)+\rho a_s^M(u)} - \partial_x Z_s^{g(u)}) \frac{g'(u)}{\partial_u x_s^g(u)} \mathbb{1}_{\{s \leq \tau_M\}} A_s^{g, \varepsilon}(x_s^g(u)) ds \right|^2 du \right]. \end{aligned}$$

Control on E_1 . By Itô's isometry and since $y \mapsto e^{-iky}$ is k -Lipschitz,

$$\begin{aligned} E_1 &\leq \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^t \sum_{k \in \mathbb{Z}} f_k^2 k^2 \left| Z_s^{g(u)+\rho a_s^M(u)} - Y_s^{\rho, M}(u) \right|^2 ds du \right] \\ &\leq C \int_0^t \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left| Z_s^{g(u)+\rho a_s^M(u)} - Y_s^{\rho, M}(u) \right|^2 du \right] ds, \end{aligned} \quad (36)$$

because $\sum_{k \in \mathbb{Z}} f_k^2 k^2 < +\infty$, since $\alpha > \frac{3}{2}$.

Control on E_2 . By (29), there is $C > 0$ such that $\int_0^{t \wedge \tau_M} \|\partial_x A_s^{g,\varepsilon}\|_{L_\infty(\mathbb{T})}^2 ds \leq C$. It follows from Cauchy-Schwarz inequality that

$$\begin{aligned} E_2 &\leq \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^{t \wedge \tau_M} \|\partial_x A_s^{g,\varepsilon}\|_{L_\infty(\mathbb{T})}^2 ds \int_0^t |Z_s^{g(u)} - Y_s^{\rho,M}(u)|^2 ds du \right] \\ &\leq C \int_0^t \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left| Z_s^{g(u) + \rho a_s^M(u)} - Y_s^{\rho,M}(u) \right|^2 du \right] ds \\ &\quad + C \int_0^1 \int_0^t \mathbb{E}^W \mathbb{E}^\beta \left[\left| Z_s^{g(u)} - Z_s^{g(u) + \rho a_s^M(u)} \right|^2 \right] ds du. \end{aligned}$$

Moreover, by inequality (28), $|\rho a_t^M(u)| \leq \rho_0 C_M^a = \frac{1}{2}$ for every $t \in [0, T]$, $u \in \mathbb{R}$ and $\rho \in (-\rho_0, \rho_0)$. Fix $u \in [0, 1]$. Let J_u be the interval $[g(u) - \frac{1}{2}, g(u) + \frac{1}{2}]$. By inequality (76) and by Kolmogorov's Lemma (see [38, p.26, Thm I.2.1]), it follows that, up to considering a modification of the process $(Z_t^x)_{x \in J_u}$, there is a constant C_{Kol} independent of u such that

$$\mathbb{E}^W \mathbb{E}^\beta \left[\sup_{x,y \in J_u, x \neq y} \frac{\sup_{t \leq T} |Z_t^x - Z_t^y|^2}{|x - y|^{1/2}} \right] \leq C_{\text{Kol}}.$$

We deduce that for every $\rho \in (-\rho_0, \rho_0)$,

$$\begin{aligned} &\mathbb{E}^W \mathbb{E}^\beta \left[\left| Z_s^{g(u)} - Z_s^{g(u) + \rho a_s^M(u)} \right|^2 \right] \\ &\leq \mathbb{E}^W \mathbb{E}^\beta \left[\mathbb{1}_{\{\rho a_t^M(u) \neq 0\}} \frac{\sup_{t \leq T} |Z_t^{g(u)} - Z_t^{g(u) + \rho a_t^M(u)}|^2}{|\rho a_t^M(u)|^{1/2}} |\rho a_t^M(u)|^{1/2} \right] \\ &\leq C C_{\text{Kol}} |\rho|^{1/2}, \end{aligned}$$

where the constants are independent of s and u . We conclude that for every $\rho \in (-\rho_0, \rho_0)$

$$E_2 \leq C \int_0^t \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left| Z_s^{g(u) + \rho a_s^M(u)} - Y_s^{\rho,M}(u) \right|^2 du \right] ds + C |\rho|^{1/2}. \quad (37)$$

Control on E_3 . By definition (24) of τ_M^2 , for every $s \leq \tau_M$, $\left\| \frac{1}{\partial_u x_s^g(\cdot)} \right\|_{L_\infty} \leq M$. Thus

$$\begin{aligned} E_3 &\leq \|g'\|_{L_\infty} M \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^t \left| \partial_x Z_s^{g(u) + \rho a_s^M(u)} - \partial_x Z_s^{g(u)} \right|^2 ds \int_0^{t \wedge \tau_M} \|A_s^{g,\varepsilon}\|_{L_\infty}^2 ds du \right] \\ &\leq C \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^t \left| \partial_x Z_s^{g(u) + \rho a_s^M(u)} - \partial_x Z_s^{g(u)} \right|^2 ds du \right], \end{aligned}$$

where the last inequality follows from (29). By inequality (78) and the fact that f is of order $\alpha > \frac{5}{2}$, we can apply as before Kolmogorov's Lemma on $\partial_x Z$ instead of Z . We get for every $\rho \in (-\rho_0, \rho_0)$,

$$\mathbb{E}^W \mathbb{E}^\beta \left[\left| \partial_x Z_s^{g(u)+\rho a_s^M(u)} - \partial_x Z_s^{g(u)} \right|^2 \right] \leq C C_{\text{Kol}} |\rho|^{1/2}.$$

Therefore, for every $\rho \in (-\rho_0, \rho_0)$, $E_3 \leq C|\rho|^{1/2}$.

Conclusion. Putting together the last inequality with (35), (36) and (37), we obtain for every $t \in [0, T]$

$$\begin{aligned} & \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |Z_t^{g(u)+\rho a_t^M(u)} - Y_t^{\rho, M}(u)|^2 du \right] \\ & \leq C \int_0^t \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left| Z_s^{g(u)+\rho a_s^M(u)} - Y_s^{\rho, M}(u) \right|^2 du \right] ds + C|\rho|^{5/2}. \end{aligned}$$

By Gronwall's inequality, the proof of Lemma 26 is complete. $\square$

Finally, the following lemma states a Bismut-Elworthy formula. Remark that the only difference with the formula of Proposition 23 is the localization by τ_M .

Lemma 27 *Let $\theta \in (0, 1)$, $g \in \mathbf{G}^{3+\theta}$ and f be of order $\alpha = \frac{7}{2} + \theta$. For every $M \geq M_0$ and for every $t \in [0, T]$,*

$$\begin{aligned} & \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \partial_\mu \phi(\mu_t^g)(x_t^g(u)) \frac{\partial_u x_t^g(u)}{g'(u)} a_t^M(u) du \right] \\ & = \mathbb{E}^W \mathbb{E}^\beta \left[\phi(\mu_t^g) \sum_{k \in \mathbb{Z}} \int_0^t \Re(\overline{\lambda_s^{k, M}} dW_s^k) \right]. \end{aligned} \quad (38)$$

Proof Take the real part of equality (26) with $y = Y_s^{\rho, M}(u)$. Recall that $A_s^{g, \varepsilon}$ and f_k are real-valued. We obtain for every $M \geq M_0$, for every $u \in \mathbb{R}$, for every $\rho \in (-\rho_0(M), \rho_0(M))$ and for every $s \in [0, T]$,

$$\mathbb{1}_{\{s \leq \tau_M\}} A_s^{g, \varepsilon}(Y_s^{\rho, M}(u)) = \sum_{k \in \mathbb{Z}} f_k \Re \left(e^{-ikY_s^{\rho, M}(u)} \lambda_s^{k, M} \right).$$

Thus, we rewrite equality (32) in the following way: for every $t \in [0, T]$

$$Y_t^{\rho, M}(u) = g(u) + \sum_{k \in \mathbb{Z}} \int_0^t f_k \Re \left(e^{-ikY_s^{\rho, M}(u)} (dW_s^k + \rho \lambda_s^{k, M} ds) \right) + \beta_t.$$

Recall that $\lambda_s^{k,M}$ is complex-valued. Define for every $t \in [0, T]$

$$\mathcal{E}_t^\rho := \exp \left(-\rho \sum_{k \in \mathbb{Z}} \int_0^t \Re(\overline{\lambda_s^{k,M}} dW_s^k) - \frac{\rho^2}{2} \sum_{k \in \mathbb{Z}} \int_0^t |\lambda_s^{k,M}|^2 ds \right).$$

Recall that by Lemma 24, there is a constant $C_{M,\varepsilon} > 0$ such that $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -almost surely, $\sum_{k \in \mathbb{Z}} \int_0^T |\lambda_s^{k,M}|^2 ds \leq C_{M,\varepsilon}$. It follows from Novikov's condition that the process $(\mathcal{E}_t^\rho)_{t \in [0, T]}$ is a $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -martingale. Let $\mathbb{P}^\rho$ be the probability measure on $\Omega^W \times \Omega^\beta$ such that $\mathbb{P}^\rho$ is absolutely continuous with respect to $\mathbb{P}^W \otimes \mathbb{P}^\beta$ with density $\frac{d\mathbb{P}^\rho}{d(\mathbb{P}^W \otimes \mathbb{P}^\beta)} = \mathcal{E}_T^\rho$. By Girsanov's Theorem, $((W_t^k + \rho \lambda_t^{k,M})_{t \in [0, T]})_{k \in \mathbb{Z}}$ is a collection of independent $\mathbb{P}^\rho$ -Brownian motions, independent of $(\beta, \mathcal{G}_0)$. By uniqueness in law of Eq. (4), the law of $(Y_t^{\rho, M})_{t \in [0, T]}$ under $\mathbb{P}^\rho$ is equal to the law of $(x_t^g)_{t \in [0, T]}$ under $\mathbb{P}^W \otimes \mathbb{P}^\beta$.

Fix $t \in [0, T]$. Recall that $\widehat{\phi}(Y) := \phi(\mathcal{L}_{[0, 1] \times \Omega^\beta}(Y))$ for every $Y \in L_2([0, 1] \times \Omega^\beta)$. Then

$$\mathbb{E}^W \mathbb{E}^\beta \left[\widehat{\phi}(Y_t^{\rho, M}) \mathcal{E}_t^\rho \right] = \mathbb{E}^W \mathbb{E}^\beta \left[\widehat{\phi}(Y_t^{\rho, M}) \frac{d\mathbb{P}^\rho}{d(\mathbb{P}^W \otimes \mathbb{P}^\beta)} \right] = \mathbb{E}^W \mathbb{E}^\beta \left[\widehat{\phi}(x_t^g) \right].$$

The r.h.s. does not depend on ρ , so we have

$$\frac{d}{d\rho} \Big|_{\rho=0} \mathbb{E}^W \mathbb{E}^\beta \left[\widehat{\phi}(Y_t^{\rho, M}) \mathcal{E}_t^\rho \right] = 0. \quad (39)$$

Let us now prove that $\frac{d}{d\rho} \Big|_{\rho=0} \mathbb{E}^W \mathbb{E}^\beta \left[\widehat{\phi}(Z_t^{g+\rho a_t^M}) \mathcal{E}_t^\rho \right] = 0$. By assumption $(\phi 2)$, $\widehat{\phi}$ is a Lipschitz-continuous function. By Lemma 26, we have for every $\rho \in (-\rho_0, \rho_0)$

$$\begin{aligned} & \left| \mathbb{E}^W \mathbb{E}^\beta \left[\widehat{\phi}(Z_t^{g+\rho a_t^M}) \mathcal{E}_t^\rho \right] - \mathbb{E}^W \mathbb{E}^\beta \left[\widehat{\phi}(Y_t^{\rho, M}) \mathcal{E}_t^\rho \right] \right| \\ & \leq \mathbb{E}^W \left[|\widehat{\phi}(Z_t^{g+\rho a_t^M}) - \widehat{\phi}(Y_t^{\rho, M})|^2 \right]^{1/2} \mathbb{E}^W \mathbb{E}^\beta \left[|\mathcal{E}_t^\rho|^2 \right]^{1/2} \\ & \leq \|\widehat{\phi}\|_{\text{Lip}} \mathbb{E}^W \left[\|Z_t^{g+\rho a_t^M} - Y_t^{\rho, M}\|_{L_2([0, 1] \times \Omega^\beta)}^2 \right]^{1/2} \mathbb{E}^W \mathbb{E}^\beta \left[|\mathcal{E}_t^\rho|^2 \right]^{1/2} \\ & \leq C_{M,\varepsilon} |\rho|^{5/4} \mathbb{E}^W \mathbb{E}^\beta \left[|\mathcal{E}_t^\rho|^2 \right]^{1/2}. \end{aligned}$$

Moreover, recalling that $\sum_{k \in \mathbb{Z}} \int_0^T |\lambda_s^{k,M}|^2 ds \leq C_{M,\varepsilon}$ (see Lemma 24)

$$\begin{aligned} & \mathbb{E}^W \mathbb{E}^\beta \left[|\mathcal{E}_t^\rho|^2 \right] \\ & \leq e^{\rho_0^2 C_{M,\varepsilon}} \mathbb{E}^W \mathbb{E}^\beta \left[\exp \left(-2\rho \sum_{k \in \mathbb{Z}} \int_0^t \Re(\overline{\lambda_s^{k,M}} dW_s^k) - \frac{(2\rho)^2}{2} \sum_{k \in \mathbb{Z}} \int_0^t |\lambda_s^{k,M}|^2 ds \right) \right] \\ & = e^{\rho_0^2 C_{M,\varepsilon}}, \end{aligned}$$

since the exponential term is a $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -martingale. Therefore,

$$\left| \mathbb{E}^W \mathbb{E}^\beta \left[\widehat{\phi}(Z_t^{g+\rho a_t^M}) \mathcal{E}_t^\rho \right] - \mathbb{E}^W \mathbb{E}^\beta \left[\widehat{\phi}(Y_t^{\rho, M}) \mathcal{E}_t^\rho \right] \right| \leq C_{M, \varepsilon} |\rho|^{5/4}. \quad (40)$$

It follows from (39) and (40) that $\frac{d}{d\rho} \big|_{\rho=0} \mathbb{E}^W \mathbb{E}^\beta \left[\widehat{\phi}(Z_t^{g+\rho a_t^M}) \mathcal{E}_t^\rho \right] = 0$. By (10), we compute

$$\begin{aligned} 0 &= \frac{d}{d\rho} \bigg|_{\rho=0} \mathbb{E}^W \mathbb{E}^\beta \left[\widehat{\phi}(Z_t^{g+\rho a_t^M}) \mathcal{E}_t^\rho \right] \\ &= \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 D\widehat{\phi}(Z_t^g)_u \partial_x Z_t^{g(u)} a_t^M(u) du \right] \\ &\quad - \mathbb{E}^W \mathbb{E}^\beta \left[\widehat{\phi}(Z_t^g) \sum_{k \in \mathbb{Z}} \int_0^t \Re(\lambda_s^{k, M}) dW_s^k \right] \\ &= \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \partial_\mu \phi(\mu_t^g)(x_t^g(u)) \frac{\partial_u x_t^g(u)}{g'(u)} a_t^M(u) du \right] \\ &\quad - \mathbb{E}^W \mathbb{E}^\beta \left[\phi(\mu_t^g) \sum_{k \in \mathbb{Z}} \int_0^t \Re(\lambda_s^{k, M}) dW_s^k \right]. \end{aligned}$$

We used Proposition 6 for the last equality. Therefore, equality (38) holds true. $\square$

Finally, we prove Proposition 23.

Proof (Proposition 23) We want to prove

$$\begin{aligned} &\mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \partial_\mu \phi(\mu_t^g)(x_t^g(u)) \frac{\partial_u x_t^g(u)}{g'(u)} a_t^M(u) du \right] \\ &= \mathbb{E}^W \mathbb{E}^\beta \left[\phi(\mu_t^g) \sum_{k \in \mathbb{Z}} \int_0^t \Re(\lambda_s^{k, M}) dW_s^k \right], \end{aligned}$$

which is equivalent to (23). In order to obtain that equality, it is sufficient to pass to the limit when $M \rightarrow +\infty$ in (38). Recall that by (25), $\mathbb{P}^W \otimes \mathbb{P}^\beta [\tau_M < T] \rightarrow_{M \rightarrow +\infty} 0$. Since $\{\tau_M < T\}_{M \geq M_0}$ is a non-increasing sequence of events, it follows that $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -almost surely, $\mathbb{1}_{\{\tau_M < T\}} \rightarrow \mathbb{1}_{\{t \leq T\}}$. Thus, it only remains to prove uniform integrability of both members of equality (38). Precisely, we want to prove:

$$\sup_{M \geq M_0} \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left(\partial_\mu \phi(\mu_t^g)(x_t^g(u)) \frac{\partial_u x_t^g(u)}{g'(u)} a_t^M(u) \right)^{3/2} du \right] < +\infty; \quad (41)$$

$$\sup_{M \geq M_0} \mathbb{E}^W \mathbb{E}^\beta \left[\left(\phi(\mu_t^g) \sum_{k \in \mathbb{Z}} \int_0^t \Re(\lambda_s^{k, M}) dW_s^k \right)^2 \right] < +\infty. \quad (42)$$

Proof of (41). For every $M \geq M_0$, by Hölder's inequality

$$\begin{aligned} & \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left(\partial_\mu \phi(\mu_t^g)(x_t^g(u)) \frac{\partial_u x_t^g(u)}{g'(u)} a_t^M(u) \right)^{3/2} du \right] \\ & \leq \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left(\partial_\mu \phi(\mu_t^g)(x_t^g(u)) \right)^2 du \right]^{3/4} \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left| \frac{\partial_u x_t^g(u)}{g'(u)} a_t^M(u) \right|^6 du \right]^{1/4}. \end{aligned}$$

By assumption $(\phi 2)$, $\mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 (\partial_\mu \phi(\mu_t^g)(x_t^g(u)))^2 du \right]$ is bounded. Moreover, by inequality (68),

$$\begin{aligned} & \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left| \frac{\partial_u x_t^g(u)}{g'(u)} a_t^M(u) \right|^6 du \right] \\ & \leq C \left\| \frac{1}{g'} \right\|_{L_\infty}^6 \|g'\|_{L_{12}}^6 \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |a_t^M(u)|^{12} du \right]^{1/2}. \end{aligned}$$

By definition (27) of a_t^M , we have

$$\mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |a_t^M(u)|^{12} du \right] \leq \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 T^{11} \int_0^t \left| \frac{g'(u)}{\partial_u x_s^g(u)} A_s^{g,\varepsilon}(x_s^g(u)) \right|^{12} ds du \right].$$

Remark that for every $s \in [0, T]$, $\|A_s^{g,\varepsilon}\|_{L_\infty} \leq \|A_s^g\|_{L_\infty} \leq \|\partial_u x_s^g\|_{L_\infty} \|h\|_{L_\infty}$. Thus

$$\begin{aligned} & \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |a_t^M(u)|^{12} du \right] \\ & \leq C \|g'\|_{L_\infty}^{12} \|h\|_{L_\infty}^{12} \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \|\partial_u x_t^g\|_{L_\infty}^{12} \sup_{t \leq T} \left\| \frac{1}{\partial_u x_t^g(\cdot)} \right\|_{L_\infty}^{12} \right] \\ & \leq C \|g'\|_{L_\infty}^{12} \|h\|_{L_\infty}^{12} \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \|\partial_u x_t^g\|_{L_\infty}^{24} \right]^{1/2} \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \left\| \frac{1}{\partial_u x_t^g(\cdot)} \right\|_{L_\infty}^{24} \right]^{1/2} \leq C, \end{aligned}$$

where the constant C does not depend on M . The last inequality is obtained by inequalities (71) and (72), because $g \in \mathbf{G}^{2+\theta}$ and $\alpha > \frac{5}{2} + \theta$. We deduce (41).

