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On some new arithmetic properties of certain restricted color partition functions

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Abstract Very recently, Pushpa and Vasuki (Arab. J. Math. 11, 355–378, 2022) have proved Eisenstein series identities of level 5 of weight 2 due to Ramanujan and some new Eisenstein identities for level 7 by the elementary way. In their paper, they introduced seven restricted color partition functions, namely $P^*(n)$, $M(n)$, $T^*(n)$, $L(n)$, $K(n)$, $A(n)$, and $B(n)$, and proved a few congruence properties of these functions. The main aim of this paper is to obtain several new infinite families of congruences modulo $2^a \cdot 5^\ell$ for $P^*(n)$, modulo 2^3 for $M(n)$ and $T^*(n)$, where $a = 3, 4$ and $\ell \geq 1$. For instance, we prove that for $n \geq 0$,

$$P^*(5^\ell(4n+3)+5^\ell-1) \equiv 0 \pmod{2^3 \cdot 5^\ell}.$$

In addition, we prove witness identities for the following congruences due to Pushpa and Vasuki:

$$M(5n+4) \equiv 0 \pmod 5, \quad T^*(5n+3) \equiv 0 \pmod 5.$$

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1 Introduction

The generating function for the unrestricted partition function $p(n)$ due to Euler is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = 1 + q^1 + 2q^2 + \cdots + 297q^{17} + \cdots .$$

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Here and throughout the article, we use the following q-product notations ($|q| < 1$):

$$(q;q)_\infty = (1-q)(1-q^2)(1-q^3)\cdots, \quad (a_1, a_2, \ldots, a_n; q)_\infty = \prod_{k=1}^{n} (a_k; q)_\infty$$

and

$$(q^{m\pm}; q^n)_\infty = (q^m, q^{n-m}; q^n)_\infty, \quad (m < n), \quad m, n \in \mathbb{N}.$$

For simplicity, we set $f_k = (q^k; q^k)_\infty$ for any integer $k \geq 1$. Using Euler's recurrence relation of $p(n)$, Percy Alexander MacMahon tabulated the first two hundred numbers, and from his table, Ramanujan in 1919 observed the following theorem which connects multiplicative number theory (divisibility) and additive number theory (partitions).

Theorem 1.1 ([15–17]). *For all* $n \geq 0$

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

Eventually, Ramanujan managed to prove all three observations. In fact, Ramanujan proved the following exact generating functions for $p(5n+4)$ and $p(7n+5)$ from which the congruences (1.1) and (1.2) follow immediately. In other words, Ramanujan proved the following *witness* identities for (1.1) and (1.2):

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{f_5^5}{f_1^6},$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{f_7^3}{f_1^4} + 49q\frac{f_7^7}{f_1^8}.$$

Ramanujan did not mention any witness identity for (1.3). Lehner [9] was one of the first to find a witness identity of (1.3). After Lehner, several mathematicians discovered witness identities of (1.3). For more details in this direction, see [1,6,12]. Very recently, Baruah et al. [2] discovered the witness identities of three congruences due to Ramanujan [16] by extending the method of Goswami et al. [6]. Motivated by Ramanujan's work on congruence properties of $p(n)$ and the above-mentioned work, several mathematicians proved Ramanujan-type congruences for several restricted partition functions. Very recently, Pushpa and Vasuki [13] introduced the following seven restricted color partition functions, namely,

$$\sum_{n=0}^{\infty} P^*(n)q^n = f_1^4 f_5^4, \tag{1.4}$$

$$\sum_{n=0}^{\infty} M(n)q^n = \frac{f_2^4 f_5^4}{(q^{1\pm}, q^{3\pm}; q^{10})_\infty}, \tag{1.5}$$

$$\sum_{n=0}^{\infty} T^*(n)q^n = f_1^4 f_{10}^4 (q^{1\pm}, q^{3\pm}; q^{10})_\infty, \tag{1.6}$$

$$\sum_{n=0}^{\infty} L(n)q^n = f_1^4 f_7^4 (q^{1\pm}, q^{3\pm}, q^{5\pm}, q^7, q^7; q^{14})_\infty, \tag{1.7}$$

$$\sum_{n=0}^{\infty} K(n)q^n = f_1^2 f_2^2 f_7^2 f_{14}^2, \tag{1.8}$$

$$\sum_{n=0}^{\infty} A(n)q^n = \frac{f_2^4 f_7^4}{(q^{1\pm}, q^{3\pm}, q^{5\pm}; q^{14})_\infty^2}, \tag{1.9}$$

$$\sum_{n=0}^{\infty} B(n)q^n = f_1^4 f_{14}^4 (q^{1\pm}, q^{3\pm}, q^{5\pm}; q^{14})_\infty^2, \tag{1.10}$$

and proved several congruences modulo 2, 4 and 5. For example, they proved that

$$P^*(An+B) \equiv 0 \pmod 2, \quad T^*(Cn+D) \equiv 0 \pmod 2, \quad L(En+F) \equiv 0 \pmod 2, \tag{1.11}$$

$$K^*(Gn+H) \equiv 0 \pmod 2, \quad A(2n+1) \equiv 0 \pmod 2, \quad B(2n+1) \equiv 0 \pmod 2, \tag{1.12}$$

$$P^*(5n+4) \equiv 0 \pmod 5, \quad M(5n+4) \equiv 0 \pmod 5, \quad T^*(5n+3) \equiv 0 \pmod 5, \tag{1.13}$$

where $(A, B) \in \{(4, 1), (4, 2), (4, 3), (5, 1), (5, 2)\}$, $(C, D) \in \{(5, 2), (5, 4)\}$, $(E, F) \in \{(7, 2), (7, 4), (7, 5)\}$ and $(G, H) \in \{(2, 1), (7, 1), (7, 3), (7, 4)\}$. The coefficients in the above-mentioned seven partition functions have nice partition-theoretic interpretations. For example, (1.4) can be read as follows, and for others, see [13, Definitions 5.3-—5.8].

