Sharp regularity near an absorbing boundary for solutions to second order SPDEs in a half-line with constant coefficients

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Abstract We prove that the weak version of the SPDE problem

$$dV_t(x) = \left[-\mu V_t'(x) + \frac{1}{2} \left(\sigma_M^2 + \sigma_I^2 \right) V_t''(x) \right] dt - \sigma_M V_t'(x) dW_t, \quad x > 0$$
$$V_t(0) = 0$$

with a specified bounded initial density, V_0 , and W a standard Brownian motion, has a unique solution in the class of finite-measure valued processes. The solution has a smooth density process which has a probabilistic representation and shows degeneracy near the absorbing boundary. In the language of weighted Sobolev spaces, we describe the precise order of integrability of the density and its derivatives near the origin, and we relate this behaviour to a two-dimensional Brownian motion in a wedge whose angle is a function of the ratio σ_M/σ_I . Our results are sharp: we demonstrate that better regularity is unattainable.

1 Introduction

Let W be a standard Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and V_0 be a deterministic bounded probability density function on the half-line $(0, \infty)$. In this paper, we consider the problem of finding a finite-measure valued process, $v = (v_t)_{t \in [0, T]}$, satisfying the weak stochastic partial differential equation (SPDE)

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$$\langle \phi, v_t \rangle = \langle \phi, V_0 \rangle + \int_0^t \left[\mu \langle \phi', v_s \rangle + \frac{1}{2} (\sigma_M^2 + \sigma_I^2) \langle \phi'', v_s \rangle \right] ds + \sigma_M \int_0^t \langle \phi', v_s \rangle dW_s \quad (1)$$

for all $t \in [0, T]$ and $\phi \in C^{\text{test}}$. Here, we use the abbreviation $\langle \phi, v_t \rangle = \int_0^\infty \phi(x) v_t(dx)$ and write C^{test} for the space of bounded, twice-differentiable functions that vanish at the origin and have bounded first and second derivatives. The coefficients $\mu \in \mathbb{R}$, $\sigma_M > 0$, and $\sigma_I > 0$ and the time horizon $T > 0$ are deterministic constants. We prove this problem has a unique solution, show how to construct this solution, and establish its regularity near the absorbing boundary.

This SPDE arises naturally as the mean-field limit of a collection of interacting particles. Specifically, suppose we have a homogeneous pool of particles each experiencing an independent noise driven by a Brownian motion with speed σ_I and drift μ , as well as a common noise $\sigma_M W_t$. If we kill each of the particles upon hitting the origin, then v_t describes the limiting spatial distribution of the particles at time t as the number of particles becomes large. In [4], this model is proposed for the pricing of portfolio credit derivatives, and in that context the common noise, W , corresponds to a market risk factor and the event that a particle hits the origin represents a default of one of the names in the portfolio. As the particles in this model are independent and identically distributed conditional on knowing the history of W , we can study the system by considering the conditional mean behaviour of a single particle. In Sect. 2 we use this idea to present a simple construction of the solution to (1).

Visualising the dynamics of the solution to (1) is straightforward, and if $\sigma_M = 0$ then the SPDE becomes the deterministic heat equation with zero boundary condition. For non-zero σ_M the solution still has this decay, but also experiences random fluctuations in location which are driven by the Brownian motion W —see Fig. 1. If we denote the density of the solution by V , then we can write (1) in differential form:

$$dV_t = \left[-\mu V_t' + \frac{1}{2} (\sigma_M^2 + \sigma_I^2) V_t'' \right] dt - \sigma_M V_t' dW_t,$$

and, thinking of this equation as a PDE with irregular coefficients, we see that V diffuses with speed $\sigma_M^2 + \sigma_I^2$ and the transport term, $\mu dt + \sigma_M dW_t$, describes the (stochastic) net movement of the density in space. Lemma 3.1 of [9] shows that the density of the solution is infinitely differentiable in $(0, \infty)$, and this is due to the smoothing effect that heat flow produces. However, the presence of the irregular fluctuations in the density's location causes mass close to the boundary to be destroyed in an abrupt fashion, and as a result the derivatives of the density blow-up at the origin. The purpose of this paper is to describe this degeneracy at the absorbing boundary.

Dirichlet boundary problems for SPDEs of this type have been studied extensively—for example [7, 8, 10]. As we are working in one-dimension with constant coefficients, the simplicity of our setting allows greater insight into the behaviour of the solution near the absorbing boundary. We take advantage of a straightforward smoothing technique to present a more elementary study of the problem than is currently offered in the standard literature, and we are able to prove uniqueness of the solution to (1) in the broad class of finite-measure valued processes. Throughout this paper, when referring

Fig. 1 On the *left* is the evolution of the system in the deterministic case $\sigma_M = 0$. The value of the density at a space-time point is represented by a *colour* increasing from *blue* ($V_t(x) = 0$) through to *red* ($V_t(x) = 1$). The initial condition is the step function $V_0 = \mathbf{1}_{(0,18,0.5)}$ and the drift, μ , is zero. Compare this with a realisation of the evolution for same parameters but with $\sigma_M > 0$, which is presented on the *right*. We still observe heat-type dispersion but with a non-smooth fluctuation in the density's location (Color figure online)

to “uniqueness” we shall always mean that if ν^1 and ν^2 are two finite-measure valued processes solving (1), then

$$\mathbb{E} \left[\int_0^T \left(\langle \phi, \nu_t \rangle - \langle \phi, \bar{\nu}_t \rangle \right)^2 dt \right] = 0, \quad \text{for every } \phi \in C^{\text{test}}.$$

The results closest to those in this paper are found in [8]; there, Krylov demonstrates the existence of a unique solution to (1) in the class of twice-differentiable function-valued processes, $U = (U_t(x))_{t \in [0, T]}$, that satisfy

$$\mathbb{E} \left[\int_0^T \left(\|U_t\|_{L^2(0, \infty)}^2 + \|U'_t\|_{L^2(0, \infty)}^2 + \|x U''_t\|_{L^2(0, \infty)}^2 \right) dt \right] < \infty.$$

Furthermore, Krylov shows that if the initial density is N -times differentiable with the function $x \mapsto x^k V_0^{(k)}(x)$ in $L^2(0, \infty)$ for $k = 0, 1, \dots, N$, then the solution is $(N + 1)$ -times differentiable with

$$\mathbb{E} \left[\int_0^T \left\| x^{k-1} U_t^{(k)} \right\|_{L^2(0, \infty)}^2 dt \right] < \infty \quad (2)$$

for $k = 0, 1, \dots, N + 1$. (For a function f on $(0, \infty)$, we will always denote the n^{th} order derivative of f by $f^{(n)}$.)

A key ingredient in [8], and the majority of the literature on this problem, is the introduction of weighted Sobolev spaces. These spaces enable us to quantify the rate at which the solution and its derivatives blow-up near the origin. In our one-dimensional setting, it suffices to use the functions

$$w_c(x) := x^c e^{-x}, \quad x > 0, \quad (3)$$

for $c \in \mathbb{R}$. For a given order of derivative of the solution to (1), we can ask: how small can c be taken in the above weighting function whilst preserving square-integrability of the product of that derivative and w_c ? From (2), we can certainly take any $c \geq k - 1$ for the k^{th} derivative of the solution. A complete answer to this question is provided by the two proceeding theorems, and a critical quantity is

$$\alpha := \frac{\pi}{2} + \arcsin \rho \in (\pi/2, \pi), \quad \text{where } \rho := \frac{\sigma_M}{\sqrt{\sigma_M^2 + \sigma_I^2}} \in (0, 1). \quad (4)$$

In the context of credit derivatives, ρ is referred to as the *correlation* in the portfolio and describes the tendency of defaults to occur simultaneously. The closer ρ is to one, the larger the value of α , and therefore, observing that $\pi/\alpha - 1 \in (0, 1)$, the following result states that *higher correlation produces less regularity near the absorbing boundary*.

Theorem 1.1 (Uniqueness and regularity) *If V_0 is bounded, then there exists a unique solution, v , to (1) in the class of finite-measure valued processes, and, for almost every $(\omega, t) \in \Omega \times [0, T]$, v_t has a density V_t on $(0, \infty)$.*

Furthermore, suppose that V_0 is N -times weakly differentiable in $(0, \infty)$, and that for $k = 0, 1, \dots, N$

$$\left\| w_{k-\beta/2} V_0^{(k)} \right\|_{L^2(0, \infty)} < \infty, \quad \text{for every } \beta \in (-\infty, \pi/\alpha - 1),$$

where $w_{(\cdot)}$ and α are defined in (3) and (4). Then, for almost all $(\omega, t) \in \Omega \times [0, T]$, V_t is $(N + 1)$ -times weakly differentiable in $(0, \infty)$, and for $k = 0, 1, \dots, N + 1$

$$\mathbb{E} \left[\int_0^T \left\| w_{k-1-\beta/2} V_t^{(k)} \right\|_{L^2(0, \infty)}^2 dt \right] < \infty, \quad \text{for every } \beta \in (-\infty, \pi/\alpha - 1).$$

This result shows that, at a time later than zero, the solution has differentiability one order higher than the initial density, and is one multiple of x more regular at the origin (recall that $w_{k-\beta/2}(x) = x^{k-\beta/2} e^{-x}$). We should notice that, because the initial density is integrable and bounded, the condition $w_{-\beta/2} V_0 \in L^2(0, \infty)$ is always satisfied. The hypotheses placed on the initial density allow its derivatives to blow-up at the origin and grow sub-exponentially towards infinity. We recover the results of [8] by taking $\beta = 0$ in Theorem 1.1, and therefore we have expanded upon the known regularity of the solution near the absorbing boundary.

