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Fixed points of Suzuki-generalized nonexpansive mappings in $CAT_p(0)$ metric spaces

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Abstract In this work, we obtain fixed point theorems and convergence theorems for Suzuki-generalized nonexpansive mappings in complete $CAT_p(0)$ metric spaces for $p \geq 2$. Our results extend and improve many results in the literature.

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1 Introduction

In 1965, Browder [3], Göhde [15], and Kirk [17] independently discovered a fundamental existence theorem for fixed points of nonexpansive mappings. Since then, fixed point theory for nonexpansive and related classes of mappings has been studied intensively. In particular, Suzuki [29] substantially weakened the assumption of nonexpansiveness in uniformly convex Banach spaces.

Let (M, d) be a metric space. Recall a mapping $T : M \to M$ is said to be Lipschitzian if there is a constant $k \geq 0$, such that for all $x, y \in M$, we have

$$d(T(x), T(y)) \leq k\, d(x, y).$$

The smallest number k for which the above holds is called the Lipschitz constant of T. Nonexpansive mappings are those mappings which have Lipschitz constant equal to one. In fact, Browder [3] and Göhde [15] proved the following fixed point theorem for nonexpansive mappings

Theorem 1.1 (Browder–Göhde's Theorem) *If K is a nonempty bounded closed convex subset of a uniformly convex Banach space E and $T : K \to K$ is a nonexpansive mapping, then T has a fixed point. Moreover, the fixed point set of T is a closed and convex subset of K.*

After that, Suzuki [29] introduced a condition on mappings, which is weaker than nonexpansiveness. Let K be a nonempty subset of a metric space. A mapping $T : K \to K$ is called a Suzuki-generalized nonexpansive mapping if $\frac{1}{2}d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$. It is interesting to note that nonexpansive mappings are continuous on their domains, but Suzuki-generalized nonexpansive mappings need not be continuous. In Theorem 5 of [29], Suzuki generalized the fixed point result of Browder [3] and Göhde [15].

Theorem 1.2 (Suzuki Theorem) *If K is a weakly compact convex subset of a uniformly convex in every direction Banach space E and $T : K \to K$ is a Suzuki-generalized nonexpansive mapping, then T has a fixed point. Moreover, the set of fixed points $Fix(T)$ of T is closed and convex subset of K.*

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Nearly, all scientific disciplines deal with nonlinear problems. Therefore, it is essential to find nonlinear versions of results from linear domains. Furthermore, investigation of numerous problems in metric spaces without linear structure has its own importance in pure and applied sciences. As nonlinear examples of metric spaces, one can consider $CAT(0)$ metric spaces [18,19,22], which are considered to be a nonlinear version of Hilbert spaces.

Recently, Nanjaras et al. [24] have obtained fixed point theorems and convergence theorems for Suzuki-generalized nonexpansive mappings in $CAT(0)$ metric spaces.

Theorem 1.3 (Nanjaras Theorem) *Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ metric space M. Suppose $T : K \to K$ is a Suzuki-generalized nonexpansive mapping, then T has a fixed point.*

On the other hand, Khamsi and Shukri in [16] have extended the Gromov geometric definition of $CAT(0)$ metric spaces. The authors of [16] called these more general metric spaces “generalized $CAT_p(0)$”.

In this work, we continue investigating the fixed point property for Suzuki-generalized nonexpansive mappings and its approximations in $CAT_p(0)$ metric spaces' setting. Our results will extend and improve the results of Suzuki, Nanjaras et. al., Khamsi and Shukri, Browder, Göhde, and Kirk.

2 $CAT_p(0)$ metric spaces

Let (M, d) be a metric space and let $\mathbb{R}$ denote the real line. We say that a mapping $c : \mathbb{R} \to M$ is a *metric embedding* of $\mathbb{R}$ into M if

$$d(c(s), c(t)) = |s - t|,$$

for all real $s, t \in \mathbb{R}$. The image of $\mathbb{R}$ under a metric embedding will be called a *metric line*. The image $c([a, b]) \subset M$ of a real interval under a metric embedding will be called a *metric segment*. Let $x, y \in M$. A metric segment $c([a, b])$ is said to join x and y if $c(a) = x$ and $c(b) = y$. We will say that (M, d) is of *hyperbolic type* in the sense of [12] if M contains a family of metric segments, such that for each pair of distinct points x and y in M, there is a unique metric line which joins x and y. We will denote by $[x, y]$ or $[y, x]$ the unique metric segment joining the two points x and y from M. Clearly, we have $[x, x] = \{x\}$, for any $x \in M$. Next, we give some basic facts about metric spaces of hyperbolic type which shows to some extent their natural similarity to the normed vector spaces.