Theorem 1.2 [13, Definition 5.2, Eq. 5.1] *For all $n \geq 0$, we have*

$$P^*(n) = p_{(d,e)}(n) - p_{(d,o)}(n),$$

where $p_{(d,e)}(n)$ (resp. $p_{(d,o)}(n)$) denote the number of distinct partitions of n with even (resp. odd) number of parts and parts congruent to 0 (mod 5) with eight colors, parts not congruent to 0 (mod 5) with four colors each.

Very few arithmetic properties of (1.4)–(1.10) are recorded by Pushpa and Vasuki in their paper. The authors did not record the witness identities of congruences mentioned in (1.13). The main aim of this paper is to prove several new congruences modulo $2^a \cdot 5^\ell$ for $P^*(n)$, modulo 2^3 for $M(n)$ and $T^*(n)$, where $a = 3, 4$ and $\ell \geq 1$. In addition, we prove witness identities of the last two congruences mentioned in (1.13). The main results of this paper are as follows:

Theorem 1.3 *For all $r, n \geq 0$, $k \in \{1, 2, 3, 4\}$, we have*

$$P^*(5^\ell(n+1)-1) \equiv 0 \pmod{5^\ell}, \tag{1.14}$$

$$P^*(5^\ell(4n+4)-1) \equiv 0 \pmod{2^3 \cdot 5^\ell}, \tag{1.15}$$

$$P^*(5^\ell(16n+12)-1) \equiv 0 \pmod{2^4 \cdot 5^\ell}, \tag{1.16}$$

$$P^*(5^\ell(160n+32k+20)-1) \equiv 0 \pmod{2^4 \cdot 5^\ell}, \tag{1.17}$$

$$P^*(5^\ell(8n+8)-1) \equiv 0 \pmod{2^4 \cdot 5^\ell}, \tag{1.18}$$

and

$$\begin{aligned} P^*(5^\ell(160n+20)-1) &\equiv P^*(5^\ell(32n+4)-1) \\ &\equiv \begin{cases} 0 \pmod{2^4 \cdot 5^\ell} & \text{if } n \neq r(r+1)/2, \\ 2^3 \cdot 5^\ell \pmod{2^4 \cdot 5^\ell} & \text{if } n = r(r+1)/2. \end{cases} \end{aligned} \tag{1.19}$$

Theorem 1.4 *Let $S = \left\{5^a\left(\frac{k(k+1)}{2} + \frac{\ell(\ell+1)}{2}\right) + a \mid k, \ell \geq 0, a = 0, 1\right\}$. If $n \notin S$, then*

$$P^*(5^\ell(8n+2)-1) \equiv 0 \pmod{2^4 \cdot 5^\ell}.$$

Let $p \geq 3$ be a prime. The Legendre symbol $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod{p}, \\ -1 & \text{if } a \text{ is a quadratic non-residue modulo } p, \\ 0 & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

Corollary 1.5 *Let p be an odd prime such that $\left(\frac{-1}{p}\right) = -1$. If p does not divide n, then for all $k, n \geq 0$*

$$P^*\left(5^\ell(8p^{2k+1}n + 2p^{2k+2}) - 1\right) \equiv 0 \pmod{2^4 \cdot 5^\ell}. \tag{1.20}$$

Theorem 1.6 *If n cannot be represented as the sum of four times a pentagonal number and five times a pentagonal number, then*

$$P^*(5^\ell(16n + 6) - 1) \equiv 0 \pmod{2^4 \cdot 5^\ell}.$$

Corollary 1.7 *Let $p > 5$ be a prime such that $\left(\frac{-5}{p}\right) = -1$. If p does not divide n, then for all $k, n \geq 0$*

$$P^*\left(5^\ell(16p^{2k+1}n + 6p^{2k+2}) - 1\right) \equiv 0 \pmod{2^4 \cdot 5^\ell}. \tag{1.21}$$

Theorem 1.8 *If n cannot be represented as the sum of a pentagonal number and twenty times a pentagonal number, then*

$$P^*(5^\ell(16n + 14) - 1) \equiv 0 \pmod{2^4 \cdot 5^\ell}.$$

Corollary 1.9 *Let $p > 5$ be a prime such that $\left(\frac{-5}{p}\right) = -1$. If p does not divide n, then for all $k, n \geq 0$*

$$P^*\left(5^\ell(16p^{2k+1}n + 14p^{2k+2}) - 1\right) \equiv 0 \pmod{2^4 \cdot 5^\ell}. \tag{1.22}$$

Theorem 1.10 *We have*

$$\sum_{n=0}^{\infty} M(5n+4)q^n = 5\frac{f_2^5 f_5^5}{f_1 f_{10}} - 5\frac{f_2 f_5^{15}}{f_1^3 f_{10}^5} + 15q\frac{f_5^{10}}{f_1^2} + 60q^3\frac{f_{10}^{10}}{f_2^2} + 80q^4\frac{f_1 f_{10}^{15}}{f_2^3 f_5^5}. \tag{1.23}$$

The second congruence in (1.13) directly follows from (1.23).

Theorem 1.11 *For all $n \geq 0$ and $1 \leq k \leq 3$, we have*

$$M(32n + 8k + 7) \equiv 0 \pmod{8}, \tag{1.24}$$

$$M(16n + 7) - 2M(8n + 3) = 24P^*(2n) - 2P^*(4n + 1), \tag{1.25}$$

$$M(16n + 15) - 2M(8n + 7) = 16P^*(2n + 1) + 16P^*(n) - 2P^*(4n + 3), \tag{1.26}$$

where $M(n)$ and $P^*(n)$ are as defined in (1.5) and (1.4), respectively.

Theorem 1.12 *If n cannot be represented as the sum of a pentagonal number and five times a pentagonal number, then $M(32n + 7) \equiv 0 \pmod{2^3}$.*

Corollary 1.13 *Let $p > 5$ be a prime such that $\left(\frac{-5}{p}\right) = -1$. If p does not divide n, then for all $k, n \geq 0$*

$$M\left(32p^{2k+1}n + 8p^{2k+2} - 1\right) \equiv 0 \pmod{2^3}.$$

Theorem 1.14 *We have*

$$\sum_{n=0}^{\infty} T^*(5n+3)q^n = 5\frac{f_2 f_5^{15}}{f_1^3 f_{10}^5} - 10q\frac{f_5^{10}}{f_1^2} - 35q^2\frac{f_5^5 f_{10}^5}{f_1 f_2} - 20q^3\frac{f_{10}^{10}}{f_2^2}. \tag{1.27}$$

Third congruence in (1.13) directly follows from (1.27).

