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B. Y. Chen-Ricci inequalities for anti-invariant Riemannian submersions in Kenmotsu space forms

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Abstract The aim of the present paper is to analyze sharp type inequalities including the scalar and Ricci curvatures of anti-invariant Riemannian submersions in Kenmotsu space forms $K_s(\varepsilon)$. We give non-trivial examples for anti-invariant Riemannian submersions, investigate some curvature relations between the total space and fibres according to vertical and horizontal cases of ξ. Moreover, we acquire Chen-Ricci inequalities on the $\ker \vartheta_*$ and $(\ker \vartheta_*)^{\perp}$ distributions for anti-invariant Riemannian submersions from Kenmotsu space forms according to vertical and horizontal cases of ξ.

Mathematics Subject Classification 53C25 · 53C15 · 53C43

1 Introduction

In [6], Chen defised the intrinsic (the Ricci curvature and the scalar curvature) and extrinsic (the squared mean curvature) invariants in a real space form $R^k(\varepsilon)$ that determinated an inequality containing Ricci curvature, the scalar curvature and squared mean curvature of a submanifold. A generalization of this inequality for arbitrary submanifolds in an arbitrary Riemannian manifold was proved by Chen in [7]. Later, this inequality has been extensively studied for different ambient spaces by some authors with some results ([1–3, 12, 13, 15, 16, 19, 20, 22–25]). Chen published a book containing all the work in this direction in 2011 [8].

As pointed out in [9, 17], an important interest in Riemannian geometry is that some geometrical properties of suitable map types between Riemannian manifolds. In this manual, O'Neill [21] and Gray [10] defined the concept of Riemannian submersions as follows:

A differentiable map $\vartheta : (K_s, g_{K_s}) \to (R_m, g_{R_m})$ between Riemannian manifolds is called a Riemannian submersion if ϑ_* is onto and $g_{R_m}(\vartheta_*\chi_1, \vartheta_*\chi_2) = g_{K_s}(\chi_1, \chi_2)$ for vector fields $\chi_1, \chi_2 \in (\ker \vartheta_*)^{\perp}$. Şahin investigated anti-invariant Riemannian submersions from almost Hermitian manifolds in [18]. In [4], Berri et al. investigated anti-invariant submersions from Kenmotsu manifolds. In [11], Gülbahar *et al.* acquired sharp inequalities involving the Ricci curvature for invariant Riemannian submersions. Inspired by the above studies, in this study we take into account anti-invariant Riemannian submersions (AIRS) from Kenmotsu manifolds to Riemannian manifolds and get sharp inequalities involving scalar curvature and Ricci curvature.

The aim of the present article is to examine the sharp type inequalities of AIRSs in Kenmotsu space forms including scalar and Ricci curvatures. The systematic of the article is prepared as follows: After remembering some basic formulas and definitions in the second section, we explore various inequalities including Ricci and scalar curvatures on $\ker \vartheta_*$ and $(\ker \vartheta_*)^{\perp}$ distributions of AIRSs in Kenmotsu space forms in the third section and finally, we acquire Chen-Ricci inequalities on $\ker \vartheta_*$ and $(\ker \vartheta_*)^{\perp}$ of AIRSs in Kenmotsu space forms.

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2 Preliminaries

Let K_s be a $(2n+1)-$dimensional smooth manifold. Then, K_s has an almost contact structure if there exist a tensor field endomorphism P of type$-(1,1)$, a vector field ξ, and 1-form η on K_s such that

$$P^2E_1 = -E_1 + \eta(E_1)\xi,\ P\xi = 0,\ \eta \circ P = 0,\ \eta(\xi) = 1. \tag{2.1}$$

If there exists a Riemannian metric g_{K_s} on an almost contact manifold K_s satisfying:

$$g_{K_s}(PE_1, PE_2) = g_{K_s}(E_1, E_2) - \eta(E_1)\eta(E_2),$$
$$g_{K_s}(E_1, PE_2) = -g_{K_s}(PE_1, E_2), \tag{2.2}$$
$$\eta(E_1) = g_{K_s}(E_1, \xi), \tag{2.3}$$

where E_1, E_2 are any vector fields on K_s, then K_s is called an almost contact metric manifold [5] with an almost contact structure (P, ξ, η, g_{K_s}) and is symbolized by $(K_s, P, \xi, \eta, g_{K_s})$. An almost contact metric manifold is called Kenmotsu if the Riemannian connection ∇^1 of g_{K_s} satisfies [14]

$$(\nabla^1_{E_1} P)E_2 = -g_{K_s}(E_1, PE_2)\xi - \eta(E_2)PE_1$$
$$\nabla^1_{E_1}\xi = E_1 - \eta(E_1)\xi. \tag{2.4}$$