Lemma 2.1 [20] *Let $x, y \in M$, $x \neq y$ and $z, w \in [x, y]$. Then*

(i) *$d(x, z) \leq d(x, y)$;*
(ii) *if $d(x, z) = d(x, w)$, then $z = w$.*

The next result brings us closer to the definition of a convex combination in metric spaces of hyperbolic type.

Lemma 2.2 [20] *Let $c : \mathbb{R} \to M$ be a metric embedding, $a \leq b \in \mathbb{R}$ and $t \in [0, 1]$. Then*

(i) *$d(c(a), c((1 - t)a + tb)) = t\, d(c(a), c(b))$;*
(ii) *$d(c(b), c((1 - t)a + tb)) = (1 - t)\, d(c(a), c(b))$.*

Proposition 2.3 [20] *Let (M, d) be a metric space of hyperbolic type. Let $x, y \in M$. For each $\alpha \in [0, 1]$, there is a unique point $z \in [x, y]$, such that*

$$d(x, z) = \alpha\, d(x, y) \ \ and \ \ d(y, z) = (1 - \alpha)\, d(x, y).$$

Such point will be denoted by $(1 - \alpha)x \oplus \alpha y$.

Obviously, we have $(1-\alpha)x \oplus \alpha x = x$, and if $x \neq y$, then for any $z \in [x, y]$, we have $z = (1-\alpha)x \oplus \alpha y$, with $\alpha = \dfrac{d(x, z)}{d(x, y)}$. This will imply that for any $z \in [x, y]$, we have $d(x, z) + d(z, y) = d(x, y)$. Also if, for some $\alpha, \beta \in [0, 1]$, we have $(1 - \alpha)x \oplus \alpha y = (1 - \beta)x \oplus \beta y$, then $\alpha = \beta$ provided $x \neq y$. Moreover, we have $(1 - \alpha)x \oplus \alpha y = \alpha y \oplus (1 - \alpha)x$. The following is a direct consequence of all these properties:

$$[x, y] = \big\{(1 - \alpha)x \oplus \alpha y;\ \ \alpha \in [0, 1]\big\}.$$

Definition 2.4 [13] We say that (M, d) is a hyperbolic metric space if

$$d\left((1-\alpha)x \oplus \alpha y, (1-\alpha)x \oplus \alpha z\right) \leq \alpha d(y, z),$$

for any $\alpha \in [0, 1]$ and all $x, y, z \in M$

In particular, every hyperbolic space is a space of hyperbolic type [20]. Note that (M, d) is hyperbolic if and only if

$$d\left((1-\alpha)x \oplus \alpha y, (1-\alpha)z \oplus \alpha w\right) \leq (1-\alpha)d(x, z) + \alpha d(y, w),$$

for any $\alpha \in [0, 1]$ and all $x, y, z, w \in M$ [13,20,22,26]. Recall that a subset C of a hyperbolic metric space M is said to be convex whenever $[x, y] \subset C$ for any $x, y \in C$.

Obviously, normed linear spaces are hyperbolic metric spaces. As nonlinear examples, one can consider the Hadamard manifolds [5], the Hilbert open unit ball equipped with the hyperbolic metric [13], and the $CAT_p(0)$ metric spaces [16] (see Example 2.6).

Definition 2.5 Let (M, d) be a hyperbolic metric space. We say that M is uniformly convex if for any $a \in M$, for every $r > 0$, and for each $\varepsilon > 0$

$$\delta(r, \varepsilon) = \inf\left\{1 - \frac{1}{r}\, d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right); d(x, a) \leq r, d(y, a) \leq r, d(x, y) \geq r\varepsilon\right\} > 0.$$

The definition of uniform convexity finds its origin in Banach spaces [7]. To the best of our knowledge, the first attempt to generalize this concept to metric spaces was done in [14]. The reader may also consult [13,26]. In the next example, we discuss $CAT_p(0)$ metric spaces.