Our results are sharp as the following result shows that we cannot take $\beta > \pi/\alpha - 1$ in Theorem 1.1:

Theorem 1.2 (Converse result) *If V_0 satisfies the hypotheses of Theorem 1.1, then for $k = 0, 1, \dots, N + 1$ we have*

$$\mathbb{E} \left[\int_0^T \left\| w_{k-1-\beta/2} V_t^{(k)} \right\|_{L^2(0,\infty)}^2 dt \right] = \infty, \quad \text{for every } \beta \in (\pi/\alpha - 1, +\infty).$$

Since the tails of the weighting functions all decay at the same rate, the only way we can obtain the blow-up observed in Theorem 1.2 is if the solution is degenerate near the absorbing boundary. With this in mind, it is clear that the first derivative of the solution cannot be bounded in a neighbourhood of the origin, otherwise we obtain the contradiction

$$\int_0^1 x^{-\beta} dx = \infty$$

for $\beta \in (\pi/\alpha - 1, 1) \subseteq (0, 1)$. In the deterministic setting ($\sigma_M = 0$) the first derivative does not have this feature, and this has been remarked upon by Krylov in [9]. If we shift the density by setting

$$f(t, x) := V_t(x + \mu t + \sigma_M W_t),$$

then, by the Itô–Wentzell formula, we have that f solves the deterministic heat equation (with speed σ_I) in the random region

$$\{(t, x) : t \in [0, T], x > -\mu t - \sigma_M W_t\}$$

with Dirichlet boundary condition. Theorem 5.1 of [9] shows that there exists $\lambda \in (0, 1)$ such that, with probability one, there is a dense subset of $[0, T]$ on which

$$\lim_{x \downarrow 0} \frac{f(t, x)}{x^\lambda} = \infty.$$

On this set it is clear that the density can have no spatial derivative at the absorbing boundary.

In the next section we present the construction of the solution to (1) and describe its properties. A proof of Proposition 2.7 is presented in Sect. 3, and this is the main probabilistic result used to prove the above two theorems in Sects. 4 and 5.

2 The probabilistic solution

We shall construct the solution to (1) in this section. We have not yet proved that (1) can admit only one solution, therefore we refer to the measure-valued process presented here as the *probabilistic solution*.

2.1 Construction

Let us introduce two independent standard Brownian motions, W^1 and W^2 , that are also independent of W . We define two test processes, X^1 and X^2 , by

$$X_t^i := X_0^i + \mu t + \sigma_M W_t + \sigma_I W_t^i,$$

where X_0^1 and X_0^2 are independent random variables with common law V_0 , and are also independent of every other random variable introduced so far. To kill the particles upon hitting zero, we set

$$\tau^i := \inf \left\{ s \in [0, T] : X_s^i \leq 0 \right\}.$$

Conditional on knowing the trajectory of W , X^1 and X^2 are independent and identically distributed, therefore we make the following definition:

Definition 2.1 (Probabilistic solution) Let $\mathcal{F}^W = \{\mathcal{F}_t^W\}_{t \in [0, T]}$ be the natural filtration of W . The *probabilistic solution*, ν , is defined to be the measure-valued process

$$\nu_t(S) := \mathbb{P} \left(X_t^1 \in S; t < \tau^1 \middle| \mathcal{F}_t^W \right)$$

for all $t \in [0, T]$ and $S \subseteq (0, \infty)$ Lebesgue measurable.

Throughout this paper “ $S \subseteq (0, \infty)$ measurable” shall always mean that S is Lebesgue measurable. The following is a useful representation:

Proposition 2.2 For all $t \in [0, T]$ and $S \subseteq (0, \infty)$ measurable we have

$$\mathbb{E} \left[\nu_t(S)^2 \right] = \mathbb{E} \left[\mathbb{E} \left[\mathbf{1}_{X_t^1 \in S, X_t^2 \in S; t < \min(\tau^1, \tau^2)} \middle| \mathcal{F}_t^W \right] \right]$$

Proof Since X^1 and X^2 are equal in distribution, we have $\nu_t(S) = \mathbb{E} \left[\mathbf{1}_{X_t^i \in S; t < \tau^i} \middle| \mathcal{F}_t^W \right]$ for $i = 1$ and 2 . Hence

$$\begin{aligned} \mathbb{E} \left[\nu_t(S)^2 \right] &= \mathbb{E} \left[\mathbb{E} \left[\mathbf{1}_{X_t^1 \in S; t < \tau^1} \middle| \mathcal{F}_t^W \right]^2 \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\mathbf{1}_{X_t^1 \in S; t < \tau^1} \middle| \mathcal{F}_t^W \right] \mathbb{E} \left[\mathbf{1}_{X_t^2 \in S; t < \tau^2} \middle| \mathcal{F}_t^W \right] \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\mathbf{1}_{X_t^1 \in S, X_t^2 \in S; t < \min(\tau^1, \tau^2)} \middle| \mathcal{F}_t^W \right] \right] \end{aligned}$$

as required, where the final line is due to the conditional independence of X^1 and X^2 . $\square$

It is a simple calculation with Itô's formula to show that ν is a solution of (1):

Proposition 2.3 *There exists a full subset of Ω on which*

$$\langle \phi, \nu_t \rangle = \langle \phi, \nu_0 \rangle + \int_0^t \left[\mu \langle \phi', \nu_s \rangle + \frac{1}{2} \left(\sigma_M^2 + \sigma_I^2 \right) \langle \phi'', \nu_s \rangle \right] ds + \sigma_M \int_0^t \langle \phi', \nu_s \rangle dW_s.$$

for all $t \in [0, T]$ and $\phi \in C^{\text{test}}$.

Proof Take $\phi \in C^{\text{test}}$, then Itô's formula gives

$$\begin{aligned} \phi \left(X_{t \wedge \tau^1}^1 \right) &= \phi \left(X_0^1 \right) + \int_0^t \left[\mu \phi' \left(X_s^1 \right) + \frac{1}{2} \left(\sigma_M^2 + \sigma_I^2 \right) \phi'' \left(X_s^1 \right) \right] \mathbf{1}_{s < \tau^1} ds \\ &\quad + \sigma_M \int_0^{t \wedge \tau^1} \phi' \left(X_s^1 \right) \mathbf{1}_{s < \tau^1} dW_s + \sigma_I \int_0^{t \wedge \tau^1} \phi' \left(X_s^1 \right) \mathbf{1}_{s < \tau^1} dW_s^1. \end{aligned} \quad (5)$$

Since $\phi(0) = 0$, we can write $\phi \left(X_{t \wedge \tau^1}^1 \right) = \phi \left(X_t^1 \right) \mathbf{1}_{t < \tau^1}$, and therefore, by taking the conditional expectation over (5) and noting that the W^1 integral then vanishes, we arrive at the required result. $\square$

Remark 2.4 (The choice of test functions) C^{test} is chosen as it is the largest space for which the previous proof remains valid. The test functions' derivatives need not be controlled at the origin, and we cannot take $C^{\text{test}} = C_0^\infty(0, \infty)$ —the space of compactly supported smooth functions on the half-line—because in Sect. 4 it will be necessary to set $T_\delta \phi$ (see Definition 4.1) into (1) despite the fact that in general $(T_\delta \phi)'(0)$ and $(T_\delta \phi)''(0)$ are non-zero.

Remark 2.5 (Connection with filtering) If we treat X^1 as a signal process and W an observation process, then the Zakai equation (see for example [1]) for the conditional distribution of the signal given the observations is exactly (1).

2.2 Properties

Conditional on knowing $\mathcal{F}_t^W$, the process X^1 is a Brownian motion started at X_0^1 with drift $t \mapsto \mu t + \sigma_M W_t$, and therefore we can write the conditional density of X^1 as

$$P_t(x; x_0) = P_t(x; x_0)(\omega) := \frac{1}{\sqrt{2\pi\sigma_I^2 t}} \exp \left\{ -\frac{(x - x_0 - \mu t - \sigma_M W_t)^2}{2\sigma_I^2 t} \right\}.$$

From Definition 2.1, it is a trivial fact that $\nu_t(S) \leq \mathbb{P}(X_t^1 \in S | \mathcal{F}_t^W)$ for any time $t \in [0, T]$ and measurable subset $S \subseteq (0, \infty)$, and this simple observation allows us to show that ν has a density process:

Proposition 2.6 (Existence of the density) *There exists a full subset of Ω on which, for every $t \in [0, T]$, v_t has a density process V_t , and, for every $1 \leq p \leq \infty$, $V_t \in L^p(0, \infty)$ and $\|V_t\|_p \leq \|V_0\|_p$.*

Proof From the above, there exists a full subset of Ω on which

$$v_t(S) \leq \mathbb{P}\left(X_t^1 \in S \mid \mathcal{F}_t^W\right) = \int_S \int_0^\infty P_t(x; x_0) V_0(x_0) dx_0 dx$$

for all $t \in [0, T]$ and $S \subseteq (0, \infty)$ measurable. Therefore, if $S \subseteq (0, \infty)$ has zero Lebesgue measure, then $v_t(S) = 0$. It follows by the Radon–Nikodym Theorem that v_t has a density for every $t \in [0, T]$, and that this density V_t satisfies

$$V_t(x) \leq \int_0^\infty P_t(x; x_0) V_0(x_0) dx_0 \quad (6)$$

for almost all $x \in (0, \infty)$.