A Kenmotsu manifold with constant $P-$holomorphic sectional curvature ε is called a Kenmotsu space form and is denoted by $K_s(\varepsilon)$. Then its curvature tensor R_{K_s} is given by [14]

$$R^{K_s}(E_1, E_2, E_3, E_4) = \frac{\varepsilon - 3}{4}\begin{bmatrix} g_{K_s}(E_2, E_3)g_{K_s}(E_1, E_4) \\ -g_{K_s}(E_1, E_3)g_{K_s}(E_2, E_4) \end{bmatrix} + \frac{\varepsilon + 1}{4}\begin{bmatrix} \eta(E_1)\eta(E_3)g_{K_s}(E_2, E_4) \\ -\eta(E_2)\eta(E_3)g_{K_s}(E_1, E_4) \\ +\eta(E_2)\eta(E_4)g_{K_s}(E_1, E_3) \\ -\eta(E_1)\eta(E_4)g_{K_s}(E_2, E_3) \\ -g_{K_s}(PE_1, E_3)g_{K_s}(PE_2, E_4) \\ +g_{K_s}(PE_2, E_3)g_{K_s}(PE_1, E_4) \\ +2g_{K_s}(E_1, PE_2)g_{K_s}(PE_3, E_4) \end{bmatrix}, \tag{2.5}$$

for all $E_1, E_2, E_3 \in \Gamma(K_s)$.

Let (K_s, g_{K_s}) and (R_m, g_{R_m}) be Riemannian manifolds such that a smooth map $\vartheta : (K_s, g_{K_s}) \to (R_m, g_{R_m})$ is a Riemannian submersion which is onto and provides the following conditions:

i. $\vartheta_{*t} : T_pK_s \to T_{\vartheta(t)}R_m$ is onto for all $t \in K_s$;

ii. the fibres ϑ_s^{-1}, $s \in R_m$, are Riemannian submanifolds of K_s;

iii. ϑ_{*t} preserves the length of the horizontal vectors.

A Riemannian submersion $\vartheta : K_s \to R_m$ defines two $(1,2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ on K_s, by the formulae [21]:

$$\mathcal{T}_{E_1}E_2 = \mathcal{T}(E_1, E_2) = h\nabla^1_{vE_1}vE_2 + v\nabla^1_{vE_1}hE_2, \tag{2.6}$$

and

$$\mathcal{A}_{E_1}E_2 = \mathcal{A}(E_1, E_2) = v\nabla^1_{hE_1}hE_2 + h\nabla^1_{hE_1}vE_2, \tag{2.7}$$

for all $E_1, E_2 \in \Gamma(K_s)$. Where h and v the horizontal and vertical projections, respectively.

Lemma 2.1 ([21]) *Let $\vartheta : (K_s, g_{K_s}) \to (R_m, g_{R_m})$ be a Riemannian submersion. Then, we have:*

$$\mathcal{A}_{\chi_1}\chi_2 = -\mathcal{A}_{\chi_2}\chi_1,$$
$$\mathcal{T}_{\gamma_1}\gamma_2 = \mathcal{T}_{\gamma_2}\gamma_1, \tag{2.8}$$
$$g_{K_s}(\mathcal{T}_{\gamma_1}E_1, E_2) = -g_{K_s}(\mathcal{T}_{\gamma_1}E_2, E_1),$$
$$g_{K_s}(\mathcal{A}_{\chi_1}E_1, E_2) = -g_{K_s}(\mathcal{A}_{\chi_1}E_2, E_1), \tag{2.9}$$

for $\chi_1, \chi_2 \in \Gamma((\ker \vartheta_*)^\perp)$, $\gamma_1, \gamma_2 \in \Gamma(\ker \vartheta_*)$, $E_1, E_2 \in \Gamma(K_s)$.

Let R^{K_s}, R^{R_m}, $R^{\ker\vartheta_*}$ and $R^{(\ker\vartheta_*)^\perp}$ represent the Riemannian curvature tensors of Riemannian manifolds K_s, R_m, $\ker\vartheta_*$ and $(\ker\vartheta_*)^\perp$, respectively.