Proof of (42). For every $M \geq M_0$ and $t \in [0, T]$

$$\begin{aligned} & \mathbb{E}^W \mathbb{E}^\beta \left[\left(\phi(\mu_t^g) \sum_{k \in \mathbb{Z}} \int_0^t \Re(\overline{\lambda_s^{k,M}} dW_s^k) \right)^2 \right] \leq \|\phi\|_{L_\infty}^2 \mathbb{E}^W \mathbb{E}^\beta \left[\sum_{k \in \mathbb{Z}} \int_0^t |\lambda_s^{k,M}|^2 ds \right] \\ & \leq \|\phi\|_{L_\infty}^2 \mathbb{E}^W \mathbb{E}^\beta \left[\sum_{k \in \mathbb{Z}} \int_0^t |\lambda_s^k|^2 ds \right], \end{aligned}$$

since $\lambda_s^{k,M} = \mathbb{1}_{\{s \leq \tau_M\}} \lambda_s^k$. By inequality (20), $\mathbb{E}^W \mathbb{E}^\beta \left[\sum_{k \in \mathbb{Z}} \int_0^t |\lambda_s^k|^2 ds \right]$ is bounded, so we deduce (42). It completes the proof of Proposition 23. $\square$

4.3 Conclusion of the analysis

Putting together Lemma 21 and Proposition 23, we conclude the proof of Proposition 19.

Proof (Proposition 19) By Cauchy-Schwarz inequality applied to (23),

$$|I_1| \leq \frac{1}{t} \|\phi\|_{L_\infty} \mathbb{E}^W \mathbb{E}^\beta \left[\sum_{k \in \mathbb{Z}} \int_0^t |\lambda_s^k|^2 ds \right]^{1/2} \leq \frac{C}{t} \|\phi\|_{L_\infty} \frac{\sqrt{t}}{\varepsilon^{3+2\theta}} C_1(g) \|h\|_{C^1},$$

where we applied inequality (18). $\square$

Remark 28 Note that we have proved a Bismut-Elworthy-Li integration by parts formula up to a remainder term. Indeed, by propositions 18 and 23, we proved that

$$\frac{d}{d\rho} \Big|_{\rho=0} P_t \phi(\mu_0^{g+\rho g'h}) = \frac{1}{t} \mathbb{E}^W \mathbb{E}^\beta \left[\phi(\mu_t^g) \sum_{k \in \mathbb{Z}} \int_0^t \Re(\overline{\lambda_s^k} dW_s^k) \right] + I_2,$$

where it should be recalled that $((\lambda_t^k)_{t \in [0, T]})_{k \in \mathbb{Z}}$, defined by (17), depends on ε . In the next section, we prove that I_2 is of order $\mathcal{O}(\varepsilon)$.

5 Analysis of I_2

In this section, we look for an upper bound of $|I_2|$. Define $H_s^{g,\varepsilon}(u) := \frac{(A_s^g - A_s^{g,\varepsilon})(x_s^g(u))}{\partial_u x_s^g(u)}$. Then I_2 rewrites as follows

$$I_2 = \frac{1}{t} \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^t \partial_\mu \phi(\mu_t^g)(x_t^g(u)) \partial_u x_t^g(u) H_s^{g,\varepsilon}(u) ds du \right]. \quad (43)$$

Moreover, let $(K_t^{g,\varepsilon})_{t \in [0, T]}$ be the process defined by:

$$K_t^{g,\varepsilon}(u) := \int_0^t H_s^{g,\varepsilon}(u) \frac{1}{t-s} \int_s^t \partial_u x_r^g(u) d\beta_r ds. \quad (44)$$

We also introduce the notation $\left[\frac{\delta \psi}{\delta m} \right]$, denoting the zero-average linear functional derivative. For every $\mu \in \mathcal{P}_2(\mathbb{R})$ and $v \in \mathbb{R}$,

$$\left[\frac{\delta \psi}{\delta m} \right] (\mu)(v) := \frac{\delta \psi}{\delta m} (\mu)(v) - \int_{\mathbb{R}} \frac{\delta \psi}{\delta m} (\mu)(v') d\mu(v'). \quad (45)$$

The main result of this section is the following proposition.

Proposition 29 *Under the same assumptions as Proposition 19, there is $C > 0$ independent of g , h and θ such that for every $t \in [0, T]$ and $\varepsilon \in (0, 1)$,*

$$|I_2| \leq \frac{C}{\sqrt{t}} \varepsilon \|h\|_{\mathcal{C}^1} C_2(g) \mathbb{E}^W \mathbb{E}^\beta \left[\left| \int_0^1 \left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(x_t^g(u)) \frac{K_t^{g,\varepsilon}(u)}{\|K_t^{g,\varepsilon}\|_{L_\infty}} du \right|^2 \right]^{1/2}. \quad (46)$$

where $C_2(g) = 1 + \|g'''\|_{L_g}^3 + \|g''\|_{L_\infty}^{12} + \|g'\|_{L_\infty}^{12} + \left\| \frac{1}{g'} \right\|_{L_\infty}^{24}$.

5.1 Progressive measurability

We start by showing that Yamada-Watanabe Theorem applies here, *i.e.* that the processes $(x_t^g)_{t \in [0, T]}$, $(\mu_t^g)_{t \in [0, T]}$ and $(H_t^{g,\varepsilon})_{t \in [0, T]}$ can all be written as progressively measurable functions of u and the noises $(W^k)_{k \in \mathbb{Z}}$ and β .

For that purpose, let $(\Theta, \mathcal{B}(\Theta))$ be the canonical space defined by $\Theta = \mathcal{C}([0, T], \mathbb{C})^{\mathbb{Z}} \times \mathcal{C}([0, T], \mathbb{R})$ and $\mathcal{B}(\Theta) = \mathcal{B}(\mathcal{C}([0, T], \mathbb{C})^{\mathbb{Z}}) \otimes \mathcal{B}(\mathcal{C}([0, T], \mathbb{R}))$. Let $\mathbf{P}$ be the probability measure on $(\Theta, \mathcal{B}(\Theta))$ defined as the distribution of $((W^k)_{k \in \mathbb{Z}}, \beta)$ on $\Omega^W \times \Omega^\beta$. Let $\mathcal{B}_t(\mathcal{C}([0, T], \mathbb{R})) := \sigma(\mathbf{x}(s); 0 \leq s \leq t)$; in other words the process $(\mathcal{B}_t(\mathcal{C}([0, T], \mathbb{R})))_{t \in [0, T]}$ is the canonical filtration on $(\mathcal{C}([0, T], \mathbb{R}), \mathcal{B}(\mathcal{C}([0, T], \mathbb{R})))$. Similarly, let $(\mathcal{B}_t(\mathcal{C}([0, T], \mathbb{C})^{\mathbb{Z}}))_{t \in [0, T]}$ be the canonical filtration on $(\mathcal{C}([0, T], \mathbb{C})^{\mathbb{Z}}, \mathcal{B}(\mathcal{C}([0, T], \mathbb{C})^{\mathbb{Z}}))$. Let $(\widehat{\mathcal{B}}_t(\Theta))_{t \in [0, T]}$ be the augmentation of the filtration $(\mathcal{B}_t(\mathcal{C}([0, T], \mathbb{C})^{\mathbb{Z}}) \otimes \mathcal{B}_t(\mathcal{C}([0, T], \mathbb{R})))_{t \in [0, T]}$ by the null sets of $\mathbf{P}$. Those notations are inspired by the textbook [26, pp.308-311]. We denote elements of Θ in bold, e.g. $((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \in \Theta$.

Lemma 30 *Let θ , ε , g , f and h be as in Proposition 29. Then*

(a) *there is a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\Theta)/\mathcal{B}(\mathcal{C}([0, T], \mathbb{R}))$ -measurable function*

$$\begin{aligned} \mathcal{X} : \mathbb{R} \times \Theta &\rightarrow \mathcal{C}([0, T], \mathbb{R}) \\ (u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) &\mapsto \mathcal{X}(u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \end{aligned}$$

which is, for every fixed $t \in [0, T]$, $\mathcal{B}(\mathbb{R}) \otimes \widehat{\mathcal{B}}_t(\Theta)/\mathcal{B}_t(\mathcal{C}([0, T], \mathbb{R}))$ -measurable, such that $\mathcal{X}$ is continuous in u for $\mathbf{P}$ -almost every fixed $((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \in \Theta$ and such that $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -almost surely, for every $u \in \mathbb{R}$ and for every $t \in [0, T]$,

$$x_t^g(u) = \mathcal{X}_t(u, (W^k)_{k \in \mathbb{Z}}, \beta); \quad (47)$$

(b) there is a $\mathcal{B}(\mathcal{C}([0, T], \mathbb{C})^{\mathbb{Z}})/\mathcal{B}(\mathcal{C}([0, T], \mathcal{P}_2(\mathbb{R})))$ -measurable function

$$\begin{aligned}\mathcal{P} : \mathcal{C}([0, T], \mathbb{C})^{\mathbb{Z}} &\rightarrow \mathcal{C}([0, T], \mathcal{P}_2(\mathbb{R})) \\ (\mathbf{w}^k)_{k \in \mathbb{Z}} &\mapsto \mathcal{P}((\mathbf{w}^k)_{k \in \mathbb{Z}})\end{aligned}$$

which is, for every fixed $t \in [0, T]$, $\mathcal{B}_t(\mathcal{C}([0, T], \mathbb{C})^{\mathbb{Z}})/\mathcal{B}_t(\mathcal{C}([0, T], \mathcal{P}_2(\mathbb{R})))$ -measurable, such that $\mathbb{P}^W$ -almost surely, for every $t \in [0, T]$,

$$\mu_t^g = \mathcal{P}_t((W^k)_{k \in \mathbb{Z}}); \quad (48)$$

(c) there is a progressively-measurable function $\mathcal{H} : [0, T] \times \mathbb{R} \times \Theta \rightarrow \mathbb{R}$, i.e. for every $t \in [0, T]$,

$$\begin{aligned}[0, t] \times \mathbb{R} \times \Theta &\rightarrow \mathbb{R} \\ (s, u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) &\mapsto \mathcal{H}_s(u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})\end{aligned}$$

is $\mathcal{B}[0, t] \otimes \mathcal{B}(\mathbb{R}) \otimes \widehat{\mathcal{B}}_t(\Theta)/\mathcal{B}(\mathbb{R})$ -measurable, such that $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -almost surely, for every $u \in \mathbb{R}$ and for every $t \in [0, T]$,

$$H_t^{g, \varepsilon}(u) = \mathcal{H}_t(u, (W^k)_{k \in \mathbb{Z}}, \beta). \quad (49)$$

Proof Consider the canonical space $(\Theta, \mathcal{B}(\Theta), (\widehat{\mathcal{B}}_t(\Theta))_{t \in [0, T]}, \mathbf{P})$. By Proposition 3, there is a strong and pathwise unique solution to (4) with initial condition $x_0^g = g$. Therefore, for every fixed $u \in \mathbb{R}$, there is a unique solution $(\mathbf{x}_t^g(u))_{t \in [0, T]}$ to

$$\mathbf{x}_t^g(u) = g(u) + \sum_{k \in \mathbb{Z}} \int_0^t f_k \Re \left(e^{-ik\mathbf{x}_s^g(u)} d\mathbf{w}_s^k \right) + \mathbf{b}_t.$$

Proof of (a). By Yamada-Watanabe Theorem, the law of $(x^g, (W^k)_{k \in \mathbb{Z}}, \beta)$ under $\mathbb{P}^W \otimes \mathbb{P}^\beta$ is equal to the law of $(\mathbf{x}^g, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})$ under $\mathbf{P}$. This result is proved in [26, Prop. 5.3.20] for a finite-dimensional noise, but the proof is the same for the infinite-dimensional noise $((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \in \Theta$. By a corollary to this theorem (see [26, Coro. 5.3.23]), it follows that for every $u \in \mathbb{Q}$, there is a $\mathcal{B}(\Theta)/\mathcal{B}(\mathcal{C}([0, T], \mathbb{R}))$ -measurable function

$$\begin{aligned}\mathcal{X}^u : \Theta &\rightarrow \mathcal{C}([0, T], \mathbb{R}) \\ ((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) &\mapsto \mathcal{X}^u((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})\end{aligned}$$

which is, for every fixed $t \in [0, T]$, $\widehat{\mathcal{B}}_t(\Theta)/\mathcal{B}_t(\mathcal{C}([0, T], \mathbb{R}))$ -measurable, such that $\mathbf{P}$ -almost surely, for every $t \in [0, T]$,

$$\mathbf{x}_t^g(u) = \mathcal{X}_t^u((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}). \quad (50)$$

Moreover, again by Proposition 3, there is a $\mathbf{P}$ -almost sure event $A \in \mathcal{B}(\Theta)$ such that for every $((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \in A$, the function $(t, u) \mapsto \mathbf{x}_t^g(u)$ is continuous on $[0, T] \times \mathbb{R}$. Up to modifying the almost-sure event A , we may assume that for every $((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \in A$ and for every $u \in \mathbb{Q}$, equality (50) holds. Therefore, we can define a continuous function in the variable $u \in \mathbb{R}$ by extending $u \in \mathbb{Q} \mapsto \mathcal{X}^u$. More precisely, define for every $u \in \mathbb{R}$, $((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \in \Theta$,

$$\mathcal{X}(u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) = \begin{cases} \lim_{\substack{u_n \rightarrow u \\ (u_n)_n \in \mathbb{Q}^{\mathbb{N}}}} \mathcal{X}^{u_n}((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) & \text{if } ((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

In the latter definition, the limit exists and for every $((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \in A$, $\mathcal{X}_t(u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) = \mathbf{x}_t^g(u)$ holds for any $t \in [0, T]$ and $u \in \mathbb{R}$. By construction, for every $((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \in \Theta$, $u \in \mathbb{R} \mapsto \mathcal{X}(u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \in \mathcal{C}([0, T], \mathbb{R})$ is continuous. It remains to show that $\mathcal{X}$ is progressively-measurable. Fix $t \in [0, T]$. By construction of $\mathcal{X}^u$, we know that for every $u \in \mathbb{Q}$,

$$\begin{aligned} [0, t] \times \Theta &\rightarrow \mathbb{R} \\ (s, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) &\mapsto \mathcal{X}_s^u((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \end{aligned}$$

is $\mathcal{B}[0, t] \otimes \widehat{\mathcal{B}}_t(\Theta) / \mathcal{B}(\mathbb{R})$ -measurable. Since $\mathcal{X}$ is the limit of $\mathcal{X}_n := \sum_{k \in \mathbb{Z}} \mathcal{X}^{k/n} \mathbb{1}_{\{u \in [\frac{k}{n}, \frac{k+1}{n}]\}}$, we deduce that for every $t \in [0, T]$,

$$\begin{aligned} [0, t] \times \mathbb{R} \times \Theta &\rightarrow \mathbb{R} \\ (s, u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) &\mapsto \mathcal{X}_s(u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \end{aligned}$$

is $\mathcal{B}[0, t] \otimes \mathcal{B}(\mathbb{R}) \otimes \widehat{\mathcal{B}}_t(\Theta) / \mathcal{B}(\mathbb{R})$ -measurable.

Recall that $\mathcal{L}^{\mathbb{P}^W \otimes \mathbb{P}^\beta}(x^g, (W^k)_{k \in \mathbb{Z}}, \beta) = \mathcal{L}^{\mathbf{P}}(\mathbf{x}^g, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})$. Since $\mathbb{P}$ -almost surely, for every $u \in \mathbb{R}$ and for every $t \in [0, T]$, $\mathbf{x}_t^g(u) = \mathcal{X}_t(u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})$, we deduce that $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -almost surely, for every $u \in \mathbb{R}$ and for every $t \in [0, T]$, equality (47) holds. It completes the proof of (a).

Proof of (b). This step is equivalent to find $\mathcal{P} : \mathcal{C}([0, T], \mathbb{C})^{\mathbb{Z}} \rightarrow \mathcal{C}([0, T], \mathcal{P}_2(\mathbb{R}))$ such that for every bounded measurable function $\Upsilon : \mathbb{R} \rightarrow \mathbb{R}$, the function

$$\begin{aligned} \langle \Upsilon, \mathcal{P} \rangle : \mathcal{C}([0, T], \mathbb{C})^{\mathbb{Z}} &\rightarrow \mathcal{C}([0, T], \mathbb{R}) \\ (\mathbf{w}^k)_{k \in \mathbb{Z}} &\mapsto \langle \Upsilon, \mathcal{P}((\mathbf{w}^k)_{k \in \mathbb{Z}}) \rangle = \int_{\mathbb{R}} \Upsilon(x) d\mathcal{P}((\mathbf{w}^k)_{k \in \mathbb{Z}})(x) \end{aligned}$$

is $\mathcal{B}_t(\mathcal{C}([0, T], \mathbb{C})^{\mathbb{Z}}) / \mathcal{B}_t(\mathcal{C}([0, T], \mathbb{R}))$ -measurable for every fixed $t \in [0, T]$. Define $\mathcal{P}$ by duality: for every $\Upsilon : \mathbb{R} \rightarrow \mathbb{R}$ bounded and measurable,

$$\langle \Upsilon, \mathcal{P}((\mathbf{w}^k)_{k \in \mathbb{Z}}) \rangle := \int_{\mathcal{C}([0, T], \mathbb{R})} \int_0^1 \Upsilon(\mathcal{X}(v, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})) dv d\mu_{\text{Wiener}}(\mathbf{b}),$$

where μ_{Wiener} denotes the Wiener measure on $\mathcal{C}([0, T], \mathbb{R})$. Thus $\mathbb{P}^W$ -almost surely, for every $t \in [0, T]$, for every $\Upsilon : \mathbb{R} \rightarrow \mathbb{R}$ bounded and measurable,

$$\langle \Upsilon, \mathcal{P}_t((W^k)_{k \in \mathbb{Z}}) \rangle = \mathbb{E}^\beta \left[\int_0^1 \Upsilon(\mathcal{X}_t(v, (W^k)_{k \in \mathbb{Z}}, \beta)) dv \right] = \langle \Upsilon, \mu_t^g \rangle,$$

where the last equality follows from Definition 7. Thus we proved equality (48).

Moreover, for every $t \in [0, T]$, by composition of two measurable functions,

$$\begin{aligned} [0, t] \times \mathbb{R} \times \Theta &\rightarrow \mathbb{R} \\ (s, u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) &\mapsto \Upsilon(\mathcal{X}_s(u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})) \end{aligned}$$

is $\mathcal{B}[0, t] \otimes \mathcal{B}(\mathbb{R}) \otimes \widehat{\mathcal{B}}_t(\Theta) / \mathcal{B}(\mathbb{R})$ -measurable. By Fubini's Theorem, it follows that for every $t \in [0, T]$,

$$\begin{aligned} [0, t] \times \mathcal{C}([0, T], \mathbb{C})^{\mathbb{Z}} &\rightarrow \mathbb{R} \\ (s, (\mathbf{w}^k)_{k \in \mathbb{Z}}) &\mapsto \int_{\mathcal{C}([0, T], \mathbb{R})} \int_0^1 \Upsilon(\mathcal{X}_s(v, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})) dv d\mu_{\text{Wiener}}(\mathbf{b}), \end{aligned}$$

is $\mathcal{B}[0, t] \otimes \mathcal{B}_t(\mathcal{C}([0, T], \mathbb{C})^{\mathbb{Z}}) / \mathcal{B}(\mathbb{R})$ -measurable. This completes the proof of (b).

