Arab. J. Math.
https://doi.org/10.1007/s40065-024-00460-5

Arabian Journal of Mathematics

Aykut Has · Beyhan Yilmaz

$\mathcal{C}_\alpha-$helices and $\mathcal{C}_\alpha-$ slant helices in fractional differential geometry

Received: 23 June 2023 / Accepted: 22 March 2024

Abstract In this study, the theory of curves is reconstructed with fractional calculus. The condition of a naturally parametrized curve is described, and the orthonormal conformable frame of the naturally parametrized curve at any point is defined. Conformable helix and conformable slant helix curves are defined with the help of conformable frame elements at any point of the conformable curve. The characterizations of these curves are obtained in parallel with the conformable analysis Finally, examples are given for a better understanding of the theories and their drawings are given with the help of Mathematics.

Mathematics Subject Classification 53A04 · 26A33

1 Introduction

The theory of curves has been one of the most important and most curious fields of differential geometry from past to present. Carl Friedrich Gauss and Gaspard Monge have done very important study on curves and surfaces. In this respect, they can be considered the founders of differential geometry. Afterwards, thanks to numerous studies, the subject of curves has shown incredible developments. Most of the studies on this subject are directed towards characterizing curves. The best way to characterize a curve is to use relations between Frenet elements. There are two known methods for this. The first is to characterize the curve using the relations between Frenet vectors. The best examples of this are curve pairs. As is known, curve pairs are characterized by the states of Frenet vectors at opposite points of the curves. Bertrand curve pair, Mannheimm curve pair and involute–evolute curve pair are the most well-known examples [5,6,10,33]. A second way to characterize a curve is to make use of the relations between its curvatures. General helices and slant helices are the best examples of curves characterized in this way. General helices characterized by M. A. Lancret in 1806 and first proved by B. de Saint Venant in 1845 as follows [22]:

$$\frac{\tau}{\kappa} = c,\ \ c \in \mathbb{R}. \tag{1}$$

Similarly, slant helices curves characterized by S. Izumiya and N. Takeuchi in 2004 as follows [19]:

$$\left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right) = \text{constant}. \tag{2}$$

Fractional calculus is the application of the classical derivative and integral concepts, which are studied in detail by many researchers today is a generalization. What is meant by fractional derivative is actually any order

A. Has (✉)· B. Yilmaz
Department of Mathematics, Faculty of Science, Kahramanmaras Sutcu Imam University, Kahramanmaras 46100, Turkey
E-mail: ahas@ksu.edu.tr
E-mail: beyhanyilmaz@ksu.edu.tr

Published online: 15 April 2024

derivative. The concepts of fractional derivative and integral are as old as the concepts of integer derivative and integral, and the term fractional derivative mentioned for the first time in Leibniz's letter to L'Hospital in 1695, as stated in many sources. Leibniz addressed in his letter to L'Hospital "Can integer decimal derivatives be extended to fractional decimal derivatives?" question can be shown as the first emergence of the concept of fractional differential. The concept of fractional calculus is attracted the attention of many mathematicians and is found a wide field of study. Fractional calculus, which is claimed to give more numerical results especially in the solutions of differential equations, is become an indispensable cornerstone of almost every subject in the field of basic sciences and engineering [1,8,24,32]. The fact that the subject of fractional calculus is extremely popular is led to the emergence of the definition of fractional calculus with different features by many researchers. Some of these are Riemann–Liouville (R–L), Caputo, Grünwald–Letnikov, Wely, Riesz fractional derivatives [25–27]. As each fractional calculus has a common feature, it also has its own unique rules. For example, none of the non-local fractional derivative types satisfy the classical Leibniz and chain rule. In addition, the derivative of the constant is not zero in any of the non-local fractional derivative except for the Caputo fractional derivative [7]. On the other hand, local fractional derivatives such as Conformable, Alternative, M-fractional and V-fractional are known to satisfy Leibniz and the chain rule. Therefore, local fractional derivatives provide an advantage in this respect in algebraically constructed subjects [20,21,28,29].

The theory of curves can be described as the study of the motion of a point in a plane or space using the techniques of linear algebra and calculus. Considering the adventure of the literature in the last ten years, it is observed that fractional calculus is started to be used for curves and surface in differential geometry. T. Yajima and K. Kamasaki are made the first study on this subject by examining surfaces with fractional calculus [34]. Later, T. Yajima et al. are obtained Frenet formulas using fractional derivatives [35]. In another study, K.A. Lazopoulos and A.K. Lazopoulos are studied fractional differentiable manifolds [23]. M.E Aydın et al. are studied plane curves in equiaffine geometry in fractional order [2]. U. Gozutok et al. are analyzed the basic concepts of curves and Frenet frame in fractional order with the help of conformable local fractional derivative [11]. On the other hand, A. Has and B. Yılmaz have investigated some special curves and curve pairs in fractional order with the help of conformable Frenet frame [13,14]. In addition, electromagnetic fields and magnetic curves are investigated under fractional derivative by A. Has and B. Yılmaz [15,36,37]. Additionally, studies on this subject are ongoing [3,4,9,17,18,31].

In this study, algebraic and calculus based properties of curves are reconstructed with the help of conformable local fractional derivative. First of all, the most basic concepts of geometry, line, plane and sphere, are redefined in fractional order. Then, the concepts of unit and orthogonality, which are the algebraic basis of curves, are defined in fractional order. Then, the $\alpha-$conformable frame of the $\alpha-$conformable naturally parameterized curve is defined. Throughout this study, definitions based on conformable analysis is denoted by $\mathcal{C}_\alpha$. For example, $\mathcal{C}_\alpha-$frame, $\mathcal{C}_\alpha-$naturally parameterized curve, etc. It should be noted here that the $\mathcal{C}_\alpha-$frame defined in this study is different from the frame discussed in the study [11]. The $\mathcal{C}_\alpha-$frame mentioned in this article is fully defined by the conformable local fractional derivative of the vectors and gives different results than the classical Frenet frame. In addition, general helices and slant helices, which are the most important concepts of the theory of curves, are discussed again in fractional order. The effects of conformable calculus on these curves are investigated and necessary characterizations are given. Finally, the concepts obtained from fractional order are given with examples and their graphs are drawn.

