Arab. J. Math. (2024) 13:103–119
https://doi.org/10.1007/s40065-023-00454-9

Arabian Journal of Mathematics

Preeti Devi · Abhishek Guleria

Classical symmetries of the Klein–Gordon–Zakharov equations with time-dependent variable coefficients

Received: 7 June 2023 / Accepted: 18 December 2023 / Published online: 23 January 2024

Abstract In this article, we employ the group-theoretic methods to explore the Lie symmetries of the Klein–Gordon–Zakharov equations, which include time-dependent coefficients. We obtain the Lie point symmetries admitted by the Klein–Gordon–Zakharov equations along with the forms of variable coefficients. From the resulting symmetries, we construct similarity reductions.The similarity reductions are further analyzed using the power series method/approach and furnished the series solutions. Additionally, the convergence of the series solutions has been reported.

Mathematics Subject Classification 35G20 · 35G50 · 35E99

1 Introduction

It is well known that various physical phenomena are characterized by nonlinear evolution equations. As mathematical models of the phenomena, the investigation of exact explicit solutions of nonlinear equations will help us to understand these phenomena in a better way. Various methods have been formulated and developed for obtaining explicit exact solutions of nonlinear partial differential equations (NLPDEs). These methods include exp-function method [31], Jacobi elliptic function method [7], Hirota's direct method [41], F-expansion method [35], tanh function method [42], modified extended tanh-function method [3], modified (G'/G) - expansion method [43], first integral method [9], (G'/G^2)-expansion method [10,46], (G'/G)-expansion method [30], $(G'/G, 1/G)$-expansion method [2], classical symmetry method [5,6,13,29] and so on. Using Ibragimov's approach, conservation laws of nonlinear equations were introduced in [37]. Lie symmetry analysis, conservation laws, and various exact invariant solutions of differential equations were also furnished in [14–19].

Sophus Lie [23] introduced the concept of Lie symmetry method. It is one of the best methods for constructing the exact solutions of the NLPDEs by reducing the number of independent variables, exploiting continuous symmetries of the nonlinear system. The obtained exact solutions are known as similarity solutions. Various mathematicians employed this method and constructed the exact solutions [8,11,24,25,27,39].

The Klein–Gordon–Zakharov equations [4] are given by

$$\phi_{tt} - \phi_{xx} + \phi + \lambda\psi\phi = 0,$$

P. Devi
Department of Mathematics, Himachal Pradesh University, Palampur, India
E-mail: palpreeti408@gmail.com

A. Guleria (✉)
Department of Physical Sciences and Languages CSKHPKV, Palampur, HP, India
E-mail: abhishekguleriahappy@gmail.com

$$\psi_{tt} - \psi_{xx} - \sigma(|\phi|^2)_{xx} = 0, \tag{1.1}$$

where σ, λ are nonzero real constants. The complex-valued function ϕ denotes the fast time scale component of electric field raised by electrons, and the real-valued function ψ represents the deviation of ion density from its equilibrium. This equation describes the interaction between the Langmuir wave and the ion-acoustic wave in high-frequency plasma physics [4]. Recently, numerous computational approaches have been taken into account to investigate the equations (1.1) such as extended hyperbolic function method [34], extended sinh-Gordon equation expansion method [4], exp-($\phi(\xi)$)-expansion method [12], solitary wave ansazt [38], bifurcation method [45], He's variational principle method [1], simplest equation method [28], [22], conservative finite difference scheme [40], sine–cosine and some extended function method [36], and so on.

In recent times, the focus on the examination of variable coefficient nonlinear equations has been increased drastically. This is due to the fact that these equations provide a more realistic description of numerous nonlinear phenomena when compared to constant-coefficient counterparts. Therefore, it is imperative to consider and incorporate these variable coefficients. Many researchers have dedicated their efforts to the study of nonlinear evolution equations that incorporate time-dependent variable coefficients, and they have put forth precise, explicit solutions as a result [20,21,26].

Herein, we will consider the variable coefficient Klein–Gordon–Zakharov equations of the following form:

$$\begin{aligned} &\phi_{tt} - \phi_{xx} + \phi + \lambda(t)\psi\phi = 0, \\ &\psi_{tt} - \psi_{xx} - \sigma(t)(|\phi|^2)_{xx} = 0, \end{aligned} \tag{1.2}$$

where the complex-valued function ϕ represents the fast time scale component of electric field, and the real-valued function ψ represents the deviation of ion density from its equilibrium [4].

This study aims to investigate the Lie symmetries of the variable coefficient Klein–Gordon–Zakharov equations (1.2) and present the exact analytical solutions.

This paper's structure involves an overview of the classical Lie symmetry in Sect. 2. In Sect. 3, we present the symmetries of Eq. (1.2) along with the forms of variable coefficients. In Sect. 4, the similarity variables and similarity reductions have been established. We observed that the similarity reductions are complex in nature and lack Lie symmetry. Therefore, in Sect. 5, we employed the power series method on the systems of reduced ODEs and proposed the exact series solutions. Concluded remarks are given in Sect. 6.

2 Classical Lie group method: an algorithmic overview

The classical Lie Symmetry approach primarily involves achieving a reduction of partial differential equations (PDEs) by identifying symmetries using determining equations derived from the requirement of invariance within the PDE system. More specifically, when a given set of PDEs is rendered invariant under a one-parameter symmetry group of transformations, it leads to an over-defined system of linear homogeneous partial differential equations. This system is employed to determine the infinitesimal elements of the transformation group. These infinitesimals of the transformations are instrumental in achieving a reduction of the PDE system. The classical Lie symmetry method involves the following key steps:

- Symmetry transformations: The first step is to find the Lie group that generates the symmetries which maintain the invariance of a given differential equation or system of differential equations. A symmetry transformation represents a change of variables that preserves the essential structure of the differential equation.
- Lie algebra: The next step is to compute the Lie algebra corresponding to the Lie group which characterizes the infinitesimal versions of these transformations, often represented as vector fields.
- Infinitesimal generators: Determine the infinitesimal generators within the Lie algebra. These generators serve as the basis for generating one-parameter groups of transformations.
- Invariance condition: Consider whether the differential equation remains invariant under the action of these infinitesimal generators. The invariance condition is met when the Lie derivative of the equation with respect to the infinitesimal generator equals zero. This gives rise to a set of determining equations.
- Solving the determining equations: solve the determining equations to identify the infinitesimal generators and, consequently, the Lie symmetries inherent to the differential equation.
- Reduction: Upon discovering the symmetries, the differential equation can be reduced, potentially leading to simpler or even separable equations. This simplification is advantageous for finding exact solutions, offering insights into the behavior of the system governed by these equations.

3 Symmetry analysis

In this section, we work out the Lie point symmetry of the equation (1.2) by adopting the group theoretic technique. Substituting $\phi = u + iv$, $\psi = w$ in Eq. (1.2). By separating the corresponding real and imaginary parts in the resulting expression, we establish a set of three equations

$$\begin{aligned} u_{tt} - u_{xx} + u + \lambda(t)uw &= 0, \\ v_{tt} - v_{xx} + v + \lambda(t)vw &= 0, \\ w_{tt} - w_{xx} - \sigma(t)(u^2 + v^2)_{xx} &= 0. \end{aligned} \tag{3.1}$$

Consider a one-parameter Lie group of infinitesimal transformations acting on the space of independent and dependent variables under which the equations (3.1) remains invariant of the form [13]:

$$\begin{aligned} x_1 &= x + \epsilon\xi(t, x, u, v, w) + O(\epsilon^2), \\ t_1 &= t + \epsilon\eta(t, x, u, v, w) + O(\epsilon^2), \\ u_1 &= u + \epsilon\tau(t, x, u, v, w) + O(\epsilon^2), \\ v_1 &= v + \epsilon\zeta(t, x, u, v, w) + O(\epsilon^2), \\ w_1 &= w + \epsilon\Theta(t, x, u, v, w) + O(\epsilon^2), \end{aligned} \tag{3.2}$$

where ϵ is a group parameter and $\xi, \eta, \zeta, \tau, \Theta$ are the group infinitesimals. On using the group of point transformations (3.2) in Eq. (3.1), and equating the coefficients of various partial derivatives of u, v and w with respect to independent variables x and t, the following list of determining equations are obtained from the first equation of (3.1),

$$\begin{aligned} &\xi_u = \xi_v = \xi_w = 0,\ \ \eta_u = \eta_v = \eta_w = 0, \\ &\tau_{uu} = \tau_{vv} = \tau_{ww} = -2\tau_{xv} = -2\tau_{xw} = 0, \\ &\tau_{vt} = \tau_{wt} = \tau_{uv} = \tau_{uw} = \tau_{vw} = \tau_w = 0, \\ &-\xi_t + \eta_x = 0,\ \ \xi_x - \eta_t = 0, \\ &2\tau_{ut} - \eta_{tt} + \eta_{xx} = 0,\ \ -\xi_{tt} - 2\tau_{xu} + \xi_{xx} = 0, \\ &\tau_{tt} - \tau_{xx} + \lambda^{'}(t)uw\eta + \tau + \lambda(t)w\tau + \lambda(t)u\Theta - u\tau_u + 2u\eta_t - \lambda(t)uw\tau_u \\ &+ 2\lambda(t)uw\eta t - v\tau_v - \lambda(t)vw\tau_v = 0. \end{aligned}$$

