Stoch PDE: Anal Comp (2021) 9:472–490
https://doi.org/10.1007/s40072-020-00172-9

Decay of the stochastic linear Schrödinger equation in $d \geq 3$ with small multiplicative noise

Chenjie Fan[1] · **Weijun Xu**[2]

Received: 16 July 2019 / Revised: 28 March 2020 / Published online: 12 May 2020

Abstract

We give decay estimates of the solution to the linear Schrödinger equation in dimension $d \geq 3$ with a small noise which is white in time and colored in space. As a consequence, we also obtain certain asymptotic behaviour of the solution. The proof relies on the bootstrapping argument used by Journé–Soffer–Sogge for decay of deterministic Schrödinger operators.

Keywords Decay · Stochastic Schrödinger equation · Small multiplicative noise

1 Introduction

1.1 Statement of the result

Let $d \geq 3$ and $V \in \mathcal{S}(\mathbb{R}^d)$ be a Schwartz function. Consider the Schrödinger equation

$$i\partial_t \Psi + \Delta \Psi = \delta V \Psi \dot{B} - \frac{i}{2}\delta^2 V^2 \Psi, \qquad \Psi(0,\cdot) = f \in \mathcal{S}(\mathbb{R}^d), \tag{1.1}$$

where B is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$, the product between Ψ and $\dot{B}$ is in the Itô sense, and $\delta > 0$ is a small number to be specified later. The linear correction term $-\frac{i}{2}\delta^2 V^2 \Psi$ is precisely the Itô-Stratonovich correction, and hence the L^2 norm of Ψ conserved pathwise (see for example [12,14,16]).

For every $\rho \geq 1$ and $q \geq 1$, we write $L^\rho_\omega L^q_x := L^\rho(\Omega, L^q(\mathbb{R}^d))$. The main estimate is the following.

✉ Weijun Xu
weijunx@gmail.com

Chenjie Fan
cjfanpku@gmail.com

[1] University of Chicago, Chicago, USA

[2] University of Oxford, Oxford, UK

Theorem 1.1 *Let Ψ be the solution to (1.1) in $d = 3$. For every $\rho \geq 1$, $q \in [2, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, there exists $\delta_0 > 0$ depending on V such that for all $\delta < \delta_0$, we have*

$$\|\Psi(t)\|_{L^\rho_\omega L^q_x} \lesssim_{\rho,q} t^{-3(\frac{1}{2}-\frac{1}{q})} \|f\|_{L^p_x} \tag{1.2}$$

for all $t > 0$ and $f \in \mathcal{S}(\mathbb{R}^3)$. The proportionality constant is independent of t and δ.

Remark 1.2 The above theorem holds for all dimension $d \geq 3$. More precisely, if Ψ is the solution to (1.1) in $\mathbb{R}^d$ ($d \geq 3$), and p, q, ρ and δ are as in Theorem 1.1, then

$$\|\Psi(t)\|_{L^\rho_\omega L^q_x} \lesssim_{\rho,q} t^{-d(\frac{1}{2}-\frac{1}{q})} \|f\|_{L^p_x}$$

for all $t > 0$ and all $f \in \mathcal{S}(\mathbb{R}^d)$. We will give detailed proof for $d = 3$. The same strategy works for $d > 3$, and we will give a sketch for that.

The estimates in Theorem 1.1 and Remark 1.2 allow us to start with initial data in L^p for any $p \in (1, 2]$. Another consequence of the decay estimate is the asymptotic behaviour of the solution.

Proposition 1.3 *Let $d = 3$. There exists $\delta_0 > 0$ depending on V such that for every $\delta < \delta_0$ and every $f \in L^2(\mathbb{R}^d)$, there exists $g \in L^\infty_\omega L^2_x$ such that the solution Ψ to (1.1) satisfies*

$$\lim_{t \to +\infty} \|\Psi(t) - e^{it\Delta} g\|_{L^\rho_\omega L^2_x} = 0$$

for every $\rho \geq 1$.

Remark 1.4 For simplicity of presentation, we choose the noise to be of the form $\dot{W}(t, x) = V(x)\dot{B}(t)$. This factorisation or finite dimensionality is not essential, and the argument still works through as long as $\dot{W}$ is white in time and sufficiently nice in space.

Remark 1.5 We shall see later that the assumption on V could be relaxed. In fact, one only needs $\delta\big(\|V\|_{L^1_x} + \|\widehat{V}\|_{L^1_x}\big)$ to be small. We however write it in terms of δ times a Schwartz function to avoid appearance of various norms in the bounds later.

1.2 Background and motivation

It is well known [11,23] that the free Schrödinger operator $e^{it\Delta}$ satisfies the dispersive estimate

$$\|e^{it\Delta}\|_{L^p_x \to L^q_x} \lesssim t^{-d(\frac{1}{2}-\frac{1}{q})}, \qquad q \in [2, +\infty], \quad \frac{1}{p} + \frac{1}{q} = 1 \tag{1.3}$$

in any dimension d. The estimate for the pair $(p, q) = (2, 2)$ is an immediate consequence of the unitarity of $e^{it\Delta}$ in L^2. The other extreme case $(1, +\infty)$ follows from the explicit representation of the integration kernel of $e^{it\Delta}$. All intermediate situations then follow from Riesz–Thorin interpolation.

For the deterministic linear operator $e^{it(\Delta+V)}$, which corresponds to the linear equation

$$i\partial_t u + \Delta u + V(x)u = 0, \tag{1.4}$$

it is known that when V is small and $d \geq 3$, one also has the dispersive estimate

$$\|e^{it(\Delta+V)}\|_{L_x^p \to L_x^q} \lesssim t^{-d(\frac{1}{2}-\frac{1}{q})}, \qquad q \in [2, +\infty], \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{1.5}$$

This estimate corresponds to a simple special case considered in [20], which could be derived via perturbation around the free bound (1.3) for the pair $(1, +\infty)$. The reason (1.5) can hold only for $d \geq 3$ is that the bound $\|e^{it\Delta}\|_{L_x^1 \to L_x^\infty}$ for the free Schrödinger operator in (1.3) is integrable for large t if and only if $d \geq 3$. There has been a lot of works regarding the dispersive estimate for (1.4), one may refer to, for example, [3,17,21].

These dispersive estimates can be used to derive Strichartz estimates via the T^*T method (see for example the survey [22] for more details), and they are essential for the study of long time behaviour of the solution to the *nonlinear* Schödinger equation.

As for the stochastic case, there have been a series of well-posedness results on the stochastic nonlinear Schrödinger equation with multiplicative noise on whole space [5,6,12–15,18,24]. However, none of the above results gives information on long time behaviour of the solution unless one puts on the noise an an extra fast decay in time (see [19] for a scattering statement for noise of finite quadratic variation in time). A first step towards the understanding the long time behaviour of the solution to the nonlinear equation would be to establish certain decay properties of the solutions to the corresponding linear equation. This is the main motivation of the current article, and we have chosen the simplest possible model in the set-up of Theorem 1.1.

It will also be of interest to establish a stochastic version of the Strichartz estimate for the solution Ψ to (1.1), which would be of the form

$$\|\Psi\|_{L_\omega^\rho L_t^q L_x^r} \lesssim \|\Psi(0,\cdot)\|_{L_x^2}$$

for admissible pairs (q, r) satisfying the Strichartz relation. But different from the deterministic case, since Brownian motion is not time reversible, it is not clear at this stage how such an estimate could be obtained from the dispersive estimate in Theorem 1.1.