Proof of (c). Define, on the canonical space $(\Theta, \mathcal{B}(\Theta))$, $\mathbf{F}_t^g = (\mathbf{x}_t^g)^{-1}$ and

$$\begin{aligned} \mathbf{A}_t^g &:= \partial_u \mathbf{x}_t^g(\mathbf{F}_t^g(\cdot)) h(\mathbf{F}_t^g(\cdot)); \\ \mathbf{A}_t^{g, \varepsilon} &:= \int_{\mathbb{R}} \mathbf{A}_t^g(\cdot - y) \varphi_\varepsilon(y) dy; \\ \mathbf{H}_t^{g, \varepsilon}(u) &:= \frac{1}{\partial_u \mathbf{x}_t^g(u)} (\mathbf{A}_t^g - \mathbf{A}_t^{g, \varepsilon})(\mathbf{x}_t^g(u)). \end{aligned}$$

In order to prove that $\mathbf{H}^{g, \varepsilon}$ can be written as a progressively measurable function of u and $((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})$, we will prove successively that this property holds for $\partial_u \mathbf{x}_t^g$, $\mathbf{F}^g$, $\mathbf{A}^g$ and $\mathbf{A}^{g, \varepsilon}$ and we will deduce the result for $\mathbf{H}^{g, \varepsilon}$ by composition of progressively measurable functions.

Let us start with $\partial_u \mathbf{x}_t^g$. By Proposition 4, since $g \in \mathbf{G}^{1+\theta}$ and $\alpha > \frac{3}{2} + \theta$, $\mathbf{P}$ -almost surely, for every $t \in [0, T]$, the map $u \mapsto \mathbf{x}_t^g(u)$ is of class $\mathcal{C}^1$. Thus there exists a $\mathbf{P}$ -almost-sure event $A \in \mathcal{B}(\theta)$ such that for every $((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \in A$, $\mathbf{x}_t^g(u) = \mathcal{X}_t(u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})$ holds for every $(t, u) \in [0, T] \times \mathbb{R}$ and such that $u \mapsto \mathbf{x}_t^g(u)$ belongs to $\mathcal{C}^1$. Define for every $((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \in A$, for every $(t, u) \in [0, T] \times \mathbb{R}$,

$$\partial_u \mathcal{X}_t(u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) := \limsup_{\eta \searrow 0} \frac{\mathcal{X}_t(u + \eta, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) - \mathcal{X}_t(u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})}{\eta}.$$

Thus for every $((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \in A$ and for every $(t, u) \in [0, T] \times \mathbb{R}$, $\partial_u \mathbf{x}_t^g(u) = \partial_u \mathcal{X}_t(u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})$. Moreover, by progressive-measurability of $\mathcal{X}$, it follows from

the definition of $\partial_u \mathcal{X}$ is also progressively measurable; more precisely, for every $t \in [0, T]$,

$$\begin{aligned} & [0, t] \times \mathbb{R} \times \Theta \rightarrow \mathbb{R} \\ & (s, u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \mapsto \partial_u \mathcal{X}_s(u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \end{aligned}$$

is $\mathcal{B}[0, t] \otimes \mathcal{B}(\mathbb{R}) \otimes \widehat{\mathcal{B}}_t(\Theta)/\mathcal{B}(\mathbb{R})$ -measurable.

Now, consider $\mathbf{F}^g$. Define for every $x \in [0, 2\pi]$

$$\widetilde{\mathbf{F}}_t^g(x) := \int_0^1 \mathbb{1}_{\{\mathbf{x}_t^g(v) - \mathbf{x}_t^g(0) \leq x\}} \mathrm{d}v. \quad (51)$$

Thus we have for every $x \in [\mathbf{x}_t(0), \mathbf{x}_t(0) + 2\pi]$

$$\widetilde{\mathbf{F}}_t^g(x - \mathbf{x}_t^g(0)) = \int_0^1 \mathbb{1}_{\{\mathbf{x}_t^g(v) \leq x\}} \mathrm{d}v = \int_0^1 \mathbb{1}_{\{v \leq \mathbf{F}_t^g(x)\}} \mathrm{d}v = \mathbf{F}_t^g(x).$$

Therefore, since for every $x \in \mathbb{R}$, $\mathbf{F}_t^g(x + 2\pi) = \mathbf{F}_t^g(x) + 1$, we have

$$\mathbf{F}_t^g(x) = \sum_{k \in \mathbb{Z}} \mathbb{1}_{\{x - 2\pi k \in [\mathbf{x}_t(0), \mathbf{x}_t(0) + 2\pi)\}} (\widetilde{\mathbf{F}}_t^g(x - 2\pi k - \mathbf{x}_t^g(0)) + k).$$

Hence it is sufficient to prove that we can write $\widetilde{\mathbf{F}}_t^g$ as a progressively measurable function of x and $((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})$. Recall that $\mathbf{P}$ -almost surely, $u \mapsto \mathbf{x}^g(u) = \mathcal{X}(u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})$ is continuous. Thus there is $\mathcal{I}$ such that $\mathbf{P}$ -almost surely, for every $v \in [0, 1]$, for every $x \in [0, 2\pi]$, $\mathbb{1}_{\{\mathbf{x}^g(v) - \mathbf{x}^g(0) \leq x\}} = \mathcal{I}_s(v, x, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})$ and such that for every $t \in [0, T]$,

$$\begin{aligned} & [0, t] \times [0, 1] \times [0, 2\pi] \times \Theta \rightarrow \mathbb{R} \\ & (s, v, x, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \mapsto \mathcal{I}_s(v, x, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \end{aligned}$$

is $\mathcal{B}[0, t] \otimes \mathcal{B}([0, 1] \times [0, 2\pi]) \otimes \widehat{\mathcal{B}}_t(\Theta)/\mathcal{B}(\mathbb{R})$ -measurable. It follows from Fubini's Theorem and from (51) that for every $t \in [0, T]$,

$$\begin{aligned} & [0, t] \times [0, 2\pi] \times \Theta \rightarrow \mathbb{R} \\ & (s, x, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \mapsto \int_0^1 \mathcal{I}_s(v, x, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}) \mathrm{d}v = \widetilde{\mathbf{F}}_s^g(x) \end{aligned}$$

is $\mathcal{B}[0, t] \otimes \mathcal{B}([0, 2\pi]) \otimes \widehat{\mathcal{B}}_t(\Theta)/\mathcal{B}(\mathbb{R})$ -measurable.

Let us conclude with $\mathbf{A}^g$, $\mathbf{A}^{g, \varepsilon}$ and $\mathbf{H}^{g, \varepsilon}$. First, remark that $\mathbf{A}^g$ is obtained by products and compositions of $\partial_u \mathbf{x}^g$, $\mathbf{F}^g$ and h , where h is a $\mathcal{C}^1$ -function. Thus $x \mapsto \mathbf{A}^g(x)$ is a progressively measurable function of x and $((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})$. It follows also that $(x, y) \mapsto \mathbf{A}^g(x - y)\varphi_\varepsilon(y)$ is a progressively measurable function of x, y and $(\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}$. By Fubini's Theorem, we deduce that $x \mapsto \mathbf{A}^{g, \varepsilon}(x)$ is a progressively

measurable function of x and $((\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b})$. Again by products and compositions, it follows that there is a progressively measurable function $\mathcal{H}$ such that $\mathbf{P}$ -almost surely, for every $u \in \mathbb{R}$ and for every $t \in [0, T]$,

$$\mathbf{H}_t^{g,\varepsilon}(u) = \mathcal{H}_t(u, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}).$$

It follows that $\mathbf{P}^W \otimes \mathbf{P}^\beta$ -almost surely, equality (49) holds. It completes the proof of (c). $\square$

5.2 Idiosyncratic noise

Coming back to equality (43), and applying the relation $\partial_\mu \phi(\mu_t^g) = \partial_v \left\{ \frac{\delta \phi}{\delta m}(\mu_t^g) \right\}$ (see Proposition 41 in “Appendix”), we have

$$\begin{aligned} I_2 &= \frac{1}{t} \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^t \partial_v \left\{ \frac{\delta \phi}{\delta m}(\mu_t^g) \right\} (x_t^g(u)) \partial_u x_t^g(u) H_s^{g,\varepsilon}(u) \, ds du \right] \\ &= \frac{1}{t} \int_0^1 \int_0^t \mathbb{E}^W \mathbb{E}^\beta \left[\partial_u \left\{ \frac{\delta \phi}{\delta m}(\mu_t^g)(x_t^g(\cdot)) \right\} (u) H_s^{g,\varepsilon}(u) \right] \, ds du. \end{aligned}$$

By Definition (45), $\left[\frac{\delta \phi}{\delta m} \right]$ is equal to $\frac{\delta \phi}{\delta m}$ up to a constant, so their derivatives are equal and

$$I_2 = \frac{1}{t} \int_0^1 \int_0^t \mathbb{E}^W \mathbb{E}^\beta \left[\partial_u \left\{ \left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(x_t^g(\cdot)) \right\} (u) H_s^{g,\varepsilon}(u) \right] \, ds du. \quad (52)$$

In the following lemma, we prove that I_2 can be expressed in terms of $\frac{\delta \phi}{\delta m}$ instead of its derivative. This key step is, as shown below, a consequence of Girsanov’s Theorem applied with respect to the idiosyncratic noise β .

Lemma 31 *Let $\theta \in (0, 1)$, $g \in \mathbf{G}^{1+\theta}$ and f be of order $\alpha > \frac{3}{2} + \theta$. Let $h \in \Delta^1$ and $\varepsilon > 0$. Fix $u \in [0, 1]$ and $s < t \in [0, T]$. Thus the following equality holds true*

$$\begin{aligned} &\mathbb{E}^W \mathbb{E}^\beta \left[\partial_u \left\{ \left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(x_t^g(\cdot)) \right\} (u) H_s^{g,\varepsilon}(u) \right] \\ &= \mathbb{E}^W \mathbb{E}^\beta \left[\left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(x_t^g(u)) H_s^{g,\varepsilon}(u) \frac{1}{t-s} \int_s^t \partial_u x_r^g(u) d\beta_r \right]. \end{aligned} \quad (53)$$

Proof Fix $u \in [0, 1]$ and $s < t \in [0, T]$. Define, for every $r \in [0, T]$, $\xi_r := \frac{1}{t-s} \int_0^r \mathbb{1}_{\{z \in [s, t]\}} dz$.

For every $v \in [-1, 1]$, denote by $(x_r^v)_{r \in [0, T]}$ the process $(x_r^{g+v\xi_r})_{r \in [0, T]}$. By Proposition 6, $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -almost surely, $x_r^v(u) = x_r^{g+v\xi_r}(u) = Z_r^{g(u)+v\xi_r}$. Apply Kunita’s

expansion (Lemma 20) to $x = g(u)$ and $\zeta_t = \xi_t$. We obtain for every $u \in \mathbb{R}$, $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -almost surely for every $r \in [0, T]$ and every $v \in [-1, 1]$:

$$x_r^v(u) = g(u) + \sum_{k \in \mathbb{Z}} f_k \int_0^r \Re \left(e^{-ikx_z^v(u)} dW_z^k \right) + \beta_r + v \int_0^r \partial_x Z_z^{g(u)+v\xi_z} \dot{\xi}_z dz.$$

Since both terms of the last equality are almost surely continuous with respect to $u \in \mathbb{R}$, that equality holds almost surely for every $u \in \mathbb{R}$.

For every $v \in [-1, 1]$, define the following stopping time

$$\sigma^v := \inf \left\{ r \geq 0 : \left| v \int_0^r \partial_x Z_z^{g(u)+v\xi_z} \dot{\xi}_z d\beta_z \right| \geq 1 \right\} \wedge T.$$

Define the process $(y_r^v)_{r \in [0, T]}$ as the solution to

$$dy_r^v(u) = \sum_{k \in \mathbb{Z}} f_k \Re \left(e^{-iky_r^v(u)} dW_r^k \right) + d\beta_r + v \mathbb{1}_{\{r \leq \sigma^v\}} \partial_x Z_r^{g(u)+v\xi_r} \dot{\xi}_r dr,$$

and $\beta_r^v := \beta_r + v \int_0^r \mathbb{1}_{\{z \leq \sigma^v\}} \partial_x Z_z^{g(u)+v\xi_z} \dot{\xi}_z dz$. Let us define for every $r \in [0, T]$

$$\mathcal{E}_r^v = \exp \left(-v \int_0^{r \wedge \sigma^v} \partial_x Z_z^{g(u)+v\xi_z} \dot{\xi}_z d\beta_z - \frac{v^2}{2} \int_0^{r \wedge \sigma^v} \left| \partial_x Z_z^{g(u)+v\xi_z} \dot{\xi}_z \right|^2 dz \right).$$

By definition of σ^v , we have $\mathcal{E}_r^v \leq \exp \left(-v \int_0^{r \wedge \sigma^v} \partial_x Z_z^{g(u)+v\xi_z} \dot{\xi}_z d\beta_z \right) \leq \exp(1)$. In particular, $(\mathcal{E}_r^v)_{r \in [0, T]}$ is a $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -martingale. Define $\mathbb{P}^v$ as the absolutely continuous probability measure with respect to $\mathbb{P}^W \otimes \mathbb{P}^\beta$ with density $\frac{d\mathbb{P}^v}{d(\mathbb{P}^W \otimes \mathbb{P}^\beta)} = \mathcal{E}_T^v$. Thus by Girsanov's Theorem, the law under $\mathbb{P}^v$ of $((W^k)_{k \in \mathbb{Z}}, \beta^v)$ is equal to the law under $\mathbb{P}^W \otimes \mathbb{P}^\beta$ of $((W^k)_{k \in \mathbb{Z}}, \beta)$. It follows that $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in [0, T]}, \mathbb{P}^v, y^v, (W^k)_{k \in \mathbb{Z}}, \beta^v)$ is a weak solution to Eq. (4).

By Lemma 30 and Yamada-Watanabe Theorem,

$$\mathcal{L}^{\mathbb{P}^v}(y^v, (W^k)_{k \in \mathbb{Z}}, \beta^v) = \mathcal{L}^{\mathbb{P}^W \otimes \mathbb{P}^\beta}(x^g, (W^k)_{k \in \mathbb{Z}}, \beta) = \mathcal{L}^{\mathbf{P}}(\mathbf{x}^g, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}),$$

and $\mathbb{P}^v$ -almost surely, for every $u \in \mathbb{R}$ and $t \in [0, T]$,

$$y_t^v(u) = \mathcal{X}_t(u, (W^k)_{k \in \mathbb{Z}}, \beta^v). \quad (54)$$

Moreover, we claim that $\mathbb{E}^v \left[\left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(y_t^v(u)) H_s^{g, \varepsilon}(u) \right]$ does not depend on v . Indeed, by (54)

$$F(v) := \mathbb{E}^v \left[\left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(y_t^v(u)) H_s^{g, \varepsilon}(u) \right]$$

$$\begin{aligned}
&= \mathbb{E}^v \left[\left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(\mathcal{X}_t(u, (W^k)_{k \in \mathbb{Z}}, \beta^v)) H_s^{g, \varepsilon}(u) \right] \\
&= \mathbb{E}^W \mathbb{E}^\beta \left[\left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(\mathcal{X}_t(u, (W^k)_{k \in \mathbb{Z}}, \beta^v)) H_s^{g, \varepsilon}(u) \frac{d\mathbb{P}^v}{d(\mathbb{P}^W \otimes \mathbb{P}^\beta)} \right].
\end{aligned}$$

Furthermore, by Lemma 30

$$\begin{aligned}
F(v) &= \mathbb{E}^W \mathbb{E}^\beta \left[\left[\frac{\delta\phi}{\delta m} \right] (\mathcal{P}_t((W^k)_{k \in \mathbb{Z}})) (\mathcal{X}_t(u, (W^k)_{k \in \mathbb{Z}}, \beta^v)) \mathcal{H}_s(u, (W^k)_{k \in \mathbb{Z}}, \beta) \frac{d\mathbb{P}^v}{d(\mathbb{P}^W \otimes \mathbb{P}^\beta)} \right] \\
&= \mathbb{E}^v \left[\left[\frac{\delta\phi}{\delta m} \right] (\mathcal{P}_t((W^k)_{k \in \mathbb{Z}})) (\mathcal{X}_t(u, (W^k)_{k \in \mathbb{Z}}, \beta^v)) \mathcal{H}_s(u, (W^k)_{k \in \mathbb{Z}}, \beta) \right].
\end{aligned}$$

Moreover, the processes (β_r) and (β_r^v) are equal on the interval $[0, s]$, because $\xi_r \equiv 0$ on $[0, s]$. Since $(\mathcal{H}_r)_{r \in [0, T]}$ is progressively measurable, then $\mathbb{P}^v$ -almost surely, $\mathcal{H}_s(u, (W^k)_{k \in \mathbb{Z}}, \beta) = \mathcal{H}_s(u, (W^k)_{k \in \mathbb{Z}}, \beta^v)$. Therefore,

$$\begin{aligned}
F(v) &= \mathbb{E}^v \left[\left[\frac{\delta\phi}{\delta m} \right] (\mathcal{P}_t((W^k)_{k \in \mathbb{Z}})) (\mathcal{X}_t(u, (W^k)_{k \in \mathbb{Z}}, \beta^v)) \mathcal{H}_s(u, (W^k)_{k \in \mathbb{Z}}, \beta^v) \right] \\
&= \mathbb{E}^W \mathbb{E}^\beta \left[\left[\frac{\delta\phi}{\delta m} \right] (\mathcal{P}_t((W^k)_{k \in \mathbb{Z}})) (\mathcal{X}_t(u, (W^k)_{k \in \mathbb{Z}}, \beta)) \mathcal{H}_s(u, (W^k)_{k \in \mathbb{Z}}, \beta) \right],
\end{aligned}$$

since the law of $((W^k)_{k \in \mathbb{Z}}, \beta^v)$ under $\mathbb{P}^v$ is equal to the law of $((W^k)_{k \in \mathbb{Z}}, \beta)$ under $\mathbb{P}^W \otimes \mathbb{P}^\beta$. The last term of that equality does not depend on v anymore, so we get finally

$$\frac{d}{dv} \Big|_{v=0} F(v) = \frac{d}{dv} \Big|_{v=0} \mathbb{E}^v \left[\left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(y_t^v(u)) H_s^{g, \varepsilon}(u) \right] = 0. \quad (55)$$

Furthermore,

$$\begin{aligned}
\mathbb{E}^v \left[\left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(y_t^v(u)) H_s^{g, \varepsilon}(u) \right] &= \mathbb{E}^W \mathbb{E}^\beta \left[\left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(y_t^v(u)) H_s^{g, \varepsilon}(u) \mathcal{E}_t^v \right] \\
&= \mathbb{E}^W \mathbb{E}^\beta \left[\left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(x_t^v(u)) H_s^{g, \varepsilon}(u) \mathcal{E}_t^v \right] + R(v),
\end{aligned} \quad (56)$$

where $R(v) = \mathbb{E}^W \mathbb{E}^\beta \left[\mathbb{1}_{\{\sigma^v < T\}} \left(\left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(y_t^v(u)) - \left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(x_t^v(u)) \right) H_s^{g, \varepsilon}(u) \mathcal{E}_t^v \right]$; we used here the fact that $\mathbb{1}_{\{\sigma^v = T\}}(x_t^v(u) - y_t^v(u)) = 0$. Let us show that $R(v) = \mathcal{O}(|v|^2)$. By Hölder's inequality and by the fact that $\mathcal{E}_t^v \leq \exp(1) = e$, we have

$$\begin{aligned}
|R(v)| &\leq 2e(\mathbb{P}^W \otimes \mathbb{P}^\beta) \left[\sigma^v < T \right]^{1/4} \mathbb{E}^W \mathbb{E}^\beta \left[H_s^{g, \varepsilon}(u)^2 \right]^{1/2} \\
&\quad \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{v \in \mathbb{R}} \left| \left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(v) \right|^4 \right]^{\frac{1}{4}}.
\end{aligned} \quad (57)$$

We control the different terms appearing on the r.h.s. of (57). By Markov's inequality, by Burkholder-Davis-Gundy inequality and by inequality (77), for every $v \in [-1, 1]$

$$\begin{aligned} \mathbb{P}^W \otimes \mathbb{P}^\beta [\sigma^v < T] &\leq \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{r \leq T} \left| v \int_0^r \partial_x Z_z^{g(u)+v\xi_z} \frac{\mathbb{1}_{\{z \in [s,t]\}}}{t-s} d\beta_z \right|^8 \right] \\ &\leq C|v|^8 \mathbb{E}^W \mathbb{E}^\beta \left[\left| \int_s^t \left| \partial_x Z_r^{g(u)+v\xi_r} \right|^2 \frac{1}{(t-s)^2} dr \right|^4 \right] \\ &\leq \frac{C}{(t-s)^5} |v|^8 \mathbb{E}^W \mathbb{E}^\beta \left[\int_s^t \left| \partial_x Z_r^{g(u)+v\xi_r} \right|^8 dr \right] \leq C|v|^8, \end{aligned}$$

where C is a constant depending on s and t changing from line to line.