2 Preliminaries

2.1 Basics parametrized curves

A regular naturally parametrization of class C^k, with $k \geq 1$ of a curve in $\mathbb{R}^3$ is a vector function $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$, $s \mapsto \mathbf{x}(s) = (\mathbf{x}_1(s), \mathbf{x}_2(s), \mathbf{x}_3(s))$ defined on an interval I which satisfies x is of class C^k and $\mathbf{x}'(s) \neq 0$ for all $s \in I$. A curve $\mathbf{x}$ is continuously differentiable if $\mathbf{x}'(s)$ exists for all $s \in I$ and the derivative $\mathbf{x}'(s)$ is a continuous function; thinking dynamically, the vector $\mathbf{x}'(s)$ is the velocity of the curve at time s. We call $\mathbf{x}(s)$ *naturally parametrized curve* if $\mathbf{x}_i(s)$ $(i = 1, 2, 3)$ is of class C^k and $\|\mathbf{x}'(s)\| = 1$, for each $s \in I$ [30].

Let $\mathbf{x}(s)$ be *biregular*, that is, $\mathbf{x}'(s) \times \mathbf{x}''(s) \neq 0$, for each $s \in I$. We consider a trihedron $\{T(s), N(s), B(s)\}$ along $\mathbf{x}(s)$, so-called *Frenet frame*, where [30]

$$T(s) = \mathbf{x}'(s), \quad N(s) = \frac{T'(s)}{\|T'(s)\|}, \quad B(s) = T(s) \times N(s).$$

The curvature κ, a non-negative scalar field, is defined by setting $\kappa(s) = \|T'(s)\|$ and torsion is defined by setting $\tau(s) = \langle N'(s), B(s)\rangle$. The naturally parametrized curve **x** has unit speed and strictly positive curvature then the following equations hold [30]:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{3}$$

2.2 Basics in conformable fractional calculus

Given $s \mapsto x(s) \in \mathbb{E}^3$, $s \in I \subset \mathbb{R}$, the *conformable derivative* of x at s is defined by [21]

$$D_\alpha(x)(s) = \lim_{\varepsilon \to 0} \frac{x(s + \varepsilon s^{1-\alpha}) - x(s)}{\varepsilon}.$$

Let $Dx(s) = dx(s)/ds$. We then notice

$$D_\alpha x(s) = s^{1-\alpha} dx(s)/ds.$$

Denote by $D_\alpha x(s)$ the αth order conformable derivative of $x(s)$ for each $s > 0$, $0 < \alpha < 1$.

It can be said that the conformable derivative provides some properties such as linearity, Leibniz rule and chain rule as in the classical derivative as follows:

1. $D_\alpha(ax + by)(s) = aD_\alpha(x)(s) + bD_\alpha(y)(s)$, for all $a, b \in \mathbb{R}$,
2. $D_\alpha(s^p) = ps^{p-\alpha}$ for all $p \in \mathbb{R}$,
3. $D_\alpha(\lambda) = 0$, for all constant functions $x(s) = \lambda$,
4. $D_\alpha(xy)(s) = x(s)D_\alpha y(s) + y(s)D_\alpha x(s)$,
5. $D_\alpha(\frac{x}{y})(s) = \frac{x(s)D_\alpha y(s) - y(s)D_\alpha x(s)}{y^2(s)}$,
6. $D_\alpha(y \circ x)(s) = x(s)^{\alpha-1} D_\alpha x(s) D_\alpha y(x(s))$,

where x, y be α-differentiable for each $s > 0$ and $0 < \alpha < 1$ [21].

The definition of the conformable integral is given as the inverse operator of the conformable derivative. The *conformable integral* of the function $x(s)$ is defined by [21]

$$I_\alpha^a f(t) = I_1^a(t^{\alpha-1} f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx.$$

The effect of conformable analysis on vector-valued functions is investigated, and the limit and derivative of these functions also are investigated. In the following theorem, the conformable fractional derivative of vector-valued functions is given.

Theorem 2.1 *Let x be a vector-valued function with n variables, and let x be a vector-valued function $x(s_1, \ldots, s_n) = (x_1(s_1, \ldots, s_n), \ldots, x_m(s_1, \ldots, s_n))$. So x is $\alpha-$differentiable at $t = (t_1, \ldots, t_n) \in \mathbb{R}$, for all $t_i > 0$ if and only if each x_i is, and* [12]

$$D_\alpha x(t) = (D_\alpha x_1(t), \ldots, D_\alpha x_m(t)).$$

3 Some concepts of fractional differential geometry

In this section, some vector operations and parameterized curves will be reconstructed with conformable calculus. First of all, let's define the concepts of conformable angle and conformable perpendicularity, which are the most important concepts of geometry, with the help of conformable calculus as follows.
The geometric interpretation of the conformable derivative draws inspiration from the principles of fractal geometry, where objects exhibit self-similarity across varying scales. In the realm of fractal geometry, this self-similarity manifests as patterns repeating themselves at different magnifications. Similarly, the conformable derivative captures this notion of self-similarity within functions by examining their local fractional variations. From a geometric standpoint, envision the conformable derivative as akin to zooming in on a function at a

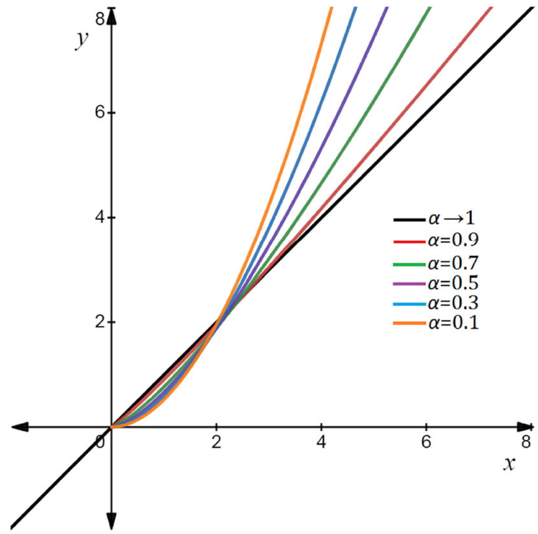

Fig. 1 Transformation from line to curve

specific point. This zooming process reveals intricate details and nuances, much like how the classical derivative unveils local linear behavior. Essentially, the conformable derivative allows us to scrutinize the function's behavior at different levels of detail and resolution, mirroring the self-similarity and scaling properties inherent in fractal objects. Moreover, we can conceptualize the conformable derivative as a metric of curvature-the extent to which a straight line or plane deviates to form a curve or surface. Figure 5 visually demonstrates this effect of the conformable calculus, illustrating how a line bends under its influence, offering a tangible representation of the concept's geometric implications.