Example 2.6 Let (M, d) be a metric space. A continuous mapping from the interval $[0, 1]$ into M is called a *path*. A path $\gamma : [0, 1] \to M$ is called a *geodesic* or (metric segment) if $d(\gamma(s), \gamma(t)) = |s - t| d(\gamma(0), \gamma(1))$, for every $s, t \in [0, 1]$. We will say that (M, d) is a *geodesic metric space* if every two points $x, y \in M$ are connected by a geodesic, i.e., there exists a geodesic $\gamma : [0, 1] \to M$, such that $\gamma(0) = x$ and $\gamma(1) = y$. In this case, we denote such geodesic by $[x, y]$. Note that, in general, such geodesic is not uniquely determined by its endpoints. The metric space (M, d) is called *uniquely geodesic* if every two points of M are connected by a unique geodesic. In this case, $[x, y]$ will denote the unique geodesic connecting x and y in M.

Recently, Khamsi and Shukri in [16] have extended the Gromov geometric definition of $CAT(0)$ metric spaces [2] to the case when the comparison triangles belong to a general Banach space. In particular, the case when the Banach space is ℓ_p, $p \geq 2$.

Recall that a *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (M, d) consists of three points x_1, x_2, x_3 in M (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (M, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Banach space l_p, for $p \geq 2$, such that $\|\overline{x}_i - \overline{x}_j\| = d\left(x_i, x_j\right)$ for $i, j \in \{1, 2, 3\}$. A point $\bar{x} \in [\bar{x}_1, \bar{x}_2]$ is called a comparison point for $x \in [x_1, x_2]$ if $d(x_1, x) = \|\bar{x}_1 - \bar{x}\|$.

Definition 2.7 [16] Let (M, d) be a geodesic metric space. M is said to be a $CAT_p(0)$ metric space if, for any geodesic triangle Δ in M, there exists a comparison triangle $\overline{\Delta}$ in ℓ_p, such that the comparison axiom is satisfied, i.e., for all $x, y \in \Delta$ and comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, we have

$$d(x, y) \leq \|\overline{x} - \overline{y}\|.$$

It is obvious that $CAT_2(0)$ metric space is exactly the classical $CAT(0)$ metric space. Note ℓ_p, $p \neq 2$, is a $CAT_p(0)$ metric space which is not a $CAT(0)$ metric space [16].

Let x, y_1, y_2 be in M, and $\dfrac{y_1 \oplus y_2}{2}$ is the midpoint of the geodesic $[y_1, y_2]$, then for $p \geq 2$, the comparison axiom implies that

$$d^p\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2} d^p(x, y_1) + \frac{1}{2} d^p(x, y_2) - \frac{1}{2^p} d^p(y_1, y_2). \tag{2.1}$$

This inequality is the (CN_p) inequality of Khamsi and Shukri [16]. As for ℓ_p, for $p > 2$, the (CN_p) inequality implies that $CAT_p(0)$ metric spaces are uniformly convex with

$$\delta(r, \varepsilon) \geq 1 - \left(1 - \frac{\varepsilon^p}{2^p}\right)^{1/p},$$

for every $r > 0$ and for each $\varepsilon > 0$.

When $p = 2$, the (CN_p) inequality reduces to the classical (CN) inequality of Bruhat and Tits [4].

3 Suzuki-generalized nonexpansive mappings

In this section, we extend Theorem 1.2 of Suzuki and Theorem 1.3 of Nanjaras et al. for Suzuki-generalized nonexpansive mappings in the setting of $CAT_p(0)$ metric spaces.

Let us discuss the behavior of type functions in a complete $CAT_p(0)$ metric space M, for $p \geq 2$. It is worth mentioning that these functions are very useful when trying to prove the existence of fixed points of mappings. Recall that $\tau : M \to \mathbb{R}_+$ is called a type if there exists a bounded sequence $\{x_n\}$ in M, such that

$$\tau(x) = \limsup_{n\to\infty} d(x, x_n).$$

Theorem 3.1 [16] *Let (M, d) be a complete $CAT_p(0)$ metric space, with $p \geq 2$. Let C be any nonempty, closed, convex, and bounded subset of M. Let τ be a type defined on C. Then, any minimizing sequence $\{x_n\} \subset C$ of τ is convergent. Its limit x is the unique minimum of τ, which is called the asymptotic center of $\{x_n\}$, and satisfies*

$$\tau^p(x) + \frac{1}{2^{p-1}}\, d^p(x, z) \leq \tau^p(z), \tag{3.1}$$

for any $z \in C$.

