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On the irreducibility of local representations of the Braid group B_n

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Abstract We prove that any homogeneous local representation $\varphi : B_n \to GL_n(\mathbb{C})$ of type 1 or 2 of dimension $n \geq 6$ is reducible. Then, we prove that any representation $\varphi : B_n \to GL_n(\mathbb{C})$ of type 3 is equivalent to a complex specialization of the standard representation τ_n. Also, we study the irreducibility of all local linear representations of the braid group B_3 of degree 3. We prove that any local representation of type 1 of B_3 is reducible to a Burau type representation and that any local representation of type 2 of B_3 is equivalent to a complex specialization of the standard representation. Moreover, we construct a representation of B_3 of degree 6 using the tensor product of local representations of type 2. Let u_i, $i = 1, 2$, be non-zero complex numbers on the unit circle. We determine a necessary and sufficient condition that guarantees the irreducibility of the obtained representation.

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1 Introduction

The braid group B_n is represented, due to Artin, in the group $\mathrm{Aut}(F_n)$ of automorphisms of the free group F_n generated by $x_1, \ldots, x_n$. To attack the linearity of the braid group B_n, the faithfulness of the braid group representations was studied. One of these represenations is the Burau representation which was, for a long time, a candidate to answer the question of faithfulness of the braid group B_n. It was proved that the Burau representation is faithful for $n \leq 3$ and not faithful for $n \geq 5$. For $n = 4$, the question of faithfulness of the Burau representation has not been answered yet. For more details, see [2] and [3].

In addition to the linearity of the braid group B_n, the classification of irreducible complex representations of B_n was of great concern. In [4], Formanek found a necessary and sufficient condition for the specialization of the reduced Burau representation to be irreducible. Moreover, Formanek classified all irreducible complex representations of the braid group B_n of degree at most $n - 1$ for $n \geq 7$. In [7], Sysoeva extended this classification to representations of degree n for $n \geq 9$. For $n = 5, 6, 7$ and 8, the classification was completed by Formanek, Lee, Sysoeva and Vazirani. For more details, see [5]. For $n \geq 10$, Sysoeva proved, in [8], that there are no irreducible representations of B_n of dimension $n + 1$.

The local representations of the braid group B_3 were studied by Mikhalchishina who proved that any local representation of the braid group B_3 into $GL_3(\mathbb{C})$ is of type 1 or 2. In addition, Mikhalchishina studied the

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n-dimensional homogeneous local representations φ of the braid group B_n and proved that φ coincides with one of the three representations φ_1, φ_2 and φ_3 which were defined. For more details, see [6].

In our work, first we study the irreducibility of the local homogeneous multi-parameter representations of types 1 and 2 of degree n of the braid group B_n. We prove that, for $n \geq 6$, any homogeneous local representation of type 1 or 2 is reducible.

Next, we consider the case of homogeneous local representations of type 3 of degree n of the braid group B_n. We prove that any homogeneous local representation of type 3 is equivalent to a complex specialization of the standard representation. Consequently, any multi-parameter homogeneous local representation of type 3 is irreducible if and only if $bc \neq 1$.

Then, we study the irreducibility of all local representations of the braid group B_3. We prove that any three-dimensional local representation of type 1 is reducible to a representation of Burau type. Also, we prove that any three-dimensional local representation of type 2 is equivalent to a complex specialization of the standard representation. Due to this equivalence, any local representation of type 2 is irrreducible if and only if $bc \neq 1$.

Finally, we find the tensor product of two complex specializations of the standard representations of B_3. We prove that the obtained nine-dimensional multi-parameter representation is a direct sum of a complex specialization of the standard representation and a six-dimensional representation φ. We consider the case when the complex numbers $u_i's$ are on the unit circle. Then, we prove that φ is irreducible if and only if $\sqrt{u_1} \neq \pm\sqrt{u_2}$.