Example 3.1 Let consider the $s \mapsto \mathbf{x}(s) = (s, \int s^{1-\alpha} ds)$, $\mathcal{C}_\alpha-$line passing through the point $P = (0, 0)$ and whose direction is $v = (s^{1-\alpha}, s^{1-\alpha})$.
Figure 1 shows the graph of the conformable line for various α values.

As seen in Fig. 1, there is no classical line in the $\mathcal{C}_\alpha-$ (fractional) system. This is only achieved when $\alpha \to 1$. Accordingly, it requires a new concept of angle in $\mathcal{C}_\alpha-$ space. This angle is called the $\mathcal{C}_\alpha-$ angle, which gives the angle between two $\mathcal{C}_\alpha-$ lines. In addition, the concept of orthogonality in this $\mathcal{C}_\alpha-$ space is different from the classical one. Because we cannot talk about classical directness, we cannot talk about steepness in the classical sense. We will explain this below.

Notation: Along the study, expressions that are equal to 1 when $\alpha \to 1$ will it denoted as $\mathbf{1}_\alpha$, and expressions that are equal to 0 when $\alpha \to 1$ will be denoted as $\mathbf{0}_\alpha$.

Suppose that $\mathbf{x}$ and $\mathbf{y}$ are $\mathcal{C}_\alpha-$unit vector, that is, they are vectors of the form $\|\mathbf{x}\| = \mathbf{1}_\alpha$ and $\|\mathbf{y}\| = \mathbf{1}_\alpha$. Then, the $\alpha-$*conformable radian measure* of $\mathcal{C}_\alpha-$*angle* between $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\theta_\alpha = \arccos\left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|}\right).$$

In this sense,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{0}_\alpha$$

when $\mathbf{x}$ and $\mathbf{y}$ are $\mathcal{C}_\alpha-$orthogonal. For example, the $\mathcal{C}_\alpha-$ vectors $u = (s^{1-\alpha}, 1-\alpha, \frac{1}{s^{1-\alpha}})$ and $v = (\frac{1-\alpha}{s^\alpha}, s^\alpha, 2-2\alpha)$ are orthogonal to each other in the $\mathcal{C}_\alpha-$ sense, and we present this in Fig. 2.

In addition, vectors u, v and $u \times v$ form the fractional orthogonal system. For example, if $u = (s^{1-\alpha}, 1-\alpha, \frac{1}{s^{1-\alpha}})$ and $v = (\frac{1-\alpha}{s^\alpha}, s^\alpha, 2-2\alpha)$, it becomes $u \times v = (2\alpha^2 - 4\alpha - s^{2\alpha-1} + 2, 2\alpha s^{1-\alpha} - 2 s^{1-\alpha} - \frac{\alpha}{s} + \frac{1}{s}, -\alpha^2 s^{-\alpha} + 2\alpha s^{-\alpha} - s^{-\alpha} + s)$. The fractional orthogonal system is shown in Fig. 3. Also fractional alternative frame definitions defined on the $\mathcal{C}_\alpha-$ parametrized curves will be given with the help of conformable fractional derivative [16].

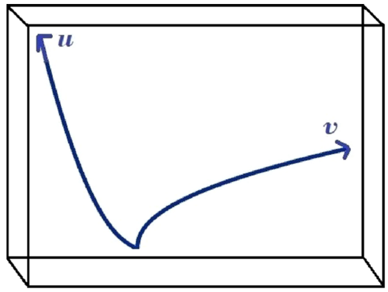

Fig. 2 $\mathcal{C}_\alpha-$orthogonal vectors

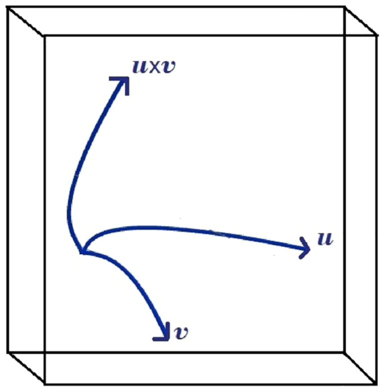

Fig. 3 $\mathcal{C}_\alpha-$orthogonal system

Definition 3.2 Let $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$ be a vector-valued function where $s \mapsto \mathbf{x}(s) = (\mathbf{x}_1(s), \mathbf{x}_2(s), \mathbf{x}_3(s))$. Then, $D_\alpha \mathbf{x}(s) = (D_\alpha \mathbf{x}_1(s), D_\alpha \mathbf{x}_2(s), D_\alpha \mathbf{x}_3(s))$. We call $\mathbf{x}(s)$ $\mathcal{C}_\alpha-$*naturally parametrized curve* if $\mathbf{x}_i(s)$ $(i = 1, 2, 3)$ is of class C^α and $\|D_\alpha \mathbf{x}(s)\| = s^{1-\alpha}$, for each $s \in I$. Here n is the maximum order that we will need.