Next, we need the following lemma

Lemma 3.2 [29] *Let C be a nonempty subset of a metric space M. Suppose $T : C \to C$ is a Suzuki-generalized nonexpansive mapping. Then*

$$d(x, Ty) \leqslant 3d(Tx, x) + d(x, y),$$

for all $x, y \in C$.

Now, we discuss the existence of fixed points of Suzuki-generalized nonexpansive mappings in $CAT_p(0)$ metric spaces.

Theorem 3.3 *Let C be a nonempty bounded closed convex subset of a complete $CAT_p(0)$ metric space M, with $p \geq 2$. Suppose $T : C \to C$ is a Suzuki-generalized nonexpansive mapping, then T has a fixed point.*

Proof Define a sequence $\{x_n\}$ by $x_1 \in C$ and $x_{n+1} = \alpha_n Tx_n \oplus (1 - \alpha_n)x_n$ for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset \left[\frac{1}{2}, 1\right)$, such that $\limsup_{n\to\infty} \alpha_n < 1$. By the convexity and boundedness assumptions of C, we deduce that $\{x_n\}$ is a bounded sequence in C. Furthermore, $\{x_n\}$ is an approximate fixed point sequence of T, i.e., $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Indeed, since $\frac{1}{2}d(x_n, Tx_n) \leq \alpha_n d(x_n, Tx_n) = d(x_n, x_{n+1})$ and since T is Suzuki-generalized nonexpansive mapping, then

$$d(Tx_n, Tx_{n+1}) \leqslant d(x_n, x_{n+1}).$$

Note that the sequence $\{d(x_n, Tx_n\}$ is decreasing. In fact

$$\begin{aligned} d(x_{n+1}, Tx_{n+1}) &\leq d(x_{n+1}, Tx_n) + d(Tx_{n+1}, Tx_n) \\ &\leq d(x_{n+1}, Tx_n) + d(x_{n+1}, x_n) \\ &= d(x_n, Tx_n). \end{aligned}$$

For any $i, n \in \mathbb{N}$ by an induction argument on i [27], we have

$$(1 + n\alpha_n)\, d(Tx_i, x_i) \le d(Tx_{n+i}, x_i) \\ -(1-\alpha_n)^{-n}\Big(d(Tx_{n+i}, x_{i+n}) - d(Tx_i, x_i)\Big).$$

Set $R = \lim_{n\to+\infty} d(Tx_n, x_n)$. Fix $n \in \mathbb{N}$ and let $i \to +\infty$ to get

$$(1 + n\alpha_n)\, R \le \delta(C),$$

where $\delta(C) = \sup\{d(x, y);\ x, y \in C\} < +\infty$. This will obviously imply $R = 0$, i.e., $\lim_{n\to+\infty} d(x_n, Tx_n) = 0$. Next, consider the type function τ generated by the sequence $\{x_n\}$. Let x be the minimum point of τ which exists using Theorem 3.1. We claim that x is a fixed point of T. Indeed, by Lemma 3.2, we have

$$d(Tx, x_n) \leqslant 3d(Tx_n, x_n) + d(x, x_n).$$

Hence,

$$\limsup_{n\to\infty} d(Tx, x_n) \leqslant 3 \limsup_{n\to\infty} d(Tx_n, x_n) + \limsup_{n\to\infty} d(x, x_n).$$

Since $\{x_n\}$ is an approximate fixed point sequence of T, we get

$$\limsup_{n\to\infty} d(Tx, x_n) \leqslant \limsup_{n\to\infty} d(x, x_n).$$

i.e., $\tau(Tx) \leqslant \tau(x)$. By the uniqueness of the asymptotic center, we have $Tx = x$.

□

Next, we study the convergence of the approximate fixed point sequence established in the above theorem. As we have seen, we were able to extend many linear properties to the case of hyperbolic metric spaces. The weak topology is still hard to capture in the nonlinear case. Note that the inequality (3.1) is similar to Opial's condition defined in Banach spaces, which is introduced in [25], to give a characterization for weak convergent sequences. Following the approach was offered by Kuczumow [21] and Lim [23] in the case of $CAT(0)$ metric spaces, an analogue to the weak convergence in ℓ_p spaces is introduced in complete $CAT_p(0)$ metric spaces, for $p \ge 2$ as follows.

Definition 3.4 [27] We shall say that $\{x_n\} \subset M$ weakly converges to a point $x \in M$ if x is the asymptotic center of each subsequence of $\{x_n\}$. We use the notation $x_n \overset{\omega}{\to} x$.