Bounding V_0 by $\|V_0\|_\infty$ in (6), we get $V_t \in L^\infty(0, \infty)$ with $\|V_t\|_\infty \leq \|V_0\|_\infty$. Suppose $1 \leq p < \infty$, then Holder's inequality applied to (6) gives

$$V_t(x)^p \leq \int_0^\infty P_t(x; x_0) V_0(x_0)^p dx_0.$$

Integrating over $x \in (0, \infty)$ and noting that $\int_0^\infty P_t(x; x_0) dx \leq 1$ gives $\|V_t\|_p \leq \|V_0\|_p$. $\square$

The most important feature of the probabilistic solution is the following estimate:

Proposition 2.7 (The $(3 + \beta)$ -condition) *Let $V_0 \in L^2(0, \infty)$ be bounded and α be defined as in (4). Then for every $\beta \in (0, \pi/\alpha - 1) \subseteq (0, 1)$ there exists constants $B > 0$ and $\gamma \in (0, 1)$ such that*

$$\mathbb{E}\left[v_t(0, \varepsilon)^2\right] \leq \frac{B}{t^\gamma} \varepsilon^{3+\beta}, \quad \text{for every } \varepsilon > 0 \text{ and } t \in [0, T].$$

A proof is presented in Sect. 3.

In Lemma 3.5 of [4], this result is presented for an initial density supported on a finite interval away from the boundary, and that assumption allows the authors to drop the singular factor of $t^{-\gamma}$. However, this is not the most natural initial condition; for any $t > 0$, V_t takes positive values on all of $(0, \infty)$, so if we stop the process at a time $t_0 > 0$ and then restart from the density V_{t_0} , we still have the estimate on $\mathbb{E}[v_t(0, \varepsilon)^2]$ for $t > t_0$ despite V_{t_0} not being supported on a finite interval away from zero. In this more general setting, we can expect a singular time factor to appear, since, for every time $t > 0$, $V_t(x) \rightarrow 0$ as $x \rightarrow 0$ (Theorem 3.2 of [9]) and so positive

values of V_0 close to zero must decay instantaneously as the system evolves in time. Since $\gamma \in (0, 1)$, $t^{-\gamma}$ is integrable over $[0, T]$ and therefore does not present technical difficulties in later proofs of L^2 -integrability.

Using Proposition 2.2, we prove Proposition 2.7 by considering the process $\mathbf{X}_t := (X_t^1, X_t^2)$, which is a two-dimensional Brownian motion with components of correlation ρ —recall (4). With α also as in (4), the map $F_\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$F_\alpha : \mathbf{x} \mapsto \begin{pmatrix} \sqrt{1-\rho^2} & \rho \\ 0 & 1 \end{pmatrix}^{-1} \mathbf{x} \quad (7)$$

transforms $\mathbf{X}$ to a Brownian motion with uncorrelated components, and maps the quadrant $(0, \infty)^2$ to the wedge

$$F_\alpha \left[(0, \infty)^2 \right] = \{(r, \theta) : 0 < r < \infty, 0 < \theta < \alpha\}.$$

We make use of explicit formulae in [5, 12] to estimate the probability that $F_\alpha(\mathbf{X}_t)$ is in a small neighbourhood of the apex of the wedge and has not exited the wedge, which corresponds to the event $X_t^1, X_t^2 \in (0, \varepsilon)$ and $t < \min(\tau^1, \tau^2)$. The kernel smoothing technique of Sect. 4, which was introduced in [4] and adapted from [11], relates this hitting probability to the behaviour of the solution near the absorbing boundary.

3 Proof of Proposition 2.7

Let us write $\mathbb{P}_{x_1, x_2}$ to indicate that $X_0^1 = x_1$ and $X_0^2 = x_2$, that is, we start the two-dimensional Brownian motion, $\mathbf{X} = (X^1, X^2)$, from (x_1, x_2) . From Proposition 2.2, we know

$$\mathbb{E}_{x_1, x_2} \left[v_t(S_1) v_t(S_2) \right] = \mathbb{E}_{x_1, x_2} \left[\mathbb{E} \left[\mathbf{1}_{X_t^1 \in S_1, X_t^2 \in S_2, t < \min(\tau^1, \tau^2)} \middle| \mathcal{F}_t^W \right] \right], \quad (8)$$

for every measurable $S_1, S_2 \subseteq (0, \infty)$. A change of measure allows us to eliminate the drift term in the dynamics of $\mathbf{X}$:

Lemma 3.1 (Eliminating drift) *Let $\tilde{\mathbb{P}}$ be defined on $(\Omega, \mathcal{F}_t)$ by the Girsanov transformation*

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = Z_t(\mu) := \exp \left\{ -\frac{\mu}{\sigma_M} W_t - \frac{\mu^2}{2\sigma_M^2} t \right\},$$

then under $\tilde{\mathbb{P}}$ $\mathbf{X}$ is a two-dimensional Brownian motion with zero drift. Let $\tilde{\mathbb{E}}$ denote the expectation under $\tilde{\mathbb{P}}$. If there exists a constant $C > 0$ such that

$$\tilde{\mathbb{E}} \left[v_t(0, \varepsilon)^2 \right] \leq \frac{C}{t^{\pi/2\alpha}} \varepsilon^{2+\pi/\alpha}, \quad \text{for all } t \in (0, T] \text{ and } \varepsilon > 0,$$

then Proposition 2.7 holds.

Proof Following the method in Lemma 3.5 of [4], for any Hölder conjugates $1 < a, b < \infty$ we have

$$\begin{aligned}\mathbb{E} \left[v_t(0, \varepsilon)^2 \right] &= \tilde{\mathbb{E}} \left[Z_t(\mu) \cdot v_t(0, \varepsilon)^2 \right] \\ &\leq \tilde{\mathbb{E}} \left[Z_t(\mu)^a \right]^{\frac{1}{a}} \tilde{\mathbb{E}} \left[v_t(0, \varepsilon)^{2b} \right]^{\frac{1}{b}} \\ &\leq \tilde{\mathbb{E}} \left[Z_t(\mu)^a \right]^{\frac{1}{a}} \tilde{\mathbb{E}} \left[v_t(0, \varepsilon)^2 \right]^{\frac{1}{b}} \\ &\leq \tilde{\mathbb{E}} \left[Z_t(\mu)^a \right]^{\frac{1}{a}} \left\{ \frac{C}{t^{\pi/2\alpha}} \varepsilon^{2+\pi/\alpha} \right\}^{\frac{1}{b}} \\ &= \mathbb{E} \left[Z_t(\mu)^{a-1} \right]^{\frac{1}{a}} \left\{ \frac{C}{t^{\pi/2\alpha}} \varepsilon^{2+\pi/\alpha} \right\}^{\frac{1}{b}},\end{aligned}$$

where one should note that $v_t(0, \varepsilon) \leq 1$. Hence

$$\mathbb{E} \left[v_t(0, \varepsilon)^2 \right] \leq \exp \left(\frac{(a-1)(a-2)\mu^2}{2\sigma_M^2 a} T \right) \left\{ \frac{C}{t^{\pi/2\alpha}} \varepsilon^{2+\pi/\alpha} \right\}^{\frac{1}{b}},$$

and, since $\alpha \in (\pi/2, \pi)$, we can therefore choose b sufficiently close to 1 so that the result holds. $\square$

Remark 3.2 We cannot take $b = 1$ above, and so, except in the case when $\mu = 0$, this change of measure removes the possibility of considering the borderline case $\beta = \pi/\alpha - 1$. Although this may seem to weaken the potential regularity available near the absorbing boundary, in proving Theorem 1.1 we make use of Lemma 4.8 which requires β to be strictly smaller than the maximal value, $\pi/\alpha - 1$, and hence it is unclear whether we could analyse the case $\beta = \pi/\alpha - 1$ regardless of this difficulty caused by eliminating the drift.

The remainder of Sect. 3 is devoted to showing that the hypothesis of Lemma 3.1 holds. Without loss of generality, from this point on we take $\mu = 0$ and $\tilde{\mathbb{P}} = \mathbb{P}$.