Theorem 1.15 *We have*

$$T^*(16n+10) \equiv 0 \pmod{2^3}, \tag{1.28}$$

$$T^*(8n+6) \equiv 4P^*(n) \pmod{2^3}. \tag{1.29}$$

Using congruences proved in this paper for $P^*(n)$ and (1.29), we can get several congruences for $T^*(n)$.

2 Preliminary results

We begin this section by defining an extraction operator. For a power series $\sum_{n=0}^{\infty} \mathcal{C}(n)q^n$ and $0 \le r \le m-1$, we define the operator $\left[q^{mn+r}\right]$ by

$$\left[q^{mn+r}\right]\left(\sum_{n=0}^{\infty} \mathcal{C}(n)q^n\right) = \sum_{n=0}^{\infty} \mathcal{C}(mn+r)q^n.$$

Ramanujan's general theta function [3, p. 34, Eq. (18.1)] is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{2.1}$$

The well-known Jacobi triple product identity [3, p. 35, Entry 19] in Ramanujan's notation is

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Replacing a by $-q$ and b by $-q^2$ in (2.1), we obtain one [3, p. 36, Entry 22 (iii)] of three special cases of (2.1).

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{2.2}$$

From [4, Theorem 1.3.9], we recall the following one of Jacobi's identities:

$$f_1^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}. \tag{2.3}$$

By binomial theorem, we have

$$f_k^{2^\ell m} \equiv f_{2k}^{2^{\ell-1}m} \pmod{2^\ell}. \tag{2.4}$$

Now, we recall the following 2-dissection of f_1^4, $\frac{f_1}{f_5}$, $\frac{f_5}{f_1}$, $f_1 f_5^3$ and $f_1^3 f_5$ which are required to prove our main results:

Lemma 2.1 *We have*

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q \frac{f_2^2 f_8^4}{f_4^2}, \tag{2.5}$$

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{40} f_{10}^3} - q \frac{f_4^2 f_{40}}{f_8 f_{10}^2}, \tag{2.6}$$

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}, \tag{2.7}$$

$$f_1 f_5^3 = f_2^3 f_{10} - q \frac{f_2^2 f_{10}^2 f_{20}}{f_4} + 2q^2 f_4 f_{20}^3 - 2q^3 \frac{f_4^4 f_{10} f_{40}^2}{f_2 f_8^2}, \tag{2.8}$$

$$f_1^3 f_5 = \frac{f_2^2 f_4 f_{10}^2}{f_{20}} + q \left(2 f_4^3 f_{20} - 5 f_2 f_{10}^3\right) + 2q^2 \frac{f_4^6 f_{10} f_{40}^2}{f_2 f_8^2 f_{20}^2}. \tag{2.9}$$

Proof Identity (2.5) follows from [7, Eq. (1.10.1)] on replacing q by $-q$ and then substituting $(-q;-q)_\infty = f_2^3/f_1 f_4$. Hirschhorn and Sellers [8] proved (2.6) and replacing q by $-q$, we obtain (2.7). The remaining two identities were proved by Mahadeva Naika et al. [11, Lemma 2.3]. □

Lemma 2.2 [14, p. 212] *We have*

$$f_1 = f_{25}\Big(T(q^5) - q - q^2 \frac{1}{T(q^5)}\Big), \tag{2.10}$$

where $T(q) = \frac{q^{1/5}}{R(q)} = \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty}$.

Lemma 2.3 [4, Eq. (7.4.14), p. 165] *We have*

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6}\Big(T^4(q^5) + qT^3(q^5) + 2q^2 T^2(q^5) + 3q^3 T(q^5) + 5q^4 - \frac{3q^5}{T(q^5)} + \frac{2q^6}{T^2(q^5)} - \frac{q^7}{T^3(q^5)} + \frac{q^8}{T^4(q^5)}\Big). \tag{2.11}$$

Lemma 2.4 [5, Lemma 1.3] *If* $x = T(q)$ *and* $y = T(q^2)$, *then*

$$xy^2 - \frac{q^2}{xy^2} = K,$$

$$\frac{x^2}{y} - \frac{y}{x^2} = \frac{4q}{K},$$

$$\frac{y^3}{x} + \frac{q^2 x}{y^3} = K + \frac{4q^2}{K} - 2q,$$

$$x^3 y + \frac{q^2}{x^3 y} = K + \frac{4q^2}{K} + 2q,$$

where $K = f_2 f_5^5 / f_1 f_{10}^5$.

From [4, Theorem 7.4.4], we recall the following identity:

$$11q + \frac{f_1^6}{f_5^6} = T^5(q) - \frac{q^2}{T^5(q)}. \tag{2.12}$$

3 Proofs of Theorems 1.3, 1.4, 1.6, 1.8 and Corollaries 1.5, 1.7, 1.9

Proof of Theorem 1.3 Substituting (2.10) in (1.4), we obtain

$$\sum_{n=0}^{\infty} P^*(n)q^n = f_5^4 f_{25}^4 \left(T(q^5) - q - \frac{q^2}{T(q^5)}\right)^4$$
$$= f_5^4 f_{25}^4 \bigg(T(q^5)^4 - 4qT(q^5)^3 + 2q^2T(q^5)^2 + 8q^3T(q^5) - 5q^4 - \frac{8q^5}{T(q^5)} + \frac{2q^6}{T(q^5)^2}$$
$$+ \frac{4q^7}{T(q^5)^3} + \frac{q^8}{T(q^5)^4}\bigg).$$