Clearly, if $x_n \to x$, then $x_n \overset{\omega}{\to} x$. If there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $x_{n_k} \overset{\omega}{\to} x$ for some $x \in M$, we say that x is a weak cluster point of the sequence $\{x_n\}$.

Lemma 3.5 [1] *Each bounded sequence $\{x_n\}$ in a complete $CAT_p(0)$ metric space M, with $p \ge 2$, has a weakly convergent subsequence, or in other words, each bounded sequence has a weak cluster point.*

The following result is similar to the demi-closed principle discovered by Göhde [15] in uniformly convex Banach spaces.

Proposition 3.6 *Let C be a nonempty bounded closed convex subset of a complete $CAT_p(0)$ metric space M, with $p \ge 2$. Suppose $T : C \to C$ is a Suzuki-generalized nonexpansive mapping. If $\{x_n\} \subset C$ is an approximate fixed point sequence of T, i.e., $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, such that $x_n \overset{\omega}{\to} x$, then x is a fixed point of T.*

Proof Since $x_n \overset{\omega}{\to} x$, then x is the asymptotic center of each subsequence $\{u_n\}$ of $\{x_n\}$. Furthermore, $\{u_n\}$ is an approximate fixed point sequence of T. Finally, in Theorem 3.3, we have seen that the asymptotic center of $\{u_n\}$ is a is a fixed point of T. Therefore, we obtain the desired result. □

Now, we are ready to prove our weak convergence result.

Theorem 3.7 *Let C be a nonempty bounded closed convex subset of a complete $CAT_p(0)$ metric space M, with $p \ge 2$. Suppose $T : C \to C$ is a Suzuki-generalized nonexpansive mapping. Define a sequence $\{x_n\}$ by $x_1 \in C$ and $x_{n+1} = \alpha_n Tx_n \oplus (1-\alpha_n)x_n$ for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset \left[\frac{1}{2}, 1\right)$, such that* $\limsup_{n\to\infty} \alpha_n < 1$. *Then, $\{x_n\}$ weakly converges to a fixed point of T.*

Proof As $\{x_n\}$ is bounded, so by Theorem 3.1, $\{x_n\}$ has a unique asymptotic center $x \in C$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ having a unique asymptotic center $u \in C$. Furthermore, since $\{u_n\}$ is bounded, by Lemma 3.5, there exists $\omega \in C$ a weak cluster point of $\{u_n\}$, i.e., there exists a subsequence $\{v_n\}$ of $\{u_n\}$ which weakly converges to $\omega \in C$. Furthermore, by Theorem 3.3, $\{x_n\}$ is an approximate fixed point sequence of T. Hence, the subsequence $\{v_n\}$ is an approximate fixed point sequence of T. Therefore, by Proposition 3.6, ω is a fixed point of T. Moreover, since $\frac{1}{2}d(\omega, T\omega) = 0 \leq d(x_n, \omega)$, we have

$$d(Tx_n, \omega) = d(Tx_n, T\omega) \leq d(x_n, \omega).$$

Hence

$$\begin{aligned} d(x_{n+1}, \omega) &= d(\alpha_n Tx_n \oplus (1-\alpha_n)x_n, \omega) \\ &\leq \alpha_n d(Tx_n, \omega) + (1-\alpha_n)d(x_n, \omega) \\ &\leq d(x_n, \omega). \end{aligned}$$

This proves that $\{d(x_n, \omega)\}$ is decreasing which implies that $\lim_{n\to\infty} d(x_n, \omega)$ exists.

Suppose $u \neq \omega$. By the uniqueness of asymptotic centers

$$\begin{aligned} \limsup_{n\to\infty} d(v_n, \omega) &< \limsup_{n\to\infty} d(v_n, u) \\ &\leq \limsup_{n\to\infty} d(u_n, u) \\ &< \limsup_{n\to\infty} d(u_n, \omega) \\ &= \limsup_{n\to\infty} d(x_n, \omega) \\ &= \limsup_{n\to\infty} d(v_n, \omega), \end{aligned}$$

a contradiction. Hence, $u = \omega$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore, ω is the asymptotic center of each subsequence of $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ converges weakly to the fixed point ω of T. □

If we assume compactness, then Theorem 3.7 implies the following result.