2 Preliminaries

Definition 2.1 [1] The braid group, B_n, is an abstract group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with the following relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \text{ for all } i, j = 1, \ldots, n-1 \text{ with } |i-j| \geq 2,$$

and

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \text{ for } i = 1, \ldots, n-2.$$

Definition 2.2 [7, Definition 2] The corank of the representation $\rho : B_n \to GL_r(\mathbb{C})$ is $rank(\rho(\sigma_i) - 1)$, where the σ_i are the standard generators of the group B_n.

Definition 2.3 [6] A representation $\varphi : B_n \longrightarrow GL_n(\mathbb{C})$ is called local if

$$\varphi(\sigma_i) = \left(\begin{array}{c|c|c} I_{i-1} & 0 & 0 \\ \hline 0 & R_i & 0 \\ \hline 0 & 0 & I_{n-i-1} \end{array}\right), \quad i = 1, 2, \ldots, n-1,$$

where I_m is the identity matrix of order m and R_i is a matrix of order 2. A local representation is called homogeneous if $R_1 = R_2 = \cdots = R_{n-1}$.

Theorem 2.4 [6, Theorem 1] *If $\varphi : B_3 \to GL_3(\mathbb{C})$ is a local representation then φ has one of the two types:*

(1) $\varphi(\sigma_1) = \left(\begin{array}{cc|c} \alpha(1-d) & \frac{(1-d)(1-\alpha+d\alpha)}{c} & 0 \\ c & d & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$, $\varphi(\sigma_2) = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \alpha & \frac{(1-\alpha)(1-d+d\alpha)}{\gamma} \\ 0 & \gamma & d(1-\alpha) \end{array}\right)$,

where $d, \alpha \neq 1$ and $c, \gamma \neq 0$;

(2) $\varphi(\sigma_1) = \left(\begin{array}{cc|c} 0 & b & 0 \\ c & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$, $\varphi(\sigma_2) = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & \frac{bc}{\gamma} \\ 0 & \gamma & 0 \end{array}\right)$, *where $bc, \gamma \neq 0$.*

Corollary 2.5 [6, Corollary to Theorem 1] *If $\varphi : B_n \longrightarrow GL_n(\mathbb{C})$, $n \geq 3$, is a homogeneous local representation, then φ coincides with one of the representations φ_1, φ_2 and φ_3 defined as follows:*

$$\varphi_j : B_n \longrightarrow GL_n(\mathbb{C}),$$

$$(1)\quad \varphi_1(\sigma_i)=\left(\begin{array}{c|ccc|c} I_{i-1} & 0 & 0 & & 0 \\ \hline 0 & \alpha & \frac{1-\alpha}{\gamma} & & 0 \\ 0 & \gamma & 0 & & 0 \\ \hline 0 & 0 & 0 & & I_{n-i-1} \end{array}\right), \quad \gamma \neq 0, i=1,2,\ldots,n-1.$$

$$(2)\quad \varphi_2(\sigma_i)=\left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{1-d}{c} & 0 \\ 0 & c & d & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right), \quad c \neq 0, i=1,2,\ldots,n-1.$$

$$(3)\quad \varphi_3(\sigma_i)=\left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right), \quad bc \neq 0, i=1,2,\ldots,n-1.$$

Definition 2.6 [7, Definition 6] The standard representation is the representation

$$\tau_n : B_n \to GL_n(\mathbb{Z}[t^{\pm 1}])$$

defined by

$$\tau_n(\sigma_i)=\left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 0 & t & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right),$$

for $i=1,2,\ldots,n-1$, where I_k is the $k \times k$ identity matrix.

Definition 2.7 [7] The complex specialization of the standard representation is defined by

$$\tau_n(u) : B_n \to GL_n(\mathbb{C}),$$

$$\tau_n(u)(\sigma_i)=\left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 0 & u & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right),$$

for $i=1,2,\ldots,n-1$, where I_k is the $k \times k$ identity matrix, and u $\in \mathbb{C}^*$.