Similarly, from the second equation of the system (3.1), the list of determining equations are as follows:

$$\begin{aligned} &\zeta_w = \zeta_{uv} = \zeta_{uu} = \zeta_{vv} = 2\zeta_{xu} = \zeta_{ut} = 0, \\ &\zeta_{ww} = \zeta_{uw} = \zeta_{vw} = \zeta_{xw} = \zeta_{tw} = 0, \\ &\xi_x - \eta_t = 0,\ \ \xi_{xx} - \xi_{tt} - 2\zeta_{xv} = 0, \\ &2\zeta_{tv} - \eta_{tt} + \eta_{xx} = 0, \\ &\zeta_{tt} - \zeta_{xx} - u\zeta_u - \lambda(t)uw\zeta_u - v\zeta_v + \zeta + \lambda^{'}(t)vw\eta + \lambda(t)v\Theta + \lambda(t)w\zeta \\ &- \lambda(t)vw\zeta_v + 2v\eta_t + 2\lambda(t)vw\eta_t = 0. \end{aligned}$$

Similarly, the third equation of the system (3.1) gives the following additional equations:

$$\begin{aligned} &\eta_x = 0, \xi_t = 0, \Theta_{tv} = \Theta_{tu} = 0, \\ &\Theta_{uu} = \Theta_{ww} = \Theta_{uv} = \Theta_{uw} = \Theta_{vw} = \Theta_{vv} = 0, \\ &2\Theta_{tw} - \eta_{tt} = 0,\ \ -2\Theta_{xw} - 4\sigma(t)u\tau_{xw} + \xi_{xx} - 4\sigma(t)v\zeta_{xw} = 0, \\ &-2\Theta_{xu} - 4\sigma(t)\tau_x - 4\sigma(t)u\tau_{xu} + 2\sigma(t)u\xi_{xx} - 4\sigma(t)v\zeta_{xu} = 0, \\ &-2\Theta_{xv} - 4\sigma(t)\zeta_x - 4\sigma(t)u\tau_{xv} - 4\sigma(t)v\zeta_{xv} + 2\sigma(t)v\xi_{xx} = 0, \\ &-\Theta_{vv} - 2\sigma^{'}(t)\eta - 4\sigma(t)\tau_u + 4\sigma(t)\xi_x + 2\sigma(t)\Theta_w - 4\sigma(t)\eta_t = 0, \\ &-\Theta_{vv} - 2\sigma^{'}(t)\eta - 4\sigma(t)\zeta_v + 4\sigma(t)\xi_x + 2\sigma(t)\Theta_w - 4\sigma(t)\eta_t = 0, \end{aligned}$$

$$
\begin{aligned}
&-2\sigma^{'}(t)v\eta-2\sigma(t)\zeta+4\sigma(t)v\xi_x-2\sigma(t)v\zeta_v+2v\sigma(t)\Theta_w-4v\sigma(t)\eta_t-2\sigma(t)u\tau_v=0,\\
&-2\sigma^{'}(t)u\eta-2\sigma(t)\tau+4\sigma(t)u\xi_x-2\sigma(t)u\tau_u+2u\sigma(t)\Theta_w-4u\sigma(t)\eta_t-2\sigma(t)v\zeta_u=0,\\
&2\xi_x-2\sigma(t)u\tau_w-2\sigma(t)v\zeta_w-2\eta_t=0,\\
&4\sigma(t)\tau_v+4\sigma(t)\zeta_u+\sigma(t)u\tau_{uv}+4\sigma(t)v\zeta_{vu}=0,\\
&\Theta_{tt}-\Theta_{xx}-2\sigma(t)u\tau_{xx}-2\sigma(t)v\zeta_{xx}-u\Theta_u-\lambda(t)uw\Theta_u-v\Theta_v-\lambda(t)vw\Theta_v=0.
\end{aligned}
$$

On simplifying the above three sets of equations, we get the following list of determining equations as follows:

$$
\begin{aligned}
&\xi=\xi_x,\ \ \eta=\eta_t,\ \ \xi_x-\eta_t=0,\ \ \xi_{xx}-2\zeta_{xv}=0,\\
&\tau_{uu}=\tau_{vv}=\tau_{vt}=\tau_{uv}=\tau_{xv}=\tau_w=\zeta_w=0,\\
&\zeta_w=\zeta_{uv}=\zeta_{uu}=\zeta_{vv}=2\zeta_{xu}=\zeta_{ut}=0,\\
&\Theta_{tv}=\Theta_{tu}=\Theta_{uu}=\Theta_{ww}=\Theta_{uv}=\Theta_{uw}=\Theta_{vw}=\Theta_{vv}=0,\\
&2\tau_{ut}-\eta_{tt}=0,\ \ -2\tau_{xu}+\xi_{xx}=0,\\
&2\zeta_{tv}-\eta_{tt}=0,\ \ 2\Theta_{tw}-\eta_{tt}=0,\\
&-2\Theta_{xw}-4\sigma(t)u\tau_{xw}+\xi_{xx}-4\sigma(t)v\zeta_{xw}=0,\\
&-2\Theta_{xu}-4\sigma(t)\tau_x-4\sigma(t)u\tau_{xu}+2\sigma(t)u\xi_{xx}-4\sigma(t)v\zeta_{xu}=0,\\
&-2\Theta_{xv}-4\sigma(t)\zeta_x-4\sigma(t)u\tau_{xv}-4\sigma(t)v\zeta_{xv}+2\sigma(t)v\xi_{xx}=0,\\
&-2\sigma^{'}(t)\eta-4\sigma(t)\tau_u+4\sigma(t)\xi_x+2\sigma(t)\Theta_w-4\sigma(t)\eta_t=0,\\
&-2\sigma^{'}(t)\eta-4\sigma(t)\zeta_v+4\sigma(t)\xi_x+2\sigma(t)\Theta_w-4\sigma(t)\eta_t=0,\\
&\Theta_{tt}-\Theta_{xx}-2\sigma(t)u\tau_{xx}-2\sigma(t)v\zeta_{xx}-u\Theta_u-\lambda(t)uw\Theta_u-v\Theta_v-\lambda(t)vw\Theta_v=0,\\
&-2\sigma^{'}(t)v\eta-2\sigma(t)\zeta+4\sigma(t)v\xi_x-2\sigma(t)v\zeta_v+2v\sigma(t)\Theta_w-4v\sigma(t)\eta_t-2\sigma(t)u\tau_v=0,\\
&-2\sigma^{'}(t)u\eta-2\sigma(t)\tau+4\sigma(t)u\xi_x-2\sigma(t)u\tau_u+2u\sigma(t)\Theta_w-4u\sigma(t)\eta_t-2\sigma(t)v\zeta_u=0,\\
&\zeta_{tt}-\zeta_{xx}-u\zeta_u-\lambda(t)uw\zeta_u-v\zeta_v+\zeta+\lambda^{'}(t)vw\eta+\lambda(t)v\Theta+\lambda(t)w\zeta-\lambda(t)vw\zeta_v\\
&+2v\eta_t+2\lambda(t)vw\eta_t=0,\\
&\tau_{tt}-\tau_{xx}+\lambda^{'}(t)uw\eta+\tau+\lambda(t)w\tau+\lambda(t)u\Theta-u\tau_u+2u\eta_t-\lambda(t)uw\tau_u\\
&+2\lambda(t)uw\eta_t-v\tau_v-\lambda(t)vw\tau_v=0.
\end{aligned}
\tag{3.3}
$$

Next, on solving the equations (3.3), we get the infinitesimals as follows:

$$
\begin{aligned}
\xi&=\alpha x+\beta,\\
\eta&=\alpha t+\delta,\\
\tau&=\rho u-\gamma v\\
\zeta&=\gamma u+\rho v,\\
\Theta&=\mu w-\frac{2\alpha}{\lambda(t)},
\end{aligned}
\tag{3.4}
$$

where α, δ, γ, ρ, β, μ are arbitrary constants and the arbitrary variable coefficients $\lambda(t)$ and $\sigma(t)$ are determined by the following equations:

$$
2\alpha\Big(\frac{\lambda(t)\lambda^{''}(t)-2(\lambda^{'}(t))^2}{\lambda^3(t)}\Big)=0,
\tag{3.5}
$$

$$
-2\alpha t\sigma^{'}(t)-2\delta\sigma^{'}(t)-4\rho\sigma(t)+2\mu\sigma(t)=0,
\tag{3.6}
$$

$$
\alpha t\lambda^{'}(t)+\delta\lambda^{'}(t)+2\alpha\lambda(t)+\mu\lambda(t)=0.
\tag{3.7}
$$

From Eq. (3.5), we consider two cases:

Case 1: For $\alpha=0$, the infinitesimals reduce to

$$
\xi=\beta,
$$

$$\begin{aligned} \eta &= \delta, \\ \tau &= \rho u - \gamma v \\ \zeta &= \gamma u + \rho v, \\ \Theta &= \mu w, \end{aligned} \tag{3.8}$$

and equations (3.6), (3.7) give the forms of variable coefficients as:

$$\begin{aligned} \sigma(t) &= Ae^{\frac{2\mu-4\rho}{2\delta}t}, \\ \lambda(t) &= De^{-\frac{\mu}{\delta}t}, \delta \neq 0. \end{aligned} \tag{3.9}$$

Case 2: For $\alpha \neq 0$ in (3.5), we get

$$\lambda(t) = -\frac{1}{\kappa_1 t + \kappa_2}. \tag{3.10}$$

On using Eq. (3.10) in Eq. (3.7), we have

$$\begin{aligned} (\alpha + \mu)\kappa_1 &= 0, \\ \kappa_1\delta - (2\alpha + \mu)\kappa_2 &= 0. \end{aligned} \tag{3.11}$$

Now, from the first equation (3.11), we consider the following two subcases:

Subcase 1: For $\kappa_1 = 0$, we get $\mu = -2\alpha$, and the infinitesimals reduce to

$$\begin{aligned} \xi &= \alpha x + \beta, \\ \eta &= \alpha t + \delta, \\ \tau &= \rho u - \gamma v \\ \zeta &= \gamma u + \rho v, \\ \Theta &= \mu w + 2\alpha\kappa_2. \end{aligned} \tag{3.12}$$