Lemma 2.2 ([21]) *Let* $\vartheta : (K_s, g_{K_s}) \to (R_m, g_{R_m})$ *be a Riemannian submersion. Then, we have:*

$$\begin{aligned} R^{K_s}(\gamma_1,\gamma_2,\gamma_3,\gamma_4) &= R^{\ker\vartheta_*}(\gamma_1,\gamma_2,\gamma_3,\gamma_4) + g_{K_s}(\mathcal{T}_{\gamma_1}\gamma_4, \mathcal{T}_{\gamma_2}\gamma_3) \\ &\quad - g_{K_s}(\mathcal{T}_{\gamma_2}\gamma_4, \mathcal{T}_{\gamma_1}\gamma_3) \end{aligned} \tag{2.10}$$

$$\begin{aligned} R^{K_s}(\chi_1,\chi_2,\chi_3,\chi_4) &= R^{(\ker\vartheta_*)^\perp}(\chi_1,\chi_2,\chi_3,\chi_4) - 2g_{K_s}(\mathcal{A}_{\chi_1}\chi_2, \mathcal{A}_{\chi_3}\chi_4) \\ &\quad + g_{K_s}(\mathcal{A}_{\chi_2}\chi_3, \mathcal{A}_{\chi_1}\chi_4) - g_{K_s}(\mathcal{A}_{\chi_1}\chi_3, \mathcal{A}_{\chi_2}\chi_4) \end{aligned} \tag{2.11}$$

$$\begin{aligned} R^{K_s}(\chi_1,\gamma_1,\chi_2,\gamma_2) &= g_{K_s}((\nabla^1_{\chi_1}\mathcal{T})(\gamma_1,\gamma_2),\chi_2) + g_{K_s}((\nabla^1_{\gamma_1}\mathcal{A})(\chi_1,\chi_2),\gamma_2) \\ &\quad - g_{K_s}(\mathcal{T}_{\gamma_1}\chi_1, \mathcal{T}_{\gamma_2}\chi_2) + g_{K_s}(\mathcal{A}_{\chi_2}\gamma_2, \mathcal{A}_{\chi_1}\gamma_1) \end{aligned} \tag{2.12}$$

for all $\chi_1, \chi_2, \chi_3, \chi_4 \in \Gamma((\ker\vartheta_*)^\perp)$ *and* $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma(\ker\vartheta_*)$.

Also, the $\mathcal{H}$ mean curvature vector field of all fiber of ϑ stated

$$\mathcal{H} = \frac{1}{\kappa}\mathcal{N}, \;\; \mathcal{N} = \sum_{p=1}^{\kappa} \mathcal{T}_{\gamma_p}\gamma_p, \tag{2.13}$$

where $\{\gamma_1, \gamma_2, \ldots, \gamma_p\}$ creates an orthonormal basis for $\ker\vartheta_*$. Further, if $\mathcal{T} = 0$ on $\ker\vartheta_*$ and $(\ker\vartheta_*)^\perp$, then ϑ has totally geodesic fibres.

Definition 2.3 [18] Let (K_s, g_{K_s}, P) and (R_m, g_{R_m}) be a Kaehler manifold and a Riemannian manifold, repectively. $\vartheta : (K_s, g_{K_s}, P) \to (R_m, g_{R_m})$ is called anti-invariant, if $\ker\vartheta_*$ is anti-invariant with respect to P, i.e. $P(\ker\vartheta_*) \subseteq (\ker\vartheta_*)^\perp$.

From above definition, we get $P(\ker\vartheta_*) \cap (\ker\vartheta_*)^\perp \neq \{0\}$. We denote the complementary orthogonal distribution to $P(\ker\vartheta_*)$ in $(\ker\vartheta_*)^\perp$ by ζ. Then, we obtain

$$(\ker\vartheta_*)^\perp = P(\ker\vartheta_*) \oplus \zeta.$$

It is straightforward to show that ζ is an invariant distribution of $(\ker\vartheta_*)^\perp$ under the endomorphism P. So, for $\chi_1 \in \Gamma(\ker\vartheta_*)^\perp$, we can state

$$P\chi_1 = \alpha\chi_1 + \beta\chi_1, \tag{2.14}$$

here $\alpha\chi_1 \in \Gamma(\ker\vartheta_*)$ and $\beta\chi_1 \in \Gamma(\zeta)$.