Theorem 3.8 *Let C be a nonempty compact convex subset of a complete $CAT_p(0)$ metric space M, with $p \geq 2$. Suppose $T : C \to C$ is a Suzuki-generalized nonexpansive mapping. Define a sequence $\{x_n\}$ by $x_1 \in C$ and $x_{n+1} = \alpha_n Tx_n \oplus (1-\alpha_n)x_n$ for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset \left[\frac{1}{2}, 1\right)$, such that* $\limsup_{n\to\infty} \alpha_n < 1$. *Then, $\{x_n\}$ strongly converges to a fixed point of T.*

Proof Since C is compact, then $\{x_n\}$ has a convergent subsequence $\{u_n\}$ in C. Let x be the limit of $\{u_n\}$. Furthermore, Theorem 3.3 implies that $\{u_n\}$ is an approximate fixed point sequence of T. Clearly, since $u_n \to x$, then $u_n \overset{\omega}{\to} x$. As a consequence to Proposition 3.6, x is a fixed point of T. Finally, according to the proof of Theorem 3.7, we conclude that $\{d(x_n, x)\}$ is bounded and decreasing which implies that $\lim_{n\to\infty} d(x_n, x)$ exists. In fact

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(u_n, x) = 0.$$

Thus, x is the strong limit of the sequence $\{x_n\}$ itself. □

Recall that a mapping $T : C \to C$ is *semi-compact* if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$, has a convergent subsequence. A sequence $\{x_n\} \subset C$ is said to be *Fejér monotone* with respect to C if $d(x_{n+1}, p) \leq d(x_n, p)$ for any fixed point p of T.

Next, we extend Theorem 3.8 as follows.

Theorem 3.9 *Let C be a nonempty bounded closed convex subset of a complete $CAT_p(0)$ metric space M, with $p \geq 2$. Suppose $T : C \to C$ is a Suzuki-generalized nonexpansive mapping, such that T is semi-compact. If $\{x_n\} \subset C$ is an approximate fixed point sequence of T, such that $\{x_n\}$ is Fejér monotone with respect to C. Then, $\{x_n\}$ strongly converges to a fixed point of T.*

Proof By the convexity and boundedness assumptions of C, we deduce that $\{x_n\}$ is a bounded sequence in C. Moreover, since $\{x_n\}$ is an approximate fixed point sequence of the semi-compact mapping T, then $\{x_n\}$ has a convergent subsequence $\{u_n\}$ in C. Let x be the limit of $\{u_n\}$. Furthermore, Lemma 3.2 implies that

$$d(Tx, u_n) \leqslant 3d(Tu_n, u_n) + d(x, u_n).$$

Letting $n \longrightarrow \infty$, we have $\{u_n\}$ converges to Tx. This implies $Tx = x$. Furthermore, since $\{x_n\}$ is a Fejér monotone sequence with respect to C, then the sequence $\{d(x_n, x)\}$ is decreasing. Finally, since $\{x_n\}$ is a bounded, we conclude that $\{d(x_n, x)\}$ is bounded which implies that $\lim_{n\to\infty} d(x_n, x)$ exists. In fact

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(u_n, x) = 0.$$

Thus, x is the strong limit of the sequence $\{x_n\}$ itself. □

4 Quasi-nonexpansive mappings

Let C be a nonempty subset of a metric space M. A mapping $T : C \to C$ is called a quasi-nonexpansive mapping if the set of fixed points $\mathrm{Fix}(T)$ of T is not empty and $d(Tx, y) \leq d(x, y)$, for all $x \in C$ and $y \in \mathrm{Fix}(T)$. The condition of quasi-nonexpansiveness was introduced in 1916 by Tricomi [30] for real functions, and was relaunched 50 years later by Diaz and Metcalf [8,9]. After that, the condition of quasi-nonexpansiveness was studied by Dotson [10,11], Senter and Dotson [28], and many others, for mappings in Hilbert and Banach spaces.

Clearly, the condition of Suzuki-generalized nonexpansiveness is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness [24]. The following proposition is easy to verify.

Proposition 4.1 *Let C be a nonempty subset of a $CAT_p(0)$ metric space M, with $p \geq 2$. Let $T : C \to C$ be a Suzuki-generalized nonexpansive mapping. If T has a fixed point, then T is a quasi-nonexpansive mapping.*

Moreover, in [6], Chaoha and Phon-on delivered a complete description of the fixed point set of quasi-nonexpansive mappings on nonempty convex subsets of a CAT(0) metric space. Next, we try to generalize the work of Chaoha and Phon-on into $\mathrm{CAT}_p(0)$ metric spaces.