From Eq. (3.6), we get

$$\sigma(t) = (\alpha t + \delta)^{-\frac{2\rho}{\alpha}-2}, \alpha \neq 0, \tag{3.13}$$

and

$$\delta(t) = -\frac{1}{\kappa_2}. \tag{3.14}$$

Subcase 2: For $\mu = -\alpha$, we have $\kappa_2 = \frac{\kappa_1\delta}{\alpha}$, and the infinitesimals reduce to

$$\begin{aligned} \xi &= \alpha x + \beta, \\ \eta &= \alpha t + \delta, \\ \tau &= \rho u - \gamma v \\ \zeta &= \gamma u + \rho v, \\ \Theta &= \mu w + 2\alpha\left(\kappa_1 t + \frac{\kappa_1\delta}{\alpha}\right). \end{aligned} \tag{3.15}$$

From Eq. (3.6), we get

$$\sigma(t) = (\alpha t + \delta)^{-\frac{2\rho}{\alpha}-1}, \alpha \neq 0, \tag{3.16}$$

and

$$\delta(t) = -\frac{1}{\kappa_1 t + \frac{\kappa_1\delta}{\alpha}}. \tag{3.17}$$

4 Similarity reductions

Here, we will wor kout the similarity variables and the similarity reductions for each case given in section (3). Similarity variables are obtained by solving the following characteristics: equations [29]

$$\frac{\mathrm{d}x}{\xi}=\frac{\mathrm{d}t}{\eta}=\frac{\mathrm{d}u}{\tau}=\frac{\mathrm{d}v}{\zeta}=\frac{\mathrm{d}w}{\Theta}, \tag{4.1}$$

where $\xi, \eta, \tau, \zeta, \Theta$ are given in section (3).

Case 1: The similarity variables corresponding to the infinitesimals (3.8) can be obtained by solving the characteristics equations (4.1) given by

$$\frac{\mathrm{d}x}{\beta}=\frac{\mathrm{d}t}{\delta}=\frac{\mathrm{d}u}{\rho u-\gamma v}=\frac{\mathrm{d}v}{\gamma u+\rho v}=\frac{\mathrm{d}w}{\mu w}, \tag{4.2}$$

which is further equal to

$$\frac{\mathrm{d}x}{\beta}=\frac{\mathrm{d}t}{\delta}=\frac{\mathrm{d}\phi}{(\rho+i\gamma)\phi}=\frac{d\psi}{\mu\psi}. \tag{4.3}$$

On solving it, we get the similarity variables as follows:

$$\begin{aligned}\theta&=\delta x-\beta t,\\ \phi&=e^{\frac{\rho+i\gamma}{\delta}t}F(\theta),\\ \psi&=e^{\frac{\mu}{\delta}}G(\theta).\end{aligned} \tag{4.4}$$

Using this equation in Eq. (1.2), we get the reduced system of ordinary differential equations (ODEs):

$$\begin{aligned}&\Big(\frac{c_3+i\gamma}{\delta}\Big)^2F-2\beta\Big(\frac{\rho+i\gamma}{\delta}t\Big)F^{'}+\beta^2F^{''}+F-\delta^2F^{''}+DFG=0,\\ &\frac{\mu^2}{\delta^2}G-\frac{2\beta\mu}{\delta}G^{'}+(\beta^2-\delta^2)G^{''}-A\delta^2(|F|^{''})^2=0.\end{aligned} \tag{4.5}$$

Next, on putting $F=U(\theta)e^{iV(\theta)}$, and on splitting the real and imaginary parts, we obtain the system of three nonlinear ODEs as follows:

$$\begin{aligned}&\Big(\frac{\rho^2-\gamma^2}{\delta^2}\Big)U-\frac{2\beta\rho}{\delta}U^{'}+\frac{2\beta\gamma}{\delta}UV^{'}+(\beta^2-\delta^2)U^{''}-(\beta^2-\delta^2)U(V^{'})^2+U+DUG=0,\\ &\frac{2\gamma\rho}{\delta^2}U-\frac{2\beta\gamma}{\delta}U^{'}-\frac{2\beta\rho}{\delta}UV^{'}+2(\beta^2-\delta^2)U^{'}V^{'}+(\beta^2-\delta^2)UV^{''}=0,\\ &\frac{\mu^2}{\delta^2}G-\frac{2\beta\mu}{\delta}G^{'}+(\beta^2-\delta^2)G^{''}-2A\delta^2UU^{''}-2A\delta^2(U^{'})^2=0.\end{aligned} \tag{4.6}$$

Case 2: For $\alpha\neq 0$, the similarity variables are given by the following subcases:

Subcase 1: The similarity variables corresponding to the infinitesimal symmetries (3.12) are given by

$$\begin{aligned}\theta&=\frac{\alpha x+\beta}{\alpha t+\delta},\\ \phi&=(\alpha t+\delta)^{\frac{\rho+i\gamma}{\alpha}}F(\theta),\\ \psi&=\kappa_2-(\alpha t+\delta)^{-2}G(\theta).\end{aligned} \tag{4.7}$$

Using this equation in Eq. (1.2), we get the reduced system of ODEs

$$\begin{aligned}&\Big(\frac{c_3+i\gamma}{\alpha}\Big)^2F-\Big(\frac{\rho+i\gamma}{\alpha}\Big)F-2(\rho+i\gamma)\alpha\theta F^{'}+\alpha^2\theta^2F^{''}+2\alpha^2\theta F^{'}-\alpha^2F^{''}+\frac{1}{\kappa_2}FG=0,\\ &6G+4\theta G^{'}+\theta^2G^{'}+(|F|^{''})^2-(1-\theta^2)G^{''}=0.\end{aligned} \tag{4.8}$$

Further, on using $F=U(\theta)e^{iV(\theta)}$ in the equations (4.8), and on separating the real and imaginary parts, we have

$$
\begin{aligned}
&\Big(\frac{\rho^2-\gamma^2}{\alpha^2}\Big)U-\frac{\rho}{\alpha}U+2\alpha\theta\rho U^{'}-2\alpha\gamma\theta UV^{'}+2\alpha^2\theta U^{'}+\alpha^2(\theta^2-1)U^{''}-\alpha^2(1-\theta^2)U(V^{'})^2+\frac{1}{\kappa_2}UG=0,\\
&\frac{2\gamma\rho}{\alpha^2}U-\frac{\gamma}{\alpha}U+2\alpha\rho\theta UV^{'}+2\alpha\gamma\theta U^{'}+2\alpha^2\theta UV^{'}+2\alpha^2(\theta^2-1)U^{'}V^{'}+\alpha^2(\theta^2-1)UV^{''}=0,\\
&6G+4\theta G^{'}+\theta^2G^{'}+2UU^{''}+2(U^{'})^2-(1-\theta^2)G^{''}=0.
\end{aligned}
\tag{4.9}
$$

Subcase 2: Similarity variable corresponding to symmetries (3.15) are given by

$$
\begin{aligned}
\theta&=\frac{\alpha x+\beta}{\alpha t+\delta},\\
\phi&=(\alpha t+\delta)^{\frac{\rho+i\gamma}{\alpha}}F(\theta),\\
\psi&=\frac{\kappa_1(\alpha t+\delta)}{\alpha}+(\alpha t+\delta)^{-1}G(\theta).
\end{aligned}
\tag{4.10}
$$

Using this equation in Eq. (1.2), we get the reduced system of ODEs:

$$
\begin{aligned}
&(\rho+i\gamma)^2F-(\rho+i\gamma)\alpha F-2(\rho+i\gamma)\alpha\theta F^{'}+\alpha^2(\theta^2-1)F^{''}+\alpha^2\theta F^{'}-\frac{\alpha}{\kappa_2}FG=0,\\
&2G+4\theta G^{'}-(|F|^{''})^2+(\theta^2-1)G^{''}=0.
\end{aligned}
\tag{4.11}
$$

On using $F=U(\theta)e^{iV(\theta)}$ in equations (4.11), and on separating the real and imaginary parts, we obtain

$$
\begin{aligned}
&(\rho^2-\gamma^2)U-\alpha\rho U-2\alpha\theta\rho U^{'}+2\alpha\gamma\theta UV^{'}+\alpha^2\theta U^{'}+\alpha^2(\theta^2-1)U^{''}-\alpha^2(\theta^2-1)U(V^{'})^2+\frac{\alpha}{\kappa_1}UG=0,\\
&2\gamma\rho U-\gamma\alpha U-2\alpha\gamma\theta U^{'}-2\alpha\rho\theta UV^{'}+\alpha^2\theta UV^{'}+2\alpha^2(\theta^2-1)U^{'}V^{'}+\alpha^2(\theta^2-1)UV^{''}=0\\
&2G+4\theta G^{'}-2UU^{''}-2(U^{'})^2+(\theta^2-1)G^{''}=0.
\end{aligned}
\tag{4.12}
$$

Subsequently, our objective is to find the exact explicit solutions of the system of nonlinear ODEs (4.6), (4.9), and (4.12) in terms of some known mathematical functions. However, this task is rendered unattainable due to the absence of Lie symmetries within these second-order nonlinear ODEs. As a result, we resort to establish the exact series solutions to these nonlinear ODEs. However, different mathematicians have utilized the power series method to extract the exact series solutions for various nonlinear ODEs [24]–[39]. Thus, in the next section, we put forward the applications of the power series method and subsequently provide the series solutions for the ODE system.

5 Power series solutions

Herein, we proposed the series solutions of the reduced ordinary differential equations by adopting the power series method.