Lemma 2.8 [7, Lemma 5.3] *If $u=1$, then $\tau_n(u)$ is reducible.*

Lemma 2.9 [7, Lemma 5.4] *If $u \neq 1$, then $\tau_n(u)$ is irreducible.*

Theorem 2.10 [7, Theorem 5.5] *Let $\rho : B_n :\to GL_r(\mathbb{C})$ be an irreducible representation of B_n for $n \geq 6$. Let $r \geq n$, and let $\rho(\sigma_1)=1+A_1$ with $rank(A_1)=2$. Then $r=n$ and ρ is equivalent to the representation $\tau_n(u)$, where $u \in \mathbb{C}^*$ and $u \neq 1$.*

Definition 2.11 [4] The complex specialization of the reduced Burau representation $\beta_3(z)$ is defined by:

$$\beta_3(z) : B_3 \to GL_2(\mathbb{C}),$$

$$\beta_3(z)(\sigma_1)=\begin{pmatrix} -z & 0 \\ -1 & 1 \end{pmatrix} \text{ and } \beta_3(z)(\sigma_2)=\begin{pmatrix} 1 & -z \\ 0 & -z \end{pmatrix}.$$

Theorem 2.12 [4, Theorem 11] *Let $\rho : B_3 \to GL_2(\mathbb{C})$ be an irreducible representation. Then ρ is equivalent to $\chi(y) \otimes \beta_3(z)$ for some $y, z \in \mathbb{C}^*$, where z is not a root of the polynomial t^2+t+1. Here $\chi(y)$ is the one dimentional representation and $\beta_3(z)$ is the reduced Burau representation. We say that ρ is of Burau type.*

Definition 2.13 The principal square root function is the function defined as follows: For all $z \in \mathbb{C}, z=(\rho,\alpha)$, $\rho \geq 0$. $\sqrt{z}=\sqrt{\rho}e^{i\frac{\alpha}{2}}$, where $-\pi < \alpha \leq \pi$.

3 Irreduciblility of homogeneous local representations of B_n for $n \geq 6$

Mikhalchishina proved, in [6, Proposition, p. 672], that type 1 and type 2 representations are not equivalent when $d \neq \alpha$. In this section, we prove that any homogeneous local representation of type 1 or 2 of dimension $n \geq 6$ is reducible. Then, we prove that any homogeneous local representation of type 3 is equivalent to a complex specialization of the standard representation τ_n.

Theorem 3.1 *The homogeneous local representations of types 1 and 2 are reducible for $n \geq 6$.*

Proof Let φ_1 and φ_2 be two homogeneous local representations of B_n of types 1 and 2 respectiveley with $n \geq 6$. Consider the matrices P_1 and P_2 defined as

$$P_1 = Diag\left(\frac{1}{\gamma^{n-1}}, \cdots, \frac{1}{\gamma}, 1\right) \text{ and } P_2 = Diag\left(\frac{1}{c^{n-1}}, \frac{1}{c^{n-2}}, \cdots, \frac{1}{c}, 1\right),$$

where $Diag(a_1, a_2, \ldots, a_n)$ is a diagonal $n \times n$ matrix, with $a_{ii} = a_i$.

By direct computations, we get

$$P_1^{-1}\varphi_1(\sigma_i)P_1 = \tilde{\varphi_1}(\sigma_i) \text{ and } P_2^{-1}\varphi_2(\sigma_i)P_2 = \tilde{\varphi_2}(\sigma_i)$$

where,

$$\tilde{\varphi_1}(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & \alpha & 1-\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right)$$

and

$$\tilde{\varphi_2}(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 0 & 1-d & 0 \\ 0 & 1 & d & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right)$$

for $i = 1, 2, \ldots, n-1$.

This implies that the representations φ_1 and φ_2 are equivalent to the representations $\tilde{\varphi_1}$ and $\tilde{\varphi_2}$ respectively. Thus, we can verify that the corank of the representations φ_1 and φ_2 is 1. This implies that the representations φ_1 and φ_2 are reducible. (See [4], Theorem 10) □

Now, we prove that any representation $\varphi_3 : B_n \to GL_n(\mathbb{C})$ of type 3 is equivalent to a complex specialization of the standard representation τ_n.