1.3 Outline of the proof

Proposition 1.3 is a simple consequence of Theorem 1.1. As for Theorem 1.1, we first note that since $\|\Psi(t)\|_{L_x^2} = \|f\|_{L_x^2}$ for all t almost surely, if we can prove the statement for all sufficiently large q (or equivalently, all p sufficiently close to 1), then the theorem will follow from interpolation ([1, Theorems 5.1.1 and 5.1.2]). Note that the interpolation statements used here are for mixed norm spaces, and are more general than Riesz–Thorin. We refer to [8,9] and [10] for more details.

It then remains to prove Theorem 1.1 for q sufficiently large such that

$$d\Big(\frac{1}{2}-\frac{1}{q}\Big) > 1. \tag{1.6}$$

This is always possible since $d \geq 3$. We follow the strategy in [20] and use a bootstrap argument. The key is to establish a bootstrap relation

$$t^{d(\frac{1}{2}-\frac{1}{q})}\|\Psi(t)\|_{L_\omega^\rho L_x^q} \leq C_1(\rho,q)\|f\|_{L_x^p} + C_2(\rho,q,\delta)\sup_{r\in[0,t]}\Big(r^{d(\frac{1}{2}-\frac{1}{q})}\|\Psi(r)\|_{L_\omega^\rho L_x^q}\Big)$$

for some C_1, C_2 independent of t, and $C_2(\rho,q,\delta) \to 0$ as $\delta \to 0$. This would allow us to absorb the second term on the right hand side into the left, and obtain the claim.

Throughout, we will frequently use the following Burkholder inequality [2,4,7] to control the stochastic integral.

Proposition 1.6 (Burkholder) *Let Φ be progressively measurable with respect to $(\mathcal{F}_t)$. Then for every $q \in [2,+\infty)$ and every $\rho \geq 2$, we have*

$$\Big\|\int_0^t e^{i(t-s)\Delta}\Phi(s)\mathrm{d}B_s\Big\|_{L_\omega^\rho L_x^q} \lesssim_{\rho,q} \Big\|\int_0^t \|e^{i(t-s)\Delta}\Phi(s)\|_{L_x^q}^2\mathrm{d}s\Big\|_{L_\omega^{\rho/2}}^{\frac{1}{2}}.$$

As a consequence, by triangle inequality, we have

$$\Big\|\int_0^t e^{i(t-s)\Delta}\Phi(s)\mathrm{d}B_s\Big\|_{L_\omega^\rho L_x^q} \lesssim_{\rho,q} \Big(\int_0^t \|e^{i(t-s)\Delta}\Phi(s)\|_{L_\omega^\rho L_x^q}^2\mathrm{d}s\Big)^{\frac{1}{2}}.$$

Remark 1.7 In order to apply Proposition 1.6 directly, we will restrict to $\rho \geq 2$ below. As for Theorem 1.1 and Proposition 1.3, the case $\rho \in [1,2)$ follows from that of $\rho \geq 2$.

Remark 1.8 The reason the end-point case $(p,q) = (1,+\infty)$ is excluded from Theorem 1.1 is that the Burkholder inequality does not hold for the space L_x^∞.

Organisation of the article

The rest of the article is organised as follows. In Sect. 2, we show how Proposition 2 follows from Theorem 1.1 and Remark 1.2. In Sect. 3, we give some preliminary lemmas needed for the proof of Theorem 1.1. Section 4 is devoted to the proof of the main theorem in $d = 3$ and q sufficiently large. We briefly explain in Sect. 5 how the arguments could be modified to cover higher dimensions. By interpolation, this completes the proof of Theorem 1.1.

Notation

We fix $d \geq 3$. We use L^q_x to denote $L^q(\mathbb{R}^d)$, and write $L^\rho_\omega L^q_x = L^\rho\big(\Omega, L^q(\mathbb{R}^d)\big)$. For any $r \in \mathbb{R}$, we write $\langle r \rangle = 1 + |r|$. Also, since the statements are for every fixed pair (p, q), we use α to denote

$$\alpha = d\Big(\frac{1}{2} - \frac{1}{q}\Big).$$

When non-commutative products are involved, we write

$$\prod_{j=1}^{m} A_j = A_m \cdots A_1. \tag{1.7}$$

Typically these A_j's will be operators. Finally, according to Remark 1.7, we assume without loss of generality that $\rho \geq 2$.

2 Proof of Proposition 1.3

We first prove the proposition for $f \in \mathcal{S}(\mathbb{R}^d)$. We need to show that $e^{-it\Delta}\Psi(t)$ has a limit in $L^\rho_\omega L^2_x$ as $t \to +\infty$, and equivalently, $\{e^{-it\Delta}\Psi(t)\}_t$ is Cauchy in $L^\rho_\omega L^2_x$. To see this, we write down the Duhamel formula

$$e^{-it\Delta}\Psi(t) - e^{-is\Delta}\Psi(s) = -i\delta \int_s^t e^{-ir\Delta}\big(V\Psi(r)\big)\mathrm{d}B_r - \frac{\delta^2}{2}\int_s^t e^{-ir\Delta}\big(V^2\Psi(r)\big)\mathrm{d}r. \tag{2.1}$$

We need to control the $L^\rho_\omega L^2_x$-norm of the two terms on the right hand side. For the first one, by Burkholder and triangle inequalities (Proposition 1.6), we have

$$\Big\| \int_s^t e^{-ir\Delta}\big(V\Psi(r)\big)\mathrm{d}B_r \Big\|^2_{L^\rho_\omega L^2_x} \lesssim_\rho \int_s^t \big\| e^{-ir\Delta}\big(V\Psi(r)\big)\big\|^2_{L^\rho_\omega L^2_x}\mathrm{d}r.$$

The integrand on the right hand side then can be controlled as

$$\|e^{-ir\Delta}\big(V\Psi(r)\big)\|^2_{L^\rho_\omega L^2_x} = \|V\Psi(r)\|^2_{L^\rho_\omega L^2_x} \lesssim_V \|\Psi(r)\|^2_{L^\rho_\omega L^q_x} \lesssim_{V,\rho} r^{-2d(\frac{1}{2}-\frac{1}{q})}\|f\|^2_{L^p_x},$$

where we have used the unitary property of $e^{-ir\Delta}$ and the decay estimates in Theorem 1.1, and the bound in the middle follows from $q \geq 2$ and Hölder. Note that the bound above holds for every $q \geq 2$. If we choose q sufficiently large such that (1.6) holds, one can immediately deduce that

$$\Big\| \int_s^t e^{-ir\Delta}\big(V\Psi(r)\big)\mathrm{d}B_r \Big\|^2_{L^\rho_\omega L^2_x} \to 0$$

as $s, t \to +\infty$. The second term on the right hand side of (2.1) can be controlled in a similar way with q in the range of (1.6). One can see the limit belongs to $L^\infty_\omega L^2_x$ since $e^{-it\Delta}$ is unitary and the L^2-norm of $\Psi(t)$ is conserved.