Theorem 4.2 *Let C be a nonempty bounded closed convex subset of a complete $CAT_p(0)$ metric space M, with $p \geq 2$. If $T : C \to C$ is a quasi-nonexpansive mapping, then $Fix(T)$ is a closed and convex subset of C.*

Proof First, we show that $\mathrm{Fix}(T)$ is a closed subset of C. Let $\{x_n\} \subset \mathrm{Fix}(T)$ be such that $x_n \to x$. Then, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that $d(x_N, x) < \dfrac{\varepsilon}{2}$. Hence

$$d(x, Tx) \leq d(x, x_N) + d(x_N, Tx) \leq 2d(x, x_N) < \varepsilon.$$

Since ε is arbitrary, we must have $d(x, Tx) = 0$, i.e., $x \in \mathrm{Fix}(T)$. Therefore, $\mathrm{Fix}(T)$ is closed.

To show the convexity of the set $\mathrm{Fix}(T)$, it is sufficient to prove that $z = \frac{1}{2}x \oplus \frac{1}{2}y \in \mathrm{Fix}(T)$, for all $x, y \in \mathrm{Fix}(T)$. By the (CN_p) inequality, we have

$$d^p\,(Tz, z) \leq \frac{1}{2}d^p(Tz, x) + \frac{1}{2}d^p(Tz, y) - \frac{1}{2^p}d^p(x, y).$$

Since $d(x, Tz) \leq d(x, z) = \frac{1}{2}d(x, y)$, and $d(y, Tz) \leq d(y, z) = \frac{1}{2}d(x, y)$, we have $d^p\,(Tz, z) = 0$. Hence, $Tz = z$. Therefore, $\mathrm{Fix}(T)$ is convex.

□

Using the above result along with Proposition 4.1 and Theorem 3.3 in the situation where the existence of a fixed point is assumed, we can obtain the following corollary.

Corollary 4.3 *Let C be a nonempty bounded closed convex subset of a complete $CAT_p(0)$ metric space M, with $p \geq 2$. Suppose $T : C \to C$ is a Suzuki-generalized nonexpansive mapping, then the set* $\mathrm{Fix}(T)$ *is nonempty closed and convex subset of C.*

Finally, we close our study of convergence of fixed point theorems of Suzuki-generalized nonexpansive mappings in $\mathrm{CAT}_p(0)$ metric spaces, with $p \geq 2$, using the concept of the nearest point projection. Let us define the nearest point projection $P_C : M \to 2^C$ by

$$P_C(x) = \Big\{c \in C;\ d(x, c) = \inf\{d(x, c) :\ c \in C\}\Big\}.$$

If $P_C(x)$ is reduced to one point, for every x in M, then C is said to be a Chebyshev (or proximinal) set. In this case, the mapping P_C is not seen as a multivalued mapping but a single-valued mapping, i.e., $P_C : M \to C$ defined by

$$d(x, P_C(x)) = \inf\{d(x, c) :\ c \in C\},$$

for any $x \in M$.

Next, we need the following lemma

Lemma 4.4 [27] *Let (M, d) be a complete $CAT_p(0)$ metric space, $p \geq 2$. Then any nonempty, closed, and convex subset C of M is a Chebyshev subset.*

Applying the above result to Corollary 4.3, we conclude the following.

Corollary 4.5 *Under the assumptions of Corollary 4.3, the set $Fix(T)$ is a Chebyshev subset.*

Next, we establish the following convergence result.

Theorem 4.6 *Let C be a nonempty bounded closed convex subset of a complete $CAT_p(0)$ metric space M, with $p \geq 2$. Suppose $T : C \to C$ is a Suzuki-generalized nonexpansive mapping. Define a sequence $\{x_n\}$ by $x_1 \in C$ and $x_{n+1} = \alpha_n T x_n \oplus (1 - \alpha_n)x_n$ for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset \left[\frac{1}{2}, 1\right)$, such that* $\limsup_n \alpha_n < 1$. *Let $P_{Fix(T)}$ be the nearest point projection of C onto* $\mathrm{Fix}(T)$. *Then, $\left\{P_{\mathrm{Fix}(T)}(x_n)\right\}$ converges strongly to a fixed point of T.*

Proof According to Theorem 3.7, $\{x_n\}$ converges weakly to $x \in \mathrm{Fix}(T)$, such that x is the unique asymptotic center of $\{x_n\}$. On the other hand, by Corollary 4.5, the sequence $\left\{P_{\mathrm{Fix}(T)}(x_n)\right\}$ is well defined. We claim that $\left\{P_{\mathrm{Fix}(T)}(x_n)\right\}$ converges strongly to x. To prove the claim, assume for contrary that there exist $\varepsilon > 0$ and a subsequence $\{P_{\mathrm{Fix}(T)}(x_{n_i})\}$, such that $d(P_{\mathrm{Fix}(T)}(x_{n_i}), x) \geq \varepsilon$, for any $n_i \geq 1$.