Case 1: Let us suppose the power series solution of the system (4.6) in the form:

$$
U(\theta)=\sum_{n=0}^{\infty}p_n\theta^n,\ \ V(\theta)=\sum_{n=0}^{\infty}q_n\theta^n,\ \ G=\sum_{n=0}^{\infty}r_n\theta^n,
\tag{5.1}
$$

where $p_n,\ q_n,\ r_n$ are real constants.

Substituting equations (5.1) into system (4.6), we get

$$\Big(\frac{\rho^2-\gamma^2}{\delta^2}\Big)\sum_{n=0}^{\infty}p_n\theta^n-\frac{2\beta\rho}{\delta}\sum_{n=0}^{\infty}(n+1)p_{n+1}\theta^n+\frac{2\beta\gamma}{\delta}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(n-j+1))p_jq_{n-j+1}\theta^n$$
$$+(\beta^2-\delta^2)\sum_{n=0}^{\infty}(n+1)(n+2)p_{n+2}\theta^n$$
$$-(\beta^2-\delta^2)\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{j}(i+1)(j-i+1)q_{i+1}q_{j-i+1}p_{n-j}\theta^n$$
$$+\sum_{n=0}^{\infty}p_n\theta^n+D\sum_{n=0}^{\infty}\sum_{j=0}^{n}p_jr_{n-j}\theta^n=0, \tag{5.2}$$

$$\frac{2\gamma\rho}{\delta^2}\sum_{n=0}^{\infty}p_n\theta^n-\frac{2\beta\gamma}{\delta}\sum_{n=0}^{\infty}(n+1)p_{n+1}\theta^n+2(\beta^2-\delta^2)\sum_{n=0}^{\infty}\sum_{j=0}^{n}(j+1)(n-j+1)p_{j+1}q_{n-j+1}\theta^n$$
$$-\frac{2\beta\rho}{\delta}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(n-j+1)p_jq_{n-j+1}\theta^n$$
$$+(\beta^2-\delta^2)\sum_{n=0}^{\infty}\sum_{j=0}^{n}(n-j+1)(n-j+2)p_jq_{n-j+2}\theta^n=0, \tag{5.3}$$

$$\frac{\mu^2}{\delta^2}\sum_{n=0}^{\infty}r_n\theta^n-\frac{2\beta\mu}{\delta}\sum_{n=1}^{\infty}(n+1)r_{n+1}\theta^n-2A\delta^2\sum_{n=0}^{\infty}\sum_{j=0}^{n}(j+1)(n-j+1)p_{j+1}p_{n-j+1}\theta^n$$
$$+(\beta^2-\delta^2)\sum_{n=0}^{\infty}(n+1)(n+2)r_{n+2}\theta^n$$
$$-2A\delta^2\sum_{n=0}^{\infty}\sum_{j=0}^{n}(n-j+1)(n-j+2)p_jp_{n-j+2}\theta^n=0. \tag{5.4}$$

On comparing the coefficients of θ^n from equations (5.2)–(5.4), for $n=0$, we get

$$p_2=\frac{1}{2(\beta^2-\delta^2)}\Big(\frac{2\beta\rho}{\delta}p_1-p_0-\frac{\rho^2-\gamma^2}{\delta^2}p_0-\frac{2\beta\gamma}{\delta}p_0q_1-p_0r_0+(\beta^2-\delta^2)p_0q_1^2\Big), \tag{5.5}$$
$$q_2=\frac{1}{(\beta^2-\delta^2)p_0}\Big(\frac{\beta\gamma}{\delta}p_1-\frac{\rho\gamma}{\delta^2}p_0+\frac{\beta\rho}{\delta}p_0q_1-(\beta^2-\delta^2)p_1q_1\Big), \tag{5.6}$$
$$r_2=\frac{1}{2(\beta^2-\delta^2)}\Big(\frac{2\beta\mu}{\delta}r_1-\frac{\mu^2}{\delta^2}r_0+2A\delta^2p_1^2+4A\delta^2p_0p_2\Big), \tag{5.7}$$

where p_2 is given by Eq. (5.5).

For $n=1,2,\ldots$, the general recurrence relations are follows:

$$p_{n+2}=\frac{1}{(\beta^2-\delta^2)(n+1)(n+2)}\Big[\frac{2\beta\rho}{\delta}(n+1)p_{n+1}-p_n-\frac{\rho^2-\gamma^2}{\delta^2}p_n-\sum_{j=0}^{n}p_jr_{n-j}$$
$$-\frac{2\beta\gamma}{\delta}\sum_{j=0}^{n}(n-j+1)p_jq_{n-j+1}+(\beta^2-\delta^2)\sum_{j=0}^{n}\sum^{j} i=0(i+1)(j-i+1)q_{i+1}q_{j-i+1}p_{n-j}\Big], \tag{5.8}$$

$$q_{n+2} = \frac{1}{(\beta^2-\delta^2)(n+1)(n+2)p_0}\Bigg[\frac{2\beta\gamma}{\delta}(n+1)p_{n+1} - (\beta^2-\delta^2)\sum_{j=1}^{n}(n-j+1)(n-j+2)p_j q_{n-j+2} - \frac{2\rho\gamma}{\delta^2}p_n + \frac{2\beta\rho}{\delta}\sum_{j=0}^{n}(n-j+1)p_j q_{n-j+1} - 2(\beta^2-\delta^2)\sum_{j=0}^{n}(j+1)(n-j+1)p_{j+1}q_{n-j+1}\Bigg], \tag{5.9}$$

$$r_{n+2} = \frac{1}{(\beta^2-\delta^2)(n+1)(n+2)}\Bigg[\frac{2\beta\mu}{\delta}(n+1)r_{n+1} - \frac{\mu^2}{\delta^2}r_n + 2A\delta^2\sum_{j=0}^{n}(j+1)(n-j+1)p_{j+1}p_{n-j+1} + 2A\delta^2\sum_{j=0}^{n}(n-j+1)(n-j+2)p_j p_{n-j+2}\Bigg]. \tag{5.10}$$

From these recurrence relations, we can find all the terms of sequences $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$, and $\{r_n\}_{n=0}^{\infty}$.

Therefore, it indicates that for system of ODEs (4.6), the power series solution (5.1) is feasible with the coefficients which are given by equations (5.8), (5.9), and (5.10). Next, we showed the convergence of series solution (5.1) to the equation (4.6).

From equations (5.8), we obtained

$$|p_{n+2}| \le M\Bigg[|p_n|+|p_{n+1}|+|\sum_{j=0}^{n}|p_j||q_{n-j+1}|+\sum_{j=0}^{n}|p_j||r_{n-j}|+\sum_{j=0}^{n}\sum_{i=0}^{j}|p_{n-j}||q_{i+1}||q_{j-i+1}|\Bigg], n=1,2\ldots \tag{5.11}$$

where $M = \max\left\{\frac{|2\beta\rho|}{|\delta||\beta^2-\delta^2|}, \frac{|\delta^2|+|\rho^2-\gamma^2|}{|\delta^2||\beta^2-\delta^2|}, \frac{2|\beta\gamma|}{|\delta||\beta^2-\delta^2|}\right\}$. From Eq. (5.9), we get

$$|q_{n+2}| \le N\Bigg[|p_n|+|p_{n+1}|+\sum_{j=0}^{n}|p_j||q_{n-j+1}|+\sum_{j=0}^{k}|p_{j+1}||q_{n-j+1}|+\sum_{j=1}^{k}|p_j||q_{n-j+2}|\Bigg], n=1,2\ldots, \tag{5.12}$$

where $N = \max\left\{\frac{|2\beta\gamma|}{|p_0||\delta||\beta^2-\delta^2|}, \frac{|2\gamma\rho|}{|p_0||\beta^2-\delta^2||\delta^2|}, \frac{|2\beta\rho|}{|p_0||\beta^2-\delta^2||\delta|}, \frac{2}{|p_0|}, \frac{1}{|p_0|}\right\}$.

Similarly, from Eq. (5.10), we obtained

$$|r_{n+2}| \le L\Bigg[|r_n|+|r_{n+1}|+\sum_{j=0}^{n}|p_{j+1}||p_{n-j+1}|+\sum_{j=0}^{n}|p_j||p_{n-j+2}|\Bigg], n=1,2\ldots, \tag{5.13}$$

where $L = \max\left\{\frac{|2\beta\mu|}{|\delta||\beta^2-\delta^2|}, \frac{|\mu^2|}{|\beta^2-\delta^2||\delta^2|}, \frac{|2A\delta^2|}{|\beta^2-\delta^2|}\right\}$.

We now consider the three power series as follows:

$$X = X(\theta) = \sum_{n=0}^{\infty}x_n\theta^n,\ \ Y = Y(\theta) = \sum_{n=0}^{\infty}y_n\theta^n,\ \ Z = Z(\theta) = \sum_{n=0}^{\infty}z_n\theta^n, \tag{5.14}$$

with $x_i = |p_i|$, $y_i = |q_i|$, and $z_i = |r_i|$ for $n = 1, 2, \ldots$. Therefore,

$$\begin{aligned} x_{n+2} &= M\Bigg[x_{n+1}+x_n+\sum_{j=0}^{n}x_j y_{n-j+1}+\sum_{j=0}^{n}x_j z_{n-j}+\sum_{j=0}^{n}\sum_{i=0}^{j}y_{i+1}y_{j-i+1}x_{n-j}\Bigg],\\ y_{n+2} &= N\Bigg[x_{n+1}+x_n+\sum_{j=0}^{n}x_j y_{n-j+1}+\sum_{j=0}^{n}x_{j+1}y_{n-j+1}+\sum_{j=1}^{n}x_j y_{n-j+2}\Bigg],\\ z_{n+2} &= L\Bigg[z_{n+1}+z_n+\sum_{j=0}^{n}x_{j+1}x_{n-j+1}+\sum_{j=0}^{n}x_j x_{n-j+2}\Bigg]. \end{aligned} \tag{5.15}$$

We can see that $|p_n| \leq x_n$, $|q_n| \leq y_n$, and $|r_n| \leq z_n$ where $n = 1, 2, \ldots$.