Lemma 2.4 *Let* $\vartheta : (K_s, g_{K_s}) \to (R_m, g_{R_m})$ *be an AIRS from a Kenmotsu manifold to a Riemannian manifold. Then following statements are true:*

i. If ξ *is vertical, then* $\beta^2\chi_1 = -\chi_1 - P\alpha\chi_1$ *and* $\alpha\beta = 0$,
ii. If ξ *is horizontal, then* $\beta^2\chi_1 = -\chi_1 + \eta(\chi_1)\xi - P\alpha\chi_1 = P^2\chi_1 - P\alpha\chi_1$ *and* $\alpha\beta = 0$,
iii. $g_{K_s}(\chi_1, PC_\sigma)g_{K_s}(PC_\sigma, \chi_1) = g_{K_s}(\chi_1, \chi_1) + g_{K_s}(\chi_1, P\alpha\chi_1)$.

Example 2.5 Let $K_s = \mathbb{R}^{2n+1}$ be an Euclidean space with the standard coordinate functions $(u_1, ...u_n, v_1, ...v_n, t)$ and its usual Kenmotsu structure (P, ξ, η, g_{K_s}) stand for

$$\begin{aligned} \eta &= dt, \\ \xi &= \frac{\partial}{\partial t}, \\ g_{K_s} &= e^{2t}\left\{\sum_{\iota=1}^{n} du_\iota \otimes du_\iota + dv_\iota \otimes dv_\iota\right\} + dt \otimes dt, \\ P\left\{U_\iota\frac{\partial}{\partial u_\iota} + V_\iota\frac{\partial}{\partial v_\iota} + T\frac{\partial}{\partial t}\right\} &= \sum_{\iota=1}^{n}\left(-V_\iota\frac{\partial}{\partial u_\iota} + U_\iota\frac{\partial}{\partial v_\iota}\right). \end{aligned}$$

Then $(\mathbb{R}^{2n+1}, P, \xi, \eta, g_{K_s})$ is a Kenmotsu space form with constant $P-$ sectional curvature $\varepsilon = 3$. The vector fields

$$E_\iota = e^{-t}\frac{\partial}{\partial v_\iota}, E_{\iota+n} = PU_\iota = e^{-t}\left(\frac{\partial}{\partial u_\iota} + v_\iota \frac{\partial}{\partial t}\right), \xi = \frac{\partial}{\partial t},$$

create a $g_{K_s}-$orthonormal basis for the contact metric structure.

Example 2.6 Let $K_s = \mathbb{R}^5(3)$ be Kenmotsu space form with the structure given in Example 2.5. The Riemannian metric $g_{R_m} = g_{\mathbb{R}^2}$ stand for $g_{\mathbb{R}^2} = e^{2t}(du \otimes du + dv \otimes dv)$. Let $\vartheta : \mathbb{R}^5(3) \to \mathbb{R}^2$ be a map given by

$$\vartheta(u_1, u_2, v_1, v_2, t) = \left(\frac{u_1 - v_1}{\sqrt{2}}, \frac{u_2 - v_2}{\sqrt{2}}\right).$$

Then the kernel of ϑ_* is

$$\ker \vartheta_* = \text{Span}\left\{\gamma_1 = \frac{1}{\sqrt{2}}(E_1 + E_3), \gamma_2 = \frac{1}{\sqrt{2}}(E_2 + E_4), \gamma_3 = E_5 = \xi\right\},$$

and

$$(\ker \vartheta_*)^\perp = \text{Span}\left\{\chi_1 = \frac{1}{\sqrt{2}}(E_1 - E_3), \chi_2 = \frac{1}{\sqrt{2}}(E_2 - E_4)\right\}.$$

Thus, ϑ is a Riemannnian submersion. Furthermore, $P\gamma_1 = -\chi_1$, $P\gamma_2 = -\chi_2$ and $P\gamma_3 = P\xi = 0$ imply that $P(\ker \vartheta_*) = (\ker \vartheta_*)^\perp$. Hence ϑ is an anti-invariant Riemannnian submersion such that ξ is vertical.