As for general $f \in L^2$, we note that the equation is linear, and hence by pathwise mass conservation, the difference of the L^2-norm of two solutions at any time is the same as the difference of the L^2-norm of the initial data. The claim then follows directly from the unitarity of $e^{it\Delta}$ and the case for Schwartz initial data. This completes the proof of Proposition 1.3.

Remark 2.1 One can see from the proof that the Cauchy property for the stochastic term only needs q to satisfy

$$2d\Big(\frac{1}{2} - \frac{1}{q}\Big) > 1,$$

while control of the third term on the right hand side of (2.1) requires q to satisfy (1.6).

3 Preliminary bounds

We give some preliminary bounds that will be used in the rest of the article. Throughout, Ψ denotes the solution to (1.1) with initial data f. We also fix arbitrary $q \in [2, +\infty)$ and p such that $\frac{1}{p} + \frac{1}{q} = 1$. Recall

$$\alpha = d\Big(\frac{1}{2} - \frac{1}{q}\Big)$$

and the dispersive estimate for the free Schrödinger operator

$$\|e^{it\Delta}\|_{L^p_x \to L^q_x} \lesssim |t|^{-\alpha}\ , \qquad t \in \mathbb{R}. \tag{3.1}$$

We do not impose conditions on α in this section. Also recall (1.7) for the non-commutative product. We have the following lemma.

Lemma 3.1 *For every $m \in \mathbb{N}$, every $\xi_1, \dots, \xi_{m-1} \in \mathbb{R}^d$ and every $u_1, \dots, u_m \in \mathbb{R}$ with $\sum_j u_j \neq 0$, there exists $\theta \in \mathbb{R}$ and $\eta, \zeta \in \mathbb{R}^d$ such that*

$$e^{iu_m\Delta} \prod_{j=1}^{m-1} \Big(e^{i\langle \xi_j, \cdot\rangle} e^{iu_j\Delta} \Big) = e^{i\theta} e^{i\langle \eta, \cdot\rangle} e^{i\left(\sum_{j=1}^m u_j\right)\Delta} e^{i\langle \zeta, \cdot\rangle}. \tag{3.2}$$

As a consequence, we have

$$\Big\| e^{iu_m\Delta} \prod_{j=1}^{m-1} \big(V_j e^{iu_j\Delta} \big) \Big\|_{L^p_x \to L^q_x} \lesssim \Big| \sum_{j=1}^m u_j \Big|^{-\alpha} \prod_{j=1}^m \|\widehat{V_j}\|_{L^1_x}. \tag{3.3}$$

Proof The assertion (3.2) is precisely [20, Lemma 2.4]. As for the second assertion (3.3), by Fourier expanding each V_j (with frequency variable ξ_j), we see that (3.3) is a direct consequence of (3.2) and the dispersive estimate (3.1) for the free Schrödinger operator. □

If we know a lower bound of $|\sum_j u_j|$, we may improve the above lemma to the following.

Lemma 3.2 *For every* $\varepsilon > 0$, *we have*

$$\Big\| e^{iu_m\Delta} \prod_{j=1}^{m} \big(V_j e^{iu_j\Delta}\big) \Big\|_{L_x^p \to L_x^q} \lesssim_{\varepsilon,m} \Big(\prod_{j=1}^{m} \langle u_j \rangle^{-\alpha} \Big) \prod_{j=1}^{m-1} \Big(\|\widehat{V}_j\|_{L^1} + \|V_j\|_{L^{\frac{pq}{q-p}}} \Big), \tag{3.4}$$

uniformly over the points $u_1, \dots, u_m \in \mathbb{R}$ *such that* $|\sum_j u_j| > \varepsilon$.

Proof This is the content of the first bound in [20, Lemma 2.6]. The only difference is that the relevant norm is $L_x^p \to L_x^q$ instead of $L_x^1 \to L_x^\infty$. The argument does not rely on that, but combines (3.3) for large $|u_j|$ and Hölder for small $|u_j|$. □

We now look at bounds that involve Ψ, the solution to (1.1). We now require all the times u_j involved are positive.

Lemma 3.3 *For* $0 \leq s \leq t$, *let*

$$F_t(s) = \sup_{\xi \in \mathbb{R}^d} \big\| e^{i(t-s)\Delta} e^{i\langle \xi, \cdot \rangle} \Psi(s) \big\|_{L_\omega^\rho L_x^q}.$$

Then, there exists $C = C(\rho, q)$ *such that*

$$F_t(s) \leq C t^{-\alpha} \|f\|_{L_x^p} \exp\Big(C\big(\delta^2 \|\widehat{V}\|_{L^1}^2 s + \delta^4 \|\widehat{V}\|_{L^1}^4 s^2\big)\Big)$$

for all $t > 0$ *and all* $s \in [0, t]$.

Proof We expand $\Psi(s)$ as

$$\Psi(s) = e^{is\Delta} f - i\delta \int_0^s e^{i(s-r)\Delta}\big(V\Psi(r)\big)\mathrm{d}B_r - \frac{\delta^2}{2}\int_0^s e^{i(s-r)\Delta}\big(V^2\Psi(r)\big)\mathrm{d}r.$$

Hence, we have the bound

$$F_t(s) \leq \text{(I)} + \text{(II)} + \text{(III)},$$

where

$$\text{(I)} = \sup_{\xi \in \mathbb{R}^d} \big\| e^{i(t-s)\Delta} e^{i\langle \xi, \cdot \rangle} e^{is\Delta} f \big\|_{L_\omega^\rho L_x^q},$$

$$\text{(II)} = \delta \sup_{\xi \in \mathbb{R}^d} \Big\| \int_0^s e^{i(t-s)\Delta} e^{i\langle \xi, \cdot \rangle} e^{i(s-r)\Delta} V\Psi(r)\mathrm{d}B_r \Big\|_{L_\omega^\rho L_x^q},$$

$$\text{(III)} = \frac{\delta^2}{2} \sup_{\xi \in \mathbb{R}^d} \Big\| \int_0^s e^{i(t-s)\Delta} e^{i\langle \xi, \cdot \rangle} e^{i(s-r)\Delta} V^2\Psi(r)\mathrm{d}r \Big\|_{L_\omega^\rho L_x^q}.$$

We control them one by one. For the first one, a direction application of Lemma 3.1 shows that it is controlled by

$$\text{(I)} \leq C t^{-\alpha} \|f\|_{L_x^p}.$$

For the third one, Fourier expanding V^2, then another application of (3.2) shows that the integrand satisfies

$$\begin{aligned}
&\big\| e^{i(t-s)\Delta} e^{i\langle \xi, \cdot\rangle} e^{i(s-r)\Delta} V^2 \Psi(r) \big\|_{L_\omega^\rho L_x^q}\\
&\quad = \Big\| \int_{\mathbb{R}^d} \widehat{V^2}(\eta) e^{i(t-s)\Delta} e^{i\langle \xi, \cdot\rangle} e^{i(s-r)\Delta} e^{i\langle \eta, \cdot\rangle} \Psi(r) \mathrm{d}\eta \Big\|_{L_\omega^\rho L_x^q}\\
&\quad \leq \|\widehat{V}\|_{L^1}^2 \sup_{\eta\in\mathbb{R}^d} \big\| e^{i(t-r)\Delta} e^{i\langle \eta, \cdot\rangle} \Psi(r) \big\|_{L_\omega^\rho L_x^q}\\
&\quad = \|\widehat{V}\|_{L^1}^2 F_t(r),
\end{aligned}$$

which is true for all $\xi \in \mathbb{R}^d$. Hence, we have

$$\text{(III)} \leq \frac{\delta^2}{2} \|\widehat{V}\|_{L^1}^2 \int_0^s F_t(r) \mathrm{d}r.$$