Since M is a uniformly convex hyperbolic space, then for every $s \geq 0$ and $\varepsilon \geq 0$, there exists $\eta(s, \varepsilon) > 0$, such that $\delta(r, \varepsilon) > \eta(s, \varepsilon) > 0$, for any $r > s$. On the other hand, it is clear that we must have $R = d(x_1, x) > 0$, otherwise $\{x_n\}$ is a constant sequence. Hence, $\delta\left(d(x_{n_i}, x), \dfrac{\varepsilon}{R}\right) > \eta$, for any $n_i \geq 1$. Furthermore, since

$$d\left(x_{n_i}, \frac{1}{2}x \oplus \frac{1}{2}P_{\mathrm{Fix}(T)}(x_{n_i})\right) \leq d(x_{n_i}, x)\left(1 - \delta\left(d(x_{n_i}, x), \frac{\varepsilon}{R}\right)\right),$$

then we have

$$d\left(x_{n_i}, \frac{1}{2}x \oplus \frac{1}{2}P_{\mathrm{Fix}(T)}(x_{n_i})\right) \leq d(x_{n_i}, x)(1 - \eta),$$

for any $n_i \geq 1$. By Corollary 4.3, the convexity of $\mathrm{Fix}(T)$ implies that $\frac{1}{2}x \oplus \frac{1}{2}P_{\mathrm{Fix}(T)}(x_{n_i}) \in \mathrm{Fix}(T)$. Using the definition of the nearest point projection $P_{\mathrm{Fix}(T)}$, we get

$$d\left(x_{n_i}, P_{\mathrm{Fix}(T)}(x_{n_i})\right) \leq d(x_{n_i}, x)(1 - \eta),$$

for any $n_i \geq 1$. For any $m \geq 1$, $n_i \geq 1$ by an induction argument on m, we have

$$d\left(x_{n_i+m}, P_{\mathrm{Fix}(T)}(x_{n_i})\right) \leq d\left(x_{n_i}, P_{\mathrm{Fix}(T)}(x_{n_i})\right).$$

Thus, we get

$$d\left(x_{n_i+m}, P_{\mathrm{Fix}(T)}(x_{n_i})\right) \leq d(x_{n_i}, x)(1 - \eta),$$

for any $n_i \geq 1$. Since $P_{\mathrm{Fix}(T)}(x_{n_i}) \in \mathrm{Fix}(T)$, we know that $\{d(x_n, P_{\mathrm{Fix}(T)}(x_{n_i}))\}$ is decreasing (in n and fixed n_i). Hence

$$\limsup_{m\to\infty} d\left(x_{n_i+m}, P_{\mathrm{Fix}(T)}(x_{n_i})\right) = \lim_{n\to\infty} d\left(x_n, P_{\mathrm{Fix}(T)}(x_{n_i})\right) \leq d(x_{n_i}, x)(1-\eta),$$

for any $n_i \geq 1$. Since x is the asymptotic center of $\{x_n\}$, we get

$$\lim_{n\to\infty} d(x_n, x) \leq \lim_{n\to\infty} d\left(x_n, P_{\mathrm{Fix}(T)}(x_{n_i})\right) \leq d(x_{n_i}, x)(1-\eta),$$

for any $n_i \geq 1$. Finally, since $x \in \mathrm{Fix}(T)$, if we let $n_i \to \infty$, we get

$$\lim_{n\to\infty} d(x_n, x) \leq \lim_{n\to\infty} d(x_n, x)(1-\eta).$$

Since $\varepsilon \leq d\left(x_{n_i}, P_{\mathrm{Fix}(T)}(x_{n_i})\right) \leq d\left(x_{n_i}, x\right)$, we conclude that $\varepsilon \leq \lim_{n\to\infty} d(x_n, x)$, which implies $1 \leq 1-\eta$ which is our desired contradiction. Therefore, $\left\{P_{\mathrm{Fix}(T)}(x_n)\right\}$ converges strongly to x.

□

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