Theorem 3.2 *Let $\varphi_3 : B_n \to GL_n(\mathbb{C})$ be a homogeneous local representation of type 3. Then, the representation φ_3 is equivalent to a complex specialization the standard representation τ_n.*

Proof Let $\varphi_3 : B_n \to GL_n(\mathbb{C})$ be a homogeneous local representation of type 3. Consider the matrix P defined by

$$P = Diag\left(\frac{1}{c^{n-1}}, \frac{1}{c^{n-2}}, \ldots, \frac{1}{c}, 1\right),$$

where $Diag(a_1, a_2, \ldots, a_n)$ is a diagonal $n \times n$ matrix with $a_{ii} = a_i$.

Direct computations show that

$$P^{-1}\sigma_i P = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 0 & bc & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right).$$

By letting $u = bc$, we find that this representation is equivalent to a complex specialization of the standard representation τ_n.

By Lemmas 2.8 and 2.9, the standard representation is irreducible if and only if $u \neq 1$. This implies that φ_3 is irreducible if and only if $bc \neq 1$. □

Thus, we state the following theorem:

Theorem 3.3 *Let $\varphi_3 : B_n \to GL_n(\mathbb{C})$ be a homogeneous local representation of type 3. Then, the representation φ_3 is irreducible if and only if $bc \neq 1$.*

4 Irreducibility of local representations of B_3

In this section, we consider all the local representations of B_3. We prove that any local representation of type 1 of B_3 is reducible to a Burau type representation. Then, we prove that any local representation of type 2 of B_3 is equivalent to a complex specialization of the standard representation.

Theorem 4.1 *Let $\varphi : B_3 \to GL_3(\mathbb{C})$ be a local representation of type 1. Then, φ is reducible to a representation of Burau type.*

Proof Let $\varphi : B_3 \to GL_3(\mathbb{C})$ be a representation of type 1, then

$$\varphi(\sigma_1) = \left(\begin{array}{cc|c} \alpha(1-d) & \frac{(1-d)(1-\alpha+d\alpha)}{c} & 0 \\ c & d & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \varphi(\sigma_2) = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \alpha & \frac{(1-\alpha)(1-d+d\alpha)}{\gamma} \\ 0 & \gamma & d(1-\alpha) \end{array}\right),$$

where $d, \alpha \neq 1$ and $c, \gamma \neq 0$.

We scale the basis using the matrix

$$P = Diag\left(\frac{1}{c}, 1, \gamma\right).$$

Thus, we get

$$P^{-1}\varphi(\sigma_1)P = \begin{pmatrix} \alpha(1-d) & (1-d)(1-\alpha+\alpha d) & 0 \\ 1 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$P^{-1}\varphi(\sigma_2)P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & (1-\alpha)(1-d+\alpha d) \\ 0 & 1 & (1-\alpha)d \end{pmatrix}.$$

We have the following two cases:

Case 1. $\alpha(1-d) = 1$. In this case, we have

$$\varphi(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \varphi(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{1-d} & \frac{-d(1-d+d^2)}{(-1+d)^2} \\ 0 & 1 & \frac{d^2}{-1+d} \end{pmatrix}.$$

It is clear that the proper subspace $S =< e_2, e_3 >$ is invariant. Consequently, the above representation φ is reducible.

Restricting φ to S, we obtain:

$$\varphi'(\sigma_1) = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \varphi'(\sigma_2) = \begin{pmatrix} \frac{1}{1-d} & \frac{-d(1-d+d^2)}{(-1+d)^2} \\ 1 & \frac{d^2}{-1+d} \end{pmatrix}.$$

This formula is well defined since $d - 1 \neq 0$.

By direct computations, there is no proper invariant subspace of dimension one if $d \neq 0$ and $d^2 - d + 1 \neq 0$. Therefore, the representation φ' is irreducible. Consequently, it is a representation of Burau type.