This indicates that the series (5.14) are the majorant series of the series (5.1). Now, we have to show that the series (5.14) have positive radius of convergence.

Hence, by some mathematical calculations, we have

$$\begin{aligned} X(\theta) &= x_0 + x_1\theta + x_2\theta^2 + x_3\theta^3 + \sum_{n=2}^{\infty} x_{n+2}\theta^{n+2} \\ &= x_0 + x_1\theta + x_2\theta^2 + x_3\theta^3 + M\Bigg[\sum_{n=2}^{\infty} x_{n+1}\theta^{n+2} + \sum_{n=2}^{\infty} x_n\theta^{n+2} + \sum_{n=2}^{\infty}\sum_{j=0}^{n} x_j y_{n-j+1}\theta^{n+2} \\ &\quad + \sum_{n=2}^{\infty}\sum_{j=0}^{n} x_j z_{n-j}\theta^{n+2} + \sum_{n=2}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{j} y_{i+1} y_{j-i+1} x_{n-j}\theta^{n+2}\Bigg], \end{aligned} \tag{5.16}$$

which can be further expressed as:

$$\begin{aligned} X(\theta) &= x_0 + x_1\theta + x_2\theta^2 + x_3\theta^3 + M\Big[\theta X - x_0\theta - x_1\theta^2 - x_2\theta^3 + \theta^2 X - x_0\theta^2 - x_1\theta^3 + \theta^2 XZ \\ &\quad - x_0 z_0\theta^2 - x_0 z_1\theta^3 - x_1 z_0\theta^3 + Y^2 X - 2y_0 XY + 2y_0^2 X - y_0^2 X - x_0 y_1^2\theta^2 - 2y_1 y_2 x_0\theta^3 \\ &\quad - x_1 y_1^2\theta^3 + \theta XY - \theta X y_0 - x_0 y_1\theta^2 - x_0 y_2\theta^3 - x_1 y_1\theta^3\Big]. \end{aligned} \tag{5.17}$$

$$\begin{aligned} Y(\theta) &= y_0 + y_1\theta + y_2\theta^2 + y_3\theta^3 + \sum_{n=2}^{\infty} x_{n+2}\theta^{n+2} \\ &= y_0 + y_1\theta + y_2\theta^2 + y_3\theta^3 + N\Bigg[\sum_{n=2}^{\infty} x_{n+1}\theta^{n+2} + \sum_{n=2}^{\infty} x_n\theta^{n+2} + \sum_{n=2}^{\infty}\sum_{j=0}^{n} x_j y_{n-j+1}\theta^{n+2} \\ &\quad + \sum_{n=2}^{\infty}\sum_{j=0}^{n} x_{j+1} y_{n-j+1}\theta^{n+2} + \sum_{n=2}^{\infty}\sum_{j=1}^{n} x_j y_{n-j+2}\theta^{n+2}\Bigg], \end{aligned} \tag{5.18}$$

which can be further expressed as:

$$\begin{aligned} Y(\theta) &= y_0 + y_1\theta + y_2\theta^2 + y_3\theta^3 + N\Big[\theta X - x_0\theta - x_1\theta^2 - x_2\theta^3 + \theta^2 X - \theta^2 x_0 - \theta^3 x_1 + XY\theta \\ &\quad - y_0 X\theta - x_0 y_1\theta^2 - x_0 y_2\theta^3 - y_1 x_1\theta^3 + 2XY + x_0 y_1 - x_1 y_1\theta^2 - x_2 y_1\theta^3 - 2y_0 X - 2x_0 Y \\ &\quad + x_0 y_0 + y_1 x_0\theta - y_1\theta X\Big]; \end{aligned} \tag{5.19}$$

similarly,

$$\begin{aligned} Z(\theta) &= z_0 + z_1\theta + z_2\theta^2 + z_3\theta^3 + \sum_{n=2}^{\infty} z_{n+2}\theta^{n+2} \\ &= z_0 + z_1\theta + z_2\theta^2 + z_3\theta^3 + L\Bigg[\sum_{n=2}^{\infty} z_{n+1}\theta^{n+2} + \sum_{n=2}^{\infty} z_n\theta^{n+2} + \sum_{n=2}^{\infty}\sum_{j=0}^{n} x_{j+1} x_{n-j+1}\theta^{n+2} \\ &\quad + \sum_{n=2}^{\infty}\sum_{j=0}^{n} x_j x_{n-j+2}\theta^{n+2}\Bigg], \end{aligned} \tag{5.20}$$

which can be further expressed as:

$$\begin{aligned} Z(\theta) &= z_0 + z_1\theta + z_2\theta^2 + z_3\theta^3 + L\Big[\theta Z - z_0\theta - z_1\theta^2 - z_2\theta^3 + \theta^2 Z - \theta^2 z_0 - z_1\theta^3 + X^2 - x_0 X - x_1 X\theta \\ &\quad - x_0 x_2\theta^2 - x_0 x_3\theta^3 + X^2 - 2x_0 X + x_0^2 - x_1^2\theta^2 - x_1 x_2\theta^3\Big]. \end{aligned} \tag{5.21}$$

Let us take the implicit functional system w. r. t. the independent variable θ:

$$
\begin{aligned}
A(\theta, X, Y, Z) = X - x_0 - x_1\theta - x_2\theta^2 - x_3\theta^3 - M\Big[\theta X - x_0\theta - x_1\theta^2 - x_2\theta^3 + \theta^2 X - x_0\theta^2 - x_1\theta^3 \\
+ \theta^2 XZ - x_0 z_0\theta^2 - x_0 z_1\theta^3 - x_1 z_0\theta^3 + Y^2 X - 2y_0 XY + 2y_0^2 X - y_0^2 X \\
- x_0 y_1^2\theta^2 - 2y_1 y_2 x_0\theta^3 - x_1 y_1^2\theta^3 + \theta XY - \theta X y_0 - x_0 y_1\theta^2 - x_0 y_2\theta^3 - x_1 y_1\theta^3\Big] = 0, \\
B(\theta, X, Y, Z) = Y - y_0 - y_1\theta - y_2\theta^2 - y_3\theta^3 - N\Big[\theta X - x_0\theta - x_1\theta^2 - x_2\theta^3 + \theta^2 X - \theta^2 x_0 - \theta^3 x_1 \\
+ XY\theta - y_0 X\theta - x_0 y_1\theta^2 - x_0 y_2\theta^3 - y_1 x_1\theta^3 + 2XY + x_0 y_1 - x_1 y_1\theta^2 \\
- x_2 y_1\theta^3 - 2y_0 X - 2x_0 Y + x_0 y_0 + y_1 x_0\theta - y_1\theta X\Big] = 0, \\
C(\theta, X, Y, Z) = Z - z_0 - z_1\theta - z_2\theta^2 - z_3\theta^3 - L\Big[\theta Z - z_0\theta - z_1\theta^2 - z_2\theta^3 + \theta^2 Z - \theta^2 z_0 - z_1\theta^3 + X^2 \\
- x_0 X - x_1 X\theta - x_0 x_2\theta^2 - x_0 x_3\theta^3 + X^2 - 2x_0 X + x_0^2 - x_1^2\theta^2 - x_1 x_2\theta^3\Big] = 0.
\end{aligned}
\tag{5.22}
$$

Since A, B, C are analytic in the neighborhood of $(0, x_0, y_0, z_0)$ and $A(0, x_0, y_0, z_0) = 0$, $B(0, x_0, y_0, z_0) = 0$, $C(0, x_0, y_0, z_0) = 0$. Therefore, the Jacobian is given by

$$
J = \frac{\partial(A, B, C)}{\partial(X, Y, Z)} = 1 \neq 0. \tag{5.23}
$$

If we assume the parameters $x_0 = |p_0|$, $y_0 = |q_0|$ and $z_0 = |r_0|$, then by using the Implicit function theorem [33], we find that $X = X(\theta)$, $Y = Y(\theta)$ and $Z = Z(\theta)$ are analytic in the neighborhood of the point $(0, x_0, y_0, z_0)$, and have positive radius of convergence. This indicates that the series solution (5.1) converges in a neighborhood of $(0, x_0, y_0, z_0)$, and hence proved. Thus, the series solution of the system of equations (4.6) is written as:

$$
\begin{aligned}
U(\theta) &= p_0 + p_1\theta + \frac{1}{2(\beta^2 - \delta^2)}\Big(\frac{2\beta c_3}{\delta} p_1 - p_0 - \frac{\rho^2 - \gamma^2}{\delta^2} p_0 - \frac{2\beta\gamma}{\delta} p_0 q_1 - p_0 r_0, \\
&\quad + (\beta^2 - \delta^2) p_0 q_1^2\Big)\theta^2 + \sum_{k=1}^{\infty} p_{k+2}\theta^{k+2}, \\
V(\theta) &= q_0 + q_1\theta + \frac{1}{(\beta^2 - \delta^2) p_0}\Big(\frac{\beta\gamma}{\delta} p_1 - \frac{\rho\gamma}{\delta^2} p_0 + \frac{\beta\rho}{\delta} p_0 q_1 - (\beta^2 - \delta^2) p_1 q_1\Big)\theta^2 + \sum_{k=1}^{\infty} q_{k+2}\theta^{k+2}, \\
G(\theta) &= r_0 + r_1\theta + \frac{1}{2(\beta^2 - \delta^2)}\Big(\frac{2\beta\mu}{\delta} r_1 - \frac{\mu^2}{\delta^2} r_0 + 2A\delta^2 p_1^2 + 4A\delta^2 p_0 p_2\Big)\theta^2 + \sum_{k=1}^{\infty} r_{k+2}\theta^{k+2}.
\end{aligned}
\tag{5.24}
$$