In the remaining part, unless otherwise specified, we will assume that $\mathbf{x}(s)$ in $\mathbb{E}^3$ is a $\mathcal{C}_\alpha-$naturally parametrized curve. Let $\mathbf{x}(s)$ be $\mathcal{C}_\alpha-$*biregular*, that is, $D_\alpha \mathbf{x}(s) \times D_\alpha^2 \mathbf{x}(s) \neq 0_\alpha$, for each $s \in I$. We consider a trihedron $\{E_1(s), E_2(s), E_3(s)\}$ along $\mathbf{x}(s)$, so-called $\mathcal{C}_\alpha-$*frame*, where

$$E_1(s) = D_\alpha \mathbf{x}(s), \quad E_2(s) = \frac{D_\alpha E_1(s)}{\|D_\alpha E_1(s)\|}, \quad E_3(s) = E_1(s) \times E_2(s), \tag{4}$$

where $\{E_1(s), E_2(s), E_3(s)\}$ trihedron is called $\mathcal{C}_\alpha-$tangent, $\mathcal{C}_\alpha-$principal normal and the $\mathcal{C}_\alpha-$binormal of the $\mathcal{C}_\alpha-$curve $\mathbf{x}$, respectively. Also, considering Eq. (4) $\mathcal{C}_\alpha-$tangent, $\mathcal{C}_\alpha-$principal normal and $\mathcal{C}_\alpha-$binormal of the $\mathcal{C}_\alpha-$curve $\mathbf{x}$, they different from the Frenet vectors by the effect of the conformable calculus. However, these vectors turn into Frenet vectors, respectively, in case $\alpha \to 1$. In addition, the set $\{E_1(s), E_2(s), E_3(s)\}$ is mutually $\mathcal{C}_\alpha-$orthogonal and $\mathcal{C}_\alpha-$unit speed vectors.

We call $\kappa_\alpha(s) = \|D_\alpha E_1(s)\|$ $\mathcal{C}_\alpha-$*curvature* and $\tau_\alpha(s) = \langle D_\alpha E_2(s),\ E_3(s)\rangle$ $\mathcal{C}_\alpha-$*torsion*. The $\mathcal{C}_\alpha-$*frame formulae* are now [16]

$$\begin{bmatrix} D_\alpha E_1 \\ D_\alpha E_2 \\ D_\alpha E_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha & 0 \\ -\kappa_\alpha & 0 & \tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}. \tag{5}$$

Conclusion 3.3 (What is the advantage of $\mathcal{C}_\alpha-$ frame?) *In noticeable that a function is α-differentiable at a point where it is not classically differentiable. For example, consider the function $f(t) = 2\sqrt{t}$. Here $f'(0)$ does not exist. However, the result $D_{\frac{1}{2}} f(0) = 1$ can be easily reached. As can be seen from this example, the function f does not have a tangent at the 0 point in the classical sense, but an approximation to this tangent can be obtained with the help of the conformable fractional derivative. As it is known, the most important vector element of the Frenet frame is the tangent of the curve. Therefore, a Frenet frame of the curve cannot be mentioned at a point where the tangent of the curve does not exist. However, the $\mathcal{C}_\alpha-$frame eliminates this problem. At the points in the $\mathcal{C}_\alpha-$frame where the tangent of the curve does not exist, fractional values*

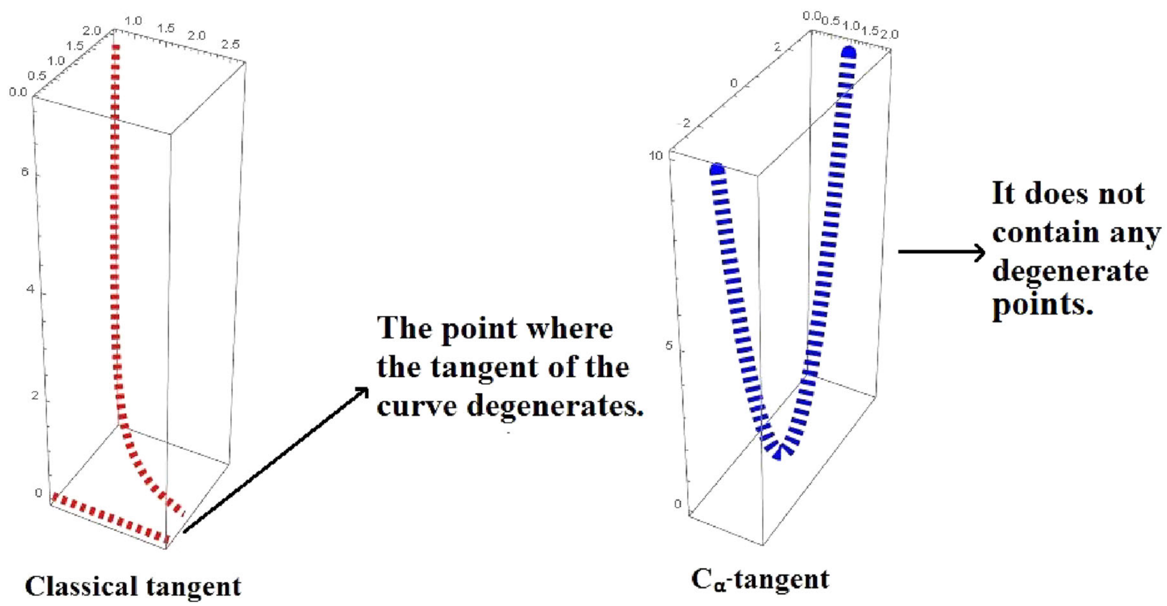

Fig. 4 Classical tangent and $\mathcal{C}_\alpha-$tangent of the curve $\mathbf{x}(s)$

are assigned to provide an approximation to the tangent at that point. Also, when $\alpha = 1$*, the* $\mathcal{C}_\alpha-$*frame is equivalent to the Frenet frame. In that case, the* $\mathcal{C}_\alpha-$*frame both includes the classical Frenet frame and provides advantages to the researchers at points not defined by the Frenet frame.*

Example 3.4 Let $\mathbf{x} : I \subset \mathbb{R} \to E^3$ be a $\mathcal{C}_\alpha-$naturally parametrized curve in $\mathbb{R}^3$ parameterized by

$$\mathbf{x}(s) = \left(2s^{\frac{1}{2}},\ s^{\frac{3}{2}},\ s^{\frac{5}{2}}\right).$$

The tangent of the curve $\mathbf{x}$ obtained with the classical derivative and the tangent obtained with the conformal fractional derivative for $\alpha = \frac{1}{2}$ and $-2 < s < 2$ are as follows, respectively.

$$T = \left(s^{\frac{-1}{2}}, \frac{3}{2}s^{\frac{1}{2}}, \frac{5}{2}s^{\frac{3}{2}}\right),$$
$$T_{\frac{1}{2}} = \left(1, \frac{3}{2}s, \frac{5}{2}s^2\right),$$

where T and $T_{\frac{1}{2}}$ are the classical tangent and $\mathcal{C}_\alpha-$tangent of the curve $\mathbf{x}(s)$, respectively. In Fig. (4) we present the graph classical tangent and $\mathcal{C}_\alpha-$tangent of the curve $\mathbf{x}(s)$.