Applying the operator $\left[q^{5n+4}\right]$ on both sides of the above, we arrive at

$$P^*(5n+4) = (-5)P^*(n).$$

From the above and by induction on $\ell \geq 1$, we have

$$P^*(5^\ell n + 5^\ell - 1) = (-5)^\ell P^*(n). \tag{3.1}$$

Congruence (1.14) follows immediately from the above. Employing (2.5) in (1.4) and then applying the operator $\left[q^{2n+1}\right]$, we obtain

$$\sum_{n=0}^{\infty} P^*(2n+1)q^n = -4\frac{f_1^2 f_4^4 f_{10}^{10}}{f_2^2 f_5^2 f_{20}^4} - 4q^2 \frac{f_2^{10} f_5^2 f_{20}^4}{f_1^2 f_4^4 f_{10}^2}. \tag{3.2}$$

Substituting (2.6) and (2.7) in (3.2), we found that

$$\sum_{n=0}^{\infty} P^*(2n+1)q^n = -4\frac{f_4^2 f_8^2 f_{10}^4 f_{20}^2}{f_{40}^2} + 8q\frac{f_4^5 f_{10}^5}{f_2 f_{20}} - 4q^2\left(\frac{f_4^8 f_{10}^6 f_{40}^2}{f_2^2 f_8^2 f_{20}^4} + \frac{f_2^6 f_8^2 f_{20}^8}{f_4^4 f_{10}^2 f_{40}^2}\right)$$
$$- 8q^3\frac{f_2^5 f_{20}^5}{f_4 f_{10}} - 4q^4\frac{f_2^4 f_4^2 f_{20}^2 f_{40}^2}{f_8^2}. \tag{3.3}$$

Applying the operator $\left[q^{2n+r}\right]$, $r = 1, 0$ in the above, we obtain

$$\sum_{n=0}^{\infty} P^*(4n+3)q^n = 8\frac{f_2^5 f_5^5}{f_1 f_{10}} - 8q\frac{f_1^5 f_{10}^5}{f_2 f_5} \tag{3.4}$$

and

$$\sum_{n=0}^{\infty} P^*(4n+1)q^n = -4\frac{f_2^2 f_4^2 f_5^4 f_{10}^2}{f_{20}^2} - 4q\left(\frac{f_1^6 f_4^2 f_{10}^8}{f_2^4 f_5^2 f_{20}^2} + \frac{f_2^8 f_5^6 f_{20}^2}{f_1^2 f_4^2 f_{10}^4}\right) - 4q^2\frac{f_1^4 f_2^2 f_{10}^2 f_{20}^2}{f_4^2} \tag{3.5}$$

From (3.4), we have

$$P^*(4n+3) \equiv 0 \pmod{2^3}.$$

Congruence (1.15) follows from the above and (3.1). From (3.4) and (2.4) with $\ell = 1$, we have

$$\sum_{n=0}^{\infty} P^*(4n+3)q^n \equiv 8f_1 f_5^3 f_8 + 8q f_1^3 f_5 f_{40} \pmod{2^4}. \tag{3.6}$$

Substituting (2.8) and (2.9) in (3.6), we find that

$$\sum_{n=0}^{\infty} P^*(4n+3)q^n \equiv 8 f_2^3 f_8 f_{10} + 8q\left(\frac{f_2^2 f_8 f_{10}^2 f_{20}}{f_4} + \frac{f_2^2 f_4 f_{10}^2 f_{40}}{f_{20}}\right) + 8q^2 f_2 f_{10}^3 f_{40} \pmod{2^4}. \quad (3.7)$$

Applying the operator $\left[q^{2n}\right]$ in the above identity, we obtain

$$\sum_{n=0}^{\infty} P^*(8n+3)q^n \equiv 8 f_1^3 f_4 f_5 + 8q f_1 f_5^3 f_{20} \pmod{2^4}. \quad (3.8)$$

Employing (2.8) and (2.9) in (3.8), we find that

$$\sum_{n=0}^{\infty} P^*(8n+3)q^n \equiv 8\frac{f_2^2 f_4^2 f_{10}^2}{f_{20}} + 8q\left(f_2 f_4 f_{10}^3 + f_2^3 f_{10} f_{20}\right) + 8q^2\frac{f_2^2 f_{10}^2 f_{20}^2}{f_4} \pmod{2^4}. \quad (3.9)$$

Applying the operator $\left[q^{2n}\right]$ and $\left[q^{2n+1}\right]$ in the above and then applying (2.4) with $\ell = 2$, we obtain

$$\sum_{n=0}^{\infty} P^*(16n+3)q^n \equiv 8 f_2^3 + 8q f_{10}^3 \pmod{2^4}, \quad (3.10)$$

and

$$P^*(16n+11) \equiv 0 \pmod{2^4},$$

respectively. From (3.1) and the above congruence, we obtain (1.16). Applying the operator $\left[q^{2n+1}\right]$ to (3.10) and then comparing the coefficients of q^{5n+k}, $k \in \{0,1,2,3,4\}$ on both sides, we find that

$$\sum_{n=0}^{\infty} P^*(160n+19)q^n \equiv 8 f_1^3 \pmod{2^4}. \quad (3.11)$$

and

$$P^*(160n+32k+19) \equiv 0 \pmod{2^4}, \quad where \quad k = 1,2,3,4. \quad (3.12)$$

Congruence (1.17) follows from the above and (3.1). It follows from (3.10), (3.11) and (2.3) that

$$P^*(160n+19) \equiv P^*(32n+3) \equiv \begin{cases} 0 \pmod{2^4} & \text{if} \quad n \neq r(r+1)/2, \quad r \geq 0 \\ 8 \pmod{2^4} & \text{if} \quad n = r(r+1)/2. \end{cases} \quad (3.13)$$

From the above and (3.1), we obtain (1.19). Substituting (2.5), (2.6) and (2.7) in (3.4), we find that

$$\sum_{n=0}^{\infty} P^*(4n+3)q^n \equiv 8\frac{f_2^3 f_8 f_{20}^{12}}{f_{10}^3 f_{40}^5} + 8q\frac{f_2^2 f_4^3 f_{20}^9}{f_{10}^2 f_{40}^3 f_8} + 24q\frac{f_{10}^2 f_{20}^3 f_4^9}{f_2^2 f_8^3 f_{40}} + 8q^2\frac{f_{10}^3 f_4^{12} f_{40}}{f_2^3 f_8^5} \pmod{2^5}. \quad (3.14)$$