Let us show that $\mathbb{E}^W \mathbb{E}^\beta [H_s^{g,\varepsilon}(u)^2] < +\infty$. Recall that $H_s^{g,\varepsilon}(u) := \frac{(A_s^g - A_s^{g,\varepsilon})(x_s^g(u))}{\partial_u x_s^g(u)}$. Thus

$$\mathbb{E}^W \mathbb{E}^\beta [H_s^{g,\varepsilon}(u)^2]^{1/2} \leq \mathbb{E}^W \mathbb{E}^\beta \left[\left\| \frac{1}{\partial_u x_s^g} \right\|_{L_\infty}^4 \right]^{1/4} \mathbb{E}^W \mathbb{E}^\beta [\|A_s^g - A_s^{g,\varepsilon}\|_{L_\infty}^4]^{1/4}.$$

By (72), $\mathbb{E}^W \mathbb{E}^\beta \left[\left\| \frac{1}{\partial_u x_s^g} \right\|_{L_\infty}^4 \right]$ is finite. Moreover, since $A_s^{g,\varepsilon} = A_s^g * \varphi_\varepsilon$, we have $\|A_s^g - A_s^{g,\varepsilon}\|_{L_\infty} \leq C\varepsilon \|\partial_x A_s^g\|_{L_\infty}$, where $C = \int_{\mathbb{R}} |y| \varphi(y) dy$. Using inequality (21) and an analogue to (22) with exponent 4 instead of 2, we check that $\mathbb{E}^W \mathbb{E}^\beta [\|\partial_x A_s^g\|_{L_\infty}]$ is finite. Thus there is C such that $\mathbb{E}^W \mathbb{E}^\beta [H_s^{g,\varepsilon}(u)^2] \leq C$.

Then show that $\mathbb{E}^W \mathbb{E}^\beta \left[\sup_{v \in \mathbb{R}} \left| \left[\frac{\delta \phi}{\delta m} \right](\mu_t^g)(v) \right|^4 \right]$ is finite. By definition (45), for every $v \in \mathbb{R}$,

$$\left| \left[\frac{\delta \phi}{\delta m} \right](\mu_t^g)(v) \right| \leq \mathbb{E}^\beta \left[\int_0^1 \left| \frac{\delta \phi}{\delta m}(\mu_t^g)(v) - \frac{\delta \phi}{\delta m}(\mu_t^g)(x_t^g(u)) \right| du \right]. \quad (58)$$

By inequality (80), there is a $C > 0$ such that $\mathbb{P}^W$ -almost surely for every $x \in [0, 2\pi]$,

$$\left| \partial_v \left\{ \frac{\delta \phi}{\delta m}(\mu_t^g) \right\}(x) \right| = |\partial_\mu \phi(\mu_t^g)(x)| \leq C(1 + 2\pi) + C \mathbb{E}^\beta \left[\int_0^1 |x_t^g(u')| du' \right]. \quad (59)$$

Thus there is $C > 0$ such that $\mathbb{P}^W$ -almost surely, for every $v, v' \in [0, 2\pi]$,

$$\left| \frac{\delta \phi}{\delta m}(\mu_t^g)(v) - \frac{\delta \phi}{\delta m}(\mu_t^g)(v') \right| \leq C + C \mathbb{E}^\beta \left[\int_0^1 |x_t^g(u')| du' \right].$$

By Proposition 45, $v \mapsto \frac{\delta\phi}{\delta m}(\mu_t^g)(v)$ is 2π -periodic, thus the latter inequality holds for every $v, v' \in \mathbb{R}$. Combining that inequality with (58), we get for every $v \in \mathbb{R}$,

$$\left| \left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(v) \right| \leq C + C\mathbb{E}^\beta \left[\int_0^1 |x_t^g(u')| du' \right].$$

This leads to

$$\mathbb{E}^W \mathbb{E}^\beta \left[\sup_{v \in \mathbb{R}} \left| \left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(v) \right|^4 \right] \leq C + C\mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |x_t^g(u')| du' \right], \quad (60)$$

which is finite. Thus $\mathbb{E}^W \mathbb{E}^\beta \left[\sup_{v \in \mathbb{R}} \left| \left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(v) \right|^4 \right] \leq C$, whence we finally deduce, in view of inequality (57), that $|R(v)| \leq C|v|^2$.

Thus $R(v) = \mathcal{O}(|v|^2)$. It follows from (55) and (56) that

$$\frac{d}{dv} \Big|_{v=0} \mathbb{E}^W \mathbb{E}^\beta \left[\left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(x_t^v(u)) H_s^{g,\varepsilon}(u) \mathcal{E}_t^v \right] = 0$$

By (60), $\left(\left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(x_t^v(u)) \right)_{v \in [-1,1]}$ is uniformly integrable. Using inequality (59), we prove in the same way that $\left(\partial_v \left\{ \left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g) \right\} (x_t^v(u)) \right)_{v \in [-1,1]}$ is also uniformly integrable. Recall that $x_t^v(u) = Z_t^{g(u)+v\xi_t}$ and that, by inequality (77), $(\partial_x Z_t^{g(u)+v\xi_t})_{v \in [-1,1]}$ is uniformly integrable. Thus we get by differentiation:

$$\begin{aligned} 0 &= \frac{d}{dv} \Big|_{v=0} \mathbb{E}^W \mathbb{E}^\beta \left[\left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(Z_t^{g(u)+v\xi_t}) H_s^{g,\varepsilon}(u) \mathcal{E}_t^v \right] \\ &= \mathbb{E}^W \mathbb{E}^\beta \left[\partial_v \left\{ \left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g) \right\} (Z_t^{g(u)}) \partial_x Z_t^{g(u)} \xi_t H_s^{g,\varepsilon}(u) \right] \\ &\quad - \mathbb{E}^W \mathbb{E}^\beta \left[\left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(Z_t^{g(u)}) H_s^{g,\varepsilon}(u) \int_0^t \partial_x Z_r^{g(u)} \dot{\xi}_r d\beta_r \right] \end{aligned}$$

Using $Z_t^{g(u)} = x_t^g(u)$ and $\partial_x Z_t^{g(u)} = \frac{\partial_u x_t^g(u)}{g'(u)}$ and recalling that $\xi_r := \frac{1}{t-s} \int_0^r \mathbb{1}_{\{z \in [s,t]\}} dz$, we have proved that

$$\begin{aligned} &\mathbb{E}^W \mathbb{E}^\beta \left[\partial_v \left\{ \left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g) \right\} (x_t^g(u)) \frac{\partial_u x_t^g(u)}{g'(u)} H_s^{g,\varepsilon}(u) \right] \\ &= \mathbb{E}^W \mathbb{E}^\beta \left[\left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(x_t^g(u)) H_s^{g,\varepsilon}(u) \frac{1}{t-s} \int_s^t \frac{\partial_u x_r^g(u)}{g'(u)} d\beta_r \right]. \end{aligned}$$

We multiply both sides by $g'(u)$ and we obtain equality (53), since $\partial_u \left\{ \left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g)(x_t^g(\cdot)) \right\} (u) = \partial_v \left\{ \left[\frac{\delta\phi}{\delta m} \right] (\mu_t^g) \right\} (x_t^g(u)) \partial_u x_t^g(u)$. $\square$

5.3 Conclusion of the analysis

We conclude the proof of Proposition 29.

Proof (Proposition 29) Putting together equalities (52) and (53) and definition (44) of $K_t^{g,\varepsilon}$,

$$I_2 = \frac{1}{t} \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(x_t^g(u)) K_t^{g,\varepsilon}(u) du \right].$$

By Cauchy-Schwarz inequality,

$$|I_2| \leq \frac{1}{t} \mathbb{E}^W \mathbb{E}^\beta \left[\|K_t^{g,\varepsilon}\|_{L_\infty}^2 \right]^{1/2} \mathbb{E}^W \mathbb{E}^\beta \left[\left| \int_0^1 \left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(x_t^g(u)) \frac{K_t^{g,\varepsilon}(u)}{\|K_t^{g,\varepsilon}\|_{L_\infty}} du \right|^2 \right]^{1/2}.$$

It remains to estimate $\mathbb{E}^W \mathbb{E}^\beta \left[\|K_t^{g,\varepsilon}\|_{L_\infty}^2 \right]$. For every $u \in [0, 1]$,

$$|K_t^{g,\varepsilon}(u)| \leq \int_0^t \|A_s^g - A_s^{g,\varepsilon}\|_{L_\infty} \frac{1}{|\partial_u x_s^g(u)|} \frac{1}{t-s} \left| \int_s^t \partial_u x_r^g(u) d\beta_r \right| ds$$

By inequality (21),

$$\|A_s^g - A_s^{g,\varepsilon}\|_{L_\infty} \leq C\varepsilon \|\partial_x A_s^g\|_{L_\infty} \leq C\varepsilon \|h\|_{C^1} \left(1 + \|\partial_u^{(2)} x_s^g\|_{L_\infty} \left\| \frac{1}{\partial_u x_s^g} \right\|_{L_\infty} \right).$$

Thus we obtain

$$\begin{aligned} \|K_t^{g,\varepsilon}\|_{L_\infty} &\leq C\varepsilon \|h\|_{C^1} \left\{ 1 + \sup_{r \leq T} \|\partial_u^{(2)} x_r^g\|_{L_\infty} \right\} \left\{ \sup_{r \leq T} \left\| \frac{1}{\partial_u x_r^g} \right\|_{L_\infty} + \sup_{r \leq T} \left\| \frac{1}{\partial_u x_r^g} \right\|_{L_\infty}^2 \right\} \\ &\quad \cdot \int_0^t \frac{1}{t-s} \left\| \int_s^t \partial_u x_r^g(\cdot) d\beta_r \right\|_{L_\infty} ds. \end{aligned}$$

By Hölder's equality, we obtain

$$\mathbb{E}^W \mathbb{E}^\beta \left[\|K_t^{g,\varepsilon}\|_{L_\infty}^2 \right]^{1/2} \leq C\varepsilon \|h\|_{C^1} E_1 E_2 E_3, \quad (61)$$

where

$$\begin{aligned} E_1 &:= 1 + \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{r \leq T} \|\partial_u^{(2)} x_r^g\|_{L_\infty}^8 \right]^{1/8}; \\ E_2 &:= \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{r \leq T} \left\| \frac{1}{\partial_u x_r^g} \right\|_{L_\infty}^8 \right]^{1/8} + \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{r \leq T} \left\| \frac{1}{\partial_u x_r^g} \right\|_{L_\infty}^{16} \right]^{1/8}; \end{aligned}$$

$$E_3 := \mathbb{E}^W \mathbb{E}^\beta \left[\left(\int_0^t \frac{1}{t-s} \left\| \int_s^t \partial_u x_r^g(\cdot) d\beta_r \right\|_{L_\infty} ds \right)^4 \right]^{1/4}.$$

Recall that g belongs to $\mathbf{G}^{3+\theta}$ and f is of order $\alpha > \frac{7}{2} + \theta$. Thus by (71) and by (72)

$$\begin{aligned} E_1 &\leq C(1 + \|g'''\|_{L_8} + \|g''\|_{L_\infty}^3 + \|g'\|_{L_\infty}^3); \\ E_2 &\leq C(1 + \|g''\|_{L_\infty}^4 + \|g'\|_{L_\infty}^4 + \|\frac{1}{g'}\|_{L_\infty}^8). \end{aligned}$$

Furthermore, $E_3 \leq E_{3,1} + E_{3,2}$, where

$$\begin{aligned} E_{3,1} &:= \mathbb{E}^W \mathbb{E}^\beta \left[\left(\int_0^t \frac{1}{t-s} \left| \int_s^t \partial_u x_r^g(0) d\beta_r \right| ds \right)^4 \right]^{1/4}; \\ E_{3,2} &:= \mathbb{E}^W \mathbb{E}^\beta \left[\left(\int_0^t \frac{1}{t-s} \int_0^1 \left| \int_s^t \partial_u^{(2)} x_r^g(v) d\beta_r \right| ds dv \right)^4 \right]^{1/4}. \end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned} E_{3,1} &\leq \mathbb{E}^W \mathbb{E}^\beta \left[\left(\int_0^t \frac{1}{|t-s|^{1/2}} ds \right)^3 \int_0^t \frac{1}{|t-s|^{5/2}} \left| \int_s^t \partial_u x_r^g(0) d\beta_r \right|^4 ds \right]^{1/4} \\ &\leq C t^{3/8} \left(\int_0^t \frac{1}{|t-s|^{5/2}} \mathbb{E}^W \mathbb{E}^\beta \left[\left| \int_s^t \partial_u x_r^g(0) d\beta_r \right|^4 \right] ds \right)^{1/4}. \end{aligned}$$

By Burkholder-Davis-Gundy inequality, it follows that

$$E_{3,1} \leq C t^{3/8} \left(\int_0^t \frac{|t-s|^2}{|t-s|^{5/2}} ds \right)^{1/4} \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{r \leq T} |\partial_u x_r^g(0)|^4 \right]^{1/4} \leq C \sqrt{t} \|g'\|_{L_\infty},$$

where the last inequality holds by (69). By the same computation,

$$E_{3,2} \leq C \sqrt{t} \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{r \leq T} \int_0^1 |\partial_u^{(2)} x_r^g(v)|^4 dv \right]^{1/4} \leq C \sqrt{t} (1 + \|g''\|_{L_\infty} + \|g'\|_{L_\infty}^2),$$

where the last inequality holds by (70). We deduce that $E_3 \leq C \sqrt{t} (1 + \|g''\|_{L_\infty} + \|g'\|_{L_\infty}^2)$. By inequality (61) and the estimates on E_i , for $i = 1, 2, 3$, we finally get:

$$\mathbb{E}^W \mathbb{E}^\beta \left[\|K_t^{g,\varepsilon}\|_{L_\infty}^2 \right]^{1/2} \leq C \sqrt{t\varepsilon} \|h\|_{C^1} C_2(g),$$

where $C_2(g) = 1 + \|g'''\|_{L_8}^3 + \|g''\|_{L_\infty}^{12} + \|g'\|_{L_\infty}^{12} + \|\frac{1}{g'}\|_{L_\infty}^{24}$. $\square$

As a conclusion of Sects. 4 and 5, we have proved the following inequality.

Corollary 32 *Let ϕ , θ and f be as in Theorem 15. Let g and h be $\mathcal{G}_0$ -measurable random variables with values respectively in $\mathbf{G}^{3+\theta}$ and Δ^1 . Let $(K_t^{g,\varepsilon})_{t \in [0,T]}$ be defined by (44). Then there is $C > 0$ independent of g , h and θ such that $\mathbb{P}^0$ -almost surely, for every $t \in (0, T]$ and $\varepsilon \in (0, 1)$,*

$$\begin{aligned} & \left| \frac{d}{d\rho} \Big|_{\rho=0} P_t \phi(\mu_0^{g+\rho g'h}) \right| \\ & \leq C \frac{\|\phi\|_{L_\infty}}{\varepsilon^{3+2\theta} \sqrt{t}} C_1(g) \|h\|_{C^1} \\ & \quad + \frac{C}{\sqrt{t}} \varepsilon \|h\|_{C^1} C_2(g) \mathbb{E}^W \mathbb{E}^\beta \left[\left| \int_0^1 \left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(x_t^g(u)) \frac{K_t^{g,\varepsilon}(u)}{\|K_t^{g,\varepsilon}\|_{L_\infty}} du \right|^2 \right]^{1/2}, \end{aligned} \quad (62)$$

where $C_1(g) = 1 + \|g'''\|_{L_4}^2 + \|g''\|_{L_\infty}^6 + \|g'\|_{L_\infty}^8 + \left\| \frac{1}{g'} \right\|_{L_\infty}^8$ and $C_2(g) = 1 + \|g'''\|_{L_8}^3 + \|g''\|_{L_\infty}^{12} + \|g'\|_{L_\infty}^{12} + \left\| \frac{1}{g'} \right\|_{L_\infty}^{24}$.

6 Proof of the main theorem

Essentially, Corollary 32 states that we can control the gradient of $P_t \phi$ by the gradient of ϕ . By iterating the inequality over successive time steps, we will conclude the proof of Theorem 15.

Definition 33 Let $\mathcal{K}_t$ be the set of $\mathcal{G}_t$ -measurable random variables taking their values $\mathbb{P}$ -almost surely in the set of continuous 1-periodic functions $k : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\|k\|_{L_\infty} = 1$.