On the other hand, if a curve does not have a tangent at any point, then there does not exist Frenet frame. As can be seen in Fig. (5), the Frenet frame is not present at $s = 0$, while the $\mathcal{C}_\alpha-$frame is present at this point.

Theorem 3.5 *Let* $\mathbf{x} = \mathbf{x}(s)$ *be* $\mathcal{C}_\alpha-$*naturally parametrized curve in the Euclidean* $3-$*space where s measures its* $\mathcal{C}_\alpha-$*arc length. When* $\alpha \to 1$*, as follows* [16]

$$\kappa_\alpha = s^{1-\alpha}\sqrt{(1-\alpha)^2 s^{-2\alpha} + s^{2-2\alpha}\kappa^2} \tag{6}$$

and

$$\tau_\alpha = \frac{s^{5-5\alpha}\kappa^2}{\kappa_\alpha^2}\tau. \tag{7}$$

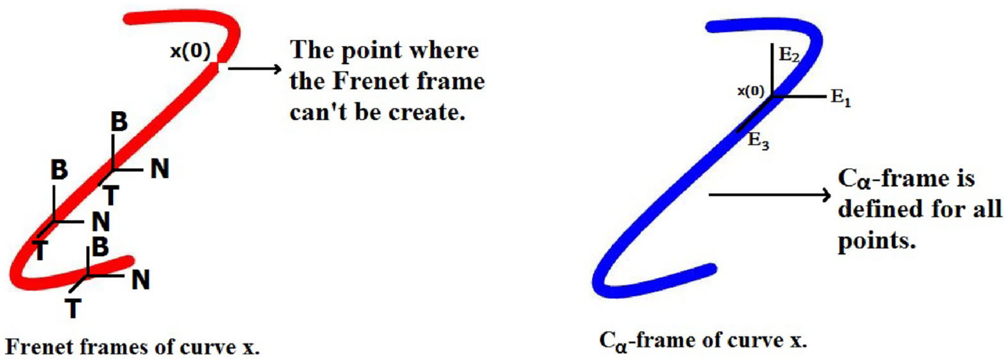

Fig. 5 Frenet frame and $\mathcal{C}_\alpha-$frame of the curve $\mathbf{x}(s)$ at $s = 0$

4 $\mathcal{C}_\alpha-$Helices and $\mathcal{C}_\alpha-$Slant Helices

Definition 4.1 Let us $\mathcal{C}_\alpha-$naturally parametrized curve $s \mapsto \mathbf{x}(s)$ in $\mathbb{E}^3$, $s \in I \subset \mathbb{R}$, where $\{E_1(s), E_2(s), E_3(s)\}$ denotes the $\mathcal{C}_\alpha-$frame. Suppose that $\mathcal{C}_\alpha-$biregular. We call the curve $\mathbf{x}(s)$ $\mathcal{C}_\alpha-$*helix*, if the $\mathcal{C}_\alpha-$tangent of the curve makes a constant angle with a constant direction $U \in \mathbb{R}^3$. In this context, it can be shown by the following equation:

$$\langle E_1, U\rangle = \cos\theta, \tag{8}$$

where θ is a constant angle.

Theorem 4.2 (The Conformable Lancret Theorem) *Let's $\mathbf{x}(s)$ be $\mathcal{C}_\alpha-$naturally parametrized curve. The $\mathcal{C}_\alpha-$curve $\mathbf{x}(s)$ with the $\mathcal{C}_\alpha-$curvature $\kappa_\alpha > 0$ is a $\mathcal{C}_\alpha-$general helix if and only if the ratio of its $\mathcal{C}_\alpha-$torsion and its $\mathcal{C}_\alpha-$ curvature is a constant.*

Proof Without loss of generality, suppose that $\mathcal{C}_\alpha-$naturally parametrized curve $s \mapsto \mathbf{x}(s)$. Taking a conformable differentiation of Eq. (8) together with considering $\mathcal{C}_\alpha-$frame formulae, we have

$$\langle D_\alpha E_1, U\rangle = 0$$

and

$$\kappa_\alpha\langle E_2, U\rangle = 0.$$

Since $\kappa_\alpha > 0$, we find

$$\langle E_2, U\rangle = 0. \tag{9}$$

If taking conformable derivative of above equation is taken together with considering $\mathcal{C}_\alpha-$frame formulae, we obtain

$$-\kappa_\alpha\langle E_1, U\rangle + \tau_\alpha\langle E_3, U\rangle = 0$$

whereupon

$$\langle E_3,\ U\rangle = \frac{\kappa_\alpha}{\tau_\alpha}\cos\theta.$$

If conformable fractional derivative of this expression is taken according to s again and considering Eq. (9), we get

$$D_\alpha\left(\frac{\kappa_\alpha}{\tau_\alpha}\right) = -\frac{\tau_\alpha\,\langle E_2,\ U\rangle}{\cos\theta}$$

and finally from this equation, we have

$$D_\alpha\left(\frac{\kappa_\alpha}{\tau_\alpha}\right) = 0$$

or in this context

$$\frac{\kappa_\alpha}{\tau_\alpha} = c,\ c \in \mathbb{R}.$$

□

Proposition 4.3 *Let the $\mathcal{C}_\alpha-$naturally parameterized curve $s \mapsto \mathbf{x}(s)$ be a helix. From this situation, the constant $U-$direction, which makes a constant angle with the $\mathcal{C}_\alpha-$tangent of the helix curve $\mathbf{x}$, as follows:*

$$U = \frac{cE_1 + E_3}{\sqrt{1+c^2}},$$

where $c \in \mathbb{R}$.