As for (II), by Burkholder and then triangle inequalities, we have

$$\begin{aligned}
\text{(II)} &\leq C\delta \sup_{\xi\in\mathbb{R}^d} \Big(\int_0^s \big\| e^{i(t-s)\Delta} e^{i\langle \xi, \cdot\rangle} e^{i(s-r)\Delta} V \Psi(r) \big\|_{L_\omega^\rho L_x^q}^2 \mathrm{d}r \Big)^{\frac{1}{2}}\\
&\leq C\delta \|\widehat{V}\|_{L^1} \Big(\int_0^s \big(F_t(r)\big)^2 \mathrm{d}r \Big)^{\frac{1}{2}},
\end{aligned}$$

where we used the previous bound for (III) with $\widehat{V^2}$ replaced by $\widehat{V}$. Combining the above three bounds, we have

$$F_t(s) \leq C\Big[t^{-\alpha} \|f\|_{L_\omega^\rho L_x^p} + \delta \|\widehat{V}\|_{L^1} \Big(\int_0^s \big(F_t(r)\big)^2 \mathrm{d}r \Big)^{\frac{1}{2}} + \delta^2 \|\widehat{V}\|_{L^1}^2 \int_0^s F_t(r) \mathrm{d}r \Big].$$

Let $K_t(s) = \big(F_t(s)\big)^2$, the above bound then implies

$$K_t(s) \leq C\Big(t^{-2\alpha} \|f\|_{L_\omega^\rho L_x^p}^2 + \delta^2 \|\widehat{V}\|_{L^1}^2 \int_0^s K_t(r) \mathrm{d}r + s\delta^4 \|\widehat{V}\|_{L^1}^4 \int_0^s K_t(r) \mathrm{d}r \Big),$$

where the additional factor s in front of the last integral comes from Hölder. The desired bound then follows from Grönwall and then taking square root of K_t. □

Combining (3.2) and Lemma 3.3, we have the following consequence.

Corollary 3.4 *For $m \geq 1$ and $u_1, \dots, u_m \in \mathbb{R}^+$, we have*

$$\begin{aligned}&\Big\| e^{iu_m\Delta} V_{m-1} \cdots V_2 e^{iu_2\Delta} V_1 \Psi(u_1) \Big\|_{L^\rho_\omega L^q_x} \\ &\quad \leq C\Big(\sum_{j=1}^m u_j\Big)^{-\alpha} \|f\|_{L^p_x} \cdot \Big(\prod_{j=1}^{m-1} \|\widehat{V}_j\|_{L^1}\Big) \exp\Big(C\big(\delta^2 \|\widehat{V}\|^2_{L^1} u_1 + \delta^4 \|\widehat{V}\|^4_{L^1} u_1^2\big)\Big),\end{aligned}$$

where $C = C(\rho, p, d)$ is independent of m and the u_j's.

The following lemma is analogous to Lemma 3.2 but with Ψ involved and requiring all u_j's being positive.

Lemma 3.5 *Let $m \geq 2$. We have*

$$\begin{aligned}&\Big\| e^{iu_m\Delta} V_{m-1} \cdots V_2 e^{iu_2\Delta} V_1 \Psi(u_1) \Big\|_{L^\rho_\omega L^q_x} \\ &\quad \lesssim_m \Big(\prod_{j=1}^m \langle u_j \rangle^{-\alpha}\Big)\Big(\prod_{j=1}^{m-1} \big(\|\widehat{V}_j\|_{L^1_x} + \|V_j\|_{L_x^{\frac{pq}{q-p}}}\big)\Big) \|f\|_{L^p_x},\end{aligned}$$

uniformly over the points $u_1, \dots, u_m \in \mathbb{R}^+$ satisfying $\sum_j u_j > 2$ and $u_1 < 1$.

Proof Note that the claim does not follow directly from Corollary 3.4 since we do not want to have a singularity in u_1 when it is small. This lemma is similar to the second bound in [20, Lemma 2.6] but not identical, so we give detailed arguments here.

We distinguish two cases: $u_2 > 2^{2m}$ and $u_2 \leq 2^{2m}$, starting with the first one. In this case, let $k_1 = 2$, and define recursively

$$k_{\ell+1} := m \wedge \inf\big\{ j > k_\ell : u_j > 2^{2m-j} \big\}$$

until it reaches m. This gives an increasing (finite) sequence of integers $2 = k_1 < \cdots < k_{N+1} = m$, and partitions $\{u_j\}_{j=2}^m$ into N blocks of the form $(u_{k_\ell}, \dots, u_{k_{\ell+1}-1})$ for $\ell = 1, \dots, N$. In each block, we have

$$\Big(\sum_{j=k_\ell}^{k_{\ell+1}-1} u_j\Big)^{-\alpha} \lesssim_m \prod_{j=k_\ell}^{k_{\ell+1}-1} \langle u_j \rangle^{-\alpha} \quad \text{and} \quad \Big(\sum_{j=1}^{k_2-1} u_j\Big)^{-\alpha} \lesssim_n \prod_{j=1}^{k_2-1} \langle u_j \rangle^{-\alpha}. \tag{3.5}$$

Hence, for $\ell \geq 2$, we have the bound

$$\begin{aligned}&\Big\| e^{iu_{k_{\ell+1}-1}\Delta} \Big(\prod_{j=k_\ell-2}^{k_{\ell+1}-2} V_j e^{iu_j\Delta}\Big) V_{k_\ell-1} \Big\|_{L^q_x \to L^q_x} \\ &\quad \lesssim_m \Big(\prod_{j=k_\ell-2}^{k_{\ell+1}-1} \langle u_j \rangle^{-\alpha}\Big) \prod_{j=k_\ell-1}^{k_{\ell+1}-2} \Big(\|\widehat{V}_j\|_{L^1_x} + \|V_j\|_{L_x^{\frac{pq}{q-p}}}\Big).\end{aligned} \tag{3.6}$$

As for $\ell = 1$, Fourier expanding $V_1, \dots, V_{k_2-2}$ and applying (3.2), Corollary 3.4 and the second bound in (3.5), we have

$$\begin{aligned}&\left\| e^{iu_{k_2-1}\Delta} V_{k_2-2} \cdots V_2 e^{iu_2\Delta} V_1 \Psi(u_1) \right\|_{L^\rho_\omega L^q_x} \\ &\lesssim_m \Big(\prod_{j=1}^{k_2-1} \langle u_j \rangle^{-\alpha} \Big) \Big(\prod_{j=1}^{k_2-1} \|\widehat{V}_j\|_{L^1_x} \Big) \|f\|_{L^p_x}.\end{aligned} \tag{3.7}$$

Muliplying (3.7) and (3.6) gives the desired bound in the case $u_2 > 2^{2m}$.