Series solutions for Case 2:

Subcase 1: Substituting equations (5.1) into system of equations (4.9), we get

$$
\begin{aligned}
&\Big(\frac{\rho^2 - \gamma^2 - \alpha\rho}{\alpha^2}\Big)(p_0 + p_1\theta) + \Big(\frac{\rho^2 - \gamma^2 - \alpha\rho}{\alpha^2}\Big)\sum_{n=2}^{\infty} p_n\theta^n + 2\rho\alpha\theta p_1 + 2\rho\alpha\sum_{n=2}^{\infty} n p_n\theta^n + 2\alpha^2\theta p_1 \\
&+ 2\alpha^2\sum_{n=2}^{\infty} n p_n\theta^n - 2\alpha\gamma\sum_{n=2}^{\infty}\sum_{j=0}^{n-1}(n - j) p_j q_{n-j}\theta^n - 2\alpha\gamma\theta p_0 q_1 + \alpha^2\sum_{n=2}^{\infty} n(n-1) p_n\theta^n - 2\alpha^2 p_2 \\
&- 6\alpha^2\theta p_3 - \alpha^2\sum_{n=2}^{\infty}(n+1)(n+2) p_{n+2}\theta^n + \alpha^2 p_0 q_1^2 + \alpha^2(p_1 q_1^2 + 4q_1 q_2 p_0)\theta + \frac{1}{\kappa_2} p_0 r_0 \\
&+ \alpha^2\sum_{n=2}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{j}(i+1)(j-i+1) q_{i+1} q_{j-i+1} p_{n-j}\theta^n + \frac{1}{\kappa_2}(p_1 r_0 + p_0 r_1)\theta + \frac{1}{\kappa_2}\sum_{n=2}^{\infty}\sum_{j=0}^{n} p_j r_{n-j}\theta^n
\end{aligned}
$$

$$-\alpha^2\sum_{n=2}^{\infty}\sum_{j=0}^{n-2}\sum_{i=0}^{j}(i+1)(j-i+1)q_{i+1}q_{j-i+1}p_{n-j-2}\theta^n=0, \tag{5.25}$$

$$\begin{aligned}
&\Big(\frac{2\gamma\rho-\alpha\gamma}{\alpha^2}\Big)(p_0+p_1\theta)+\Big(\frac{2\gamma\rho-\alpha\gamma}{\alpha^2}\Big)\sum_{n=2}^{\infty}p_n\theta^n+2\alpha\gamma p_1\theta+2\alpha\gamma\sum_{n=2}^{\infty}np_n\theta^n+2\alpha\rho p_0p_1\theta\\
&+2\alpha\rho\sum_{n=2}^{\infty}\sum_{j=0}^{n-1}(n-j)p_jp_{n-j}\theta^n+2\alpha^2p_0q_1\theta+2\alpha^2\sum_{n=2}^{\infty}\sum_{j=0}^{n-1}(n-j)p_jq_{n-j}\theta^n-2\alpha^2p_1q_1\\
&-4\alpha^2(p_1q_2+q_1p_2)\theta+2\alpha^2\sum_{n=2}^{\infty}\sum_{j=0}^{n-2}(j+1)(n-j-1)p_{j+1}q_{n-j-1}\theta^n\\
&-2\alpha^2\sum_{n=2}^{\infty}\sum_{j=0}^{n}(j+1)(n-j+1)p_{j+1}q_{n-j+1}\theta^n\\
&-2p_0q_2\alpha^2-\alpha^2(6p_0q_3+2p_1q_2)\theta+\alpha^2\sum_{n=2}^{\infty}\sum_{j=0}^{n-2}(n-j)(n-j-1)p_jq_{n-j}\theta^n\\
&-\alpha^2\sum_{n=2}^{\infty}\sum_{j=0}^{n}(n-j+1)(n-j+2)p_jq_{n-j+2}\theta^n=0,
\end{aligned} \tag{5.26}$$

$$\begin{aligned}
&6r_0+6r_1\theta+6\sum_{n=2}^{\infty}r_n\theta^n+4r_1\theta+4\sum_{n=2}^{\infty}nr_n\theta^n+\sum_{n=2}^{\infty}n(n-1)r_n\theta^n+\sum_{n=2}^{\infty}(n-1)r_{n-1}\theta^n\\
&-2r_2-6r_3\theta-\sum_{n=2}^{\infty}(n+1)(n+2)r_{n+2}\theta^n+2p_1^2+8p_1p_2\theta+4p_0p_2+12p_0p_3\theta\\
&+2\sum_{n=2}^{\infty}\sum_{j=0}^{n}(j+1)(n-j+1)p_{j+1}p_{n-j+1}\theta^n+2\sum_{n=0}^{\infty}\sum_{j=0}^{n}(n-j+1)(n-j+2)p_jp_{n-j+2}\theta^n=0.
\end{aligned} \tag{5.27}$$

On comparing the coefficients of θ^n from equations (5.25)–(5.27), for $n=0$, we get

$$p_2=\frac{1}{2\alpha^2}\Big[\Big(\frac{\rho^2-\gamma^2-\alpha\rho}{\alpha^2}\Big)p_0-\alpha^2p_0q_1^2+\frac{p_0r_0}{\kappa_2}\Big], \tag{5.28}$$

$$q_2=\frac{1}{2\alpha^2p_0}\Big[\Big(\frac{2\gamma\rho-\alpha\gamma}{\alpha^2}\Big)p_0-2\alpha^2p_1q_1^2\Big], \tag{5.29}$$

$$r_2=\frac{1}{2}\Big[6r_0+4p_0p_2+2p_1^2\Big], \tag{5.30}$$

where p_2 is given by Eq. (5.28).

Next, for $n\geq 1$, the general recurrence relations are as follows:

$$\begin{aligned}
p_{n+2}=\frac{1}{\alpha^2(n+1)(n+2)}\Big[&\Big(\frac{\rho^2-\gamma^2-\alpha\rho}{\alpha^2}\Big)p_n+2\alpha\rho np_n+2\alpha^2np_n-2\alpha\gamma\sum_{j=0}^{n-1}(n-j)p_jp_{n-j}\\
&+\alpha^2n(n-1)p_n+\alpha^2\sum_{j=0}^{n}\sum_{i=0}^{j}(i+1)(j-i+1)q_{i+1}q_{j-i+1}p_{n-j}+\frac{1}{\kappa_2}\sum_{j=0}^{n}p_jr_{n-j}\\
&-\alpha^2\sum_{j=0}^{n-2}\sum_{i=0}^{j}(i+1)(j-i+1)q_{i+1}q_{j-i+1}p_{n-2-j}\Big],
\end{aligned} \tag{5.31}$$

$$q_{n+2}=\frac{1}{\alpha^2(n+1)(n+2)p_0}\Bigg[\Big(\frac{2\gamma\rho-\alpha\gamma}{\alpha^2}\Big)p_n+2\alpha\gamma n p_n+2\alpha\rho\sum_{j=0}^{n-1}(n-j)p_jp_{n-j}$$
$$+2\alpha^2\sum_{j=0}^{n-1}(n-j)p_jq_{n-j}+2\alpha^2\sum_{j=0}^{n-2}(j+1)(n-j-1)p_{j+1}q_{n-j-1}$$
$$-2\alpha^2\sum_{j=0}^{n}(j+1)(n-j+1)p_{j+1}q_{n-j+1}$$
$$+\alpha^2\sum_{j=0}^{n-2}(n-j)(n-j-1)p_jq_{n-j}-\alpha^2\sum_{j=1}^{n}(n-j+1)(n-j+2)p_jq_{n-j+2}\Bigg], \tag{5.32}$$

$$r_{n+2}=\frac{1}{(n+1)(n+2)}\Bigg[6r_n+4nr_n+n(n-1)r_n+(n-1)r_{n-1}+2\sum_{j=0}^{n}(j+1)(n-j+1)p_{j+1}p_{n-j+1}$$
$$+2\sum_{j=0}^{n}(n-j+1)(n-j+2)p_jp_{n-j+2}\Bigg]. \tag{5.33}$$

Using recurrence relations, we can calculate the other terms of the sequences $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$. Hence, this indicates that for equations (4.9), a series solutions (5.1) is feasible with the coefficients which are given by equations (5.31), (5.32), and (5.33). Thus, the power series solution of equations (4.9) is given by

$$U(\theta)=p_0+p_1\theta+\frac{1}{2\alpha^2}\Bigg[\Big(\frac{\rho^2-\gamma^2-\alpha\rho}{\alpha^2}\Big)p_0-\alpha^2p_0q_1^2+\frac{p_0r_0}{\kappa_2}\Bigg]\theta^2+\sum_{k=1}^{\infty}p_{k+2}\theta^{k+2},$$
$$V(\theta)=q_0+q_1\theta+\frac{1}{2\alpha^2p_0}\Bigg[\Big(\frac{2\gamma\rho-\alpha\gamma}{\alpha^2}\Big)p_0-2\alpha^2p_1q_1^2\Bigg]\theta^2+\sum_{k=1}^{\infty}q_{k+2}\theta^{k+2},$$
$$G(\theta)=r_0+r_1\theta+\frac{1}{2}\Big[6r_0+4p_0p_2+2p_1^2\Big]\theta^2+\sum_{k=1}^{\infty}r_{k+2}\theta^{k+2}. \tag{5.34}$$