In view of (2.6) and (2.7), one can observe that

$$24\frac{f_5^2 f_{10}^3 f_2^9}{f_1^2 f_4^3 f_{20}} + 8\frac{f_1^2 f_2^3 f_{10}^9}{f_5^2 f_{20}^3 f_4} \equiv 0 \pmod{2^4}.$$

Applying the operator $\left[q^{2n+1}\right]$ to (3.14) and then using the above fact, we find that

$$P^*(8n+7) \equiv 0 \pmod{2^4}.$$

Congruence (1.18) immediately follows from the above and (3.1). □

Proof of Theorem 1.4 We rewrite (3.5) as

$$\sum_{n=0}^{\infty} P^*(4n+1)q^n = -4\frac{f_2^2 f_4^2 f_{10}^2}{f_{20}^2} f_5^4 - 4q\left(\frac{f_4^2 f_8^{10}}{f_2^4 f_{20}^2}\left(\frac{f_1}{f_5}\right)^2 f_1^4 + \frac{f_2^8 f_{20}^2}{f_4^2 f_{10}^4}\left(\frac{f_5}{f_1}\right)^2 f_5^4\right) - 4q^2\frac{f_2^2 f_{10}^2 f_{20}^2}{f_4^2} f_1^4.$$

Substituting (2.5), (2.6) and (2.7) in the above, under modulo 16, we have

$$\begin{aligned}\sum_{n=0}^{\infty} P^*(4n+1)q^n \equiv & -4\frac{f_2^2 f_{10}^2 f_4^2}{f_{20}^2}\left(\frac{f_{20}^{10}}{f_{10}^2 f_{40}^4}\right) - 4q\frac{f_4^2 f_{10}^8}{f_2^4 f_{20}^2}\left(\frac{f_2 f_8 f_{20}^3}{f_4 f_{40} f_{10}^3} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}\right)^2\left(\frac{f_4^{10}}{f_2^2 f_8^4}\right) \\ & - 4q\frac{f_2^8 f_{20}^2}{f_{10}^4 f_4^2}\left(\frac{f_{20}^{10}}{f_{10}^2 f_{40}^4}\right)\left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}\right)^2 - 4q^2\frac{f_{10}^2 f_2^2 f_{20}^2}{f_4^2}\left(\frac{f_4{}^{10}}{f_2^2 f_8^4}\right) \\ \equiv & 12\frac{f_2^2 f_4^2 f_{20}^8}{f_{40}^4} + \left(\frac{f_4^{10} f_{10}^2 f_{20}^4}{f_2^4 f_8^2 f_{40}^2} + \frac{f_2^4 f_8^2 f_{20}^{16}}{f_4^2 f_{10}^6 f_{40}^6}\right)12q + \left(12\frac{f_4^8 f_{10}^2 f_{20}^2}{f_8^4} + 8\frac{f_4^{13} f_{10}^3 f_{20}^2}{f_2^5 f_8^4}\right. \\ & \left. + 8\frac{f_2^3 f_4 f_{20}^{13}}{f_{10}^5 f_{40}^4}\right)q^2 + \left(\frac{f_4^{16} f_{10}^4 f_{40}^2}{f_2^6 f_8^6 f_{20}^2} + \frac{f_2^2 f_4^4 f_{20}^{10}}{f_8^2 f_{10}^4 f_{40}^2}\right)12q^3. \end{aligned} \tag{3.15}$$

Applying (2.4) in the above and then applying the operator $\left[q^{2n}\right]$ to the resulting identity, we deduce that

$$\sum_{n=0}^{\infty} P^*(8n+1)q^n \equiv 12 f_1^6 + 12q f_5^6 \pmod{2^4}.$$

In view of (2.3), we can rewrite the above congruence as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} P^*(8n+1)q^n \equiv 12 & \sum_{k,\ell=0}^{\infty} (-1)^{k+\ell}(2\ell+1)(2k+1) \\ & \times\left\{q^{\frac{k(k+1)}{2}+\frac{\ell(\ell+1)}{2}} + q^{5\left(\frac{k(k+1)}{2}+\frac{\ell(\ell+1)}{2}\right)+1}\right\} \pmod{2^4}. \end{aligned} \tag{3.16}$$

From the above, we have

$$P^*(8n+1) \equiv 0 \pmod{2^4},$$

where $n \notin S = \left\{5^a\left(\frac{k(k+1)}{2}+\frac{\ell(\ell+1)}{2}\right) + a \mid k,\ell \geq 0, a = 0,1\right\}$. From (3.1) and the above, we obtain the required congruence. □

Proof of Corollary 1.5 Congruence (3.16) is equivalent to

$$\begin{aligned}\sum_{n=0}^{\infty} P^*(8n+1)q^{8n+2} \equiv 12 & \sum_{k,\ell=0}^{\infty} (-1)^{k+\ell}(2\ell+2k+1)q^{(2k+1)^2+(2\ell+1)^2} \\ & + 12\sum_{k,\ell=0}^{\infty} (-1)^{k+\ell}(2\ell+2k+1)q^{5((2k+1)^2+(2\ell+1)^2)} \pmod{2^4}. \end{aligned}$$

From the above, we observe that if $8n+2$ is not of the form $(2k+1)^2+(2\ell+1)^2$ and $5((2k+1)^2+(2\ell+1)^2)$, then $P^*(8n+1) \equiv 0 \pmod{2^4}$. Let $M = r^2+s^2$ and let $V_p(M)$ be the highest power of p that divides M.