Proof From Eq. (9), we have

$$\langle E_2, U \rangle = 0.$$

It can be easily seen that E_2 and U are perpendicular. Therefore, we can say that the constant $U-$direction is in the plane spanned by the vectors E_1 and E_2. So the constant $U-$direction can be expressed as

$$U = \cos\theta E_1 + \sin\theta E_3. \tag{10}$$

Taking the conformable differentiation of this equation and considering $\mathcal{C}_\alpha-$frame formulae, we get

$$D_\alpha U = (\kappa_\alpha \cos\theta - \tau_\alpha \sin\theta) E_2.$$

From the last equation, we obtain

$$\frac{\tau_\alpha}{\kappa_\alpha} = \cot\theta = c,$$

where $c \in \mathbb{R}$. So, we have the following equations:

$$\cos\theta = \frac{c}{\sqrt{1+c^2}} \text{ and } \sin\theta = \frac{1}{\sqrt{1+c^2}}.$$

If this results is used in Eq. (10), we can easily see that

$$U = \frac{cE_1 + E_3}{\sqrt{1+c^2}}.$$

□

Definition 4.4 Let us $\mathcal{C}_\alpha-$naturally parametrized curve $s \mapsto \mathbf{x}(s)$ in $\mathbb{E}^3$, $s \in I \subset \mathbb{R}$, where $\{E_1(s), E_2(s), E_3(s)\}$ denotes the $\mathcal{C}_\alpha-$frame. Suppose that $\mathcal{C}_\alpha-$biregular. We call the curve $\mathbf{x}(s)$ $\mathcal{C}_\alpha-$*slant helix,* if the $\mathcal{C}_\alpha-$principal normal vector of the curve makes a constant angle with a constant direction and $\mathcal{C}_\alpha-$unit vector $U \in \mathbb{R}^3$. In this context, it can be shown by the following equation:

$$\langle E_2, U \rangle = \cos\theta, \tag{11}$$

where θ is a constant angle.

Theorem 4.5 *Let $\mathcal{C}_\alpha-$naturally parametrized curve $\mathbf{x}$ with the $\kappa_\alpha(s) \neq 0$. Then $\mathbf{x}$ is a $\mathcal{C}_\alpha-$slant helix if and only if*

$$\frac{\kappa_\alpha^2}{(\kappa_\alpha^2 + \tau_\alpha^2)^{3/2}} D_\alpha \left(\frac{\kappa_\alpha}{\tau_\alpha} \right) \tag{12}$$

is constant function.

Proof Suppose that $\mathcal{C}_\alpha-$naturally parametrized curve $s \mapsto \mathbf{x}(s)$ is a $\mathcal{C}_\alpha-$*slant helix*. Taking a conformable derivative of Eq. (11), we get

$$\langle D_\alpha E_2, U \rangle = 0 \tag{13}$$

and

$$-\kappa_\alpha \langle E_1, U \rangle + \tau_\alpha \langle E_3, U \rangle = 0.$$

As can be seen from the elements of $\mathcal{C}_\alpha-$frame and (11), since there is a constant angle between E_2 and fixed direction U, there is also a constant angle between E_3 and fixed direction U. Then there is the following equations:

$$\langle E_1, U \rangle = \frac{\tau_\alpha}{\kappa_\alpha} c, \tag{14}$$

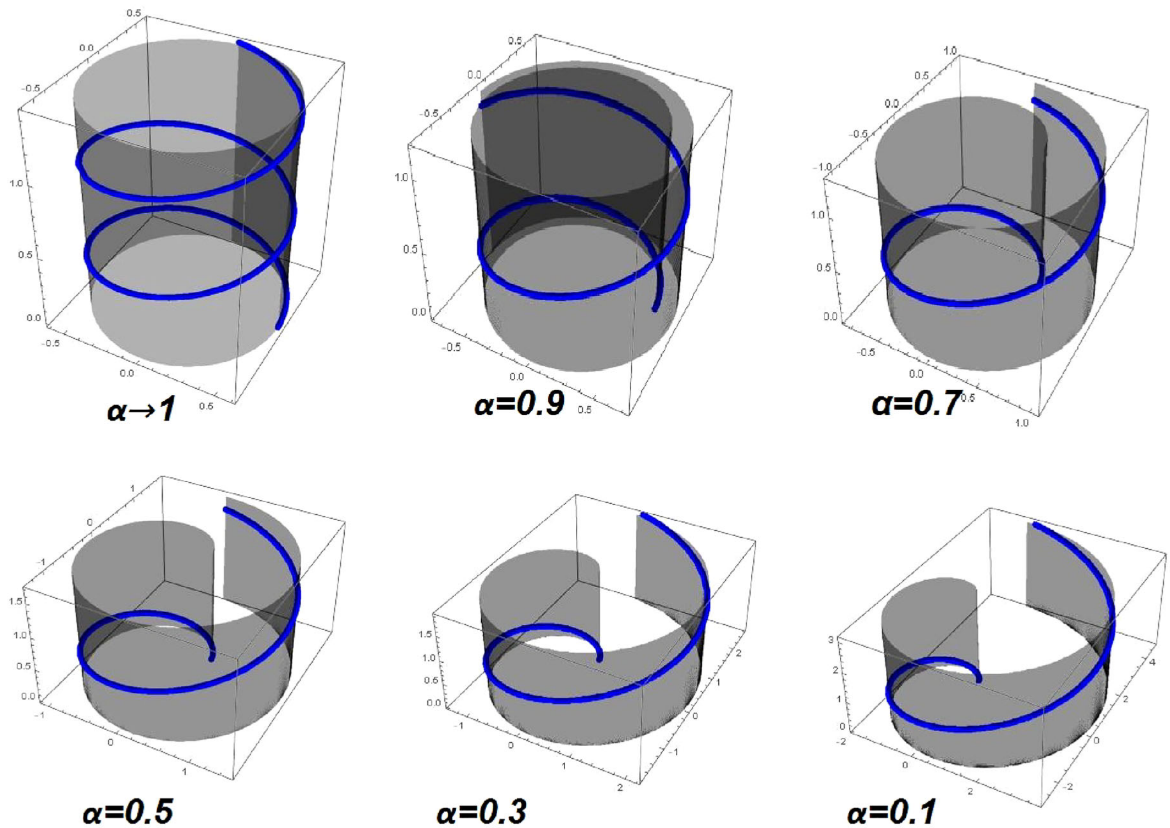

Fig. 6 $\mathcal{C}_\alpha-$cylindrical helices

$$\langle E_3, U\rangle = c,\ \ c \in \mathbb{R}. \tag{15}$$

The constant direction U from Eqs. (11), (14) and (15) is obtained as follows:

$$U = \frac{\tau_\alpha}{\kappa_\alpha} c E_1 + \cos\theta E_2 + c E_3. \tag{16}$$

Since U is the $\mathcal{C}_\alpha-$unit vector, taking the norm of both sides of the above equation:

$$1_\alpha = \left(\frac{\tau_\alpha}{\kappa_\alpha}\right)^2 c^2 1_\alpha + \cos^2\theta 1_\alpha + c^2 1_\alpha$$

or

$$1 = \left(\frac{\tau_\alpha}{\kappa_\alpha}\right)^2 c^2 + \cos^2\theta + c^2.$$

If this equation is arranged, we get

$$\left(\frac{\tau_\alpha^2}{\kappa_\alpha^2} + 1\right) c^2 = \sin^2\theta$$

whenupon, it follows that

$$c = \mp \frac{\kappa_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} \sin\theta.$$

Therefore, U can be written as

$$U = \mp E_1 \frac{\tau_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} \sin\theta + E_2 \cos\theta \mp E_3 \frac{\kappa_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} \sin\theta. \tag{17}$$

Since U vector is also a constant vector, by taking a conformable differentiation in Eq. (13), as follows:

$$\langle D_\alpha^2 E_2, U\rangle = 0. \tag{18}$$

If we use $\mathcal{C}_\alpha-$frame and from Eq. (17), we have

$$\left\langle D_\alpha\kappa_\alpha E_1 + (\kappa_\alpha^2+\tau_\alpha^2)E_2 - D_\alpha\tau_\alpha E_3, \mp\frac{\tau_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}\sin\theta E_1 + \cos\theta E_2 \mp \frac{\kappa_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}\sin\theta E_3\right\rangle = 0,$$

$$\mp\frac{\tau_\alpha D_\alpha\kappa_\alpha - \kappa_\alpha D_\alpha\tau_\alpha}{(\kappa_\alpha^2+\tau_\alpha^2)^{3/2}}\tan\theta + 1 = 0,$$

$$\tan\theta = \mp\frac{(\kappa_\alpha^2+\tau_\alpha^2)^{3/2}}{\tau_\alpha D_\alpha\kappa_\alpha - \kappa_\alpha D_\alpha\tau_\alpha}$$

and finally, we get

$$\mp\frac{\kappa_\alpha^2}{(\kappa_\alpha^2+\tau_\alpha^2)^{3/2}}D_\alpha\left(\frac{\kappa_\alpha}{\tau_\alpha}\right) = \cot\theta = c, \quad c\in\mathbb{R}.$$

□

Example 4.6 Let $\mathbf{x}: I\subset\mathbb{R}\to E^3$ $\mathcal{C}_\alpha-$naturally parametrized curve in $\mathbb{R}^3$ parameterized by

$$\mathbf{x}(s) = \left(-\frac{2\sqrt{13}}{7}\int s^{\alpha-1}\sin 2s ds, \frac{2\sqrt{13}}{7}\int s^{\alpha-1}\cos 2s ds, \frac{6}{7}\int s^{\alpha-1}ds\right).$$

In Fig. (6) we present the graph of the $\mathcal{C}_\alpha-$cylindrical helices for different α values

5 Conclusion

At the beginning of the most popular topics in the field of mathematics are different types of calculus. While this is done the mathematical concepts obtained by classical analysis are redefined from different types of analysis and compared with their classical results. The first thing that comes to mind when we say different calculus is fractional calculus. Today, fractional analysis is used in almost all basic sciences, especially in Physics, Chemistry and Engineering. The reason for this is the claim that fractional calculus gives more numerical results than classical calculus. This makes fractional calculus more advantageous than other calculus. In this direction this article, the basic concepts of geometry is re-examined with conformable fractional analysis, which is a local fractional calculus. The difference of this study from the others is that geometric concepts are defined with conformable fractional calculus, unlike the classical ones. The $\mathcal{C}_\alpha-$ frame is constructed differently from previous similar studies and different from the classical Frenet frame. The advantage of this frame is that when $\alpha\to 1$ it gives the classical Frenet frame, it also gives the opportunity to examine the frame of the curve for all cases in the range of $0<\alpha<1$. In other words, it exhibits a more general situation compared to the classical Frenet frame. In this context, curves defined according to the $\mathcal{C}_\alpha-$frame take on a different variation of the curve for each α value. In addition, general helices and slant helices are introduced according to the $\mathcal{C}_\alpha-$frame and their necessary characterizations are given. As seen in Fig. 6, for each α value, the $\mathcal{C}_\alpha-$helix curve corresponds to a point on the $\mathcal{C}_\alpha-$cylindrical. Thus, we can obtain an infinite number of variations of the $\mathcal{C}_\alpha-$cylindrical helices curve obtained by classical calculus. In addition, we can make approximations as close as we want to the helices curve obtained by classical methods.

Acknowledgements We are grateful to Kahramanmaras Sutcu Imam University Research unit (2022/7-14M) for the financial support.

Author contributions Each part of the article was prepared jointly by the authors.

Funding This article is supported by Kahramanmaras Suutcu Imam University within the scope of project 2022/7-14M.

Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors do not have any conflict of interest.