Subcase 2: Substituting equations (5.1) into system of equations (4.12), we get

$$(\rho^2-\gamma^2)(p_0+p_1\theta)+(\rho^2-\gamma^2)\sum_{n=2}^{\infty}p_n\theta^n-\rho\alpha p_0-\rho\alpha\theta p_1-\rho\alpha\sum_{n=2}^{\infty}p_n\theta^n-2\alpha\rho\theta p_1$$
$$-2\alpha\rho\sum_{n=2}^{\infty}np_n\theta^n+2\alpha\gamma p_0q_1\theta+\alpha^2p_1\theta+2\alpha\gamma\sum_{n=2}^{\infty}\sum_{j=0}^{n-1}(n-j)p_jq_{n-j}\theta^n+\alpha^2\sum_{n=2}^{\infty}np_n\theta^n-2\alpha^2p_2$$
$$-6\alpha^2\theta p_3+\alpha^2\sum_{n=2}^{\infty}n(n-1)p_n\theta^n-\alpha^2\sum_{n=2}^{\infty}(n+1)(n+2)p_{n+2}\theta^n-\frac{\alpha}{\kappa_1}p_0r_0$$
$$-\frac{\alpha}{\kappa_1}p_0r_1\theta-\frac{\alpha}{\kappa_1}p_1r_0\theta+\alpha^2p_0q_1^2+\alpha^2(p_1q_1^2+4q_1q_2p_0)\theta+\frac{1}{\kappa_2}p_0r_0$$
$$+\alpha^2\sum_{n=2}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{j}(i+1)(j-i+1)q_{i+1}q_{j-i+1}p_{n-j}\theta^n-\frac{\alpha}{\kappa_1}\sum_{n=2}^{\infty}\sum_{j=0}^{n}p_jr_{n-j}\theta^n$$
$$-\alpha^2\sum_{n=2}^{\infty}\sum_{j=0}^{n-2}\sum_{i=0}^{j}(i+1)(j-i+1)q_{i+1}q_{j-i+1}p_{n-j-2}\theta^n=0, \tag{5.35}$$

$$
\begin{aligned}
&2\gamma\rho(p_0+p_1\theta)+2\gamma\rho\sum_{n=2}^{\infty}p_n\theta^n-\gamma\alpha(p_0+p_1\theta)-\gamma\alpha\sum_{n=2}^{\infty}p_n\theta^n-2\alpha\gamma p_1\theta-2\alpha\gamma\sum_{n=2}^{\infty}np_n\theta^n-2\alpha\rho p_0q_1\theta\\
&-2\alpha\rho\sum_{n=2}^{\infty}\sum_{j=0}^{n-1}(n-j)p_jq_{n-j}\theta^n+\alpha^2p_0q_1\theta+\alpha^2\sum_{n=2}^{\infty}\sum_{j=0}^{n-1}(n-j)p_jq_{n-j}\theta^n-2\alpha^2p_1q_1-4\alpha^2(p_1q_2+q_1p_2)\theta\\
&+2\alpha^2\sum_{n=2}^{\infty}\sum_{j=0}^{n-2}(j+1)(n-j-1)p_{j+1}q_{n-j-1}\theta^n-2\alpha^2\sum_{n=2}^{\infty}\sum_{j=0}^{n}(j+1)(n-j+1)p_{j+1}q_{n-j+1}\theta^n\\
&-2p_0q_2\alpha^2-\alpha^2(6p_0q_3+2p_1q_2)\theta+\alpha^2\sum_{n=2}^{\infty}\sum_{j=0}^{n-2}(n-j)(n-j-1)p_jq_{n-j}\theta^n\\
&-\alpha^2\sum_{n=2}^{\infty}\sum_{j=0}^{n}(n-j+1)(n-j+2)p_jq_{n-j+2}\theta^n=0,
\end{aligned}\tag{5.36}
$$

$$
\begin{aligned}
&2r_0+2r_1\theta+2\sum_{n=2}^{\infty}r_n\theta^n+4r_1\theta+4\sum_{n=2}^{\infty}nr_n\theta^n+\sum_{n=2}^{\infty}n(n-1)r_n\theta^n-2r_2-6r_3\theta\\
&-\sum_{n=2}^{\infty}(n+1)(n+2)r_{n+2}\theta^n-2p_1^2-8p_1p_2\theta-4p_0p_2-12p_0p_3\theta\\
&-\sum_{n=2}^{\infty}\sum_{j=0}^{n}(j+1)(n-j+1)p_{j+1}p_{n-j+1}\theta^n-2\sum_{n=2}^{\infty}\sum_{j=0}^{n}(n-j+1)(n-j+2)p_jp_{n-j+2}\theta^n=0.
\end{aligned}\tag{5.37}
$$

On comparing the coefficients of θ^n from Eqs. (5.35)–(5.37), for $n=0$, we obtain

$$p_2=\frac{1}{2\alpha^2}\Big[(\rho^2-\gamma^2)p_0p_0-\rho\alpha p_0-\frac{\alpha}{\kappa_1}p_0r_0+\alpha^2p_0q_1^2\Big],\tag{5.38}$$

$$q_2=\frac{1}{2\alpha^2p_0}\Big[2\gamma\rho p_0-\gamma\alpha p_0-2\alpha^2p_1q_1\Big],\tag{5.39}$$

$$r_2=\frac{1}{2}\Big[2r_0-4p_0p_2-2p_1^2\Big],\tag{5.40}$$

where p_2 is given by Eq. (5.38).

Next, for $n=1,2,\ldots$, we presented the general recurrence relations as follows:

$$
\begin{aligned}
p_{n+2}=\frac{1}{\alpha^2(n+1)(n+2)}\Big[&(\rho^2-\gamma^2)p_n-2\alpha\rho np_n+\alpha^2p_n-\rho\alpha p_n+2\alpha\gamma\sum_{j=0}^{n-1}(n-j)p_jp_{n-j}\\
&+\alpha^2n(n-1)p_n+\alpha^2\sum_{j=0}^{n}\sum_{i=0}^{j}(i+1)(j-i+1)q_{i+1}q_{j-i+1}p_{n-j}-\frac{\alpha}{\kappa_1}\sum_{j=0}^{n}p_jr_{n-j}\\
&-\alpha^2\sum_{j=0}^{n-2}\sum_{i=0}^{j}(i+1)(j-i+1)q_{i+1}q_{j-i+1}p_{n-2-j}\Big],
\end{aligned}\tag{5.41}
$$

$$
\begin{aligned}
q_{n+2}=\frac{1}{\alpha^2(n+1)(n+2)p_0}\Big[&2\gamma\rho p_n-2\alpha\gamma np_n-\alpha\gamma p_n-2\alpha\rho\sum_{j=0}^{n-1}(n-j)p_jq_{n-j}\\
&+\alpha^2\sum_{j=0}^{n-1}(n-j)p_jq_{n-j}+2\alpha^2\sum_{j=0}^{n-2}(j+1)(n-j-1)p_{j+1}q_{n-j-1}
\end{aligned}
$$

$$
\begin{aligned}
& -2\alpha^2 \sum_{j=0}^{n} (j+1)(n-j+1) p_{j+1} q_{n-j+1} \\
& + \alpha^2 \sum_{j=0}^{n-2} (n-j)(n-j-1) p_j q_{n-j} - \alpha^2 \sum_{j=1}^{n} (n-j+1)(n-j+2) p_j q_{n-j+2} \Bigg], \qquad (5.42)
\end{aligned}
$$

$$
\begin{aligned}
r_{n+2} = {} & \frac{1}{(n+1)(n+2)} \Bigg[2r_n + 4nr_n + n(n-1)r_n - 2\sum_{j=0}^{n} (j+1)(n-j+1) p_{j+1} p_{n-j+1} \\
& -2\sum_{j=0}^{n} (n-j+1)(n-j+2) p_j p_{n-j+2} \Bigg]. \qquad (5.43)
\end{aligned}
$$

Using recurrence relations, we can find the other terms of the sequences $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$, and $\{r_n\}_{n=0}^{\infty}$. This indicates that for system (4.12), a series solutions (5.1) is feasible with the coefficients given by Eqs. (5.41), (5.42), and (5.43).

Therefore, the power series solution of equations (4.12) is given by

$$
\begin{aligned}
U(\theta) &= p_0 + p_1\theta + \frac{1}{2\alpha^2}\Big[(\rho^2-\gamma^2)p_0p_0 - \rho\alpha p_0 - \frac{\alpha}{\kappa_1}p_0r_0 + \alpha^2 p_0 q_1^2\Big]\theta^2 + \sum_{k=1}^{\infty} p_{k+2}\theta^{k+2}, \\
V(\theta) &= q_0 + q_1\theta + \frac{1}{2\alpha^2 p_0}\Big[2\gamma\rho p_0 - \gamma\alpha p_0 - 2\alpha^2 p_1 q_1\Big]\theta^2 + \sum_{k=1}^{\infty} q_{k+2}\theta^{k+2}, \\
G(\theta) &= r_0 + r_1\theta + \frac{1}{2}\Big[2r_0 - 4p_0p_2 - 2p_1^2\Big]\theta^2 + \sum_{k=1}^{\infty} r_{k+2}\theta^{k+2}, \qquad (5.44)
\end{aligned}
$$

where p_i, and $q_i (i = 0, 1)$ are arbitrary constants & $p_{k+2}, q_{k+2}, r_{k+2} (k \geq 1)$ are given by Eqs. (5.41), (5.42) and (5.43). □

Remark: Convergence of the power series solution in *Case 2* can be proved similar to *Case 1*.