Since $\left(\frac{-1}{p}\right) = -1$, then $V_p(M)$ must be even. If $n = p^{2k+1}m + \frac{p^{2k+2}-1}{4}$, then clearly $8n+2$ is not of the form r^2+s^2, since $V_p(8n+2) = 2k+1$. Therefore, we have

$$P^*\left(8p^{2k+1}n + 2p^{2k+2} - 1\right) \equiv 0 \pmod{2^4}.$$

Congruence (1.20) follows from the above and (3.1). This completes the proof. □

Proof of Theorem 1.6 Applying the operator $\left[q^{2n+1}\right]$ to (3.15) and then by (2.4), we obtain

$$\sum_{n=0}^{\infty} P^*(8n+5)q^n \equiv 8f_8f_{10} + 8qf_2f_{40} \pmod{2^4}. \tag{3.17}$$

From the above and (2.2), we find that

$$\sum_{n=0}^{\infty} P^*(16n+5)q^n \equiv 8f_4f_5 = 8\sum_{m,n=-\infty}^{\infty}(-1)^{n+m}q^{4\frac{n(3n+1)}{2}+5\frac{m(3m+1)}{2}} \pmod{2^4}. \tag{3.18}$$

Theorem 1.6 follows from the above and (3.1). □

Proof of Corollary 1.7 We rewrite (3.18) as follows:

$$\sum_{n=0}^{\infty} P^*(16n+5)q^{24n+9} \equiv 8\sum_{m,n=-\infty}^{\infty}(-1)^{n+m}q^{(12n+2)^2+5(6m+1)^2} \pmod{2^4}, \tag{3.19}$$

and the rest of the proof is similar to the proof of Corollary 1.5, so we omit the details here. □

Proof The proofs of Theorem 1.8 and Corollary 1.9 are similar to the proofs of Theorem 1.4 and Corollary 1.7, respectively. Here we start with the terms involving odd powers of q in (3.17) instead of terms involving even powers of q.

4 Proofs of Theorems 1.10, 1.11, 1.12 and Corollary 1.13

Proof of Theorem 1.10 We rewrite (1.5) as follows:

$$\sum_{n=0}^{\infty} M(n)q^n = \frac{f_2^4f_5^4}{(q,q^9,q^3,q^7;q^{10})_\infty} = f_2^4f_5^4\frac{(q^5;q^{10})_\infty}{(q;q^2)_\infty} = \frac{f_2^5f_5^5}{f_1f_{10}}. \tag{4.1}$$

Substituting (2.10) and (2.11) in to the above and then applying the operator $\left[q^{5n+4}\right]$, we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} M(5n+4)q^n &= \frac{f_5^5f_{10}^5}{f_1f_2}\left[q^{5n+4}\right]\left\{\left(T(q^{10}) - q^2 - \frac{q^4}{T(q^{10})}\right)^5\left(T(q^5)^4 + qT(q^5)^3 + 2q^2T(q^5)^2\right.\right.\\ &\left.\left.+3q^3T(q^5) + 5q^4 - 3\frac{q^5}{T(q^5)} + 2\frac{q^6}{T(q^5)^2} - \frac{q^7}{T(q^5)^3} + \frac{q^8}{T(q^5)^4}\right)\right\}\\ &= \frac{f_5^5f_{10}^5}{f_1f_2}\left[5T(q^2)^3T(q)^4 - 10T(q^2)^4T(q)^2 + 5T(q^2)^5 + 5q\left(\frac{T(q^2)^4}{T(q)^3} + 6T(q^2)^2T(q)\right.\right.\\ &\left.-3\frac{T(q^2)^3}{T(q)} - 3T(q^2)T(q)^3\right) + 5q^2\left(-11 + \frac{6T(q)^2}{T(q^2)} + \frac{2T(q^2)^2}{T(q)^4} - \frac{6T(q^2)}{T(q)^2} + \frac{2T(q)^4}{T(q^2)^2}\right)\\ &-5q^3\left(\frac{3}{T(q)^3T(q^2)} + \frac{6}{T(q)T(q^2)^2} + \frac{3T(q)}{T(q^2)^3} + \frac{T(q)^3}{T(q^2)^4}\right) - 5q^4\left(\frac{1}{T(q)^4T(q^2)^3}\right.\end{aligned}$$

$$+\frac{2}{T(q)^2T(q^2)^4}+\frac{1}{T(q^2)^5}\Big)\Big)\Big],$$

which is equivalent to

$$\sum_{n=0}^{\infty} M(5n+4)q^n = \frac{f_5^5 f_{10}^5}{f_1 f_2}\Bigg[5\left(x^4y^3-\frac{q^4}{x^4y^3}\right)-10\left(x^2y^4+\frac{q^4}{x^2y^4}\right)+5\left(y^5-\frac{q^4}{y^5}\right)$$
$$+5q\left(\left(\frac{y^4}{x^3}-\frac{q^2x^3}{y^4}\right)-3\left(x^3y+\frac{q^2}{x^3y}\right)-3\left(\frac{y^3}{x}+\frac{q^2x}{y^3}\right)+6\left(xy^2-\frac{q^2}{xy^2}\right)\right)$$
$$+5q^2\left(6\left(\frac{x^2}{y}-\frac{y}{x^2}\right)+2\left(\frac{y^2}{x^4}+\frac{x^4}{y^2}\right)-11\right)\Bigg], \tag{4.2}$$

where $x=T(q)$ and $y=T(q^2)$. It is easy to see the following equalities.

$$x^4y^3-\frac{q^4}{x^4y^3}=\left(xy^2-\frac{q^2}{xy^2}\right)\left(x^3y+\frac{q^2}{x^3y}\right)+q^2\left(\frac{x^2}{y}-\frac{y}{x^2}\right), \quad \frac{y^2}{x^4}+\frac{x^4}{y^2}=\left(\frac{x^2}{y}-\frac{y}{x^2}\right)^2+2,$$
$$x^2y^4+\frac{q^4}{x^2y^4}=\left(xy^2-\frac{q^2}{xy^2}\right)^2+2q^2, \quad \frac{y^4}{x^3}-\frac{q^2x^3}{y^4}=\left(xy^2-\frac{q^2}{xy^2}\right)-\left(\frac{x^2}{y}-\frac{y}{x^2}\right)\left(\frac{y^3}{x}+\frac{q^2x}{y^3}\right).$$

Substituting all of these in (4.2) and then applying Lemma 2.4 and (2.12), we obtain (1.23). ◻