Proposition 34 *Let ϕ , θ , f and g be as in Theorem 15. Let $t, s \in [0, T]$ such that $t + s \leq T$. For each $\mathcal{G}_s$ -measurable function h with values in Δ^1 satisfying $\mathbb{P}$ -almost surely $\|h\|_{C^1} \leq 4$, there exists $C_g > 0$ independent of s , t and h such that*

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^1 \left[\frac{\delta P_t \phi}{\delta m} \right] (\mu_s^g)(x_s^g(u)) h'(u) du \right|^2 \right]^{1/2} \\ & \leq C_g \frac{\|\phi\|_{L_\infty}}{t^{2+\theta}} \\ & \quad + \frac{1}{2^{3+\theta}} \sup_{k \in \mathcal{K}_{t+s}} \mathbb{E} \left[\left| \int_0^1 \left[\frac{\delta \phi}{\delta m} \right] (\mu_{t+s}^g)(x_{t+s}^g(u)) k(u) du \right|^2 \right]^{1/2}. \end{aligned} \quad (63)$$

Proof By equality (12),

$$\frac{d}{d\rho} \Big|_{\rho=0} P_t \phi(\mu_0^{g+\rho g'h})$$

$$\begin{aligned}
&= - \int_0^1 \frac{\delta P_t \phi}{\delta m} (\mu_0^g)(g(u)) h'(u) du \\
&= - \int_0^1 \left[\frac{\delta P_t \phi}{\delta m} (\mu_0^g)(g(u)) - \int_0^1 \frac{\delta P_t \phi}{\delta m} (\mu_0^g)(g(u')) du' \right] h'(u) du \\
&= - \int_0^1 \left[\frac{\delta P_t \phi}{\delta m} \right] (\mu_0^g)(g(u)) h'(u) du, \tag{64}
\end{aligned}$$

where the second equality follows from the fact that h is 1-periodic and the last equality follows from (45). Apply now inequality (62) with $\varepsilon_0 = \frac{1}{2^{\frac{7}{2}+\theta}} \frac{\sqrt{t}}{C \|h\|_{C^1} C_2(g)}$. For every $\mathcal{G}_0$ -measurable g and h ,

$$\begin{aligned}
&\left| \int_0^1 \left[\frac{\delta P_t \phi}{\delta m} \right] (\mu_0^g)(g(u)) h'(u) du \right| \\
&\leq C \frac{\|\phi\|_{L^\infty}}{t^{2+\theta}} C_3(g) \|h\|_{C^1}^{4+2\theta} \\
&\quad + \frac{1}{2^{\frac{7}{2}+\theta}} \mathbb{E}^W \mathbb{E}^\beta \left[\left| \int_0^1 \left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(x_t^g(u)) \frac{K_t^{g, \varepsilon_0}(u)}{\|K_t^{g, \varepsilon_0}\|_{L^\infty}} du \right|^2 \right]^{\frac{1}{2}},
\end{aligned}$$

where $C_3(g) = C_1(g)C_2(g)^{3+2\theta}$. Moreover, $\mathbb{E}^W \mathbb{E}^\beta [\cdot] = \mathbb{E}[\cdot | \mathcal{G}_0]$, since for any random variable X on Ω and any $\mathcal{G}_0$ -measurable Y , $\mathbb{E}[XY] = \mathbb{E}^0 \mathbb{E}^W \mathbb{E}^\beta [XY] = \mathbb{E}^0 [\mathbb{E}^W \mathbb{E}^\beta [X] Y]$. Thus it follows from the latter inequality that:

$$\begin{aligned}
&\mathbb{E} \left[\left| \int_0^1 \left[\frac{\delta P_t \phi}{\delta m} \right] (\mu_0^g)(g(u)) h'(u) du \right|^2 \middle| \mathcal{G}_0 \right] \\
&\leq C \frac{\|\phi\|_{L^\infty}^2}{t^{4+2\theta}} C_3(g)^2 \|h\|_{C^1}^{8+4\theta} \\
&\quad + \frac{2}{2^{7+2\theta}} \mathbb{E} \left[\left| \int_0^1 \left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(x_t^g(u)) \frac{K_t^{g, \varepsilon_0}(u)}{\|K_t^{g, \varepsilon_0}\|_{L^\infty}} du \right|^2 \middle| \mathcal{G}_0 \right].
\end{aligned}$$

Now, consider a deterministic function g and a $\mathcal{G}_s$ -measurable h , where $s \leq T - t$. Then, repeating the whole argument with the $\mathcal{G}_s$ -measurable variables x_s^g and h instead of g and a $\mathcal{G}_0$ -measurable h , respectively, we get:

$$\begin{aligned}
&\mathbb{E} \left[\left| \int_0^1 \left[\frac{\delta P_t \phi}{\delta m} \right] (\mu_s^g)(x_s^g(u)) h'(u) du \right|^2 \middle| \mathcal{G}_s \right] \\
&\leq C \frac{\|\phi\|_{L^\infty}^2}{t^{4+2\theta}} C_3(x_s^g)^2 \|h\|_{C^1}^{8+4\theta}
\end{aligned}$$

$$+ \frac{1}{2^{6+2\theta}} \mathbb{E} \left[\left| \int_0^1 \left[\frac{\delta \phi}{\delta m} \right] (\mu_{t+s}^{s, x_s^g})(x_{t+s}^{s, x_s^g}(u)) \frac{K_{t+s}^{s, x_s^g, \varepsilon_s}(u)}{\|K_{t+s}^{s, x_s^g, \varepsilon_s}\|_{L_\infty}} du \right|^2 \middle| \mathcal{G}_s \right],$$

where $x_{t+s}^{s, x_s^g}(u)$ denotes the value at time $t+s$ and at point u of the unique solution to (4) which is equal to x_s^g at time s and where ε_s is $\mathcal{G}_s$ -measurable. By strong uniqueness of (4), we have the following flow property: $x_{t+s}^{s, x_s^g} = x_{t+s}^g$ and $\mu_{t+s}^{s, x_s^g} = \mu_{t+s}^g$. Therefore,

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^1 \left[\frac{\delta P_t \phi}{\delta m} \right] (\mu_s^g)(x_s^g(u)) h'(u) du \right|^2 \middle| \mathcal{G}_s \right] \\ & \leq C \frac{\|\phi\|_{L_\infty}^2}{t^{4+2\theta}} C_3(x_s^g)^2 \|h\|_{C^1}^{8+4\theta} \\ & \quad + \frac{1}{2^{6+2\theta}} \mathbb{E} \left[\left| \int_0^1 \left[\frac{\delta \phi}{\delta m} \right] (\mu_{t+s}^g)(x_{t+s}^g(u)) \frac{K_{t+s}^{s, x_s^g, \varepsilon_s}(u)}{\|K_{t+s}^{s, x_s^g, \varepsilon_s}\|_{L_\infty}} du \right|^2 \middle| \mathcal{G}_s \right]. \end{aligned}$$

Remark that $u \mapsto K_{t+s}^{s, x_s^g, \varepsilon_s}(u) / \|K_{t+s}^{s, x_s^g, \varepsilon_s}\|_{L_\infty}$ belongs to $\mathcal{K}_{t+s}$. Thus, taking the expectation of the latter inequality, there is $C > 0$ so that for every $\mathcal{G}_s$ -measurable function h satisfying $\|h\|_{C^1} \leq 4$

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^1 \left[\frac{\delta P_t \phi}{\delta m} \right] (\mu_s^g)(x_s^g(u)) h'(u) du \right|^2 \right]^{\frac{1}{2}} \\ & \leq C \frac{\|\phi\|_{L_\infty}}{t^{2+\theta}} \mathbb{E} \left[C_3(x_s^g)^2 \right]^{\frac{1}{2}} \\ & \quad + \frac{1}{2^{3+\theta}} \sup_{k \in \mathcal{K}_{t+s}} \mathbb{E} \left[\left| \int_0^1 \left[\frac{\delta \phi}{\delta m} \right] (\mu_{t+s}^g)(x_{t+s}^g(u)) k(u) du \right|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

In order to prove inequality (63), it remains to show that there is C_g such that $\mathbb{E} [C_3(x_s^g)^2] \leq C_g$. Since $C_3(g) = C_1(g)C_2(g)^{3+2\theta}$, we have:

$$\begin{aligned} \mathbb{E} [C_3(x_s^g)^2] &= \mathbb{E} \left[\left(1 + \|\partial_u^{(3)} x_s^g\|_{L_4}^2 + \|\partial_u^{(2)} x_s^g\|_{L_\infty}^6 + \|\partial_u x_s^g\|_{L_\infty}^8 + \left\| \frac{1}{\partial_u x_s^g} \right\|_{L_\infty}^8 \right)^2 \right. \\ & \quad \cdot \left. \left(1 + \|\partial_u^{(3)} x_s^g\|_{L_8}^3 + \|\partial_u^{(2)} x_s^g\|_{L_\infty}^{12} + \|\partial_u x_s^g\|_{L_\infty}^{12} + \left\| \frac{1}{\partial_u x_s^g} \right\|_{L_\infty}^{24} \right)^{6+4\theta} \right]. \end{aligned}$$

We refer to (70), (71) and (72) to argue that the r.h.s. is bounded by a uniform constant in $s \in [0, T]$ and depending polynomially on $\|g'''\|_{L_\infty}$, $\|g''\|_{L_\infty}$, $\|g'\|_{L_\infty}$ and $\| \frac{1}{g'} \|_{L_\infty}$. The constant is finite since g belongs to $\mathbf{G}^{3+\theta}$. $\square$

Corollary 35 *Let ϕ , θ , f and g satisfy the same assumption as in Proposition 34. Let $t, s \in [0, T]$ such that $2t + s \leq T$. For any $h : \mathbb{R} \rightarrow \mathbb{R}$ be a $\mathcal{G}_s$ -measurable random*

variable with values in Δ^1 satisfying $\mathbb{P}$ -almost surely $\|h\|_{C^1} \leq 4$, there exists $C_g > 0$ independent of s , t and h such that

$$\begin{aligned} \mathbb{E} \left[\left| \int_0^1 \left[\frac{\delta P_{2t}\phi}{\delta m} \right] (\mu_s^g)(x_s^g(u)) h'(u) du \right|^2 \right]^{\frac{1}{2}} &\leq C_g \frac{\|\phi\|_{L^\infty}}{t^{2+\theta}} \\ &+ \frac{1}{2^{3+\theta}} \sup_{k \in \mathcal{K}_{t+s}} \mathbb{E} \left[\left| \int_0^1 \left[\frac{\delta P_t\phi}{\delta m} \right] (\mu_{t+s}^g)(x_{t+s}^g(u)) k(u) du \right|^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (65)$$

Proof We get the above inequality by applying (63) to $P_t\phi$ instead of ϕ . We note that $P_t(P_t\phi) = P_{2t}\phi$ and that $\|P_t\phi\|_{L^\infty} \leq \|\phi\|_{L^\infty}$. $\square$

Fix $t_0 \in (0, T]$. For every $t \in (0, t_0]$, define

$$S_t := \sup_{k \in \mathcal{K}_{t_0-t}} \mathbb{E} \left[\left| \int_0^1 \left[\frac{\delta P_t\phi}{\delta m} \right] (\mu_{t_0-t}^g)(x_{t_0-t}^g(u)) k(u) du \right|^2 \right]^{\frac{1}{2}},$$

where $\mathcal{K}_{t_0-t}$ is defined by Definition 33.

Proposition 36 Let ϕ , θ , f and g be as in Theorem 15. For every $t \in (0, \frac{t_0}{2}]$, we have:

$$S_{2t} \leq C_g \frac{\|\phi\|_{L^\infty}}{t^{2+\theta}} + \frac{1}{2^{3+\theta}} S_t. \quad (66)$$

Proof Fix $t \in (0, \frac{t_0}{2}]$ and $k \in \mathcal{K}_{t_0-2t}$. Hence k is a continuous 1-periodic function and a $\mathcal{G}_{t_0-2t}$ -measurable random variable so that $\mathbb{P}$ -almost surely, $\|k\|_{L^\infty} = \sup_{u \in [0,1]} |k(u)| = 1$.

Let us denote by h the map defined for every $u \in \mathbb{R}$ by $h(u) := \int_0^u (k(v) - \bar{k}) dv$, where $\bar{k} = \int_0^1 k(v) dv$. We check that h is a $\mathcal{G}_{t_0-2t}$ -measurable 1-periodic C^1 -function. Moreover, $\|h\|_{L^\infty} \leq 2$ and $\|\partial_u h\|_{L^\infty} \leq 2$; thus $\|h\|_{C^1} \leq 4$. Therefore, the assumptions of Corollary 35 are satisfied, with $s = t_0 - 2t$. We apply (65) with $s = t_0 - 2t$:

$$\mathbb{E} \left[\left| \int_0^1 \left[\frac{\delta P_{2t}\phi}{\delta m} \right] (\mu_{t_0-2t}^g)(x_{t_0-2t}^g(u)) h'(u) du \right|^2 \right]^{\frac{1}{2}} \leq C_g \frac{\|\phi\|_{L^\infty}}{t^{2+\theta}} + \frac{1}{2^{3+\theta}} S_t.$$

Moreover, $h'(u) = k(u) - \bar{k}$ and by definition (45), $\int_0^1 \left[\frac{\delta P_{2t}\phi}{\delta m} \right] (\mu_{t_0-2t}^g)(x_{t_0-2t}^g(u)) \cdot \bar{k} du = 0$. Thus

$$\mathbb{E} \left[\left| \int_0^1 \left[\frac{\delta P_{2t}\phi}{\delta m} \right] (\mu_{t_0-2t}^g)(x_{t_0-2t}^g(u)) k(u) du \right|^2 \right]^{\frac{1}{2}} \leq C_g \frac{\|\phi\|_{L^\infty}}{t^{2+\theta}} + \frac{1}{2^{3+\theta}} S_t,$$

and by taking the supremum over all k in $\mathcal{K}_{t_0-2t}$, we get $S_{2t} \leq C_g \frac{\|\phi\|_{L^\infty}}{t^{2+\theta}} + \frac{1}{2^{3+\theta}} S_t$. $\square$

We complete the proof of Theorem 15.

Proof (Theorem 15) It follows from Proposition 36 that for every $t \in (0, \frac{t_0}{2}]$,

$$(2t)^{2+\theta} S_{2t} \leq 2^{2+\theta} C_g \|\phi\|_{L_\infty} + \frac{1}{2} t^{2+\theta} S_t.$$

Therefore, denoting by $\mathbf{S} := \sup_{t \in (0, t_0]} t^{2+\theta} S_t$, we have $\mathbf{S} \leq 2^{2+\theta} C_g \|\phi\|_{L_\infty} + \frac{1}{2} \mathbf{S}$. Since $\mathbf{S}$ is finite, we obtain $\mathbf{S} \leq 2^{3+\theta} C_g \|\phi\|_{L_\infty}$. Thus for every $t_0 \in (0, T]$, $t_0^{2+\theta} S_{t_0} \leq 2^{3+\theta} C_g \|\phi\|_{L_\infty}$. Therefore, for any deterministic 1-periodic function $k : \mathbb{R} \rightarrow \mathbb{R}$ and for every $t \in (0, T]$, we have

$$\left| \int_0^1 \left[\frac{\delta P_t \phi}{\delta m} \right] (\mu_0^g)(g(u)) k(u) du \right| \leq C_g \frac{\|\phi\|_{L_\infty}}{t^{2+\theta}} \|k\|_{L_\infty}.$$

Let $h \in \Delta^1$. Thus $k = \partial_u \left(\frac{h}{g'} \right)$ is a 1-periodic function and we deduce that

$$\begin{aligned} & \left| \int_0^1 \left[\frac{\delta P_t \phi}{\delta m} \right] (\mu_0^g)(g(u)) \partial_u \left(\frac{h}{g'} \right) (u) du \right| \\ & \leq C_g \frac{\|\phi\|_{L_\infty}}{t^{2+\theta}} \left\| \partial_u \left(\frac{h}{g'} \right) \right\|_{L_\infty} \leq C_g \frac{\|\phi\|_{L_\infty}}{t^{2+\theta}} \|h\|_{C^1}, \end{aligned}$$

for a new constant C_g . Applying equality (64) with $\frac{h}{g'}$ instead of g' , we obtain

$$\begin{aligned} & \left| \frac{d}{d\rho} \Big|_{\rho=0} P_t \phi(\mu_0^{g+\rho h}) \right| = \left| \int_0^1 \left[\frac{\delta P_t \phi}{\delta m} \right] (\mu_0^g)(g(u)) \partial_u \left(\frac{h}{g'} \right) (u) du \right| \\ & \leq C_g \frac{\|\phi\|_{L_\infty}}{t^{2+\theta}} \|h\|_{C^1}, \end{aligned}$$

which concludes the proof of the theorem. $\square$

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A Appendix

A.1 Density functions and quantile functions on the torus

We define the set of positive densities on the torus.

Definition 37 Let $\mathcal{P}^+$ be the set of continuous functions $p : \mathbb{T} \rightarrow \mathbb{R}$ such that for every $x \in \mathbb{T}$, $p(x) > 0$ and $\int_{\mathbb{T}} p = 1$. $\mathcal{P}^+$ can also be seen as the set of 2π -periodic and continuous functions $p : \mathbb{R} \rightarrow (0, +\infty)$ such that $\int_0^{2\pi} p(x)dx = 1$.

Let $p \in \mathcal{P}^+$ and $x_0 \in \mathbb{T}$ be an arbitrary point on the torus. Define a cumulative distribution function (c.d.f.) $F_0 : \mathbb{R} \rightarrow \mathbb{R}$ by $F_0(x) = \int_{x_0}^x p(y)dy$, for each $x \in \mathbb{R}$. Since p is 2π -periodic and $\int_0^{2\pi} p = 1$, F_0 satisfies $F_0(x + 2\pi) = F_0(x) + 1$ for each $x \in \mathbb{R}$. It follows from the continuity and from the positivity of p that F_0 is a C^1 -function and for every $x \in \mathbb{R}$, $F'_0(x) = p(x) > 0$, so that F_0 is strictly increasing. Therefore, the inverse function $g_0 := F_0^{-1} : \mathbb{R} \rightarrow \mathbb{R}$ is well defined. The following properties of g_0 are straightforward:

- for every $x \in \mathbb{R}$, $g_0 \circ F_0(x) = x$ and for every $u \in \mathbb{R}$, $F_0 \circ g_0(u) = u$;
- g_0 is a strictly increasing C^1 -function and for each $u \in \mathbb{R}$, $g'_0(u) = \frac{1}{p(g_0(u))}$;
- $g_0(0) = x_0$ and for every $u \in \mathbb{R}$, $g_0(u + 1) = g_0(u) + 2\pi$ (we say that g_0 is pseudo-periodic);
- $g'_0 : \mathbb{R} \rightarrow \mathbb{R}$ is positive everywhere and is a 1-periodic function.

Proposition 38 *There is a one-to-one correspondence between the set $\mathcal{P}^+$ and the set $\mathbf{G}^1$ of Definition 2.*

Proof Let $\iota : \mathcal{P}^+ \rightarrow \mathbf{G}^1$ be the map such that for every $p \in \mathcal{P}^+$, $\bar{g} = \iota(p)$ is the equivalence class given by the above construction. We show that ι is one-to-one.

First, ι is injective. Indeed, let $p_1, p_2 \in \mathcal{P}^+$ such that $\iota(p_1) = \iota(p_2)$. Let $x_0 \in \mathbb{T}$ and define, for $i = 1, 2$, $F_i(x) = \int_{x_0}^x p_i(y)dy$ and $g_i = F_i^{-1}$. Then by construction $\bar{g}_1 = \iota(p_1) = \iota(p_2) = \bar{g}_2$. Therefore, there is $c \in \mathbb{R}$ such that $g_2(\cdot) = g_1(\cdot + c)$. Thus for every $x \in \mathbb{R}$,

$$F_1(x) = F_1(g_2 \circ F_2(x)) = F_1(g_1(F_2(x) + c)) = F_2(x) + c.$$

Thus F_1 and F_2 share the same derivative: $p_1 = p_2$.

Second, ι is surjective. Let $\bar{g} \in \mathbf{G}^1$ and g be a representative of the class $\bar{g}$. It is a C^1 -function such that $g'(u) > 0$ for every $u \in \mathbb{R}$ and, since $g(u + 1) = g(u) + 2\pi$ for every $u \in \mathbb{R}$, g' is 1-periodic. Define $F := g^{-1} : \mathbb{R} \rightarrow \mathbb{R}$. In particular, F is a C^1 -function such that $F' > 0$ and for every $x \in \mathbb{R}$, $F(x + 2\pi) = F(x) + 1$. Thus $p := F'$ is a continuous function with values in $(0, +\infty)$ and for every $x \in \mathbb{R}$, $p(x) = \frac{1}{g'(F(x))}$.