References

1. Akgül, A.; Khoshnawb, S.H.A.: Application of fractional derivative on non-linear biochemical reaction models. Int. J. Intell. Netw. **1**, 52–58 (2020)
2. Aydin, M.E.; Mihai, A.; Yokus, A.: Applications of fractional calculus in equiaffine geometry: plane curves with fractional order. Math. Method Appl. Sci. **44**(17), 13659–13669 (2021)
3. Aydin, M.E.; Bektas, M.; Ögrenmiş, A.O.; Yokuş, A.: Differential geometry of curves in euclidean 3-space with fractional order. Int. Electron. J. Geom. **14**(1), 132–144 (2021)
4. Aydin, M.E.; Kaya, S.: Fractional equiaffine curvatures of curves in 3-dimensional affine space. Int. J. Maps Math. **6**(1), 67–82 (2023)
5. Bertrand, J.: Mémoire sur la théorie des courbes à double courbure. Journal de Mathématiques Pures et Appliquées **36**, 332–350 (1850)
6. Boyer, C.: A History of Mathematics. Wiley, New York (1968)
7. Caputo, M.; Mainardi, F.: Linear models of dissipation in anelastic solids. La Rivista del Nuovo Cimento **1**(2), 161–198 (1971)
8. Chen, W.; Sun, H.; Li, X.: Fractional Derivative Modeling in Mechanics and Engineering. Springer, Singapore (2022)
9. Durmaz, H.; Özdemir, Z.; Şekerci, Y.: Fractional approach to evolution of the magnetic field lines near the magnetic null points. Physica Scripta **99**(2), 025239 (2024)
10. Fuchs, D.: Evolutes and involutes of spatial curves. Am. Math. Mon. **120**, 217–231 (2013)
11. Gözütok, U.; Çoban, H.A.; Sagiroglu, Y.: Frenet frame with respect to conformable derivative. Filomat **33**(6), 1541–1550 (2019)
12. Gözütok, N.Y.; Gözütok, U.: Multivariable conformable fractional calculus. Filomat **32**(2), 45–53 (2018)
13. Has, A.; Yılmaz, B.: Special fractional curve pairs with fractional calculus. Int. Electron. J. Geom. **15**(1), 132–144 (2022)
14. Has, A.; Yılmaz, B.; Akkurt, A.; Yıldırım, B.: Conformable special curves in Euclidean 3-Space. Filomat **36**(14), 4687–4698 (2022)
15. Has, A.; Yılmaz, B.: Effect of fractional analysis on magnetic curves. Revista Mexicana de Fisica **68**(4), 1–15 (2022)
16. Has, A.; Yılmaz, B.: $\mathcal{C}_\alpha-$curves and their $\mathcal{C}_\alpha-$frame in fractional differential Geometry (In Press)
17. Has, A.; Yılmaz, B.; Ayvacı, K.H.: C_α ruled surfaces respect to direction curve in fractional differential geometry. J. Geom. **115**, 11 (2024)
18. Has, A.; Yılmaz, B.; Baleanu, D.: On the geometric and physical properties of conformable derivative. Math. Sci. Appl. E-Notes **12**(2), 60–70 (2024)
19. Izumiya, S.; Takeuchi, N.: New special curves and developable surfaces. Turk. J. Math. **28**, 153–163 (2004)
20. Katugampola, U.N.: A New Fractional Derivative with Classical Properties (2014). arXiv:1410.6535v2
21. Khalil, R.; Khalil, M.; Yousef, A.; Sababheh, M.: A new definition of fractional derivative. J. Comput. Appl. Math. **264**, 65–70 (2014)
22. Lancret, M.A.: Mémoire sur les courbes à double courbure. Mémoires présentés à l'Institut. **1**, 416–454 (1806)
23. Lazopoulos, K.A.; Lazopoulos, A.K.: Fractional differential geometry of curves and surfaces. Prog. Fract. Differ. Appl. **2**(3), 169–186 (2016)
24. Magin, R.L.: Fractional calculus in bioengineering. Crit. Rev. Biomed. Eng. **32**(1), 1–104 (2004)
25. Miller, K.S.; Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
26. Oldham, K.B.; Spanier, J.: The Fractional Calculus. Academic Pres, New York (1974)
27. Podlubny, I.: Fractional Differential Equations. Academic Pres, New York (1999)
28. Sousa, J.V.C.; de Oliveira, E.C.: Mittag–Leffler functions and the truncated V-fractional derivative. Mediterr. J. Math. **14**(6), 244 (2017)
29. Sousa, J.V.C.; de Oliveira, E.C.: On the local M-derivative. Progr. Fract. Differ. Appl. **4**(4), 479–492 (2018)
30. Struik, D.J.: Lectures on Dlassical Diferential Geometry. Dover Publications, New York (1988)
31. Taşdemir, M.; Canfes, E.Ö.; Uzun, B.: On Caputo fractional Bertrand curves in E^3 and E^3_1. Filomat **38**(5), 1681–1702 (2024)
32. Uchaikin, V.V.: Fractional Derivatives for Physicists and Engineers. Springer, Berlin, Heidelberg (2013)
33. Wang, F.; Liu, H.: Mannheim partner curves in 3-Euclidean space. Math. Pract. Theory **37**, 141–143 (2007)
34. Yajima, T.; Yamasaki, K.: Geometry of surfaces with Caputo fractional derivatives and applications to incompressible two-dimensional flows. J. Phys. A: Math. Theor. **45**, 065201 (2012)
35. Yajima, T.; Oiwa, S.; Yamasaki, K.: Geometry of curves with fractional-order tangent vector and Frenet–Serret formulas. Fract. Calc. Appl. Anal. **21**(6), 1493–1505 (2018)
36. Yılmaz, B.; Has, A.: Obtaining fractional electromagnetic curves in optical fiber using fractional alternative moving frame. Optik - Int. J. Light Electron Opt. **260**(8), 169067 (2022)
37. Yılmaz, B.: A new type electromagnetic curves in optical fiber and rotation of the polarization plane using fractional calculus. Optik - Int. J. Light Electron Opt. **247**(30), 168026 (2021)