Proof of Theorem 1.11 Substituting (2.7) and (2.5) with q replaced by q^5 in (4.1) and then applying the operator $\left[q^{2n+1}\right]$, we obtain

$$\sum_{n=0}^{\infty} M(2n+1)q^n = \frac{f_1^2 f_2^3 f_{10}^9}{f_4 f_5^2 f_{20}^3} - 4q^2 f_1^3 f_4 f_5 f_{20}^3.$$

It follows from (2.6), (2.9) and the above that

$$\sum_{n=0}^{\infty} M(4n+3)q^n = -2f_1^4 f_5^4 - 8q f_2^4 f_{10}^4 + 20q f_1 f_2 f_5^3 f_{10}^3. \tag{4.3}$$

Applying (2.4) with $\ell=2$ to the first term and substituting (2.8) in the third term of (4.3), we find that

$$\sum_{n=0}^{\infty} M(4n+3)q^n \equiv 6 f_2^2 f_{10}^2 + 4q f_2^4 f_{10}^4 + 4q^2 \frac{f_2^3 f_{10}^5 f_{20}}{f_4} \pmod{2^3}. \tag{4.4}$$

It follows from (2.4) and the above that

$$\sum_{n=0}^{\infty} M(8n+7)q^n \equiv 4 f_4 f_{20} \pmod{2^3}, \tag{4.5}$$

from which, the congruence (1.24) follows. We rewrite (4.3) as follows:

$$\sum_{n=0}^{\infty}\left(M(4n+3)+2P^*(n)\right)q^n + 8\sum_{n=0}^{\infty} P^*(n)q^{2n+1} = 20q f_1 f_2 f_5^3 f_{10}^3. \tag{4.6}$$

Substituting (2.8) in the above, we obtain

$$\sum_{n=0}^{\infty}\left(M(4n+3)+2P^*(n)\right)q^n + 8\sum_{n=0}^{\infty} P^*(n)q^{2n+1}$$
$$= 20q {f_2}^4 {f_{10}}^4 - 20q^2 \frac{f_2^3 f_{10}^5 f_{20}}{f_4} + 40q^3 f_2 f_4 {f_{10}}^3 {f_{20}}^3 - 40q^4 \frac{f_4^4 f_{10}^4 f_{40}^2}{f_8^2}. \tag{4.7}$$

Applying the operator $\left[q^{2n+1}\right]$ to the above and then using (4.6) and (4.1), we obtain

$$\sum_{n=0}^{\infty}\left(M(8n+7)+2P^*(2n+1)\right)q^n+8\sum_{n=0}^{\infty}P^*(2n+1)q^{2n+1}$$
$$=2\left(\sum_{n=0}^{\infty}\left(M(4n+3)+2P^*(n)\right)q^n+8\sum_{n=0}^{\infty}P^*(n)q^{2n+1}\right)+20\sum_{n=0}^{\infty}P^*(n)q^n.$$

Equating the coefficients involving q^{2n} and q^{2n+1} on both sides of the above, we obtain (1.25) and (1.26), respectively. □

Proof of Theorem 1.12 and Corollary 1.13 Using (4.5) and (2.2), we can prove these and the proofs are similar to the proofs of Theorem 1.6 and Corollary 1.9. Therefore, we omit the details here. □

5 Proof of Theorems 1.14 and 1.15

Proof of Theorem 1.14 From (1.6), we have

$$\sum_{n=0}^{\infty}T^*(n)q^n=f_1^4f_{10}^4(q^1,q^9,q^3,q^7;q^{10})_\infty=f_1^4f_{10}^4\frac{(q;q^2)_\infty}{(q^5;q^{10})_\infty}=\frac{f_1^5f_{10}^5}{f_2f_5}. \tag{5.1}$$

Substituting (2.10) and (2.11) in the above and then applying the operator $\left[q^{5n+3}\right]$, we obtain

$$\sum_{n=0}^{\infty}T^*(5n+3)q^n=5\frac{f_5^5f_{10}^5}{f_1f_2}\Big\{2T(q)^2T(q^2)^4-T(q)^4T(q^2)^3+\Big(T(q)^5+3T(q)^3T(q^2)$$
$$-6T(q)T(q^2)^2+\frac{3T(q^2)^3}{T(q)}-\frac{T(q^2)^4}{T(q)^3}\Big)q+\Big(-\frac{2T(q)^4}{T(q^2)^2}-\frac{6T(q)^2}{T(q^2)}$$
$$-11+\frac{6T(q^2)}{T(q)^2}-\frac{2T(q^2)^2}{T(q)^4}\Big)q^2+\Big(\frac{T(q)^3}{T(q^2)^4}+\frac{3T(q)}{T(q^2)^3}+\frac{6}{T(q)T(q^2)^2}$$
$$+\frac{3}{T(q)^3T(q^2)}-\frac{1}{T(q)^5}\Big)q^3+\Big(\frac{2}{T(q)^2T(q^2)^4}+\frac{1}{T(q)^4T(q^2)^3}\Big)q^4\Big\},$$

which is equivalent to

$$\sum_{n=0}^{\infty}T^*(5n+3)q^n=5\frac{f_5^5f_{10}^5}{f_1f_2}\Big\{q\left(x^5-\frac{q^2}{x^5}\right)-\left(x^4y^3-\frac{q^4}{x^4y^3}\right)+2\left(x^2y^4+\frac{q^4}{x^2y^4}\right)$$
$$+3q\left(x^3y+\frac{q^3}{x^3y}\right)-2q^2\left(\frac{x^4}{y^2}+\frac{y^2}{x^4}\right)-6q\left(xy^2-\frac{q^2}{xy^2}\right)-6q^2\left(\frac{x^2}{y}-\frac{y}{x^2}\right)$$
$$+3q\left(\frac{y^3}{x}+q^2\frac{x}{y^3}\right)-q\left(\frac{y^4}{x^3}-q^2\frac{x^3}{y^4}\right)-11q^2\Big\}. \tag{5.2}$$