Since the representation φ' is of Burau type, thus, by Theorem 3, it is equivalent to $\chi(y) \otimes \beta_3(z)$ for some $y, z \in \mathbb{C}^*$. This implies that φ' and $\chi(y) \otimes \beta_3(z)$ have the same eigenvalues which are $(1, d)$ and $(y, -yz)$ respectively. Thus, these representations are equivalent for

$$(y, z) = \left(d, -\frac{1}{d}\right) \text{ and } (y, z) = (1, -d).$$

On the other hand, if $d^2 - d + 1 = 0$, then the subspace $< (0, 1) >$ is invariant. This implies that the representation is reduced to a one dimensional representation.

Case 2. $\alpha(1 - d) \neq 1$.
Consider the subspace $S =< u, v >$ where

$$u = e_1 + \frac{1}{-1 + \alpha - \alpha d} e_2 \text{ and } v = e_1 + \frac{\alpha}{-1 + \alpha - \alpha d} e_2 + \frac{1}{-1 + \alpha - \alpha d} e_3.$$

By direct computations, we have

- $\varphi(\sigma_1)(u) = (-1 + \alpha + d - \alpha d)u$,
- $\varphi(\sigma_1)(v) = -u + v$,
- $\varphi(\sigma_2)(u) = v$ and
- $\varphi(\sigma_2)(v) = (-1 - \alpha - d + \alpha d)u + (\alpha + d - \alpha d)v$.

Thus, the proper subspace S is invariant. Therefore, the representation φ is reducible.

By restricting φ to S, we obtain the representation φ' defined as follows:

$$\varphi'(\sigma_1) = \begin{pmatrix} -1 + \alpha + d - \alpha d & -1 \\ 0 & 1 \end{pmatrix} \text{ and } \varphi'(\sigma_2) = \begin{pmatrix} 0 & 1 - \alpha - d + \alpha d \\ 1 & \alpha + d - \alpha d \end{pmatrix}.$$

Let $t = -1 + \alpha + d - \alpha d$. The representation φ' is reducible if and only if the matrices $\varphi'(\sigma_1)$ and $\varphi'(\sigma_2)$ have a common eigenvector. Direct computations show that the representation φ' is irreducible if and only if $t^3 \neq \pm 1$.
Therefore, any representation of type 1 is reduced to a Burau type representation (See Theorem 2.12).

Also, the representation φ' is of Burau type, thus, by Theorem 3, it is equivalent to $\chi(y) \otimes \beta_3(z)$ for some $y, z \in \mathbb{C}^*$. Using the same argument of case 1, these representations are equivalent for:

$$(y, z) = \left(-1 + \alpha + d - \alpha d, \frac{1}{1 - \alpha - d + \alpha d}\right) \text{ and } (y, z) = (1, 1 - \alpha - d + \alpha d).$$

□

Proposition 4.2 *Let $\varphi : B_3 \to GL_3(\mathbb{C})$ be a local representation of type 2 of B_3, then φ is equivalent to a complex specialization of the standard representation.*

Proof Let $\varphi : B_3 \to GL_3(\mathbb{C})$ be a local representation of type 2 of B_3. We scale the basis using the matrix

$$P = Diag\left(\frac{1}{c}, 1, \gamma\right).$$

Thus, we get

$$P^{-1}\varphi(\sigma_1)P = \begin{pmatrix} 0 & bc & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad and \quad P^{-1}\varphi(\sigma_2)P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & bc \\ 0 & 1 & 0 \end{pmatrix}.$$

By letting $u = bc$, we notice that the representation φ is equivalent to a complex specialization of the standard representation. □

Now, we state the following theorem:

Theorem 4.3 *Let $\varphi : B_3 \to GL_3(\mathbb{C})$ be a local representation of type 2 of B_3, then φ is irreducible if and only if $bc \neq 1$.*

5 Representations of dimension 6 of B_3

In this section, we study the irreducibility of the tensor product of two irreducible local representations of type 2 of B_3.

Consider two irreducible local representations $\rho_1 = \varphi(b_1, c_1, \gamma_1)$ and $\rho_2 = \varphi(b_2, c_2, \gamma_2)$ of type 2 of the braid group B_3.