Thus for every $x \in \mathbb{R}$, $p(x + 2\pi) = p(x)$ and $\int_0^{2\pi} p = 1$. Therefore p belongs to $\mathcal{P}^+$. We check that $\bar{g} = \iota(p)$. Let x_0 be an arbitrary point in $\mathbb{T}$, F_0 be defined by $F_0(x) = \int_{x_0}^x p(y)dy$ and $g_0 := F_0^{-1}$. Since $F'_0 = p = F'$, there is $c \in \mathbb{R}$ such that $F_0(\cdot) = F(\cdot) + c$. Therefore, $g_0(\cdot) = g(\cdot + c)$, whence $g_0 \sim g$. This completes the proof. $\square$

A.2 Properties of the diffusion on the torus

We prove in this paragraph several properties of the diffusion constructed in Paragraph 2.1 of the main text. First, let us show propositions 3 and 4.

Proof (Proposition 3) Strong existence and uniqueness hold by a fixed-point argument: we refer to the proof of [34, Prop. 3]. The additional Brownian motion $(\beta_t)_{t \in [0, T]}$ does not add any difficulty to that proof, since it does not depend on the initial condition g nor on the variable u . By a standard application of Kolmogorov's Lemma, we also obtain the existence of a version in $\mathcal{C}(\mathbb{R} \times [0, T])$, see the proof of [34, Prop. 5]. Moreover, the fact that the map $u \mapsto x_t^g(u)$ is strictly increasing is obtained by the study of the process $(x_t^g(u_2) - x_t^g(u_1))_{t \in [0, T]}$ for every $u_1 < u_2$ as in [34, Prop. 6]. The fact that it holds $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -almost surely, for every $0 \leq u_1 < u_2 \leq 1$ follows from the continuity of x^g , see [34, Cor. 7]. $\square$

Proof (Proposition 4) Assume that g belongs to $\mathcal{G}^{1+\theta}$ and that f is of order $\alpha > \frac{3}{2} + \theta$. This second assumption ensures that $\sum_{k \in \mathbb{Z}} \langle k \rangle^{2+2\theta} |f_k|^2$ converges. By differentiating formally (w.r.t. variable u) Eq. (4), consider a solution $(z_t(u))_{t \in [0, T], u \in \mathbb{R}}$ to:

$$z_t(u) = g'(u) + \sum_{k \in \mathbb{Z}} f_k \int_0^t z_s(u) \Re \left(-i k e^{-i k x_s^g(u)} dW_s^k \right). \quad (67)$$

Using the fact that g' is θ -Hölder continuous and that $\sum_{k \in \mathbb{Z}} \langle k \rangle^{2+2\theta} |f_k|^2 < +\infty$, we prove by standard arguments that for each $u \in \mathbb{R}$, the solution $(z_t(u))_{t \in [0, T]}$ exists, is unique and that the map $u \mapsto z_t(u) \in L_2(\Omega, \mathcal{C}[0, T])$ is θ' -Hölder continuous for each $\theta' < \theta$.

Furthermore, for each $u \in \mathbb{R}$, $\frac{x_s^g(u+\varepsilon) - x_s^g(u)}{\varepsilon} \rightarrow_{\varepsilon \rightarrow 0} z_s(u)$ in $L_2(\Omega, \mathcal{C}[0, T])$. Indeed, define for each $t \in [0, T]$ and $\varepsilon \neq 0$, $E_\varepsilon(t) := \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{s \leq t} \left| \frac{x_s^g(u+\varepsilon) - x_s^g(u)}{\varepsilon} - z_s(u) \right|^2 \right]$. We easily get a constant C depending on $\|g\|_{\mathcal{C}^{1+\theta}}$ and on $\sum_{k \in \mathbb{Z}} \langle k \rangle^{2+2\theta} |f_k|^2$ such that for each $t \in [0, T]$, $E_\varepsilon(t) \leq C|\varepsilon|^{2\theta} + C \int_0^t E_\varepsilon(s) ds$. By Gronwall's Lemma, it follows that $E_\varepsilon(T) \leq C|\varepsilon|^{2\theta}$, thus $E_\varepsilon(T) \rightarrow 0$. Therefore, using the continuity of z , we get almost surely for every $u \in \mathbb{R}$, for every $\varepsilon \neq 0$ and for every $t \in [0, T]$:

$$\frac{x_t^g(u + \varepsilon) - x_t^g(u)}{\varepsilon} = \int_0^1 z_t(u + \lambda \varepsilon) d\lambda.$$

Thus almost surely, $\partial_u x_t^g(u) = z_t(u)$ for every $u \in \mathbb{R}$ and $t \in [0, T]$ and furthermore it is given by the exponential form (5). The statements for higher derivatives are obtained similarly. For a detailed version of this proof with every computation of the inequalities mentioned above, see [33, Lemmas II.12 and II.13]. $\square$

By the previous proof, we know that $\partial_u x^g$ satisfies Eq. (67). It follows that we can control the L_p -norms of $\partial_u x^g$ and of higher derivatives with respect to the initial condition g .

Lemma 39 Let $\theta \in (0, 1)$, $j \geq 1$, $\kappa > 0$ and let g be a $\mathcal{G}_0$ -measurable random variable belonging $\mathbb{P}^0$ -a.s. to $\mathbf{G}^\kappa$. Assume that f is of order $\alpha > \kappa + \frac{1}{2}$. Then there

are constants⁶ C_p and $C_{p,j}$ such that $\mathbb{P}^0$ -a.s.

$$\text{if } \kappa \geq 1 + \theta, \quad \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \|\partial_u x_t^g\|_{L_p[0,1]}^p \right] \leq C_p \|g'\|_{L_p[0,1]}^p, \quad (68)$$

$$\text{if } \kappa \geq 1 + \theta, \quad \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} |\partial_u x_t^g(0)|^p \right] \leq C_p g'(0)^p, \quad (69)$$

$$\begin{aligned} \text{if } \kappa \geq j + \theta, \quad \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \|\partial_u^{(j)} x_t^g\|_{L_p[0,1]}^p \right] \\ \leq C_{p,j} \left\{ 1 + \|\partial_u^{(j)} g\|_{L_p[0,1]}^p + \sum_{k=1}^{j-1} \|\partial_u^{(k)} g\|_{L_\infty[0,1]}^{jp} \right\}, \end{aligned} \quad (70)$$

$$\begin{aligned} \text{if } \kappa \geq j + 1 + \theta, \quad \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \|\partial_u^{(j)} x_t^g\|_{L_\infty}^p \right] \\ \leq C_{p,j} \left\{ 1 + \|\partial_u^{(j+1)} g\|_{L_p}^p + \sum_{k=1}^j \|\partial_u^{(k)} g\|_{L_\infty}^{(j+1)p} \right\}, \end{aligned} \quad (71)$$

$$\begin{aligned} \text{if } \kappa \geq 2 + \theta, \quad \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \left\| \frac{1}{\partial_u x_t^g} \right\|_{L_\infty}^p \right] \\ \leq C_p \left\{ 1 + \frac{1}{g'(0)^p} + \left\| \frac{1}{g'} \right\|_{L_{4p}}^{4p} + \|g''\|_{L_{2p}}^{2p} + \|g'\|_{L_\infty}^{4p} \right\}. \end{aligned} \quad (72)$$

Proof The proofs of those five inequalities are pretty similar. Let us show inequality (68) in details. Let $M > 0$ be chosen large enough so that $\int_0^1 |g'(v)|^p dv < M$. Define the stopping time $\sigma^M := \inf\{t \geq 0 : \int_0^1 |\partial_u x_t^g(v)|^p dv \geq M\}$. Since $\partial_u x^g$ satisfies (67), it follows from Burkholder-Davis-Gundy inequality that for every $t \in [0, T]$,

$$\begin{aligned} \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{s \leq t \wedge \sigma^M} \int_0^1 |\partial_u x_s^g(v)|^p dv \right] \\ \leq C_p \|g'\|_{L_p[0,1]}^p \\ + C_p \int_0^1 \mathbb{E}^W \mathbb{E}^\beta \left[\left| \sum_{k \in \mathbb{Z}} f_k^2 \int_0^{t \wedge \sigma^M} |\partial_u x_s^g(v)|^2 |(-ik)e^{-ikx_s^g(v)}|^2 ds \right|^{p/2} \right] dv \\ \leq C_p \|g'\|_{L_p[0,1]}^p \\ + C_p \left(\sum_{k \in \mathbb{Z}} f_k^2 k^2 \right)^{p/2} T^{p/2-1} \int_0^t \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{r \leq s \wedge \sigma^M} \int_0^1 |\partial_u x_r^g(v)|^p dv \right] ds. \end{aligned} \quad (73)$$

⁶ The constants C_p and $C_{p,j}$ appearing in this lemma are independent of θ and of g .

Since f is of order $\alpha > \frac{3}{2}$, $\sum_{k \in \mathbb{Z}} f_k^2 k^2$ converges. By Gronwall's Lemma, we deduce that there is a constant C_p such that for every $M > \|g'\|_{L_p[0,1]}^p$,

$$\mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq \sigma^M} \int_0^1 |\partial_u x_t^g(v)|^p dv \right] \leq C_p \|g'\|_{L_p[0,1]}^p. \quad (74)$$

Moreover,

$$\mathbb{P}^W \otimes \mathbb{P}^\beta \left[\sigma^M < T \right] \leq \mathbb{P}^W \otimes \mathbb{P}^\beta \left[\sup_{t \leq \sigma^M} \int_0^1 |\partial_u x_t^g(v)|^p dv \geq M \right] \leq \frac{C_p}{M} \|g'\|_{L_p[0,1]}^p,$$

hence we deduce $\mathbb{P}^W \otimes \mathbb{P}^\beta [\bigcup_M \{\sigma^M = T\}] = 1$. Thus, we let M tend to $+\infty$ in (74) and inequality (68) follows by Fatou's Lemma.

Inequality (69) is obtained by writing Eq. (67) at $u = 0$ and by mimicking the previous proof. We get inequality (70) by induction over $j \geq 1$; we write for each $j \geq 1$ the equation satisfied by $\partial_u^{(j)} x^g$ and we apply successively Burkholder-Davis-Gundy inequality and Gronwall's Lemma. Similarly, we also prove that $\mathbb{P}^0$ -a.s.

$$\mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \left| \partial_u^{(j)} x_t^g(0) \right|^p \right] \leq C_{p,j} \left\{ 1 + |\partial_u^{(j)} g(0)|^p + \sum_{k=1}^{j-1} \|\partial_u^{(k)} g\|_{L_\infty[0,1]}^{jp} \right\}. \quad (75)$$

Moreover, since for every 1-periodic function f , $\|f\|_{L_\infty} \leq |f(0)| + \int_0^1 |\partial_u f(v)| dv$, we have for every $p \geq 2$,

$$\begin{aligned} \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \left\| \partial_u^{(j)} x_t^g \right\|_{L_\infty}^p \right] &\leq C_p \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} |\partial_u^{(j)} x_t^g(0)|^p \right] \\ &\quad + C_p \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \int_0^1 \left| \partial_u^{(j+1)} x_t^g(v) \right|^p dv \right], \end{aligned}$$

which together with (70) and (75) leads to inequality (71).

Furthermore, recall that $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -a.s., for every $t \in [0, T]$ and for every $v \in [0, 1]$, $\partial_u x_t^g(v) > 0$. By Itô's formula, it follows from Eq. (67) that

$$\frac{1}{\partial_u x_t^g(v)} = \frac{1}{g'(v)} - \sum_{k \in \mathbb{Z}} f_k \int_0^t \frac{1}{\partial_u x_s^g(v)} \Re \left(-ik e^{-ikx_s^g(v)} dW_s^k \right) + \sum_{k \in \mathbb{Z}} f_k^2 k^2 \int_0^t \frac{1}{\partial_u x_s^g(v)} ds.$$

Again, applying successively Burkholder-Davis-Gundy inequality and Gronwall's Lemma, we get $\mathbb{P}^0$ -a.s.

$$\mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \int_0^1 \frac{1}{|\partial_u x_t^g(v)|^p} dv \right] \leq C_p \left\| \frac{1}{g'} \right\|_{L_p[0,1]}^p,$$

$$\mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \frac{1}{|\partial_u x_t^g(0)|^p} \right] \leq \frac{C_p}{g'(0)^p}.$$

Then, using once more the 1-periodicity of $\partial_u x^g$ and $\|f\|_{L_\infty} \leq |f(0)| + \int_0^1 |\partial_u f(v)| dv$, we obtain

$$\begin{aligned} \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \left\| \frac{1}{\partial_u x_t^g} \right\|_{L_\infty}^p \right] &\leq C_p \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \frac{1}{|\partial_u x_t^g(0)|^p} \right] \\ &\quad + C_p \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} \int_0^1 \left| \frac{\partial_u^{(2)} x_t^g(v)}{\partial_u x_t^g(v)^2} \right|^p dv \right]. \end{aligned}$$

and we finally prove (72) by using previous inequalities together with Cauchy-Schwarz inequality. $\square$

We obtain the same estimates for the solution to the parametric SDE (6).

Proposition 40 *Let f be of order $\alpha > \frac{3}{2} + \theta$ for some $\theta \in (0, 1)$. Then there is a collection $(Z_t^x)_{t \in [0, T], x \in \mathbb{R}}$ such that for each $x \in \mathbb{R}$, $(Z_t^x)_{t \in [0, T]}$ is the unique solution to (6). Moreover, $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -almost surely, the map $x \mapsto Z_t^x$ is differentiable on $\mathbb{R}$ for every $t \in [0, T]$ and there is a continuous version of the map continuous version of the map $(t, x) \in [0, T] \times \mathbb{R} \mapsto \partial_x Z_t^x$. Furthermore, for every $p \geq 2$, there is $C_p > 0$ such that for every $x, y \in \mathbb{R}$,*

$$\mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} |Z_t^x - Z_t^y|^p \right] \leq C_p |x - y|^p \quad (76)$$

$$\mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} |\partial_x Z_t^x|^p \right] \leq C_p, \quad (77)$$

$$\mathbb{E}^W \mathbb{E}^\beta \left[\sup_{t \leq T} |\partial_x Z_t^x - \partial_x Z_t^y|^p \right] \leq C_p |x - y|^{p\theta}. \quad (78)$$

If $\alpha > \frac{5}{2}$, then (78) holds with θ equal to 1.

Proof Let us focus on the proof of inequality (78), which is the only one which is new with respect to what has been proved in Lemma 39 for $(x_t^g(u))_{t \in [0, T]}$. For every $x, y \in \mathbb{R}$,

$$\begin{aligned} &\mathbb{E}^W \mathbb{E}^\beta \left[\sup_{s \leq t} |\partial_x Z_s^x - \partial_x Z_s^y|^p \right] \\ &\leq C_p \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{s \leq t} \left| \sum_{k \in \mathbb{Z}} f_k \int_0^s (\partial_x Z_r^x - \partial_x Z_r^y) \Re \left(-ik e^{-ik Z_r^x} dW_r^k \right) \right|^p \right] \\ &\quad + C_p \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{s \leq t} \left| \sum_{k \in \mathbb{Z}} f_k \int_0^s \partial_x Z_r^y \Re \left(-ik (e^{-ik Z_r^x} - e^{-ik Z_r^y}) dW_r^k \right) \right|^p \right]. \end{aligned}$$

By Burkholder-Davis-Gundy inequality, we deduce that

$$\begin{aligned} & \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{s \leq t} |\partial_x Z_s^x - \partial_x Z_s^y|^p \right] \\ & \leq C_p \left(\sum_{k \in \mathbb{Z}} f_k^2 k^2 \right)^{p/2} \int_0^t \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{r \leq s} |\partial_x Z_r^x - \partial_x Z_r^y|^p \right] dr \\ & \quad + C_p \left(\sum_{k \in \mathbb{Z}} f_k^2 |k|^{2+2\theta} \right)^{p/2} \int_0^t \mathbb{E}^W \mathbb{E}^\beta \left[|\partial_x Z_s^y|^p |Z_s^x - Z_s^y|^{p\theta} \right] ds. \end{aligned}$$

Furthermore, by (76) and (77), we obtain for every $s \in [0, T]$ and for every x and y ,

$$\begin{aligned} \mathbb{E}^W \mathbb{E}^\beta \left[|\partial_x Z_s^y|^p |Z_s^x - Z_s^y|^{p\theta} \right] & \leq \mathbb{E}^W \mathbb{E}^\beta \left[|\partial_x Z_s^y|^{\frac{p}{1-\theta}} \right]^{1-\theta} \mathbb{E}^W \mathbb{E}^\beta \left[|Z_s^x - Z_s^y|^p \right]^\theta \\ & \leq C_{p,\theta} |x - y|^{p\theta}. \end{aligned}$$

Since f is of order $\alpha > \frac{3}{2} + \theta$, the series $\sum_{k \in \mathbb{Z}} f_k^2 |k|^{2+2\theta}$ is finite. By Gronwall's Lemma, we deduce (78). By Kolmogorov's Lemma, it follows from (78) (with p larger than $\frac{1}{\theta}$) that there is a continuous version of $(t, x) \mapsto \partial_x Z_t^x$. $\square$

Proof (Proposition 6) Fix $g \in \mathbf{G}^1$ and $u \in \mathbb{R}$. The processes $(Z_t^{g(u)})_{t \in [0, T]}$ and $(x_t^g(u))_{t \in [0, T]}$ both satisfy the same SDE (4) with same initial condition. Since f is of order $\alpha > \frac{3}{2}$, pathwise uniqueness holds for this equation. Hence for every $u \in \mathbb{R}$, $Z_t^{g(u)} = x_t^g(u)$ holds almost surely. Moreover, since $u \in \mathbb{R} \mapsto x_t^g(u)$ and $x \in \mathbb{R} \mapsto Z_t^x$ are $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -almost surely continuous, and g is continuous, we deduce that $Z_t^{g(u)} = x_t^g(u)$ holds almost surely for every $u \in \mathbb{R}$.

Moreover, differentiating the Eq.s (4) and (6), we have $\mathbb{P}^W \otimes \mathbb{P}^\beta$ -almost surely, for every $u \in [0, 1]$ and for every $t \in [0, T]$,

$$\begin{aligned} \partial_u x_t^g(u) &= g'(u) \exp \left(\sum_{k \in \mathbb{Z}} f_k \int_0^t \Re \left(-i k e^{-i k x_s^g(u)} dW_s^k \right) - \frac{t}{2} \sum_{k \in \mathbb{Z}} f_k^2 k^2 \right); \\ \partial_x Z_t^{g(u)} &= \exp \left(\sum_{k \in \mathbb{Z}} f_k \int_0^t \Re \left(-i k e^{-i k Z_s^{g(u)}} dW_s^k \right) - \frac{t}{2} \sum_{k \in \mathbb{Z}} f_k^2 k^2 \right). \end{aligned}$$

Using equality (7), we get: $\partial_u x_t^g(u) = g'(u) \partial_x Z_t^{g(u)}$. $\square$

A.3 Differential calculus on the Wasserstein space

Let us recall a few results about differentiation of real-valued functions ϕ defined on $\mathcal{P}_2(\mathbb{R})$. We refer to [30], [8, Chap. 5] or [6] for a complete introduction to those differential calculus.