The following equalities can be verified easily:

$$\left(x^4y^3-\frac{q^4}{x^4y^3}\right)=\left(xy^2-\frac{q^2}{xy^2}\right)\left(x^3y+\frac{q^2}{x^3y}\right)+q^2\left(\frac{x^2}{y}-\frac{y}{x^2}\right),$$
$$\left(x^2y^4+\frac{q^4}{x^2y^4}\right)=\left(xy^2-\frac{q^2}{xy^2}\right)^2+2q^2,\qquad\left(\frac{x^4}{y^2}+\frac{y^2}{x^4}\right)=\left(\frac{x^2}{y}-\frac{y}{x^2}\right)^2+2,$$
$$\left(\frac{y^4}{x^3}-\frac{x^3q^2}{y^4}\right)=\left(xy^2-\frac{q^2}{xy^2}\right)-\left(\frac{x^2}{y}-\frac{y}{x^2}\right)\left(\frac{y^3}{x}+\frac{q^2x}{y^3}\right).$$

Substituting these in (5.2) and then applying the Lemma 2.4, we arrive at (1.27). □

Proof of Theorem 1.15 Substituting (2.5) and (2.6) in (1.6) and then applying the operator $\left[q^{2n}\right]$, we obtain

$$\sum_{n=0}^{\infty} T^*(2n)q^n = \frac{f_2^9 f_5^2 f_{10}^3}{f_1^2 f_4^3 f_{20}} + 4q f_1 f_4^3 f_5^3 f_{20}.$$

In view of (2.7), (2.8) and the above, it follows that

$$\sum_{n=0}^{\infty} T^*(4n+2)q^n \equiv 2 f_1^4 f_5^4 + 4 f_1^3 f_2^3 f_5 f_{10} \equiv 2 f_2^2 f_{10}^2 + 4 f_1^3 f_2^3 f_5 f_{10} \pmod{2^3}.$$

Substituting (2.9) in the above and then applying the operator $\left[q^{2n+1}\right]$ and $\left[q^{2n}\right]$, we obtain (1.29) and

$$\sum_{n=0}^{\infty} T^*(8n+2)q^n \equiv 2 f_1^2 f_5^2 + 4\frac{f_1^5 f_2 f_5^3}{f_{10}} \equiv 2 f_2^2 \frac{f_5^2}{f_1^2} + 4\frac{f_1 f_2 f_4 f_5^3}{f_{10}} \pmod{2^3}.$$

From (2.7), (2.8) and the above, we have

$$\begin{aligned}\sum_{n=0}^{\infty} T^*(8n+2)q^n &\equiv 4 f_4 f_2^4 + 2\frac{f_8^2 f_{20}^4}{f_2^2 f_{40}^2} + 4q\left(\frac{f_{20} f_4^3 f_{10}}{f_2^3} + f_2^3 f_{20} f_{10}\right)\\ &\quad + 2q^2\frac{f_4^6 f_{10}^2 f_{40}^2}{f_2^4 f_8^2 f_{20}^2} \pmod{2^3}.\end{aligned} \tag{5.3}$$

Since

$$\frac{f_{20} f_4^3 f_{10}}{f_2^3} + f_2^3 f_{20} f_{10} \equiv 2 f_2^3 f_{10}^3 \equiv 0 \pmod{2}, \tag{5.4}$$

congruence (1.28) follows from (5.3). □

6 Simple proofs of congruences in (1.11) and (1.12)

Pushpa and Vasuki have proved all congruences mentioned in (1.11) and (1.12) by using 2-dissection formulas of $f_1^2, f_1^4, 1/f_1^2$, 5-dissection formulas of $f_1, 1/f_1, f_2^2/f_1, f_1^2/f_2$ and 7-dissection formulas of f_1^2/f_2 and f_2^2/f_1. Simple proofs of these congruences without using the dissection formulas are as follows: From (1.4)–(1.10), (2.3) and (2.4), we have

$$\sum_{n=0}^{\infty} P^*(n)q^n \equiv f_2^2 f_{10}^2 \pmod{2^2},$$

$$\sum_{n=0}^{\infty} P^*(n)q^n \equiv f_4 f_{20} \pmod{2},$$

$$\sum_{n=0}^{\infty} T^*(n)q^n \equiv f_2^{24} \pmod{5},$$

$$\sum_{n=0}^{\infty} K(n)q^n \equiv f_2^3 f_{14}^3 \pmod{2},$$

$$\sum_{n=0}^{\infty} A(n)q^n \equiv f_2^5 f_{14} \pmod{2},$$

$$\sum_{n=0}^{\infty} B(n)q^n \equiv f_2 f_{14}^5 \pmod{2},$$

$$\sum_{n=0}^{\infty} T^*(n)q^n \equiv f_1^3 f_5^9 = f_5^9 \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} \pmod 2,$$
$$\sum_{n=0}^{\infty} L(n)q^n = \frac{f_1^5 f_7^5}{f_2 f_{14}} \equiv f_1^3 f_7^3 \equiv f_7^3 \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} \pmod 2.$$

Also note that $n(n+1)/2 \equiv 0, 1, 3, 6 \pmod 7$ and $n(n+1)/2 \equiv 0, 1, 3 \pmod 5$ for all $n \geq 0$. From these congruences, we obtain the congruences mentioned in (1.11) and (1.12) including the congruence $P^*(2n+1) \equiv 0 \pmod 4$, which was proved by using the 2-dissection of f_1^4 in [13, Theorem 5.12].

7 Concluding remarks

In this paper, we have proved several infinite families of congruences modulo $2^a \cdot 5^\ell$, where $a = 3, 4$ for $P^*(n)$, modulo 2^3 for $M(n)$ and $T^*(n)$ by employing 2, 5-dissections of some theta functions. Only a few congruence properties are known for $L(n)$, $K(n)$, $A(n)$ and $B(n)$ and they are due to Pushpa and Vasuki [13]. It will be interesting to establish the congruences modulo integers other than 2 for these partition functions. We end this section with an unexpected congruence, which we pose as a conjecture on the basis of numerical evidence.

Conjecture 7.1 For $n \geq 0$ and $\alpha \geq 1$, we have

$$K(7^\alpha n + 7^\alpha - 2) \equiv 0 \pmod{7^\alpha}.$$

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