Conditioning on the start point of $\mathbf{X}$, we have

$$\mathbb{E} \left[v_t(0, \varepsilon)^2 \right] = \int_0^\infty \int_0^\infty \mathbb{E}_{x_1, x_2} \left[\mathbb{E} \left[\mathbf{1}_{X_t^1 \in (0, \varepsilon), X_t^2 \in (0, \varepsilon)} \middle| \mathcal{F}_t^W \right] \right] V_0(x_1) V_0(x_2) dx_1 dx_2.$$

Under the transformation F_α , the region $(0, \varepsilon)^2$ is mapped *into* a region of the form

$$\{(r, \theta) : 0 < r < c\varepsilon, 0 < \theta < \alpha\},$$

for some numerical constant c . The estimate in Proposition 2.7 is unchanged under the map $\varepsilon \mapsto c\varepsilon$ (modulo multiplicative constants), and so it is no loss of generality to assume $c = 1$. Using that V_0 is bounded, that the Jacobian of the transform $F_\alpha : \mathbb{R}^2 \rightarrow$

$\mathbb{R}^2$ is a numerical constant depending only on ρ , and formula (8) of [5] (corroborated in [12]), we arrive at

$$\begin{aligned} & \mathbb{E} \left[v_t(0, \varepsilon)^2 \right] \\ & \leq C \int_0^\infty \int_0^\alpha \int_0^\alpha \int_0^\varepsilon \frac{rr_0}{t} e^{-\frac{r^2+r_0^2}{2t}} \sum_{n=1}^\infty \sin\left(\frac{n\pi\theta}{\alpha}\right) \sin\left(\frac{n\pi\theta_0}{\alpha}\right) I_{n\pi/\alpha}\left(\frac{rr_0}{2t}\right) dr d\theta dr_0 d\theta_0 \\ & = C \int_0^\varepsilon \frac{r^{\pi/\alpha+1}}{t^{\pi/2\alpha}} e^{-\frac{r^2}{2t}} \sum_{n,m=0}^\infty \frac{\Gamma\left(m + \frac{(2n+1)\pi}{2\alpha} + 1\right)}{(2n+1)^2 m! \Gamma\left(m + \frac{(2n+1)\pi}{\alpha} + 1\right)} \left(\frac{r}{2t}\right)^{m+n\pi/\alpha} dr, \end{aligned}$$

for $C > 0$ some numerical constant. As $\sup_{x \geq 0} \{x^c e^{-x}\} = c^c e^{-c}$, we have

$$\mathbb{E} \left[v_t(0, \varepsilon)^2 \right] \leq C \int_0^\varepsilon \frac{r^{\pi/\alpha+1}}{t^{\pi/2\alpha}} dr \sum_{n,m=0}^\infty \frac{\Gamma\left(m + \frac{(2n+1)\pi}{2\alpha} + 1\right) \left(m + \frac{n\pi}{\alpha}\right)^{m+\frac{n\pi}{\alpha}} e^{-(m+\frac{n\pi}{\alpha})}}{(2n+1)^2 m! \Gamma\left(m + \frac{(2n+1)\pi}{\alpha} + 1\right)}, \quad (9)$$

and therefore we are done provided the double sum on the right-hand is finite. Using Stirling's approximation, the inequality

$$\frac{\Gamma(a+c+1)}{\Gamma(b+c+1)} \leq \frac{\Gamma(a+1)}{\Gamma(b+1)}$$

for $a, b, c > 0$, and the asymptotic behaviour $\Gamma(m + \pi/2\alpha + 1) \sim m! m^{\pi/2\alpha}$ for large m , we have that, for large m and $m+n$, the summand in (9) is dominated by a constant multiple of

$$\frac{1}{(2n+1)^2 m^{\pi/2\alpha+1/2}}.$$

Since $\pi/2\alpha + 1/2 > 1$, this expression is summable over m and n , and so we have the result. $\square$

4 Proof of Theorem 1.1

We begin by proving the regularity result for the probabilistic solution. Let us write

$$p_\delta(x) = \frac{1}{\sqrt{2\pi\delta}} \exp\left\{-\frac{x^2}{2\delta}\right\}, \quad \text{for } z \in \mathbb{R} \text{ and } \delta > 0. \quad (10)$$

Definition 4.1 (Heat kernels) The *absorbing* and *reflected heat kernels* are defined to be

$$G_\delta(x, y) := p_\delta(x-y) - p_\delta(x+y) \quad \text{and} \quad G_\delta^r(x, y) := p_\delta(x-y) + p_\delta(x+y)$$

for $x, y > 0$ and $\delta > 0$. The *smoothed density* and *reflected smoothed density* are then defined to be

$$T_{t,\delta}(x) := \int_0^\infty G_\delta(x, y) V_t(y) dy \quad \text{and} \quad T_{t,\delta}^r(x) := \int_0^\infty G_\delta^r(x, y) V_t(y) dy$$

for $t \in [0, T]$, $x > 0$, and $\delta > 0$.

Take any smooth compactly supported function, $\phi \in C_0^\infty(0, \infty)$, then, for every $\delta > 0$, $x \mapsto T_\delta \phi(x)$ (that is the function obtained by convolving the heat kernel, G_δ , with ϕ) is in C^{test} , so putting $T_\delta \phi$ and v into (1) gives the following strong equation for the evolution of the smoothed density:

Proposition 4.2 (Strong smoothed evolution equation) *For $\delta > 0$, $t \in [0, T]$, and $x > 0$*

$$\begin{aligned} T_{t,\delta}(x) = & T_{0,\delta}(x) + \int_0^t \left[-\mu T'_{s,\delta}(x) + \frac{\sigma_M^2 + \sigma_I^2}{2} T''_{s,\delta}(x) \right] ds - \sigma_M \int_0^t T'_{s,\delta}(x) dW_s \\ & + \mu \int_0^t R'_{s,\delta}(x) ds + \sigma_M \int_0^t R'_{s,\delta}(x) dW_s, \end{aligned} \quad (11)$$

where $R_{t,\delta}$ is the remainder function:

$$R_{t,\delta}(x) := T_{t,\delta}(x) - T_{t,\delta}^r(x) = 2 \int_0^\infty p_\delta(x+y) V_t(y) dy.$$

Proof This is simply a matter of observing that

$$\begin{aligned} \langle \phi', T_{t,\delta} \rangle &= - \langle \phi, (T_{t,\delta}^r)' \rangle \\ \langle \phi'', T_{t,\delta} \rangle &= \langle \phi, T_{t,\delta}'' \rangle. \end{aligned}$$

One also needs to apply the stochastic Fubini theorem [3] to switch integration in the space dimension with integration with respect to W (and likewise with respect to time). To this end, it is enough to note that the tails of $T_{t,\delta}$ and $T_{t,\delta}^r$ and their derivatives decay exponentially, and so the conditions of the (stochastic) Fubini theorem are met. $\square$

Differentiating Eq. 11 n times (or integrating once for the case $n = -1$ —the result is no different) and applying Itô's formula to $\left(T_{t,\delta}^{(n)}\right)^2$ gives

$$d \left(T_{t,\delta}^{(n)} \right)^2 = -2\mu T_{t,\delta}^{(n)} T_{t,\delta}^{(n+1)} dt + \left(\sigma_M^2 + \sigma_I^2 \right) T_{t,\delta}^{(n)} T_{t,\delta}^{(n+2)} dt + \sigma_M^2 \left(T_{t,\delta}^{(n+1)} \right)^2 dt$$

$$\begin{aligned} & + \sigma_M^2 \left(R_{t,\delta}^{(n+1)} \right)^2 dt + 2\mu T_{t,\delta}^{(n)} R_{t,\delta}^{(n+1)} dt \\ & - 2\sigma_M T_{t,\delta}^{(n)} T_{t,\delta}^{(n+1)} dW_t + 2\sigma_M T_{t,\delta}^{(n)} R_{t,\delta}^{(n+1)} dW_t. \end{aligned} \quad (12)$$

From this equation, we shall proceed to a proof by induction in Sect. 4.2. Before continuing, we introduce several technical lemmas in the next section.

4.1 Some technical lemmas

Lemma 4.3 *There exists a full subset of Ω on which $x^{-c} T_{t,\delta}(x) \rightarrow 0$ as $x \rightarrow 0$ for every $c \in (0, 1)$, $\delta > 0$, and $t \in (0, T]$.*

Proof This is straightforward since, for $x < 1$, we have

$$\begin{aligned} 0 \leq T_{t,\delta}(x) &= \frac{1}{\sqrt{2\pi\delta}} \int_0^\infty \left[\exp\left\{-\frac{(x-y)^2}{2\delta}\right\} - \exp\left\{-\frac{(x+y)^2}{2\delta}\right\} \right] V_t(y) dy \\ &\leq \frac{2\|V_0\|_\infty x}{\delta\sqrt{2\pi\delta}} \int_0^\infty y \exp\left\{-\frac{(y-1)^2}{2\delta}\right\} dy \\ &= D_\delta x, \end{aligned}$$

for a numerical constant $D_\delta > 0$. Note that we used the estimate $1 - e^{-z} \leq z$. $\square$

Lemma 4.4 *Let $n \geq 0$ be an integer and suppose that for $k = 0, 1, \dots, n$*

$$\liminf_{\delta \rightarrow 0} \mathbb{E} \left[\int_0^T \|w_{k-1-\beta/2}(T_{t,\delta})^{(k)}\|_2^2 dt \right] < \infty.$$

Then, for almost every $(\omega, t) \in \Omega \times [0, T]$, the density V_t is n -times weakly differentiable with

$$\mathbb{E} \left[\int_0^T \|w_{k-1-\beta/2} V_t^{(k)}\|_2^2 dt \right] < \infty, \quad \text{for } k = 0, 1, \dots, n. \quad (13)$$

Proof Let $\langle \cdot, \cdot \rangle$ denote the usual $L^2(0, \infty)$ inner product: $\langle f, g \rangle = \int_0^\infty f(x) g(x) dx$. Let $\{\phi_i\}_{i=1}^\infty$ be a basis of $L^2(0, \infty)$ for which each ϕ_i is smooth and bounded.