We now turn to the case $u_2 \le 2^{2m}$. If $u_j \le 2^{2m+j}$ for all j, then we have

$$\Big(\sum_{j=1}^m u_j \Big)^{-\alpha} \lesssim_m \prod_{j=1}^m \langle u_j \rangle^{-\alpha}, \tag{3.8}$$

and the bound follows. If not, then let

$$k = \inf \big\{ j \ge 3 : u_j > 2^{2m+j} \big\}.$$

If $k = m$, or $k < m$ and $|u_j| \le 1$ for all $k < j \le m$, then we still have (3.8), and the bound follows in the same way. Otherwise, let

$$k' = \inf \big\{ j > k : u_j > 1 \big\}.$$

Then $k' < m$. Since $|u_{k'}| > 1$, by Lemma 3.2, we have

$$\begin{aligned}&\left\| e^{iu_m\Delta} \Big(\prod_{j=k'}^{m-1} V_j e^{iu_j\Delta} \Big) V_{k'-1} \right\|_{L^q_x \to L^q_x} \\ &\lesssim_m \left(\prod_{j=k'}^{m} \langle u_j \rangle^{-\alpha} \right) \prod_{j=k'-1}^{m-1} \Big(\|\widehat{V}_j\|_{L^1} + \|V_j\|_{L^{\frac{pq}{q-p}}} \Big).\end{aligned}$$

For the term $\|e^{iu_{k'-1}\Delta} V_{u_{k'-2}} \cdots V_2 e^{iu_2\Delta} V_1 \Psi(u_1)\|_{L^\rho_\omega L^q_x}$, one can control it in the same way as (3.7) since we have

$$\Big(\sum_{j=1}^{k'-1} u_j \Big)^{-\alpha} \lesssim_m \prod_{j=1}^{k'-1} \langle u_j \rangle^{-\alpha}.$$

This completes the proof. □

4 Proof of Theorem 1.1

We are now ready to prove Theorem 1.1. We consider $d = 3$ in this section, and will briefly explain in the next section how the situation for general d can be established with minor modification of the argument. Recall the notation

$$\alpha = d\Big(\frac{1}{2} - \frac{1}{q}\Big), \qquad d = 3.$$

As mentioned in Sect. 1.3, it suffices to consider (arbitrary) q in the range (1.6). Also since $q < +\infty$, we have

$$1 < \alpha < \frac{3}{2}. \tag{4.1}$$

Theorem 1.1 is equivalent to the bound

$$\sup_{t \in \mathbb{R}^+} \Big(t^{\alpha} \|\Psi(t)\|_{L_\omega^\rho L_x^q} \Big) \lesssim_{\rho, q} \|f\|_{L_x^p}. \tag{4.2}$$

To achieve this, we need to establish an inequality of the form

$$t^{\alpha} \|\Psi(t)\|_{L_\omega^\rho L_x^q} \leq C_1(\rho, q) \|f\|_{L_x^p} + C_2(\rho, q, \delta) \sup_{r \in [0,t]} \Big(r^{\alpha} \|\Psi(r)\|_{L_\omega^\rho L_x^q} \Big) \tag{4.3}$$

for all t, where both C_1 and C_2 are independent of t, and $C_2(\rho, q, \delta) \to 0$ as $\delta \to 0$ for every fixed ρ and q. This will allow us to absorb the second term on the right hand side into the left and establish (4.2).

Using Corollary 3.4 with $m = 1$ and $u_1 = t$, we see the bound (4.3) is true with $C_2 = 0$ if $t \leq 2$. Hence, we only need to prove (4.3) for $t > 2$. Following the strategy in [20], we expand Ψ with Duhamel formula twice to obtain

$$\begin{aligned} \Psi(t) = {} & e^{it\Delta} f - i\delta \int_0^t e^{i(t-s)\Delta} V e^{is\Delta} f \,\mathrm{d}B_s - \frac{\delta^2}{2} \int_0^t e^{i(t-s)\Delta} V^2 e^{is\Delta} f \,\mathrm{d}s \\ & + \mathrm{(I)} + \mathrm{(II)} + \mathrm{(III)} + \mathrm{(IV)}, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \mathrm{(I)} &= \frac{\delta^4}{4} \int_0^t \int_0^s e^{i(t-s)\Delta} V^2 e^{i(s-r)\Delta} V^2 \Psi(r) \,\mathrm{d}r \,\mathrm{d}s, \\ \mathrm{(II)} &= \frac{i\delta^3}{2} \int_0^t \int_0^s e^{i(t-s)\Delta} V^2 e^{i(s-r)\Delta} V \Psi(r) \,\mathrm{d}B_r \,\mathrm{d}s, \\ \mathrm{(III)} &= \frac{i\delta^3}{2} \int_0^t \int_0^s e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V^2 \Psi(r) \,\mathrm{d}r \,\mathrm{d}B_s, \\ \mathrm{(IV)} &= -\delta^2 \int_0^t \int_0^s e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V \Psi(r) \,\mathrm{d}B_r \,\mathrm{d}B_s. \end{aligned} \tag{4.5}$$

We need to show that each term in the right hand side of (4.4) is bounded by the right hand side of (4.3). Since ρ and q are fixed, for simplicity of notation, in what follows we write "$\lesssim$" instead of "$\lesssim_{\rho,q}$". Also, we restrict to $t > 2$ from now on.

4.1 "Constant" terms

We first treat the terms without $\Psi(r)$, that is, those on the first line of the right hand side of (4.4). The bound for $e^{it\Delta}f$ is the standard dispersive estimate for $e^{it\Delta}$.

As for the second term, since $\rho \geq 2$, by Burkholder and triangle inequalities (Proposition 1.6), we have

$$\Big\| \int_0^t e^{i(t-s)\Delta} V e^{is\Delta} f \mathrm{d}B_s \Big\|_{L_\omega^\rho L_x^q} \lesssim \Big(\int_0^t \| e^{i(t-s)\Delta} V e^{is\Delta} f \|_{L_\omega^\rho L_x^q}^2 \mathrm{d}s \Big)^{\frac{1}{2}}.$$

Since $t > 2$, by Lemma 3.2, the integrand satisfies the bound

$$\| e^{i(t-s)\Delta} V e^{is\Delta} f \|_{L_x^q} \lesssim \langle t-s \rangle^{-\alpha} \langle s \rangle^{-\alpha} \| f \|_{L_x^p}.$$

Since $2\alpha > 1$, we get

$$\Big\| \int_0^t e^{i(t-s)\Delta} V e^{is\Delta} f \mathrm{d}B_s \Big\|_{L_\omega^\rho L_x^q} \lesssim t^{-\alpha} \| f \|_{L_x^p}.$$

The third term can be controlled in the same way (but requiring $\alpha > 1$). This completes the three terms in the first line of the right hand side of (4.4).

The rest of the section is devoted to the four terms in (4.5). We start with the first one.