6 Conclusions

In this work, the Klein–Gordon–Zakharov equations with time-dependent variable coefficients have been examined using the Lie group method of infinitesimals. This approach is efficient and feasible for the analysis of nonlinear evolution equations. We furnished the Lie symmetries of the Eq. (1.2) alongwith the forms of variable coefficients. The obtained symmetries reduce the Eq. (1.2) to the system of nonlinear ordinary differential equations. Due to the lack of Lie point symmetries of the obtained system of second-order nonlinear ODEs, we employed the power series method on the nonlinear ODEs (4.6), (4.9), and (4.12) which allowed us to present the exact series solutions. We also demonstrated the convergence of the series solutions.

Funding Not applicable.

Data availability Not applicable.

Declarations

Conflict of interest The authors declare no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Akbari, M.; Taghizadeh, N.: Applications of He's variational principle method and modification of truncated expansion method to the coupled Klein-Gordon-Zakharov equations. Ain Shams Engineering Journal **5**, 979–983 (2014)
2. Akbulut, A., Kaplan, M., Tascan, F.: Conservation laws and Exact Solutions of Phi-Four (Phi-4) Equation via the $(G^{'}/G, 1/G)$-Expansion Method, (2016)
3. Akbulut, A., Tascan, F.: Application of conservation theorem and modified extended tanh-function method to (1+1)-dimensional nonlinear coupled Klein-Gordon-Zakharov equation, Chaos, Solitons & Fractals, vol.104, pp.-40, (2017)
4. Baskonus, H.; Sulaiman, T.; Bulut, H.: On the new wave behavior to the Klein-Gordon-Zakharov equations in plasma physics. Indian Journal Physics **93**, 393–399 (2019)
5. Bluman, G.W., Anco, S.C.: Symmetry and integration methods for differential equations, Applied Mathematical sciences, 154, Springer. (2002)
6. Bluman, G.W., Cole, J.D.: Similarity methods for differential equations, Springer. (1974)
7. Chen, Y.; Wang, Q.: Extended Jacobi elliptic function rational expansion method and abundant families of Jacobi elliptic function solutions to (1+1) dimensional dispersive long wave equation. Chaos, Solitons Fractals **24**, 745–757 (2005)
8. Clarkson, P.: New similarity reductions for the modified Boussinesq equation. J. Phys. A: Math. Gen. **22**, 2355–2367 (1989)
9. Devi, P.; Singh, K.: Exact solutions of the (2+1)-dimensional Boiti-Leon-Pempinelli system by first integral method. AIP Conf. Proc. **2061**, 020014 (2019)
10. Devi, P.; Singh, K.: Exact traveling wave solutions of the (2+1)-dimensional Boiti-Leon-Pempinelli system using $(G^{'}/G^2)$ expansion method. AIP Conf. Proc. **2214**, 020030 (2020)
11. Devi, P.; Singh, K.: Classical Lie symmetries and similarity reductions of the (2+1)-dimensional dispersive long wave system. Asian European Journal of Mathematics **14**(4), 2150052 (2021)
12. Hafez, M.; Akbar, M.: New exact traveling wave solutions to the (1+1)-dimensional Klein-Gordon-Zakharov equation for wave propagation in plasma using the exp-$(\phi(\xi))$-expansion method. Propulsion and Power Research **4**, 31–39 (2015)
13. Hill, J.M.: Solution of differential equations by means of one-parameter groups, Pitman Advanced Publishing Progam, (1982)
14. Ibragimov (Ed.), N.H.: CRC handbook of Lie group analysis of differential equations, CRC Press, Boca Raton, Florida, Vols. I, II, III, (1994-1996)
15. Ibragimov, N.H.: CRC handbook of Lie group analysis of differential equations: Symmetries, Exact solutions, and Conservation laws, CRC Press, Inc, Boca Raton, Florida, vol. 1, (1994)
16. Ibragimov, N.H.: Transformation groups applied to mathematical physics, Mathematics and its Applications, (1985)
17. Ibragimov, N.H.: CRC Handbook of Lie group analysis of differential equations. CRC Press, Boca Raton (1995)
18. Ibragimov, N.H.: A new conservation theorem. J. Math. Anal. Appl. **333**(1), 311–328 (2007)
19. Ibragimov, R.N.; Ibragimov, N.H.; Galiakberova, L.R.: Conservation laws and invariant solutions of the non-linear governing equations associated with a thermodynamic model of a rotating detonation engines with Korobeinikov's chemical source term. Int. J. Non-Linear Mech. **78**, 29–34 (2016)
20. Johnpillai, A.; Khalique, C.: Group analysis of KdV equation with time dependent coefficients. Appl. Math. Comput. **216**, 3761–3771 (2010)
21. Johnpillai, A.; Khalique, C.; Biswas, A.: Exact solutions of the mKdV equation with time-dependent coefficients. Math. Commun. **16**, 509–518 (2011)
22. Kudryashov, N.A.: Simplest equation method to look for exact solutions of nonlinear differential equations. Chaos, Solitons & Fractals **24**, 1217–1231 (2005)
23. Lie, S.: Vorlesungenuber differentialgleichungen mit Bekannten infinitesimalen transformationen. Teubner, Leipzig (1912)
24. Liu, H.; Geng, Y.: Symmetry reductions and exact solutions to the systems of carbon nanotubes conveying fluid. J. Differential Equations **254**, 2289–2303 (2013)
25. Liu, H.; Li, J.: Lie symmetry analysis and exact solutions for the short pulse equation. Nonlinear Anal. **71**, 2126–2133 (2009)
26. Liu, H.; Li, J.; Liu, L.: Painlevé analysis, Lie symmetries, and exact solutions for the time-dependent coefficients Gardner equations. Nonlinear Dyn. **59**, 497–502 (2010)
27. Liu, H.; Li, J.; Zhang, Q.: Lie symmetry analysis and exact explicit solutions for general Burger's equation. J. Comput. Appl. Math. **228**, 1–9 (2009)
28. Mhlanga, I., Khalique, C.: Exact solutions of the symmetric regularized long wave equation and the Klein-Gordon-Zakharov equations, Abstract and Applied Analysis, vol. Article ID 679016, 7 pages. (2014)
29. Olver, P.J.: Applications of Lie Groups to Differential Equations. Springer, New York (1986)
30. Quan, X.; Cheng, X.: Exact Solutions of (2 + 1)-dimensional Boiti-Leon-Pempinelle equation with $(G^{'}/G)$ - expansion method. Commun. Theor. Phys. **54**(1), 35–37 (2010)
31. Ravi, L., Saha, Ray, S., Sahoo, S.: New exact solutions of coupled Boussinesq-Burger's equations by exp-function method, Journal of Ocean Engineering and Science, vol. 2(1), pp. 34-46, (2017)
32. Rosa, R., Gandarias, M., Bruzón, M.: Symmetries and conservation laws of a fifth-order KdV equation with time-dependent coefficients and linear damping, Nonlinear Dynamics, vol. 84, pp. 135-141, (2016)
33. Rudin, W.: Principles of Mathematical Analysis. China Machine Press, Beijing (2004)
34. Shang, Y.; Huang, Y.; Yuan, W.: New exact traveling wave solutions for the Klein-Gordon-Zakharov equations. Comput. Math. Appl. **56**, 1441–1450 (2008)

35. Shi, Y.; Dai, Z.; Li, D.: The correct traveling wave solutions for the high-order dispersive nonlinear Schrödinger equation. Appl. Math. Comput. **216**(5), 1583–1591 (2010)
36. Shi, Q.; Xiao, Q.; Liu, X.: Extended wave solutions for a nonlinear Klein-Gordon-Zakharov system. Appl. Math. Comput. **218**, 9922–9929 (2012)
37. Tascan, F.; Yakut, A.: Conservation Laws and Exact Solutions with Symmetry Reduction of Nonlinear Reaction Diffusion Equations. International Journal of Nonlinear Sciences and Numerical Simulation **16**(3–4), 191–196 (2015)
38. Triki, H.; Boucerredj, N.: Soliton solutions of the Klein-Gordon-Zakharov equations with power law nonlinearity. Appl. Math. Comput. **227**, 341–346 (2014)
39. Tu, J.; Tian, S.; Xu, M.; Zhang, T.: On Lie symmetries, optimal systems and explicit solutions to the Kudryashov-Sinelshchikov equation. Appl. Math. Comput. **275**, 345–352 (2016)
40. Wang, T.; Chen, J.; Zhang, L.: Conservative difference methods for the Klein-Gordon-Zakharov equations. J. Comput. Appl. Math. **205**, 430–452 (2007)
41. Wazwaz, A.M.: The Hirota's direct method and the tanh-coth method for multiple-soliton solutions of the Sawada-Kotera-Ito seventh-order equation. Appl. Math. Comput. **199**(1), 133–138 (2008)
42. Zahran, E.; Khater, M.: Modified extended tanh-function method and its applications to the Bogoyavlenskii equation. Appl. Math. Model. **40**, 1769–1775 (2016)
43. Zayed, E.M., Joudi, S.: Applications of an extended $(\frac{G'}{G})$-expansion method to find exact solutions of nonlinear PDEs in mathematical physics, Mathematical Problems in Engineering, Article ID 768573, 19 pages, (2010).
44. Zhang, Y.; Li, J.; Lv, Y.N.: The exact solution and integrable properties to the variable coefficient modified Korteweg-de Vries equation. Ann. Phys. **323**, 3059–3064 (2008)
45. Zhang, Z.; Xi, F.; Li, X.: Bifurcation analysis and the travelling wave solutions of the Klein-Gordon-Zakharov equations. Pramana Journal of Physics **80**, 41–59 (2013)
46. Zhang, Y.; Zhang, L.; Pang, J.: Application of $(\frac{G'}{G^2})$-expansion method for solving Schrödinger's equation with three-order dispersion. Adv. Appl. Math. **6**, 212–217 (2017)