Firstly, for any $\phi \in C^\infty(0, \infty)$ and $m > k \geq 1$, $\lim_{x \downarrow 0} (w_{m-\beta/2} \phi(x))^{(k)} = 0$, since $\partial_x^k w_{m-\beta/2}$ is a linear combination of $w_{i-\beta/2}$'s and $w_{i-\beta/2} \rightarrow 0$ for $m-k \leq i \leq m$. So for $k \geq 1$, integrating by parts gives

$$\begin{aligned} \left\langle (w_{k-1-\beta/2}\phi)^{(k)}, T_{t,\delta} \right\rangle &= (-1)^k \left\langle \phi, w_{k-1-\beta/2} T_{t,\delta}^{(k)} \right\rangle \\ &\quad - \lim_{x \downarrow 0} \left\{ T_{t,\delta}(x) (w_{k-1-\beta/2}\phi)^{(k-1)} \right\}. \end{aligned}$$

Applying Leibniz's rule to $(w_{k-1-\beta/2}\phi)^{(k-1)}$, we see that

$$\lim_{x \downarrow 0} \left\{ T_{t,\delta}(x) (w_{k-1-\beta/2}\phi)^{(k-1)}(x) \right\} = C_k \lim_{x \downarrow 0} \left\{ T_{t,\delta}(x) w_{-\beta/2}(x) \phi(x) \right\},$$

for some numerical constant C_k , and that the right-hand side vanishes by Lemma 4.3, because $w_{-\beta/2}(x) = \mathcal{O}(x^{-\beta/2})$ for small x . Hence we conclude

$$\left\langle (w_{k-1-\beta/2}\phi)^{(k)}, T_{t,\delta} \right\rangle = (-1)^k \left\langle \phi, w_{k-1-\beta/2} T_{t,\delta}^{(k)} \right\rangle.$$

The remaining case, $k = 0$, is trivial.

By the above results and Fatou's Lemma, we have

$$\begin{aligned} &\mathbb{E} \left[\int_0^T \sum_{i=1}^{\infty} \left| \left\langle (w_{k-1-\beta/2}\phi_i)^{(k)}, V_t \right\rangle \right|^2 dt \right] \\ &= \mathbb{E} \left[\int_0^T \sum_{i=1}^{\infty} \lim_{\delta \rightarrow 0} \left| \left\langle (w_{k-1-\beta/2}\phi_i)^{(k)}, T_{t,\delta} \right\rangle \right|^2 dt \right] \\ &\leq \liminf_{\delta \rightarrow 0} \mathbb{E} \left[\int_0^T \sum_{i=1}^{\infty} \left| \left\langle \phi_i, w_{k-1-\beta/2} T_{t,\delta}^{(k)} \right\rangle \right|^2 dt \right] \\ &= \liminf_{\delta \rightarrow 0} \mathbb{E} \left[\int_0^T \left\| w_{k-1-\beta/2} T_{t,\delta}^{(k)} \right\|_2^2 dt \right] < \infty, \end{aligned}$$

for $k = 0, 1, \dots, n$, and therefore we can define the processes

$$W_{t,k}(x) := \sum_{i=1}^{\infty} \left\langle (w_{k-1-\beta/2}\phi_i)^{(k)}, V_t \right\rangle \phi_i(x)$$

which satisfy

$$\mathbb{E} \left[\int_0^T \left\| W_{t,k} \right\|_2^2 dt \right] < \infty.$$

For $k = 0$, let $f_0 = w_{-1-\beta/2}$ and take any $\psi \in C_0^\infty(0, \infty)$, then we calculate

$$\left\langle \psi, \frac{1}{f_0} W_{t,0} \right\rangle = \sum_{i=1}^{\infty} \langle f_0 \phi_i, V_t \rangle \left\langle \frac{\psi}{f_0}, \phi_i \right\rangle = \left\langle f_0 \sum_{i=1}^{\infty} \left\langle \frac{\psi}{f_0}, \phi_i \right\rangle \phi_i, V_t \right\rangle = \langle \psi, V_t \rangle,$$

and so $V_t = W_{t,0}/f_0$. Hence we have demonstrated (13) for $k = 0$.

Take $k > 0$, and set $f_k = w_{k-1-\beta/2}$. Using $V_t = W_{t,0}/f_0$ we can compute

$$\begin{aligned} \left\langle \psi, \frac{1}{f_k} W_{t,k} \right\rangle &= \sum_{i=1}^{\infty} \left\langle (f_k \phi_i)^{(k)}, V_t \right\rangle \left\langle \frac{\psi}{f_k}, \phi_i \right\rangle \\ &= \sum_{i,j=1}^{\infty} \left\langle (f_k \phi_i)^{(k)}, \frac{\phi_j}{f_0} \right\rangle \langle f_0 \phi_j, V_t \rangle \left\langle \frac{\psi}{f_k}, \phi_i \right\rangle \\ &= \sum_{i,j,m=1}^{\infty} \left\langle (f_k \phi_i)^{(k)}, \phi_m \right\rangle \left\langle \frac{\phi_j}{f_0}, \phi_m \right\rangle \langle f_0 \phi_j, V_t \rangle \left\langle \frac{\psi}{f_k}, \phi_i \right\rangle \\ &= \sum_{i,j,m=1}^{\infty} (-1)^k \left\langle \phi_i, f_k \phi_m^{(k)} \right\rangle \left\langle \frac{\phi_j}{f_0}, \phi_m \right\rangle \langle f_0 \phi_j, V_t \rangle \left\langle \frac{\psi}{f_k}, \phi_i \right\rangle \\ &= \sum_{j,m=1}^{\infty} (-1)^k \left\langle \psi, \phi_m^{(k)} \right\rangle \left\langle \frac{\phi_j}{f_0}, \phi_m \right\rangle \langle f_0 \phi_j, V_t \rangle \\ &= \sum_{j=1}^{\infty} \left\langle \psi^{(k)}, \frac{\phi_j}{f_0} \right\rangle \langle f_0 \phi_j, V_t \rangle \\ &= \left\langle \psi^{(k)}, V_t \right\rangle, \end{aligned}$$

hence $(-1)^k W_{t,k}/f_k$ is the k^{th} weak derivative of V_t , and this completes the proof. $\square$

Lemma 4.5 *There exists a full subset of Ω on which*

$$\int_0^\infty w_{n-\beta/2}^2 T_{t,\delta}^{(n)}(x) T_{t,\delta}^{(n+1)}(x) dx = \left\| w_{n-\beta/2} T_{t,\delta}^{(n)} \right\|_2^2 - \left(n - \frac{\beta}{2} \right) \left\| w_{n-1/2-\beta/2} T_{t,\delta}^{(n)} \right\|_2^2,$$

and

$$\begin{aligned} \int_0^\infty w_{n-\beta/2}^2 T_{t,\delta}^{(n)}(x) T_{t,\delta}^{(n+2)}(x) dx &= 2 \left\| w_{n-\beta/2} T_{t,\delta}^{(n)} \right\|_2^2 - \left\| w_{n-\beta/2} T_{t,\delta}^{(n+1)} \right\|_2^2 \\ &\quad - 4 \left(n - \frac{\beta}{2} \right) \left\| w_{n-1/2-\beta/2} T_{t,\delta}^{(n)} \right\|_2^2 \\ &\quad + 2 \left(n - \frac{\beta}{2} \right) \left(n - \frac{\beta+1}{2} \right) \left\| w_{n-1-\beta/2} T_{t,\delta}^{(n)} \right\|_2^2, \end{aligned}$$

for every $n \geq -1$, $\delta > 0$, and $t \in [0, T]$.

Proof The result is an integration-by-parts exercise provided the following limits vanish:

$$\lim_{x \downarrow 0} \left\{ w_{n-1/2-\beta/2}(x) T_{t,\delta}^{(n)}(x) \right\}$$

$$\lim_{x \downarrow 0} \left\{ w_{n-\beta/2}^2(x) T_{t,\delta}^{(n)}(x) T_{t,\delta}^{(n+1)}(x) \right\}$$

for $n \geq -1$. If $n \geq 1$, then $w_{n-1/2-\beta/2}, w_{n-\beta/2} \rightarrow 0$ as $x \rightarrow 0$, and the result is immediate. In the case $n = 0$, Lemma 4.3 gives $T_{t,\delta}(x) = \mathcal{O}(x)$, which is sufficient for the limit to be zero. This also implies $T_{t,\delta}^{(-1)}(x) = \mathcal{O}(x^2)$, so the result holds for $n = -1$ too. $\square$

Lemma 4.6 With p_δ defined as in (10), for every integer $n \geq 0$ there exist constants c_i^n for $n/2 \leq i \leq n$ such that

$$\frac{\partial^n p_\delta}{\partial x^n}(x) = \sum_{n/2 \leq i \leq n} c_i^n x^{2i-n} \delta^{-i} p_\delta(x), \quad \text{for every } \delta > 0 \text{ and } x > 0.$$

Proof This is a simple inductive argument. $\square$

Lemma 4.7 For $n = -1, 0, 1, \dots, N$ and $\beta < \pi/\alpha - 1$

$$\liminf_{\delta \rightarrow 0} \|w_{n-\beta/2} T_{0,\delta}^{(n)}\|_2^2 < \infty.$$

Proof The cases $n = 0$ and $n = -1$ are simple. Firstly, observe from Definition 4.1 that $\|T_{0,\delta}\|_\infty \leq \|V_0\|_\infty$, so for all $\delta > 0$

$$\|w_{-\beta/2} T_{0,\delta}\|_2^2 \leq \|V_0\|_\infty \int_0^\infty x^{-\beta} e^{-2x} dx < \infty,$$

and therefore we have the result for $n = 0$. The case $n = -1$ follows by noting that $0 \leq T_{0,\delta}^{-1}(x) \leq \|V_0\|_\infty x$ and applying the same argument.