4.2 Term (I)

Let

$$G_t(r,s) = \big\| e^{i(t-s)\Delta} V^2 e^{i(s-r)\Delta} V^2 \Psi(r) \big\|_{L_\omega^\rho L_x^q},$$

so we have

$$t^\alpha \| (\mathrm{I}) \|_{L_\omega^\rho L_x^q} \leq \delta^4 t^\alpha \int_0^t \int_0^s G_t(r,s) \mathrm{d}r \mathrm{d}s. \tag{4.6}$$

Recall that $t > 2$. In the domains $r \in [0,1]$, $r \in [1, t-1]$ and $r \in [t-1, t]$, by Lemmas 3.5, 3.2 and 3.1 respectively, we can bound the integrand G_t pointwise by

$$G_t(r,s) \lesssim \begin{cases} \langle t-s \rangle^{-\alpha} \langle s-r \rangle^{-\alpha} \| f \|_{L_x^p}, & r \in [0,1] \\ \langle t-s \rangle^{-\alpha} \langle s-r \rangle^{-\alpha} \| \Psi(r) \|_{L_\omega^\rho L_x^q}, & r \in [1, t-1] \\ (t-r)^{-\alpha} \| \Psi(r) \|_{L_\omega^\rho L_x^q}, & r \in [t-1, t]. \end{cases} \tag{4.7}$$

All the proportionality constants above are independent of r, s and t. Note that the factor in the last bound is $(t-r)^{-\alpha}$ rather than $\langle t-r\rangle^{-\alpha}$, so it has a singularity at $r \approx t$.

According to (4.7), we decompose the integral on the right hand side of (4.6) into three disjoint regions as

$$\int_0^t \int_0^s G_t \mathrm{d}r\mathrm{d}s = \int_0^t \int_0^{s\wedge 1} G_t \mathrm{d}r\mathrm{d}s + \int_1^t \int_1^{s\wedge(t-1)} G_t \mathrm{d}r\mathrm{d}s + \int_{t-1}^t \int_{t-1}^s G_t \mathrm{d}r\mathrm{d}s.$$

For the first one, using the first bound in (4.7) and $\alpha > 1$, we have

$$\begin{aligned}\int_0^t \int_0^{s\wedge 1} G_t \mathrm{d}r\mathrm{d}s &\lesssim \|f\|_{L_x^p} \int_0^t \int_0^{s\wedge 1} \langle t-s\rangle^{-\alpha}\langle s-r\rangle^{-\alpha}\mathrm{d}r\mathrm{d}s \\ &\lesssim \|f\|_{L_x^p} \int_0^t \langle t-s\rangle^{-\alpha}\langle s\rangle^{-\alpha}\mathrm{d}s \\ &\lesssim t^{-\alpha}\|f\|_{L_x^p}\ ,\end{aligned} \tag{4.8}$$

where the last inequality uses $\alpha > 1$. For the second one, using the second bound in (4.7), we get

$$\begin{aligned}\int_1^t \int_1^{s\wedge(t-1)} G_t \mathrm{d}r\mathrm{d}s &\lesssim \int_1^t \int_1^{s\wedge(t-1)} \langle t-s\rangle^{-\alpha}\langle s-r\rangle^{-\alpha}\|\Psi(r)\|_{L_\omega^\rho L_x^q}\mathrm{d}r\mathrm{d}s \\ &\leq \sup_{r\in[1,t-1]}\Big(\langle r\rangle^{\alpha}\|\Psi(r)\|_{L_\omega^\rho L_x^q}\Big)\int_1^t \int_1^s \langle t-s\rangle^{-\alpha}\langle s-r\rangle^{-\alpha}\langle r\rangle^{-\alpha}\mathrm{d}r\mathrm{d}s \\ &\lesssim t^{-\alpha}\sup_{r\in[0,t]}\Big(r^{\alpha}\|\Psi(r)\|_{L_\omega^\rho L_x^q}\Big),\end{aligned} \tag{4.9}$$

where in the last bound we have also used $\alpha > 1$. For the third term, we have

$$\begin{aligned}t^{\alpha}\int_{t-1}^t \int_{t-1}^s G_t \mathrm{d}r\mathrm{d}s &\lesssim t^{\alpha}\int_{t-1}^t \int_{t-1}^s (t-r)^{-\alpha}\|\Psi(r)\|_{L_\omega^\rho L_x^q} \\ &\lesssim \sup_{r\in[t-1,t]}\Big(r^{\alpha}\|\Psi(r)\|_{L_\omega^\rho L_x^q}\Big)\int_{t-1}^t \int_{t-1}^s (t-r)^{-\alpha}\mathrm{d}r\mathrm{d}s \\ &\lesssim \sup_{r\in[t-1,t]}\Big(r^{\alpha}\|\Psi(r)\|_{L_\omega^\rho L_x^q}\Big).\end{aligned} \tag{4.10}$$

Here, for the second line above, we used $r \in [t-1,t]$ and $t > 2$ so that we replaced t^{α} with r^{α} with a universal proportionality constant. Also, since $\alpha < 2$, the integral on the second line above is finite if and only if $\alpha < 2$, which is indeed the case. Hence one obtains the last bound.

Combining (4.6), (4.8), (4.9) and (4.10), we conclude that

$$t^{\alpha}\|(\mathrm{I})\|_{L_\omega^\rho L_x^q} \lesssim \delta^4\|f\|_{L_x^p} + \delta^4\sup_{r\in[0,t]}\Big(r^{\alpha}\|\Psi(r)\|_{L_\omega^\rho L_x^q}\Big),$$

which is of the form (4.3). This completes Term (I).

4.3 Term (II)

We now turn to the second term in (4.5). By Burkholder and triangle inequalities, we have the bound

$$\|(\mathrm{II})\|_{L^\rho_\omega L^q_x} \lesssim \delta^3 \int_0^t \Big(\int_0^s |G_t(r,s)|^2 \mathrm{d}r \Big)^{\frac{1}{2}} \mathrm{d}s,$$

where this time G_t has the expression

$$G_t(r,s) = \|e^{i(t-s)\Delta} V^2 e^{i(s-r)\Delta} V \Psi(r)\|_{L^\rho_\omega L^q_x},$$

but still satisfies exactly the same bound as in (4.7). Similar as before but with an inequality, we split the integral by

$$\begin{aligned}\int_0^t \Big(\int_0^s G_t^2 \mathrm{d}r \Big)^{\frac{1}{2}} \mathrm{d}s \leq \int_0^t \Big(\int_0^{s\wedge 1} G_t^2 \mathrm{d}r \Big)^{\frac{1}{2}} \mathrm{d}s + \int_1^t \Big(\int_1^{s\wedge(t-1)} G_t^2 \mathrm{d}r \Big)^{\frac{1}{2}} \mathrm{d}s \\ + \int_{t-1}^t \Big(\int_{t-1}^s G_t^2 \mathrm{d}r \Big)^{\frac{1}{2}} \mathrm{d}s \;.\end{aligned}$$

The first two terms above can be controlled in exactly the same way as in Term (I) since they contain no singularity. For the third one, also similar as before, we have

$$t^\alpha \int_{t-1}^t \Big(\int_{t-1}^s G_t^2 \mathrm{d}r \Big)^{\frac{1}{2}} \mathrm{d}s \lesssim \sup_{r\in[t-1,t]} \Big(r^\alpha \|\Psi(r)\|_{L^\rho_\omega L^q_x} \Big) \int_{t-1}^t \Big(\int_{t-1}^s (t-r)^{-2\alpha} \mathrm{d}r \Big)^{\frac{1}{2}} \mathrm{d}s.$$

This time, the integral on the right hand side above has a worse singularity, but it is still integrable

$$\frac{1}{2}(2\alpha - 1) < 1 \leftrightarrow \alpha < \frac{3}{2}.$$

Hence, $t^\alpha \|(\mathrm{II})\|_{L^\rho_\omega L^q_x}$ is also bounded by the right hand side of (4.3) with C_1 and C_2 proportional to δ^3. This completes the bound for Term (II).