These representations are defined as:

$$\rho_1(\sigma_1) = \left(\begin{array}{cc|c} 0 & b_1 & 0 \\ c_1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \rho_1(\sigma_2) = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & \frac{b_1c_1}{\gamma_1} \\ 0 & \gamma_1 & 0 \end{array}\right)$$

and

$$\rho_2(\sigma_1) = \left(\begin{array}{cc|c} 0 & b_2 & 0 \\ c_2 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \rho_2(\sigma_2) = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & \frac{b_2c_2}{\gamma_2} \\ 0 & \gamma_2 & 0 \end{array}\right),$$

where $b_1c_1 \neq 0$, $b_2c_2 \neq 0$, $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$.

By Proposition 4.2, the representations ρ_1 and ρ_2 are equivalent to the standard representations τ_1 and τ_2 defined by:

$$\tau_1(\sigma_1) = \begin{pmatrix} 0 & u_1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau_1(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & u_1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$\tau_2(\sigma_1) = \begin{pmatrix} 0 & u_2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau_2(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & u_2 \\ 0 & 1 & 0 \end{pmatrix}.$$

where $u_1 = b_1c_1$, $u_2 = b_2c_2$, $u_1 \neq 1$ and $u_2 \neq 1$.

Definition 5.1 Consider the tensor product $\tau_1 \otimes \tau_2$ defined by $(\tau_1 \otimes \tau_2)(\sigma_i) = \tau_1(\sigma_i) \otimes \tau_2(\sigma_i)$, $i = 1, 2$. We get the following matrices:

$$(\tau_1 \otimes \tau_2)(\sigma_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & u_1u_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_1 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$(\tau_1 \otimes \tau_2)(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_1u_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For simplicity, we denote $(\tau_1 \otimes \tau_2)$ by ρ.

We now show that the representation ρ is reducible.

Proposition 5.2 *The representation ρ is reducible.*

Proof By choosing a different basis for $\mathbb{C}^9$, namely $\{e_1, e_5, e_9, e_2, e_3, e_4, e_6, e_7, e_8\}$, the representation ρ is equivalent to the representation ψ whose matrices are given by:

$$\psi(\sigma_1) = \begin{pmatrix} 0 & u_1u_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & u_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\psi(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_1u_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_2 & 0 & 0 \end{pmatrix}.$$

It is clear from the form of the martices of the generators σ_1 and σ_2 that the representation ψ is a direct sum of a standard representation and a representation φ of dimension 6. □

Definition 5.3 We define the representation $\varphi : B_3 \to GL_6(\mathbb{C})$ of B_3 of dimension 6 by:

$$\varphi(\sigma_1) = \begin{pmatrix} 0 & 0 & u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 \\ u_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\varphi(\sigma_2) = \begin{pmatrix} 0 & u_2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_2 & 0 & 0 \end{pmatrix}.$$

We diagonalize the matrix corresponding to $\varphi(\sigma_1)$ by an invertible matrix, say T, and conjugate the matrix $\varphi(\sigma_2)$ by the same matrix T.

The invertible matrix T is given by

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{\sqrt{u_1}}{\sqrt{u_2}} & \frac{\sqrt{u_1}}{\sqrt{u_2}} \\ -\sqrt{u_1} & \sqrt{u_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{u_2} & \sqrt{u_2} & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

In fact, computations show that

$$T^{-1}\sigma_1 T = \begin{pmatrix} -\sqrt{u_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{u_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{u_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{u_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{u_1}\sqrt{u_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{u_1}\sqrt{u_2} \end{pmatrix}.$$