Now fix $n \geq 1$. Begin by noting that $\partial_x^n G_\delta(x, y) = \partial_y^n G_\delta^{r^n}(x, y)$, where $G^{r^n} = G$ if n is even and $G^{r^n} = G^r$ if n is odd. Splitting the range of integration and integrating by parts in the definition of $T_{0,\delta}$ gives

$$T_{0,\delta}^{(n)}(x) = \sum_{i=0}^{n-1} (-1)^{n-1-i} \partial_y^{n-1-i} G_\delta^{r^n}(x, x/2) V_0^{(i)}(x/2) + \int_{x/2}^\infty G_\delta^{r^n}(x, y) V_0^{(n)}(y) dy$$

$$+ \int_0^{x/2} \partial_x^n G_\delta(x, y) V_0(y) dy. \quad (14)$$

We proceed by considering the weighted L^2 -norm of these three components.

Firstly by Lemma 4.6, there exists a numerical constant $D > 0$ such that

$$\begin{aligned} & \left| \partial_y^{n-1-i} G_\delta^{r^n}(x, x/2) V_0^{(i)}(x/2) w_{n-\beta/2}(x) \right| \\ & \leq D \sum_{\frac{n-1-i}{2} \leq j \leq n-1-i} x^{2j-n+1+i} \delta^{-j-1/2} e^{-\frac{x^2}{8\delta}} \left| V_0^{(i)}(x/2) \right| w_{i-\beta/2}(x) x^{n-i}, \end{aligned}$$

therefore integrating over $x \in (0, \infty)$ and applying the Cauchy–Schwarz inequality gives

$$\begin{aligned} & \left\| \partial_y^{n-1-i} G_\delta^{r^n}(\cdot, \cdot/2) V_0^{(i)}(\cdot/2) w_{n-\beta/2} \right\|_2^2 \\ & \leq 2D \sum_{\frac{n-1-i}{2} \leq j \leq n-1-i} \int_0^\infty x^{4j+2} \delta^{-2j-1} e^{-\frac{x^2}{4\delta}} dx \left\| V_0^{(i)}(\cdot/2) w_{i-\beta/2} \right\|_2^2 \\ & \leq D' \sum_{\frac{n-1-i}{2} \leq j \leq n-1-i} \delta^{1/2} 2^{2i-\beta} \left\| V_0^{(i)} w_{i-\beta/2} \right\|_2^2 \rightarrow 0 \end{aligned} \quad (15)$$

as $\delta \rightarrow 0$, where $D' > 0$ is a further numerical constant.

For the second term, we apply Cauchy–Schwarz to obtain

$$\begin{aligned} & \left\{ \int_{x/2}^\infty G_\delta^{r^n}(\cdot, y) V_0^{(n)}(y) dy \right\}^2 \\ & \leq \left\{ \int_0^\infty G_\delta^{r^n}(\cdot, y) w_{n-\beta/2}(y)^2 V_0^{(n)}(y)^2 dy \right\} \left\{ \int_{x/2}^\infty G_\delta^{r^n}(\cdot, y) y^{-2n+\beta} e^{2y} dy \right\} \\ & \leq 4 \cdot 2^{2n-\beta} x^{-2n+\beta} \left\{ \int_0^\infty p_\delta(x-y) w_{n-\beta/2}(y)^2 V_0^{(n)}(y)^2 dy \right\} \\ & \quad \times \left\{ \int_{x/2}^\infty p_\delta(x-y) e^{2y} dy \right\}. \end{aligned}$$

For $\delta < 1$ we have $\int_{x/2}^\infty p_\delta(x-y) e^{2y} dy \leq D' e^{2x}$, for some numerical constant $D' > 0$, therefore

$$\left\| w_{n-\beta/2} \int_{\cdot/2}^\infty G_\delta^{r^n}(\cdot, y) V_0^{(n)}(y) dy \right\|_2^2$$

$$\begin{aligned}
&\leq 2^{2n-\beta+2} D' \int_0^\infty \int_0^\infty p_\delta(x-y) dx w_{n-\beta/2}(y)^2 V_0^{(n)}(y)^2 dy \\
&\leq 2^{2n-\beta+2} D' \left\| w_{n-\beta/2} V_0^{(n)} \right\|_2^2.
\end{aligned} \tag{16}$$

Finally, using the fact that V_0 is bounded and Lemma 4.6, there exists a numerical constant $D'' > 0$ such that

$$\left| \int_0^{x/2} \partial_x^n G_\delta(x, y) V_0(y) dy \right| \leq D'' \|V_0\|_\infty x \sum_{n/2 \leq i \leq n} x^{2i-n} \delta^{-i-1/2} e^{-\frac{x^2}{8\delta}},$$

and therefore

$$\begin{aligned}
&\left\| w_{n-\beta/2} \int_0^{x/2} \partial_x^n G_\delta(\cdot, y) V_0(y) dy \right\|_2^2 \\
&\leq 2 (D'')^2 \|V_0\|_\infty^2 \sum_{n/2 \leq i \leq n} \int_0^\infty x^{4i-2n+2} \delta^{-2i-1} e^{-\frac{x^2}{8\delta}} dx \\
&\leq D''' \delta^{(1-\beta)/2} \rightarrow 0
\end{aligned} \tag{17}$$

as $\delta \rightarrow 0$, since $\beta < 1$. Here $D''' > 0$ is a further numerical constant.

By using (14), results (15–17) complete the proof. $\square$

Lemma 4.8 For $n \geq 0$ and $\beta < \pi/\alpha - 1$

$$\liminf_{\delta \rightarrow 0} \mathbb{E} \left[\int_0^t \left\| w_{n-\beta/2} R_{s,\delta}^{(n+1)} \right\|_2^2 ds \right] = 0.$$

Proof Interchanging differentiation and integration and applying Lemma 4.6 gives

$$\begin{aligned}
\left| R_{s,\delta}^{(n+1)}(x) \right| &= \left| 2 \int_0^\infty \frac{\partial^{n+1} p_\delta}{\partial x^{n+1}}(x+y) V_s(y) dy \right| \\
&\leq D \sum_{\frac{n+1}{2} \leq i \leq n+1} \int_0^\infty (x+y)^{2i-n-1} \delta^{-i-1/2} e^{-\frac{x^2}{2\delta}} e^{-\frac{y^2}{2\delta}} V_s(y) dy,
\end{aligned}$$

with $D > 0$ a numerical constant. Let $\eta \in (0, 1)$ and split the region of integration at $y = \delta^{\frac{1}{2}(1-\eta)}$, then for $\delta < 1$ we have $\delta^{-i} \leq \delta^{-n-1}$ and $e^{-\frac{x^2}{2\delta}} \leq e^{-\frac{y^2}{4}} e^{-1/4\delta\eta}$ on $y > \delta^{\frac{1}{2}(1-\eta)}$, and so we have

$$\begin{aligned}
 & \left| w_{n-\beta/2}(x) R_{s,\delta}^{(n+1)}(x) \right| \\
 & \leq \left\{ D \sum_{\frac{n+1}{2} \leq i \leq n+1} \int_0^\infty (x+y)^{2i-n-1} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{4}} dy \right\} \|V_0\|_\infty \delta^{-n-1/2} e^{-1/4\delta^\eta} w_{n-\beta/2}(x) \\
 & + D \sum_{\frac{n+1}{2} \leq i \leq n+1} \left(x + \delta^{(1-\eta)/2} \right)^{2i-n-1} \delta^{-i-1/2} v_s \left(0, \delta^{(1-\eta)/2} \right) w_{n-\beta/2}(x) e^{-\frac{x^2}{2\delta}}.
 \end{aligned} \tag{18}$$