Remark 4.1 As we can see, if there were no square root for the inner-integral, then bound for $\int_{t-1}^t \int_{t-1}^s G_t^2 \mathrm{d}r \mathrm{d}s$ would have a non-integrable singularity. This is precisely the case for Term (IV), for which we need to expand one more time to reduce this singularity to make it integrable.

4.4 Term (III)

For Term (III), we have

$$\|(\mathrm{III})\|_{L^\rho_\omega L^q_x} \lesssim \delta^3 \Big[\int_0^t \Big(\int_0^s G_t(r,s) \mathrm{d}r \Big)^2 \mathrm{d}s \Big]^{\frac{1}{2}},$$

where

$$G_t(r,s) = \|e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V^2 \Psi(r)\|_{L^\rho_\omega L^q_x}$$

satisfies the same bound as in (4.7). Again, we split (the square of) the integral above as

$$\int_0^t \Big(\int_0^s G_t \mathrm{d}r\Big)^2 \mathrm{d}s \lesssim \int_0^t \Big(\int_0^{s\wedge 1} G_t \mathrm{d}r\Big)^2 \mathrm{d}s + \int_1^t \Big(\int_1^{s\wedge(t-1)} G_t \mathrm{d}r\Big)^2 \mathrm{d}s + \int_{t-1}^t \Big(\int_{t-1}^s G_t \mathrm{d}r\Big)^2 \mathrm{d}s.$$

The first two terms on the right hand side can be controlled in the same way as in (I) and (II). The third one is also essentially the same except a slightly different but still integrable singularity. It can be controlled by

$$t^{2\alpha} \int_{t-1}^t \Big(\int_{t-1}^s G_t \mathrm{d}r\Big)^2 \mathrm{d}s \lesssim \sup_{r\in[t-1,t]} \Big(r^{2\alpha} \|\Psi(r)\|^2_{L^\rho_\omega L^q_x}\Big) \int_{t-1}^t \Big(\int_{t-1}^s (t-r)^{-\alpha} \mathrm{d}r\Big)^2 \mathrm{d}s.$$

The integral on the right hand side above is finite since $2(\alpha-1) < 1$. Term (III) is then complete by taking square root of the above bounds.

4.5 Term (IV)

As explained in Remark 4.1, if we control Term (IV) in the same way as before, then we will end up a non-integrable singularity at $s \approx t$, so we need to expand the term one more time to make it integrable.

Expanding $\Psi(r)$ as

$$\Psi(r) = e^{ir\Delta} f - i\delta \int_0^r e^{i(r-u)\Delta} V \Psi(u) \mathrm{d}B_u - \frac{\delta^2}{2} \int_0^r e^{i(r-u)\Delta} V^2 \Psi(u) \mathrm{d}u,$$

and substituting it into (IV) in (4.5), we get

$$\text{(IV)} = \text{(IV-i)} + \text{(IV-ii)} + \text{(IV-iii)},$$

where

$$\begin{aligned}
\text{(IV-i)} &= -\delta^2 \int_0^t \int_0^s e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V e^{ir\Delta} f \mathrm{d}B_r \mathrm{d}B_s, \\
\text{(IV-ii)} &= i\delta^3 \int_0^t \int_0^s \int_0^r e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V e^{i(r-u)\Delta} V \Psi(u) \mathrm{d}B_u \mathrm{d}B_r \mathrm{d}B_s, \qquad (4.11) \\
\text{(IV-iii)} &= \frac{\delta^4}{2} \int_0^t \int_0^s \int_0^r e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V e^{i(r-u)\Delta} V^2 \Psi(u) \mathrm{d}u \mathrm{d}B_r \mathrm{d}B_s.
\end{aligned}$$

Term (IV-i) is straightforward. Indeed, by Burkholder and triangle inequalities, we have

$$\|(\text{IV-i})\|_{L^\rho_\omega L^q_x} \lesssim \delta^2 \bigg(\int_0^t \int_0^s \|e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V e^{ir\Delta} f\|^2_{L^\rho_\omega L^q_x} \mathrm{d}r \mathrm{d}s \bigg)^{\frac{1}{2}}.$$

Since $t > 2$, by Lemma 3.2, the integrand satisfies a pointwise bound

$$\|e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V e^{ir\Delta} f\|_{L^\rho_\omega L^q_x} \lesssim \langle t-s \rangle^{-\alpha} \langle s-r \rangle^{-\alpha} \langle r \rangle^{-\alpha} \|f\|_{L^p_x},$$

and hence the desired bound for (IV-i) follows immediately. As for (IV-ii), we have

$$\|(\text{IV-ii})\|_{L^\rho_\omega L^q_x} \lesssim \delta^3 \bigg(\int_0^t \int_0^s \int_0^r |G_t(u,r,s)|^2 \mathrm{d}u \mathrm{d}r \mathrm{d}s \bigg)^{\frac{1}{2}},$$

where again by Lemmas 3.5, 3.2 and 3.1, the integrand

$$G_t(u,r,s) = \|e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V e^{i(r-u)\Delta} V \Psi(u)\|_{L^\rho_\omega L^q_x}$$

satisifes the pointwise bound

$$G_t(u,r,s) \lesssim \begin{cases} \langle t-s \rangle^{-\alpha} \langle s-r \rangle^{-\alpha} \langle r-u \rangle^{-\alpha} \|f\|_{L^p_x}, & u \in [0,1] \\ \langle t-s \rangle^{-\alpha} \langle s-r \rangle^{-\alpha} \langle r-u \rangle^{-\alpha} \|\Psi(u)\|_{L^\rho_\omega L^q_x}, & u \in [1, t-1] \\ (t-u)^{-\alpha} \|\Psi(u)\|_{L^\rho_\omega L^q_x}\ , & u \in [t-1, t]. \end{cases} \tag{4.12}$$

We now split the integration into three disjoint sub-domains: $\{u \leq 1\}$, $\{u \in [1, t-1]\}$ and $\{u \geq t-1\}$. The bound for the first two domains are similar as before. The only one containing a singularity is the third one, which we have the bound

$$\begin{aligned} &t^{2\alpha} \int_{t-1<u<r<s<t} G_t^2 \mathrm{d}u \mathrm{d}r \mathrm{d}s \\ &\lesssim \sup_{u \in [t-1,t]} \Big(u^{2\alpha} \|\Psi(u)\|^2_{L^\rho_\omega L^q_x} \Big) \int_{t-1}^t \int_{t-1}^s \int_{t-1}^r (t-u)^{-2\alpha} \mathrm{d}u \mathrm{d}r \mathrm{d}s. \end{aligned}$$

This time, the integral on the right hand side above is finite since the exponent of the singularity satisfies $2\alpha - 2 < 1$. One can then conclude the desired bound for (IV-ii).

Finally, the bound for Term (IV-iii) is essentially the same. This completes the case for (IV) as well as the proof of Theorem 1.1 in $d = 3$.

5 Higher dimensions

The argument for higher dimensions is essentially the same, except that when the inner-most integration variable is close to t, the singularity is worse since α becomes larger. Hence, we need to expand more times with the Duhamel formula in order to make the singularity integrable. We give a brief sketch here (see also page 587 of [20] for a sketch for the extension to high dimensions in the deterministic case).