After conjugation, we get

$$T^{-1}\sigma_2 T = \begin{pmatrix} 0 & 0 & \frac{u_1}{2} & \frac{u_1}{2} & \frac{1}{2\sqrt{u_2}} & -\frac{1}{2\sqrt{u_2}} \\ 0 & 0 & \frac{u_1}{2} & \frac{u_1}{2} & -\frac{1}{2\sqrt{u_2}} & \frac{1}{2\sqrt{u_2}} \\ \frac{u_2}{2} & \frac{u_2}{2} & 0 & 0 & -\frac{1}{2\sqrt{u_2}} & -\frac{1}{2\sqrt{u_2}} \\ \frac{u_2}{2} & \frac{u_2}{2} & 0 & 0 & \frac{1}{2\sqrt{u_2}} & \frac{1}{2\sqrt{u_2}} \\ \frac{u_2\sqrt{u_2}}{2} & -\frac{u_2\sqrt{u_2}}{2} & -\frac{u_1\sqrt{u_2}}{2} & \frac{u_1\sqrt{u_2}}{2} & 0 & 0 \\ -\frac{u_2\sqrt{u_2}}{2} & \frac{u_2\sqrt{u_2}}{2} & -\frac{u_1\sqrt{u_2}}{2} & \frac{u_1\sqrt{u_2}}{2} & 0 & 0 \end{pmatrix}.$$

For simplicity, we denote $T^{-1}\sigma_1 T$ by σ_1 and $T^{-1}\sigma_2 T$ by σ_2.

Assume that u_i, $i = 1, 2$, are non-zero complex numbers on the unit circle. We determine a sufficient condition for the irreducibility of the representation φ of B_3 of dimension 6.

Lemma 5.4 *Let u_i, $i = 1, 2$, be non-zero complex numbers on the unit circle. The representation $\varphi : B_3 \to GL_6(\mathbb{C})$ is irreducible if $\sqrt{u_1} \neq \pm\sqrt{u_2}$.*

Proof Direct computations show that $\sigma_i \sigma_i^* = I_6$, where $i = 1, 2$, * denotes the complex conjugate transpose, and I_6 denotes the 6×6 identity matrix.

Therefore, the representation is unitary. Consequently, if S is an invariant subspace then the orthogonal complement of S is also invariant.

Thus, it is sufficient to prove that there is no possible proper invariant subspace of dimensions 1,2,3.

Since the representations ρ_1 and ρ_2 are irreducible, then $u_1 \neq 1$ and $u_2 \neq 1$ (Lemmas 1 and 2). This implies that $\sqrt{u_1}\sqrt{u_2} \neq \pm\sqrt{u_1}$ and $\sqrt{u_1}\sqrt{u_2} \neq \pm\sqrt{u_2}$.

Let S be an invariant subspace of dimension ≤ 3.

We have the following cases:

Case 1. $S =< e_i >, i = 1, ..., 6$.
Case 2. $S =< e_i, e_j >, i, j = 1, ..., 6, i \neq j$.
Case 3. $S =< e_i, e_j, e_k >, i, j, k = 1, ..., 6, i \neq j \neq k$.

In all the above cases, it is clear that $e_i \in S$ for some $1 \leq i \leq 6$.
As S is invariant, this implies that $\sigma_2(e_i) \in S$.

On the other hand, by direct computations, we have the following:

- $\sigma_2(e_1) = \frac{u_2}{2}(e_3 + e_4) + \frac{u_2\sqrt{u_2}}{2}(e_5 - e_6)$
- $\sigma_2(e_2) = \frac{u_2}{2}(e_3 + e_4) + \frac{u_2\sqrt{u_2}}{2}(-e_5 + e_6)$
- $\sigma_2(e_3) = \frac{u_1}{2}(e_1 + e_2) - \frac{u_1\sqrt{u_2}}{2}(e_5 + e_6)$
- $\sigma_2(e_4) = \frac{u_1}{2}(e_1 + e_2) + \frac{u_1\sqrt{u_2}}{2}(e_5 + e_6)$
- $\sigma_2(e_5) = \frac{1}{2\sqrt{u_2}}(e_1 - e_2 - e_3 + e_4)$
- $\sigma_2(e_6) = \frac{1}{2\sqrt{u_2}}(-e_1 + e_2 - e_3 + e_4)$

In all the above cases, $\sigma_2(e_i) \notin S$ for all $1 \leq i \leq 6$. This gives a contradiction.