The first term on the right-hand side of (18) vanishes as $\delta \rightarrow 0$, hence we have

$$\begin{aligned}
 & \liminf_{\delta \rightarrow 0} \mathbb{E} \left[\int_0^t \|w_{n-\beta/2} R_{s,\delta}^{(n+1)}\|_2^2 ds \right] \\
 & \leq \liminf_{\delta \rightarrow 0} D' \sum_{\frac{n+1}{2} \leq i \leq n+1} \left\{ \int_0^\infty (x+1)^{4i-2n-2} x^{2n-\beta} e^{-x^2} dx \right. \\
 & \quad \times \delta^{(1-\eta)(2i-n-1)} \delta^{-2i-1} \delta^{\frac{1}{2}(1-\eta)(3+\beta+\varepsilon)} \delta^{1/2} \delta^{n-\beta/2} \Big\}
 \end{aligned} \tag{19}$$

where we have used Proposition 2.7 with β replaced by $\beta + \varepsilon < \pi/\alpha - 1$ for small $\varepsilon > 0$, and $D' > 0$ is a numerical constant. (Note as we integrate over $s \in (0, t)$, the singular factor of $s^{-\gamma}$ from Proposition 2.7 integrates to a finite value because $\gamma \in (0, 1)$.) The δ term in (19) is bounded by

$$\delta^{\frac{1}{2}(1-\eta)\varepsilon - \eta(2n+5/2+\beta/2)}$$

for small δ , and, since we choose η small enough so that

$$\frac{1}{2}(1-\eta)\varepsilon > \eta(2n+5/2+\beta/2),$$

the limit on the right-hand side of (19) vanishes. $\square$

4.2 The inductive proof for the regularity result

4.2.1 Initial case

The initial case is when $n = -1$, and the result we shall demonstrate is:

$$\liminf_{\delta \rightarrow 0} \mathbb{E} \left[\int_0^T \|w_{-2-\beta/2} T_{t,\delta}^{(-1)}\|_2^2 dt \right] < \infty, \quad \text{for every } \beta < \pi/\alpha - 1. \tag{20}$$

By truncating the range of integration, we have

$$T_{t,\delta}^{(-1)}(x) = \int_0^\infty \int_0^x G_\delta(z, y) dz V_t(y) dy = v_t(0, x) + I_{t,\delta}(x),$$

where $I_{t,\delta}(x) := \int_{2x}^\infty \int_0^x G_\delta(z, y) dz V_t(y) dy$. From Proposition 2.7, it is clear that

$$\mathbb{E} \left[\int_0^T \|w_{-2-\beta/2} v_t(0, \cdot)\|_2^2 dt \right] < \infty$$

and therefore we are done provided $\liminf_{\delta \rightarrow 0} \mathbb{E} \left[\int_0^T \|w_{-2-\beta/2} I_{t,\delta}\|_2^2 dt \right] < \infty$. Using the inequality

$$G_\delta(z, y) \leq \frac{2zy}{\sqrt{2\pi}\delta^3} \exp \left\{ -(z-y)^2/2\delta \right\},$$

we have a numerical constant $C > 0$ such that

$$I_{t,\delta}(x) \leq C\delta^{-3/2}x^2e^{-x^2/2\delta} \int_0^\infty (y+2x)e^{-y^2/2\delta} V_t(y+2x) dy,$$

and so

$$\begin{aligned} & \|w_{-2-\beta/2} I_{t,\delta}\|_2^2 \\ & \leq \frac{C^2}{\delta^3} \int_0^\infty \int_0^\infty \int_0^\infty x^{-\beta} (y_1+2x)(y_2+2x) \\ & \quad \times e^{-\frac{x^2+y_1^2+y_2^2}{2\delta}} V_t(y_1+2x) V_t(y_2+2x) dy_1 dy_2 dx. \end{aligned}$$

Now let $\eta \in (0, 1)$. Splitting the region of integration on $(0, \delta^{(1-\eta)/2})^3 \subseteq (0, \infty)^3$ and its complement, we see that on the complement the triple integral on the right-hand side above is bounded by a multiple of e^{-1/δ^η} , and so vanishes as $\delta \rightarrow 0$. Therefore we need only consider the region of integration $(0, \delta^{(1-\eta)/2})^3$, and so we have a numerical constant $C' > 0$ such that

$$\begin{aligned} \liminf_{\delta \rightarrow 0} \mathbb{E} \left[\int_0^T \|w_{-2-\beta/2} I_{t,\delta}\|_2^2 dt \right] & \leq C' \liminf_{\delta \rightarrow 0} \delta^{-(3+\beta)/2} \int_0^T \mathbb{E} \left[v_t \left(0, \delta^{(1-\eta)/2} \right)^2 \right] dt \\ & \leq B \liminf_{\delta \rightarrow 0} \delta^{(1-\eta)(3+\beta+\varepsilon)/2 - (3+\beta)/2}, \end{aligned}$$

where the second line follows by Proposition 2.7 and $\varepsilon > 0$ is sufficiently small. Clearly we may choose $\eta > 0$ small enough so that the exponent of δ is positive, whereby the right-hand side vanishes and we have (20).

4.2.2 Inductive step

Assume that for some $-1 \leq n \leq N - 1$ we have

$$\liminf_{\delta \rightarrow 0} \mathbb{E} \left[\int_0^T \|w_{k-1-\beta/2} T_{t,\delta}^{(k)}\|_2^2 dt \right] < \infty, \quad \text{for every } \beta < \pi/\alpha - 1 \quad (21)$$

for $k = -1, 0, 1, \dots, n$. (We demonstrated the case $n = -1$ above.) The proof will be complete if we can show condition (21) holds for $k = n + 1$, since then, by induction, the condition holds for all $k = 0, \dots, N$ and we can apply Lemma 4.4 to give the regularity result of Theorem 1.1 for the probabilistic solution v .

Let us begin by multiplying throughout Eq. (12) by the square of the weighting function $w_{n-\beta/2}^2$ and integrating over $(0, \infty)$. To do so we apply Lemma 4.5 and arrive at

$$\begin{aligned} & \sigma_I^2 \mathbb{E} \left[\int_0^t \|w_{n-\beta/2} T_{s,\delta}^{(n+1)}\|_2^2 ds \right] \\ &= \|w_{n-\beta/2} T_{0,\delta}^{(n)}\|_2^2 - \mathbb{E} \left[\|w_{n-\beta/2} T_{t,\delta}^{(n)}\|_2^2 \right] - 2\mu \mathbb{E} \left[\int_0^t \|w_{n-\beta/2} T_{s,\delta}^{(n)}\|_2^2 ds \right] \\ &+ 2\mu \left(n - \frac{\beta}{2} \right) \mathbb{E} \left[\int_0^t \|w_{n-1/2-\beta/2} T_{s,\delta}^{(n)}\|_2^2 ds \right] \\ &+ 2 \left(\sigma_M^2 + \sigma_I^2 \right) \mathbb{E} \left[\int_0^t \|w_{n-\beta/2} T_{s,\delta}^{(n)}\|_2^2 ds \right] \\ &- 4 \left(\sigma_M^2 + \sigma_I^2 \right) \left(n - \frac{\beta}{2} \right) \mathbb{E} \left[\int_0^t \|w_{n-1/2-\beta/2} T_{s,\delta}^{(n)}\|_2^2 ds \right] \\ &+ 2 \left(\sigma_M^2 + \sigma_I^2 \right) \left(n - \frac{\beta}{2} \right) \left(n - \frac{\beta+1}{2} \right) \mathbb{E} \left[\int_0^t \|w_{n-1-\beta/2} T_{s,\delta}^{(n)}\|_2^2 ds \right] \\ &+ \sigma_M^2 \mathbb{E} \left[\int_0^t \|w_{n-\beta/2} R_{s,\delta}^{(n+1)}\|_2^2 ds \right] + 2\mu \mathbb{E} \left[\int_0^t \int_0^\infty w_{n-\beta/2}^2 T_{s,\delta}^{(n)} R_{s,\delta}^{(n+1)} ds \right] \quad (22) \end{aligned}$$

Eliminating the negative terms from the right-hand side and applying the Cauchy–Schwarz inequality gives

$$\begin{aligned}
 & \sigma_I^2 \mathbb{E} \left[\int_0^t \|w_{n-\beta/2} T_{s,\delta}^{(n+1)}\|_2^2 ds \right] \\
 & \leq \|w_{n-\beta/2} T_{0,\delta}^{(n)}\|_2^2 - 2\mu \mathbb{E} \left[\int_0^t \|w_{n-\beta/2} T_{s,\delta}^{(n)}\|_2^2 ds \right] \\
 & \quad + 2\mu \left(n - \frac{\beta}{2} \right) \mathbb{E} \left[\int_0^t \|w_{n-1/2-\beta/2} T_{s,\delta}^{(n)}\|_2^2 ds \right] \\
 & \quad + 2 \left(\sigma_M^2 + \sigma_I^2 \right) \mathbb{E} \left[\int_0^t \|w_{n-\beta/2} T_{s,\delta}^{(n)}\|_2^2 ds \right] \\
 & \quad + 2 \left(\sigma_M^2 + \sigma_I^2 \right) \left(n - \frac{\beta}{2} \right) \left(n - \frac{\beta+1}{2} \right) \mathbb{E} \left[\int_0^t \|w_{n-1-\beta/2} T_{s,\delta}^{(n)}\|_2^2 ds \right] \\
 & \quad + \sigma_M^2 \mathbb{E} \left[\int_0^t \|w_{n-\beta/2} R_{s,\delta}^{(n+1)}\|_2^2 ds \right] \\
 & \quad + 2 |\mu| \left\{ \mathbb{E} \left[\int_0^t \|w_{n-\beta/2} T_{s,\delta}^{(n)}\|_2^2 ds \right] \right\}^{1/2} \left\{ \mathbb{E} \left[\int_0^t \|w_{n-\beta/2} R_{s,\delta}^{(n+1)}\|_2^2 ds \right] \right\}^{1/2}.
 \end{aligned} \tag{23}$$

We are done provided the seven terms on the right-hand side of inequality (23) have a finite limit-infimum as $\delta \downarrow 0$. This is true of the second to fifth terms by the inductive hypothesis (21), and, with this, the final two terms vanish by Lemma 4.8. From Lemma 4.7, the first term remains finite in the limit-infimum, hence we have shown the probabilistic solution v satisfies the regularity properties in Theorem 1.1.