In dimension d, we expand $\Psi(t)$ to the d-th order so that it can be expressed as a linear combination of terms of the form

$$\left\{ \int_{0<s_1<\cdots<s_j<t} e^{i(t-s_j)\Delta} V_j \cdots V_2 e^{i(s_2-s_1)\Delta} V_1 e^{is_1\Delta} f \mathrm{d}X_{s_1} \cdots \mathrm{d}X_{s_j} \right\}_{j=1,\dots,d} \tag{5.1}$$

and

$$\left\{ \int_{0<s_1<\cdots<s_d<t} e^{i(t-s_d)\Delta} V_d \cdots V_2 e^{i(s_2-s_1)\Delta} V_1 \Psi(s_1) \mathrm{d}X_{s_1} \cdots \mathrm{d}X_{s_d} \right\}, \tag{5.2}$$

where each X_{s_j} is either s_j or B_{s_j}, and each V_j is either δV or $\delta^2 V^2$. To control these terms, we first note that if q is sufficiently close to 1, then

$$\alpha = d\Big(\frac{1}{2} - \frac{1}{q}\Big) = \frac{d}{2} - \kappa$$

for some sufficiently small $\kappa > 0$.

The bound for the terms in (5.1) are straightforward. As for (5.2), when $s_1 < t-1$, the proof are exactly the same as before since the pointwise decay estimates are even better. Singularity occurs when $s_1 > t-1$, and the worst case is that all X_{s_j} in (5.2) are B_{s_j}, and we need to control

$$\delta^d t^\alpha \left(\int_{t-1<s_1<\cdots<s_d<t} \| e^{i(t-s_d)\Delta} V \cdots V e^{i(s_2-s_1)\Delta} V \Psi(s_1) \|_{L_\omega^\rho L_x^q}^2 \mathrm{d}s_1 \cdots \mathrm{d}s_d \right)^{\frac{1}{2}}. \tag{5.3}$$

By Lemma 3.1, we have the pointwise bound

$$\| e^{i(t-s_d)\Delta} V \cdots V e^{i(s_2-s_1)\Delta} V_1 \Psi(s_1) \|_{L_\omega^\rho L_x^q}^2 \lesssim (t-s_1)^{-2\alpha} \| \Psi(s_1) \|_{L_\omega^\rho L_x^q}^2$$

for the integrand, so that (5.3) is controlled by

$$\delta^d \sup_{r\in[t-1,t]} \Big(r^\alpha \| \Psi(r) \|_{L_\omega^\rho L_x^q} \Big) \cdot \left(\int_{t-1<s_1<\cdots<s_d<t} (t-s_1)^{-2\alpha} \mathrm{d}s_1 \cdots \mathrm{d}s_k \right)^{\frac{1}{2}}.$$

The integral on the right hand side above is finite since $-2\alpha + (d-1) > -1$. This bound is of the form (4.3) with both C_2 proportional to δ^d. The proof is then complete.

Acknowledgements We thank Zihua Guo and Carlos Kenig for discussions, in particular on interpolation. WX gratefully acknowledges the support from the Engineering and Physical Sciences Research Council through the fellowship EP/N021568/1. We thank two reviewers for comments and suggestions that help improve the presentation of the article.

References

1. Bergh, J., Löfström, J.: Interpolation Spaces: An Introduction. Springer, Berlin (2012)
2. Brzeźniak, Z., Peszat, S.: Space-time continuous solutions to SPDE's driven a homogeneous Wiener process. Stud. Math. **137**(3), 261–290 (1999)
3. Burq, N., Planchon, F., Stalker, J.G., Tahvildar-Zadeh, A.S.: Strichartz estimates for the wave and schrödinger equations with the inverse-square potential. J. Funct. Anal. **203**(2), 519–549 (2003)
4. Brzeźniak, Z.: On stochastic convolution in Banach spaces and applications. Stochast. Stochast. Rep. **61**, 245–295 (1997)
5. Barbu, V., Röckner, M., Zhang, D.: Stochastic nonlinear Schrödinger equations with linear multiplicative noise: rescaling approach. J. Nonlinear Sci. **24**(3), 383–409 (2014)
6. Barbu, V., Röckner, M., Zhang, D.: Stochastic nonlinear Schrödinger equations. Nonlinear Anal. Theory Methods Appl. **136**, 168–194 (2016)
7. Burkholder, D.: Distribution function inequalities for martingales. Ann. Prob. **1**(1), 19–42 (1973)
8. Calderón, A.-P.: Intermediate spaces and interpolation. Stud. Math. **1**, 31–34 (1963)
9. Calderón, A.-P.: Intermediate spaces and interpolation, the complex method. Stud. Math. **24**, 113–190 (1964)
10. Calderón, A.-P.: Spaces between l^1 and l^∞ and the theorem of marcinkiewicz. Stud. Math. **26**, 273–299 (1966)
11. Cazenave, T.: Semilinear Schrödinger Equations. Courant Lecture Notes in Mathematics. American Mathematical Society, Providence (2003)
12. de Bouard, A., Debussche, A.: A stochastic nonlinear Schrödinger equation with multiplicative noise. Commun. Math. Phys. **205**(1), 161–181 (1999)
13. de Bouard, A., Debussche, A.: The stochastic nonlinear Schrödinger equation in H^1. Stoch. Anal. Appl. **21**(1), 97–126 (2003)
14. Fan, C., Xu, W.: Global well-posedness for the defocusing mass-critical stochastic nonlinear Schrödinger equation on $\mathbb{R}$ at L^2 regularity. *ArXiv e-prints* (2018). arXiv:1810.07925
15. Fan, C., Xu, W.: Subcritical approximations to stochastic defocusing mass-critical nonlinear Schrödinger equation on $\mathbb{R}$. *ArXiv e-prints* (2018). arXiv:1810.09407
16. Fan, C., Xu, W.: A wong-zakai theorem for mass critical nls. *ArXiv e-prints* (2019). arXiv:1906.06616
17. Goldberg, M., Vega, L., Visciglia, N.: Counterexamples of strichartz inequalities for schrödinger equations with repulsive potentials. Int. Math. Res. Not. **2006**, (2006)
18. Hornung, F.: The nonlinear stochastic Schrödinger equation via stochastic Strichartz estimates. J. Evol. Equ. **18**(3), 1085–1113 (2018)
19. Herr, S., Röckner, M., Zhang, D.: Scattering for stochastic nonlinear Schrödinger equations. *ArXiv e-prints* (2018). arXiv:1804.10429
20. Journé, J.-L., Soffer, A., Sogge, C.: Decay estimates for Schrödinger operators. Commun. Pure Appl. Math. **44**(5), 573–604 (1991)
21. Rodnianski, I., Schlag, W.: Time decay for solutions of schrödinger equations with rough and time-dependent potentials. Invent. Math. **155**(3), 451–513 (2004)
22. Schlag, W.: Dispersive estimates for schrödinger operators: a survey. Ann. Math. Stud. **163**, 255–285 (2007)
23. Tao, T.: Nonlinear Dispersive Equations: Local and Global Analysis. CBMS Regional Conference Series in Mathematics. American Mathematical Society, Providence (2006)
24. Zhang, D.: Stochastic nonlinear Schrödinger equations in the defocusing mass and energy critical cases. *ArXiv e-prints* (2018). arXiv:1811.00167