Therefore, there is no possible invariant subspace of dimension ≤ 3.

Thus, the representation $\varphi : B_3 \to GL_6(\mathbb{C})$ is irreducible if $\sqrt{u_1} \neq \pm\sqrt{u_2}$. □

Now, we determine a necessary condition for irreducibility of the representation $\varphi : B_3 \to GL_6(\mathbb{C})$.

Lemma 5.5 *Let u_i, $i = 1, 2$, be non-zero complex numbers on the unit circle. If $\sqrt{u_1} = \pm\sqrt{u_2}$, then the representation $\varphi : B_3 \to GL_6(\mathbb{C})$ is reducible.*

Proof We consider the following cases:

Case 1: $\sqrt{u_1} = \sqrt{u_2}$.

Let $S = < e_5, u, v >$ where $u = e_1 - e_3$ and $v = e_2 - e_4$.

Direct computations show that:

- $\sigma_1(e_5) = -u_1 e_5$
- $\sigma_2(e_5) = \frac{1}{2\sqrt{u_1}}(u - v)$
- $\sigma_1(u) = -\sqrt{u_1}u$
- $\sigma_2(u) = -\frac{u_1}{2}u - \frac{u_1}{2}v + u_1\sqrt{u_1}e_5$
- $\sigma_1(v) = \sqrt{u_1}v$
- $\sigma_2(v) = -\frac{u_1}{2}u - \frac{u_1}{2}v - u_1\sqrt{u_1}e_5$

Thus, the subspace S is invariant.

Case 2: $\sqrt{u_1} = -\sqrt{u_2}$

Let $S = < e_5, u, v >$ where $u = e_1 + e_4$ and $v = e_2 + e_3$.

Direct computations show that:

- $\sigma_1(e_5) = u_1 e_5$
- $\sigma_2(e_5) = -\frac{1}{2\sqrt{u_1}}(u - v)$
- $\sigma_1(u) = -\sqrt{u_1}u$
- $\sigma_2(u) = \frac{u_1}{2}u + \frac{u_1}{2}v - u_1\sqrt{u_1}e_5$
- $\sigma_1(v) = \sqrt{u_1}v$
- $\sigma_2(v) = \frac{u_1}{2}u + \frac{u_1}{2}v + u_1\sqrt{u_1}e_5$

Thus, the subspace S is invariant.

Therefore, the representation $\varphi : B_3 \to GL_6(\mathbb{C})$ is reducible if $\sqrt{u_1} = \pm\sqrt{u_2}$. □

We state now the theorem of irreducibility of the considered representation.

Theorem 5.6 *Let u_i, $i = 1, 2$, be non-zero complex numbers on the unit circle. The representation $\varphi : B_3 \to GL_6(\mathbb{C})$ is irreducible if and only if $\sqrt{u_1} \neq \pm\sqrt{u_2}$.*

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References

1. Artin, E.: The Collected Papers of Emil Artin. Addison-Wesley Publishing Company Inc., Reading (1965)
2. Birman, J.: Braids, Links and Mapping Class Groups, vol. 82. Annals of Mathematical Studies. Princeton University Press, Princeton (1975)
3. Chreif, M.; Abdulrahim, M.: Attacks on the faithfulness of the burau representation of the braid group B_4. J. Math. Res. **8**(1), 5 (2016)
4. Formanek, E.: Braid group representations of low degree. Proc. Lond. Math. Soc. **73**, 279–322 (1996)
5. Formanek, E.; Lee, W.; Sysoeva, I.; Vazirani, M.: The irreducible complex representations of the braid group on n strings of degree ≤n. J. Algebra Appl. **02**(03), 317–333 (2003)
6. Mikhalchishina, Y.: Local representations of braid groups. Sib. Math. J. **54**(4), 666–678 (2013)
7. Sysoeva, I.: Dimension n representations of the braid group on n strings. J. Algebra **243**, 518–538 (2001)
8. Sysoeva, I.: Irreducible representations of braid group B_n of dimension $n + 1$. J. Group Theory **24**(1), 39–78 (2021)