Lions-derivative or L-derivative. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space rich enough so that for any probability measure μ on any Polish space, we can construct on

$(\Omega, \mathcal{F}, \mathbb{P})$ a random variable with distribution μ ; a sufficient condition is that $(\Omega, \mathcal{F}, \mathbb{P})$ is Polish and atomless. Let $L_2(\Omega)$ be the set of square integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, modulo the equivalence relation of almost sure equality. For any $\phi : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$, we define $\widehat{\phi} : L_2(\Omega) \rightarrow \mathbb{R}$ by $\widehat{\phi}(X) = \phi(\mathcal{L}(X))$, where $\mathcal{L}(X)$ denotes the law of X . If $\widehat{\phi} : L_2(\Omega) \rightarrow \mathbb{R}$ is Fréchet-differentiable, there is a measurable function $\partial_\mu \phi(\mu) : \mathbb{R} \rightarrow \mathbb{R}$, called L-derivative of ϕ , such that for every X with distribution μ , $D\widehat{\phi}(X) = \partial_\mu \phi(\mu)(X)$.

Linear functional derivative. Basically, it is nothing but the notion of differentiability we would use for $\phi : \mathcal{M}(\mathbb{R}) \rightarrow \mathbb{R}$ if it were defined on the whole $\mathcal{M}(\mathbb{R})$, where $\mathcal{M}(\mathbb{R})$ is the linear space of signed measures on $\mathbb{R}$. Note that a subset K of $\mathcal{P}_2(\mathbb{R})$ is said to be bounded if there is M such that for every $\mu \in K$, $\int_{\mathbb{R}} |x|^2 d\mu(x) \leq M$. A function $\phi : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ is said to have a linear functional derivative if there exists a function

$$\begin{aligned} \frac{\delta \phi}{\delta m} : \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} &\rightarrow \mathbb{R} \\ (m, v) &\mapsto \frac{\delta \phi}{\delta m}(m)(v), \end{aligned}$$

jointly continuous in (m, v) , such that for any bounded subset K of $\mathcal{P}_2(\mathbb{R})$, the function $v \mapsto \frac{\delta \phi}{\delta m}(m)(v)$ is at most of quadratic growth in v uniformly in m for $m \in K$, and such that for all $m, m' \in \mathcal{P}_2(\mathbb{R})$, $\phi(m') - \phi(m) = \int_0^1 \int_{\mathbb{R}} \frac{\delta \phi}{\delta m}(\lambda m' + (1 - \lambda)m)(v) d(m' - m)(v) d\lambda$. Note that $\frac{\delta \phi}{\delta m}$ is uniquely defined up to an additive constant only.

Link between both derivatives.

Proposition 41 *Let $\phi : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ be L-differentiable on $\mathcal{P}_2(\mathbb{R})$, such that the Fréchet derivative of its lifted function $D\widehat{\phi} : L_2(\Omega) \rightarrow L_2(\Omega)$ is uniformly Lipschitz-continuous. Assume also that for each $\mu \in \mathcal{P}_2(\mathbb{R})$, there is a version of $v \in \mathbb{R} \mapsto \partial_\mu \phi(\mu)(v)$ such that the map $(v, \mu) \in \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \mapsto \partial_\mu \phi(\mu)(v)$ is continuous.*

Then ϕ has a linear functional derivative and for every $\mu \in \mathcal{P}_2(\mathbb{R})$,

$$\partial_\mu \phi(\mu)(\cdot) = \partial_v \left\{ \frac{\delta \phi}{\delta m}(\mu) \right\}(\cdot).$$

A.4 Functions of probability measures on the torus

We use in this paper some well-known properties of the L-derivative, which are recalled in this paragraph. Moreover, since we work in the particular case of probability measures on the torus, it leads naturally to the periodicity of the L-derivative, see Proposition 43 and foll. Finally, we prove Proposition 13.

Lemma 42 *Let $\phi : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ be a function satisfying the ϕ -assumptions. Then there is a constant $C > 0$ such that for every $\mu \in \mathcal{P}_2(\mathbb{R})$, we can redefine $\partial_\mu \phi(\mu)(\cdot)$ on a μ -negligible set in such a way that for every $v, v' \in \mathbb{R}$,*

$$|\partial_\mu \phi(\mu)(v) - \partial_\mu \phi(\mu)(v')| \leq C|v - v'|, \quad (79)$$

and $(\mu, v) \mapsto \partial_\mu \phi(\mu)(v)$ is continuous at any point (μ, v) such that v belongs to the support of μ . Furthermore, there is $C > 0$ such that for every $\mu \in \mathcal{P}_2(\mathbb{R})$, for every $v \in \mathbb{R}$,

$$|\partial_\mu \phi(\mu)(v)| \leq C(1 + |v|) + C \int_{\mathbb{R}} |x| d\mu(x). \quad (80)$$

Proof By [8, Proposition 5.36], inequality (79) is a consequence of assumption $(\phi 3)$. The proof of the continuity of $(\mu, v) \mapsto \partial_\mu \phi(\mu)(v)$ at any point where v belongs to the support of μ is given in [8, Corollary 5.38]. Moreover, it follows by (79) that

$$\begin{aligned} & \left| \partial_\mu \phi(\mu)(v) - \int_{\mathbb{R}} \partial_\mu \phi(\mu)(v') d\mu(v') \right| \\ & \leq \int_{\mathbb{R}} |\partial_\mu \phi(\mu)(v) - \partial_\mu \phi(\mu)(v')| d\mu(v') \\ & \leq C \int_{\mathbb{R}} |v - v'| d\mu(v') \leq C|v| + C \int_{\mathbb{R}} |v'| d\mu(v'). \end{aligned}$$

By assumption $(\phi 2)$, $|\int_{\mathbb{R}} \partial_\mu \phi(\mu)(v') d\mu(v')| \leq (\int_{\mathbb{R}} |\partial_\mu \phi(\mu)(v')|^2 d\mu(v'))^{\frac{1}{2}} \leq C$. Thus we deduce inequality (80). $\square$

Proposition 43 Let $\phi : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ be a function satisfying the ϕ -assumptions. Let $\mu \in \mathcal{P}_2(\mathbb{R})$. Then, up to redefining $\partial_\mu \phi(\mu)(\cdot)$ on a μ -negligible set, the map $v \mapsto \partial_\mu \phi(\mu)(v)$ is 2π -periodic.

Proof Let $X \in L_2(\Omega)$ be a random variable with distribution μ . For any $Y \in L_2(\Omega)$ and for any $\mathbb{Z}$ -valued random variable K , we have

$$\begin{aligned} & \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \widehat{\phi}(X + 2K\pi + \varepsilon Y) \\ & = \mathbb{E} [D\widehat{\phi}(X + 2K\pi) Y] = \mathbb{E} [\partial_\mu \phi(\mathcal{L}(X + 2K\pi))(X + 2K\pi) Y]. \end{aligned}$$

Moreover, for any ε , $\widehat{\phi}(X + 2K\pi + \varepsilon Y) = \widehat{\phi}(X + \varepsilon Y)$ because $\mathcal{L}(X + 2K\pi + \varepsilon Y) \sim \mathcal{L}(X + \varepsilon Y)$. Therefore,

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \widehat{\phi}(X + 2K\pi + \varepsilon Y) = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \widehat{\phi}(X + \varepsilon Y) = \mathbb{E} [\partial_\mu \phi(\mathcal{L}(X))(X) Y].$$

Thus for any $Y \in L_2(\Omega)$, $\mathbb{E} [\partial_\mu \phi(\mathcal{L}(X + 2K\pi))(X + 2K\pi) Y] = \mathbb{E} [\partial_\mu \phi(\mathcal{L}(X))(X) Y]$. We deduce that for any $\mathbb{Z}$ -valued random variable K , almost surely,

$$\partial_\mu \phi(\mathcal{L}(X + 2K\pi))(X + 2K\pi) = \partial_\mu \phi(\mathcal{L}(X))(X). \quad (81)$$

First, assume that the support of $\mu = \mathcal{L}(X)$ is equal to $\mathbb{R}$. For every $\delta \in (0, 1)$, let K^δ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ independent of X with a Bernoulli distribution

of parameter δ . Thus it follows from (81) that

$$1 = (1 - \delta) \mathbb{P} [\partial_\mu \phi(\mathcal{L}(X + 2K^\delta \pi))(X) = \partial_\mu \phi(\mathcal{L}(X))(X)] \\ + \delta \mathbb{P} [\partial_\mu \phi(\mathcal{L}(X + 2K^\delta \pi))(X + 2\pi) = \partial_\mu \phi(\mathcal{L}(X))(X)].$$

We deduce that $\mathbb{P} [\partial_\mu \phi(\mathcal{L}(X + 2K^\delta \pi))(X + 2\pi) = \partial_\mu \phi(\mathcal{L}(X))(X)] = 1$ for any $\delta \in (0, 1)$. Since the support of $\mathcal{L}(X)$ is equal to $\mathbb{R}$, it follows from Lemma 42 that $(v, x) \mapsto \partial_\mu \phi(v)(x)$ is continuous at (μ, x) for every $x \in \mathbb{R}$. Moreover $\mathcal{L}(X + 2K^\delta \pi)$ tends in L_2 -Wasserstein distance to $\mathcal{L}(X) = \mu$ when $\delta \rightarrow 0$. So, there exists an event $\tilde{\Omega}$ of probability one such that for every $\omega \in \tilde{\Omega}$ and every $\delta \in (0, 1) \cap \mathbb{Q}$, $\partial_\mu \phi(\mathcal{L}(X + 2K^\delta \pi))(X(\omega) + 2\pi) = \partial_\mu \phi(\mu)(X(\omega))$. Thus for every $\omega \in \tilde{\Omega}$, $\partial_\mu \phi(\mu)(X(\omega) + 2\pi) = \partial_\mu \phi(\mu)(X(\omega))$. Since $\mathbb{P} [\tilde{\Omega}] = 1$ and the support of μ is $\mathbb{R}$, we deduce that $\partial_\mu \phi(\mu)(x + 2\pi) = \partial_\mu \phi(\mu)(x)$ holds with every x in a dense subset of $\mathbb{R}$. By continuity, of $\partial_\mu \phi(\mu)(\cdot)$, the last equality holds for every $x \in \mathbb{R}$. We deduce that $\partial_\mu \phi(\mu)(\cdot)$ is 2π -periodic.

Then, consider a general $\mu \in \mathcal{P}_2(\mathbb{R})$. Let Z be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ independent of X with normal distribution $\mathcal{N}(0, 1)$ and $(a_n)_{n \in \mathbb{N}}$ be a sequence such that for all $n \in \mathbb{N}$, $a_n \in (0, 1)$ and $a_n \rightarrow_{n \rightarrow +\infty} 0$. For every $n \in \mathbb{N}$, the support of the distribution $\mathcal{L}(X + a_n Z)$ is equal to $\mathbb{R}$. Thus for every $n \in \mathbb{N}$, $v \mapsto \partial_\mu \phi(\mathcal{L}(X + a_n Z))(v)$ is 2π -periodic.

By (79), the sequence of continuous functions $(\partial_\mu \phi(\mathcal{L}(X + a_n Z)))_{n \in \mathbb{N}}$ is equicontinuous. Furthermore, (80) implies that for every $v \in [0, 2\pi]$,

$$|\partial_\mu \phi(\mathcal{L}(X + a_n Z))(v)| \leq C(1 + |v|) + C\mathbb{E}[|X + a_n Z|].$$

Since $(a_n)_{n \in \mathbb{N}}$ is bounded by 1 and $X \in L_2(\Omega)$, there exists $C > 0$ such that for every $n \in \mathbb{N}$ and for every $v \in [0, 2\pi]$

$$|\partial_\mu \phi(\mathcal{L}(X + a_n Z))(v)| \leq C(1 + |v|) \leq C(1 + 2\pi). \quad (82)$$

Recall that $v \mapsto \partial_\mu \phi(\mathcal{L}(X + a_n Z))(v)$ is 2π -periodic for every $n \in \mathbb{N}$. Thus the sequence $(\partial_\mu \phi(\mathcal{L}(X + a_n Z)))_{n \in \mathbb{N}}$ is uniformly bounded on $\mathbb{R}$. By Arzela-Ascoli's Theorem, up to extracting a subsequence, $(\partial_\mu \phi(\mathcal{L}(X + a_n Z)))_{n \in \mathbb{N}}$ converges uniformly to a limit $u : \mathbb{R} \rightarrow \mathbb{R}$. In particular, u is a 2π -periodic function.

Moreover, we prove that the following quantity tends to zero. Let $Y \in L_2(\Omega)$.

$$\begin{aligned} & \left| \mathbb{E} [\partial_\mu \phi(\mathcal{L}(X + a_n Z))(X + a_n Z)Y] - \mathbb{E} [u(X)Y] \right| \\ & \leq \left| \mathbb{E} [\partial_\mu \phi(\mathcal{L}(X + a_n Z))(X + a_n Z)Y] - \mathbb{E} [\partial_\mu \phi(\mathcal{L}(X + a_n Z))(X)Y] \right| \\ & \quad + \left| \mathbb{E} [\partial_\mu \phi(\mathcal{L}(X + a_n Z))(X)Y] - \mathbb{E} [u(X)Y] \right| \\ & \leq Ca_n \mathbb{E}[|ZY|] + \left| \mathbb{E} [\partial_\mu \phi(\mathcal{L}(X + a_n Z))(X)Y] - \mathbb{E} [u(X)Y] \right|, \end{aligned}$$

by inequality (79). Since ZY is integrable, $Ca_n \mathbb{E}[|ZY|] \rightarrow_{n \rightarrow +\infty} 0$. Moreover, let us show that $|\mathbb{E} [\partial_\mu \phi(\mathcal{L}(X + a_n Z))(X)Y] - \mathbb{E} [u(X)Y]| \rightarrow_{n \rightarrow +\infty} 0$. Remark that

$(\partial_\mu \phi(\mathcal{L}(X + a_n Z)))_{n \in \mathbb{N}}$ converges uniformly to u , hence it converges pointwise to u . Moreover, we have a uniform integrability property. Indeed, by 2π -periodicity of $\partial_\mu \phi(\mathcal{L}(X + a_n Z))$ and by (82)

$$\begin{aligned} \mathbb{E} \left[|\partial_\mu \phi(\mathcal{L}(X + a_n Z))(X)Y|^{3/2} \right] &\leq \mathbb{E} \left[|\partial_\mu \phi(\mathcal{L}(X + a_n Z))(X)|^6 \right]^{1/4} \mathbb{E} \left[|Y|^2 \right]^{3/4} \\ &= \mathbb{E} \left[|\partial_\mu \phi(\mathcal{L}(X + a_n Z))(\{X\})|^6 \right]^{1/4} \mathbb{E} \left[|Y|^2 \right]^{3/4} \\ &\leq C \mathbb{E} \left[|Y|^2 \right]^{3/4}. \end{aligned}$$

Since Y is square integrable, $\sup_{n \in \mathbb{N}} \mathbb{E} \left[|\partial_\mu \phi(\mathcal{L}(X + a_n Z))(X)Y|^{3/2} \right] < +\infty$. Thus the sequence $(\partial_\mu \phi(\mathcal{L}(X + a_n Z))(X)Y)_{n \in \mathbb{N}}$ is uniformly integrable. By Fatou's Lemma, it follows that $\mathbb{E} \left[|u(X)Y|^{3/2} \right] < +\infty$, thus we conclude that $\mathbb{E} \left[\partial_\mu \phi(\mathcal{L}(X + a_n Z))(X)Y \right] \rightarrow_{n \rightarrow +\infty} \mathbb{E} [u(X)Y]$. Therefore,

$$\mathbb{E} \left[\partial_\mu \phi(\mathcal{L}(X + a_n Z))(X + a_n Z)Y \right] \rightarrow_{n \rightarrow +\infty} \mathbb{E} [u(X)Y].$$

On the other hand,

$$\mathbb{E} \left[\partial_\mu \phi(\mathcal{L}(X + a_n Z))(X + a_n Z)Y \right] = \mathbb{E} \left[D\widehat{\phi}(X + a_n Z)Y \right] \xrightarrow{n \rightarrow +\infty} \mathbb{E} \left[D\widehat{\phi}(X)Y \right]$$

because $D\widehat{\phi}$ is Lipschitz by assumption $(\phi 3)$. We deduce that $\mathbb{E} [u(X)Y] = \mathbb{E} [D\widehat{\phi}(X)Y]$ for every $Y \in L_2(\Omega)$, hence almost surely, $u(X) = \partial_\mu \phi(\mu)(X)$. Recall that u is continuous and 2π -periodic. Therefore, up to redefining $\partial_\mu \phi(\mu)(\cdot)$ on a μ -negligible set, $v \mapsto \partial_\mu \phi(\mu)(v)$ is continuous and 2π -periodic. $\square$

For every $\mu \in \mathcal{P}_2(\mathbb{R})$, we define $\tilde{\mu}$ the measure satisfying $\tilde{\mu}(A) = \mu(A + 2\pi\mathbb{Z})$ for every $A \in \mathcal{B}[0, 2\pi]$. Clearly, $\tilde{\mu}$ belongs to $\mathcal{P}(\mathbb{T})$ and $\tilde{\mu} = \tilde{v}$ for every $\mu \sim v$ in the sense of Definition 10.

Corollary 44 *Let $\phi : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ be a function satisfying the ϕ -assumptions. Let $\mu \in \mathcal{P}_2(\mathbb{R})$ and assume that $\tilde{\mu}$ has a density belonging to $\mathcal{P}^+$ in the sense of Definition 37. Then there is a unique 2π -periodic and continuous version of $\partial_\mu \phi(\mu)(\cdot)$. Furthermore, for every $v \in [0, 2\pi]$, $\partial_\mu \phi(\mu)(v) = \partial_\mu \phi(\tilde{\mu})(v)$.*

Proof Let $X \in L_2(\Omega)$ with distribution μ . Then the law of $\{X\}$ is $\tilde{\mu}$, seen as an element of $\mathcal{P}_2(\mathbb{R})$ with support included in $[0, 2\pi]$. Since the density of $\tilde{\mu}$ belongs to $\mathcal{P}^+$, the support of $\tilde{\mu}$ is equal to $[0, 2\pi]$.

Furthermore, by equality (81) applied to $K = \frac{1}{2\pi}(\{X\} - X)$, i.e. $X + 2K\pi = \{X\}$, the following equality holds almost surely: $\partial_\mu \phi(\mathcal{L}(\{X\}))(\{X\}) = \partial_\mu \phi(\mathcal{L}(X))(X)$. By Proposition 43, $\partial_\mu \phi(\mathcal{L}(X))(\cdot)$ is 2π -periodic, so $\partial_\mu \phi(\mathcal{L}(X))(X) = \partial_\mu \phi(\mathcal{L}(X))(\{X\})$. Since the support of $\mathcal{L}(\{X\})$ is equal to $[0, 2\pi]$, we deduce that for every $v \in [0, 2\pi]$, $\partial_\mu \phi(\mathcal{L}(X))(v) = \partial_\mu \phi(\mathcal{L}(\{X\}))(v)$. This shows that there is a unique 2π -periodic and continuous version $\partial_\mu \phi(\mathcal{L}(X))(\cdot)$. $\square$

In the light of Proposition 41, we prove that the linear functional derivative is also 2π -periodic:

Proposition 45 *Let $\phi : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ be a function satisfying the ϕ -assumptions. Let $\mu \in \mathcal{P}_2(\mathbb{R})$ be such that $\tilde{\mu}$ has a density belonging to $\mathcal{P}^+$ in the sense of Definition 37. Then $\int_{\mathbb{T}} \partial_{\mu} \phi(\mu)(v) dv = 0$. In other words, $v \mapsto \frac{\delta \phi}{\delta m}(\mu)(v)$ is 2π -periodic.*

Proof By Corollary 44, it is sufficient to prove that $\int_{\mathbb{T}} \partial_{\mu} \phi(\tilde{\mu})(v) dv = 0$. Let Y_0 be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with distribution equal to $\tilde{\mu}$. Let $p : \mathbb{R} \rightarrow \mathbb{R}$ denote its density, extended by 2π -continuity. By assumption, $p(v) > 0$ for every $v \in [0, 2\pi]$, hence it holds for every $v \in \mathbb{R}$.