4.3 Uniqueness

It remains to show that the probabilistic solution is indeed the unique finite-measure valued solution of (1). To do so, suppose we have another finite-measure valued solution $\bar{v}$, and define the signed-measure valued process, Δ , as the difference between the solutions:

$$\Delta_t(S) := v_t(S) - \bar{v}_t(S).$$

Let us write $I(\zeta)$ for the expected value of a random measure, ζ , that is

$$I(\zeta)(S) := \mathbb{E}[\zeta(S)],$$

for $S \subseteq (0, \infty)$ measurable, then, by taking expectation in (1), we have that $I(\nu)$ and $I(\bar{\nu})$ solve

$$d\langle \phi, \zeta_t \rangle = \frac{\sigma_M^2 + \sigma_I^2}{2} \langle \phi'', \zeta_t \rangle dt, \quad \zeta_0 = V_0. \quad (24)$$

Also, if we define the expected product measure to be the unique extension of

$$J(\zeta)(S_1 \times S_2) := \mathbb{E}[\zeta(S_1)\zeta(S_2)]$$

on $(0, \infty)^2$, applying Itô's formula to the product $\langle \phi_1, \nu_t \rangle \langle \phi_2, \nu_t \rangle$ using (1) gives that $J(\nu)$ and $J(\bar{\nu})$ solve

$$d\langle \phi, \zeta_t \rangle = \frac{\sigma_M^2 + \sigma_I^2}{2} \left\langle \left\{ \partial_{xx} + 2\rho^2 \partial_{xy} + \partial_{yy} \right\} \phi, \zeta_t \right\rangle dt, \quad \zeta_0(x, y) = V_0(x) V_0(y) \quad (25)$$

for $\phi \in C_0^\infty((0, \infty)^2)$. Equations (24) and (25) are just the deterministic heat equations with zero boundary conditions, hence their solutions are unique so we conclude $I(\nu) = I(\bar{\nu})$ and $J(\nu) = J(\bar{\nu})$ —that is, the first and second moments of ν and $\bar{\nu}$ are equal in distribution.

By the linearity of (1), we have that the smoothed measure $T_\delta \Delta_t = \int_0^\infty G_\delta(\cdot, y) \Delta_t(dy)$ solves

$$dT_\delta \Delta_t = -\mu (T_\delta^r \Delta_t)' dt + \frac{1}{2} (\sigma_M^2 + \sigma_I^2) (T_\delta \Delta_t)'' dt - \sigma_M (T_\delta^r \Delta_t)' dW_t$$

with $T_{t,\delta} \Delta \equiv 0$. Using the work in (12) and integrating over $\Omega \times (0, \infty)$ gives

$$\begin{aligned} \mathbb{E} \left[\|T_\delta \Delta_t\|_2^2 \right] &\leq \int_0^t \sigma_M^2 \mathbb{E} \left[\left\| (R_{s,\delta} \Delta)' \right\|_2^2 \right] ds \\ &\quad + 2|\mu| \int_0^t \mathbb{E} \left[\|T_\delta \Delta_s\|_2^2 \right]^{\frac{1}{2}} \mathbb{E} \left[\left\| (R_{s,\delta} \Delta)' \right\|_2^2 \right]^{\frac{1}{2}} ds, \end{aligned}$$

then, since

$$\mathbb{E} \left[|\Delta_t(0, \varepsilon)|^2 \right] \leq \mathbb{E} \left[|\nu_t(0, \varepsilon)|^2 \right] + \mathbb{E} \left[|\bar{\nu}_t(0, \varepsilon)|^2 \right] = 2\mathbb{E} \left[|\nu_t(0, \varepsilon)|^2 \right],$$

Lemma 4.8 can be applied to $R_{s,\delta}\Delta$ and gives that $\liminf_{\delta \rightarrow 0} \mathbb{E} [\|T_\delta \Delta_t\|_2^2] = 0$. We conclude that we have uniqueness, since, for every $\phi \in L^2(0, \infty)$,

$$\begin{aligned} \mathbb{E} \left[\int_0^T \left(\langle \phi, v_t \rangle - \langle \phi, \tilde{v}_t \rangle \right)^2 dt \right] &= \mathbb{E} \left[\int_0^T \lim_{\delta \rightarrow 0} \langle \phi, T_\delta \Delta_t \rangle^2 dt \right] \\ &\leq \|\phi\|_2^2 \int_0^T \liminf_{\delta \rightarrow 0} \mathbb{E} [\|T_\delta \Delta_t\|_2^2] dt = 0, \end{aligned}$$

and so, by approximation, the result holds for every $\phi \in C^{\text{test}}$. $\square$

5 Proof of Theorem 1.2

This theorem relies on the fact that the estimate in Proposition 2.7 is sharp. Specifically, Lemma 5.1 of [6] states that if V_0 is supported on a finite interval away from zero, then for every $\eta > 0$ there exist $C(\eta) > 0$ and $\varepsilon_0(\eta) > 0$ such that

$$\mathbb{E} [v_t(0, \varepsilon)^2] \geq C(\eta) \varepsilon^{2+\pi/\alpha+\eta}, \quad \text{for all } 0 < \varepsilon < \varepsilon_0(\eta) \text{ and } t \in [0, T]. \quad (26)$$

It is clear that if $\tilde{V}_0 \leq V_0$, then the empirical measure, $\tilde{v}$, obtained from starting the system from $\tilde{V}_0$, satisfies $\tilde{v}_t \leq v_t$. Since we can certainly find a constant $a > 0$ such that $\tilde{V}_0 := V_0 \mathbf{1}_{(a, \infty)}$ is non-zero, we see that (26) remains valid if we drop the assumption on the support of V_0 .

To complete the proof, suppose, for a contradiction, that we have $\beta > \pi/\alpha - 1$ and a $k \in \{0, 1, \dots, N\}$ such that

$$\mathbb{E} \left[\int_0^T \|w_{k-1-\beta/2} V_t^{(k)}\|_2^2 dt \right] < \infty.$$

Then

$$\mathbb{E} \left[\int_0^T \int_0^1 x^{2(k-1)-\tilde{\beta}} V_t^{(k)}(x)^2 dx dt \right] < \infty$$

for any $\tilde{\beta} < \beta$, and let us fix a $\tilde{\beta} \in (\pi/\alpha - 1, 1)$ and take $\varepsilon > 0$ small enough so that we have $\tilde{\beta} - \varepsilon \in (\pi/\alpha - 1, 1)$. Then for $k \geq 2$

$$\int_0^1 x^{2(k-2)-\tilde{\beta}+\varepsilon} dx < \infty,$$

and, since

$$\begin{aligned} \left(V_t^{(k-1)}(x) - V_t^{(k-1)}(1) \right)^2 &= \left(\int_x^1 V_t^{(k)}(y) dy \right)^2 \\ &\leq \int_0^1 x^{2(k-1)-\tilde{\beta}} V_t^{(k)}(x)^2 dx \cdot \int_x^1 y^{-2k+2+\tilde{\beta}} dy \\ &\leq \int_0^1 x^{2(k-1)-\tilde{\beta}} V_t^{(k)}(x)^2 dx \cdot \frac{x^{-2k+3+\tilde{\beta}}}{2k-3-\tilde{\beta}}, \quad (27) \end{aligned}$$

applying the triangle inequality, multiplying by $x^{2k-4-\tilde{\beta}+\varepsilon}$, and integrating over $\Omega \times [0, T] \times (0, 1)$ gives

$$\mathbb{E} \left[\int_0^T \int_0^1 x^{2(k-2)-\tilde{\beta}+\varepsilon} V_t^{(k-1)}(x)^2 dx dt \right] < \infty.$$

Repeating this argument eventually yields

$$\mathbb{E} \left[\int_0^T \int_0^1 x^{-\tilde{\beta}+\varepsilon} V_t'(x)^2 dx dt \right] < \infty$$

for all allowable ε . If we return to (27) with $k = 1$ and take the upper limit of the integral to be x and the lower limit 0, then, since $V_t(0) = 0$, we arrive at

$$\mathbb{E} \left[\int_0^T \int_0^1 x^{-2-\tilde{\beta}+\varepsilon} V_t(x)^2 dx dt \right] < \infty.$$

Repeating this manipulation once more gives the following for $x < 1$:

$$\begin{aligned} \mathbb{E} \left[\int_0^T v_t(0, x)^2 dt \right] &\leq \mathbb{E} \left[\int_0^T \int_0^1 x^{-2-\tilde{\beta}+\varepsilon} V_t(x)^2 dx dt \right] \int_0^x y^{2+\tilde{\beta}-\varepsilon} dy \\ &= D(\tilde{\beta}, \varepsilon) x^{3+\tilde{\beta}-\varepsilon}, \end{aligned}$$

with $D(\tilde{\beta}, \varepsilon) \in (0, \infty)$ a constant. This is a contradiction of (26) for the case $\eta = (1 - \pi/\alpha) + (\tilde{\beta} - \varepsilon) > 0$, and hence we have Theorem 1.2. $\square$

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