Define the following ordinary differential equation:

$$\dot{Y}_t = \frac{1}{p(Y_t)},$$

with initial condition Y_0 . Denoting by $F := x \mapsto \int_0^x p(v) dv$ and by $g = F^{-1}$ respectively the c.d.f. and the quantile function associated to p , we have $\frac{d}{dt} F(Y_t) = 1$. Thus for every $t \geq 0$, $F(Y_t) = F(Y_0) + t$ and $Y_t = g(F(Y_0) + t) = g_t(F(Y_0))$, where $g_t(\cdot) = g(\cdot + t)$.

Fix $t \geq 0$. Since $F(Y_0)$ has a uniform distribution on $[0, 1]$, $Y_t = g_t(F(Y_0))$ implies that g_t is the quantile function of the random variable Y_t . According to Definition 2, $g_t \sim g$, thus we deduce that the law of $\{Y_t\}$ is $\tilde{\mu}$. Since ϕ is $\mathbb{T}$ -stable, $\widehat{\phi}(Y_t) = \widehat{\phi}(\{Y_t\}) = \widehat{\phi}(Y_0)$ for every $t \geq 0$. Thus $\frac{d}{dt} \widehat{\phi}(Y_t) = 0$. Thus

$$\begin{aligned} 0 &= \frac{d}{dt} \widehat{\phi}(Y_t) = \mathbb{E} [D\widehat{\phi}(Y_0) \dot{Y}_0] = \mathbb{E} \left[D\widehat{\phi}(Y_0) \frac{1}{p(Y_0)} \right] \\ &= \int_{\mathbb{R}} \partial_{\mu} \phi(\tilde{\mu})(v) \frac{1}{p(v)} d\tilde{\mu}(v) = \int_0^{2\pi} \partial_{\mu} \phi(\tilde{\mu})(v) \frac{p(v)}{p(v)} dv = \int_0^{2\pi} \partial_{\mu} \phi(\tilde{\mu})(v) dv, \end{aligned}$$

since p is the density of the measure $\tilde{\mu}$. The statement of the proposition follows. $\square$

Let us finally prove Proposition 13.

Proof (Proposition 13) Fix $t \in [0, T]$. Let us check the three assumptions of Definition 11.

Assumption (ϕ 1): We start by proving that $P_t \phi$ is $\mathbb{T}$ -stable. Let $\mu \sim \nu$ in the sense of Definition 10 and $X, Y \in L_2[0, 1]$ satisfy $\mathcal{L}_{[0,1]}(X) = \mu$ and $\mathcal{L}_{[0,1]}(Y) = \nu$. Recall that $\{x\}$ denotes the unique number in $[0, 2\pi)$ such that $x - \{x\} \in 2\pi\mathbb{Z}$. Since $\mathcal{L}_{[0,1]}(X) \sim \mathcal{L}_{[0,1]}(Y)$, we have $\mathcal{L}_{[0,1]}(\{X\}) = \mathcal{L}_{[0,1]}(\{Y\})$. By Proposition 5, it follows that $\mathbb{P}^W$ -almost surely, the laws of $\{Z_t^X\}$ and of $\{Z_t^Y\}$ under $[0, 1] \times \Omega^{\beta}$ are equal. Therefore, $\mathbb{P}^W$ -almost surely, $\widehat{\phi}(\{Z_t^X\}) = \widehat{\phi}(\{Z_t^Y\})$. Since ϕ is $\mathbb{T}$ -stable, it implies that $\mathbb{P}^W$ -almost surely, $\widehat{\phi}(Z_t^X) = \widehat{\phi}(Z_t^Y)$. By Definition 8, $P_t \phi(\mu) = \widehat{P_t \phi}(X) = \mathbb{E}^W[\widehat{\phi}(Z_t^X)] = \mathbb{E}^W[\widehat{\phi}(Z_t^Y)] = \widehat{P_t \phi}(Y) = P_t \phi(\nu)$. Thus $P_t \phi$ is $\mathbb{T}$ -stable.

By Definition 8, it is clear that $P_t \phi$ is bounded on $\mathcal{P}_2(\mathbb{R})$, because ϕ is bounded. Furthermore, $P_t \phi$ is continuous on $\mathcal{P}_2(\mathbb{R})$, and even Lipschitz-continuous. Indeed, let

$\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ and $X, Y \in L_2[0, 1]$ be the quantile functions respectively associated with μ and ν : $\mu = \mathcal{L}_{[0,1]}(X)$ and $\nu = \mathcal{L}_{[0,1]}(Y)$; in other words, X (resp. Y) is the increasing rearrangement of μ (resp. ν). A classical result in optimal transportation (see e.g. [41, Theorem 2.18]) states that (X, Y) realises the optimal coupling in the definition of the L_2 -Wasserstein distance: $W_2(\mu, \nu)^2 = \int_0^1 |X(u) - Y(u)|^2 du$.

By Remark 12, $\widehat{\phi}$ is Lipschitz-continuous, thus:

$$\begin{aligned} |P_t \phi(\mu) - P_t \phi(\nu)| &= |\widehat{P_t \phi}(X) - \widehat{P_t \phi}(Y)| \\ &\leq \mathbb{E}^W \left[|\widehat{\phi}(Z_t^X) - \widehat{\phi}(Z_t^Y)| \right] \\ &\leq \|\widehat{\phi}\|_{\text{Lip}} \mathbb{E}^W \left[\|Z_t^X - Z_t^Y\|_{L_2([0,1] \times \Omega^\beta)} \right] \\ &\leq \|\widehat{\phi}\|_{\text{Lip}} \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |Z_t^{X(u)} - Z_t^{Y(u)}|^2 du \right]^{\frac{1}{2}}. \end{aligned}$$

By (76), we have

$$\mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |Z_t^{X(u)} - Z_t^{Y(u)}|^2 du \right] \leq C_2 \int_0^1 |X(u) - Y(u)|^2 du = W_2(\mu, \nu)^2.$$

Thus $|P_t \phi(\mu) - P_t \phi(\nu)| \leq C W_2(\mu, \nu)$; in particular, $P_t \phi$ is continuous.

Assumption (ϕ 2) and proof of equality (10): Let us prove that $P_t \phi$ is L -differentiable. Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ and $X, Y \in L_2[0, 1]$ such that $\mathcal{L}_{[0,1]}(X) = \mu$ and $\mathcal{L}_{[0,1]}(Y) = \nu$. We prove that the Fréchet derivative of $\widehat{P_t \phi}$ at point X is given by

$$\partial_\mu P_t \phi(\mu)(X) \cdot Y = D\widehat{P_t \phi}(X) \cdot Y = \int_0^1 \mathbb{E}^W \mathbb{E}^\beta \left[D\widehat{\phi}(Z_t^X)_u \partial_x Z_t^{X(u)} \right] Y(u) du,$$

which implies (10) using identities (7) and (8).

Assume that $\|Y\|_{L_2} \leq 1$. We compute

$$\begin{aligned} &\widehat{P_t \phi}(X + Y) - \widehat{P_t \phi}(X) \\ &= \mathbb{E}^W \left[\widehat{\phi}(Z_t^{X+Y}) - \widehat{\phi}(Z_t^X) \right] = \mathbb{E}^W \left[\int_0^1 \frac{d}{d\lambda} \widehat{\phi}(Z_t^{X+\lambda Y}) d\lambda \right] \\ &= \mathbb{E}^W \left[\int_0^1 D\widehat{\phi}(Z_t^{X+\lambda Y}) \cdot \partial_x Z_t^{X+\lambda Y} Y d\lambda \right] \\ &= \mathbb{E}^W \left[\int_0^1 \mathbb{E}^\beta \left[\int_0^1 D\widehat{\phi}(Z_t^{X+\lambda Y})_u \partial_x Z_t^{(X+\lambda Y)(u)} Y(u) du \right] d\lambda \right], \end{aligned}$$

since $\widehat{\phi}$ is Fréchet-differentiable. Therefore

$$\left| \widehat{P_t \phi}(X + Y) - \widehat{P_t \phi}(X) - \int_0^1 \mathbb{E}^W \mathbb{E}^\beta \left[D\widehat{\phi}(Z_t^X)_u \partial_x Z_t^{X(u)} \right] Y(u) du \right| \leq |D_1| + |D_2|,$$

where

$$D_1 := \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^1 \left(D\widehat{\phi}(Z_t^{X+\lambda Y})_u - D\widehat{\phi}(Z_t^X)_u \right) \partial_x Z_t^{X(u)} Y(u) \, d\lambda du \right];$$

$$D_2 := \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^1 D\widehat{\phi}(Z_t^{X+\lambda Y})_u \left(\partial_x Z_t^{(X+\lambda Y)(u)} - \partial_x Z_t^{X(u)} \right) Y(u) \, d\lambda du \right].$$

Let us start by estimating D_1 . By Cauchy-Schwarz inequality,

$$|D_1| \leq \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^1 \left| D\widehat{\phi}(Z_t^{X+\lambda Y})_u - D\widehat{\phi}(Z_t^X)_u \right|^2 d\lambda du \right]^{\frac{1}{2}}$$

$$\mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |\partial_x Z_t^{X(u)} Y(u)|^2 du \right]^{\frac{1}{2}}.$$

On the one hand, by assumption $(\phi 3)$ and by (76), we have (with constants modified from a line to the next):

$$\begin{aligned} & \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^1 \left| D\widehat{\phi}(Z_t^{X+\lambda Y})_u - D\widehat{\phi}(Z_t^X)_u \right|^2 d\lambda du \right] \\ & \leq C \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^1 \left| Z_t^{(X+\lambda Y)(u)} - Z_t^{X(u)} \right|^2 d\lambda du \right] \\ & \leq C \int_0^1 \int_0^1 \lambda^2 |Y(u)|^2 d\lambda du \leq C \|Y\|_{L_2}^2. \end{aligned}$$

On the other hand

$$\mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |\partial_x Z_t^{X(u)} Y(u)|^2 du \right] = \int_0^1 \mathbb{E}^W \mathbb{E}^\beta \left[|\partial_x Z_t^{X(u)}|^2 \right] |Y(u)|^2 du.$$

By (77), there is $C > 0$ such that for every $u \in [0, 1]$, $\mathbb{E}^W \mathbb{E}^\beta \left[|\partial_x Z_t^{X(u)}|^2 \right] \leq C$. Thus $\mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |\partial_x Z_t^{X(u)} Y(u)|^2 du \right] \leq C \|Y\|_{L_2}^2$. Finally, we get

$$|D_1| \leq C \|Y\|_{L_2}^2. \quad (83)$$

Moreover, compute D_2 . By Cauchy-Schwarz inequality, $|D_2| \leq |D_{2,1}|^{\frac{1}{2}} \cdot |D_{2,2}|^{\frac{1}{2}}$ where

$$D_{2,1} := \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^1 \left| D\widehat{\phi}(Z_t^{X+\lambda Y})_u Y(u) \right|^2 d\lambda du \right];$$

$$D_{2,2} := \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^1 \left| \partial_x Z_t^{(X+\lambda Y)(u)} - \partial_x Z_t^{X(u)} \right|^2 d\lambda du \right].$$

On the one hand,

$$D_{2,1} = \int_0^1 \int_0^1 \mathbb{E}^W \mathbb{E}^\beta \left[\left| D\widehat{\phi}(Z_t^{X+\lambda Y})_u \right|^2 \right] |Y(u)|^2 d\lambda du.$$

Recall that ϕ is $\mathbb{T}$ -stable. It follows that for any random variables $U, V \in L_2([0, 1] \times \Omega^\beta)$, $D\widehat{\phi}(U) \cdot V = \lim_{\varepsilon \rightarrow 0} \frac{\widehat{\phi}(U+\varepsilon V) - \widehat{\phi}(U)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{\widehat{\phi}(\{U\} + \varepsilon V) - \widehat{\phi}(\{U\})}{\varepsilon} = D\widehat{\phi}(\{U\}) \cdot V$. Hence for every $U \in L_2([0, 1] \times \Omega^\beta)$, $D\widehat{\phi}(U) = D\widehat{\phi}(\{U\})$.

Let us denote by $\xi := \mathcal{L}_{[0,1] \times \Omega^\beta}(\{Z_t^{X+\lambda Y}\})$. Then

$$\begin{aligned} & \mathbb{E}^W \mathbb{E}^\beta \left[\left| D\widehat{\phi}(\{Z_t^{X+\lambda Y}\})_u \right|^2 \right] \\ &= \mathbb{E}^W \mathbb{E}^\beta \left[\left| \partial_\mu \phi(\xi)(\{Z_t^{X+\lambda Y}(u)\}) \right|^2 \right] \leq \mathbb{E}^W \left[\sup_{v \in \mathbb{R}} |\partial_\mu \phi(\xi)(v)|^2 \right]. \end{aligned}$$

By Proposition 43, $v \mapsto \partial_\mu \phi(\xi)(v)$ is 2π -periodic. By inequality (80), it follows that

$$\begin{aligned} \sup_{v \in \mathbb{R}} |\partial_\mu \phi(\xi)(v)| &= \sup_{v \in [0, 2\pi]} |\partial_\mu \phi(\xi)(v)| \\ &\leq C(1 + 2\pi) + C \int_{\mathbb{R}} |x| d\xi(x) \\ &= C(1 + 2\pi) + C \mathbb{E}^\beta \left[\int_0^1 |\{Z_t^{X+\lambda Y}(u)\}| du \right] \leq C, \end{aligned}$$

since $\{Z_t^{X+\lambda Y}(u)\}$ takes values in $[0, 2\pi)$. Thus $\mathbb{E}^W \left[\sup_{v \in \mathbb{R}} |\partial_\mu \phi(\xi)(v)|^2 \right] \leq C$, where C is independent of X, λ and Y . We deduce that $D_{2,1} \leq \int_0^1 \int_0^1 C |Y(u)|^2 d\lambda du \leq C \|Y\|_{L_2}^2$.

On the other hand, since f is of order $\alpha > \frac{5}{2}$, inequality (78) holds with $\theta = 1$. Thus $D_{2,2} \leq C \int_0^1 \int_0^1 \lambda^2 |Y(u)|^2 d\lambda du \leq C \|Y\|_{L_2}^2$. We finally obtain that $|D_2| \leq C \|Y\|_{L_2}^2$. It follows from the latter inequality and from (83) that for every $\|Y\|_{L_2} \leq 1$,

$$\left| \widehat{P_t \phi}(X + Y) - \widehat{P_t \phi}(X) - \int_0^1 \mathbb{E}^W \mathbb{E}^\beta \left[D\widehat{\phi}(Z_t^X)_u \partial_x Z_t^{X(u)} \right] Y(u) du \right| \leq C \|Y\|_{L_2}^2.$$

Thus $\widehat{P_t \phi}$ is Fréchet-differentiable at point X and the derivative is given by (10).

Moreover, prove that

$$\sup_{\mu \in \mathcal{P}_2(\mathbb{R})} \int_{\mathbb{R}} |\partial_\mu P_t \phi(\mu)(x)|^2 d\mu(x) < +\infty.$$

Observe that $\sup_{\mu \in \mathcal{P}_2(\mathbb{R})} \int_{\mathbb{R}} |\partial_\mu P_t \phi(\mu)(x)|^2 d\mu(x) = \sup_{X \in L_2[0,1]} \int_0^1 |D\widehat{P_t \phi}(X)_u|^2 du$.

Let us apply (10) with $Y = D\widehat{P_t \phi}(X)$. We obtain

$$\begin{aligned}
& \int_0^1 |\widehat{DP_t\phi}(X)_u}|^2 du \\
&= \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 D\widehat{\phi}(Z_t^X)_u \partial_x Z_t^{X(u)} \widehat{DP_t\phi}(X)_u du \right] \\
&\leq \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |D\widehat{\phi}(Z_t^X)_u|^2 du \right]^{\frac{1}{2}} \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |\partial_x Z_t^{X(u)} \widehat{DP_t\phi}(X)_u|^2 du \right]^{\frac{1}{2}}. \quad (84)
\end{aligned}$$

By (77),

$$\begin{aligned}
\mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |\partial_x Z_t^{X(u)} \widehat{DP_t\phi}(X)_u|^2 du \right] &= \int_0^1 \mathbb{E}^W \mathbb{E}^\beta \left[|\partial_x Z_t^{X(u)}|^2 \right] |\widehat{DP_t\phi}(X)_u|^2 du \\
&\leq C \int_0^1 |\widehat{DP_t\phi}(X)_u|^2 du. \quad (85)
\end{aligned}$$

It follows from (84) and (85) that

$$\begin{aligned}
\int_0^1 |\widehat{DP_t\phi}(X)_u}|^2 du &\leq C \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |D\widehat{\phi}(Z_t^X)_u|^2 du \right] \\
&= C \mathbb{E}^W \left[\int_{\mathbb{R}} |\partial_\mu \phi(\xi)(x)|^2 d\xi(x) \right],
\end{aligned}$$

where $\xi = \mathcal{L}_{[0,1] \times \Omega^\beta}(Z_t^{X(u)})$. The last term is bounded by a constant independent of ξ because by assumption $(\phi 2)$, $\sup_{\mu \in \mathcal{P}_2(\mathbb{R})} \int_{\mathbb{R}} |\partial_\mu \phi(\mu)(x)|^2 d\mu(x) < +\infty$.

Assumption $(\phi 3)$: Let us prove that for every $X_1, X_2, Y \in L_2[0, 1]$,

$$|\widehat{DP_t\phi}(X_1) \cdot Y - \widehat{DP_t\phi}(X_2) \cdot Y| \leq C \|X_1 - X_2\|_{L_2[0,1]} \|Y\|_{L_2[0,1]}. \quad (86)$$

By formula (10), $|\widehat{DP_t\phi}(X_1) \cdot Y - \widehat{DP_t\phi}(X_2) \cdot Y| \leq |D_3| + |D_4|$, where

$$\begin{aligned}
D_3 &:= \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 (D\widehat{\phi}(Z_t^{X_1})_u - D\widehat{\phi}(Z_t^{X_2})_u) \partial_x Z_t^{X_1(u)} Y(u) du \right]; \\
D_4 &:= \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 D\widehat{\phi}(Z_t^{X_2})_u (\partial_x Z_t^{X_1(u)} - \partial_x Z_t^{X_2(u)}) Y(u) du \right].
\end{aligned}$$

Up to replacing X and $X + \lambda Y$ by X_1 and X_2 , D_3 and D_4 are equivalent to D_1 and D_2 . Thus we get by the same computations as for D_1 and D_2 :

$$\begin{aligned}
|D_3| &\leq C \|X_1 - X_2\|_{L_2[0,1]} \|Y\|_{L_2[0,1]} \\
|D_4| &\leq C \|X_1 - X_2\|_{L_2[0,1]} \|Y\|_{L_2[0,1]}.
\end{aligned}$$

This completes the proofs of (86) and of the proposition. $\square$

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