Example 2.7 Let $K_s = \mathbb{R}^5(3)$ be Kenmotsu space form with the structure given in Example 2.5. The Riemannian metric $g_{R_m} = g_{\mathbb{R}^3}$ stand for $g_{\mathbb{R}^2} = e^{2t}(du \otimes du + dv \otimes dv) + dt \otimes dt$. Let $\vartheta : \mathbb{R}^5(3) \to \mathbb{R}^3$ be a map given by

$$\vartheta(u_1, u_2, v_1, v_2, t) = \left(\frac{u_1 - v_1}{\sqrt{2}}, \frac{u_2 - v_2}{\sqrt{2}}, t\right).$$

Then the kernel of ϑ_* is

$$\ker \vartheta_* = \text{Span}\left\{\gamma_1 = \frac{1}{\sqrt{2}}(E_1 + E_3), \gamma_2 = \frac{1}{\sqrt{2}}(E_2 + E_4)\right\},$$

and

$$(\ker \vartheta_*)^\perp = \text{Span}\left\{\chi_1 = \frac{1}{\sqrt{2}}(E_1 - E_3), \chi_2 = \frac{1}{\sqrt{2}}(E_2 - E_4), \chi_3 = E_5 = \xi\right\}.$$

Thus, ϑ is a Riemannnian submersion. Furthermore, $P\gamma_1 = -\chi_1$, $P\gamma_2 = -\chi_2$ imply that $P(\ker \vartheta_*) \subset (\ker \vartheta_*)^\perp = P(\ker \vartheta_*) \oplus \{\xi\}$. Thus, ϑ is an anti-invariant Riemannnian submersion such that ξ is horizontal.

3 Basic inequalities

For basic inequalities, we first give the following result. Since ϑ is an AIRS, and using (2.10) and (2.5) we obtain:

Lemma 3.1 *$(K_s(\varepsilon), g_{K_s})$ and (R_m, g_{R_m}) indicate a Kenmotsu space form and a Riemannian manifold and let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})$ be an AIRS such that ξ is vertical . Then, any for $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma(\ker\vartheta_*)$ we obtain*

$$\begin{aligned} R^{\ker\vartheta_*}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) &= \frac{\varepsilon - 3}{4}\{g_{K_s}(\gamma_1, \gamma_4)g_{K_s}(\gamma_2, \gamma_3) - g_{K_s}(\gamma_1, \gamma_3)g_{K_s}(\gamma_2, \gamma_4)\} \\ &+ \frac{\varepsilon + 1}{4}\{\eta(\gamma_1)\eta(\gamma_3)g_{K_s}(\gamma_2, \gamma_4) - \eta(\gamma_2)\eta(\gamma_3)g_{K_s}(\gamma_1, \gamma_4) \\ &+ \eta(\gamma_2)\eta(\gamma_4)g_{K_s}(\gamma_1, \gamma_3) - \eta(\gamma_1)\eta(\gamma_4)g_{K_s}(\gamma_2, \gamma_3)\} \\ &- g_{K_s}(\mathcal{T}_{\gamma_1}\gamma_4, \mathcal{T}_{\gamma_2}\gamma_3) + g_{K_s}(\mathcal{T}_{\gamma_2}\gamma_4, \mathcal{T}_{\gamma_1}\gamma_3), \qquad (3.1) \\ K^{\ker\vartheta_*}(\gamma_1, \gamma_2) &= \frac{\varepsilon - 3}{4}\{g^2_{K_s}(\gamma_1, \gamma_2) - \|\gamma_1\|^2\|\gamma_2\|^2\} \\ &+ \frac{\varepsilon + 1}{4}\left\{\begin{array}{c}\eta(\gamma_1)^2\|\gamma_2\|)^2 - 2\eta(\gamma_1)\eta(\gamma_2)g_{K_s}(\gamma_1, \gamma_2) \\ +\eta(\gamma_2)^2\|\gamma_1\|)^2\end{array}\right\} \\ &- \|\mathcal{T}_{\gamma_1}\gamma_2\|^2 + g_{K_s}(\mathcal{T}_{\gamma_2}\gamma_2, \mathcal{T}_{\gamma_1}\gamma_1), \qquad (3.2) \end{aligned}$$

here $K^{\ker\vartheta_*}$ is called bi-sectional curvature of $\ker\vartheta_*$.

Lemma 3.2 *$(K_s(\varepsilon), g_{K_s})$ and (R_m, g_{R_m}) indicate a Kenmotsu space form and a Riemannian manifold and let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})$ be an AIRS such that ξ is horizontal . Then, any for $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma(\ker\vartheta_*)$ we obtain*

$$\begin{aligned} R^{\ker\vartheta_*}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) &= \frac{\varepsilon - 3}{4}\{g_{K_s}(\gamma_1, \gamma_4)g_{K_s}(\gamma_2, \gamma_3) - g_{K_s}(\gamma_1, \gamma_3)g_{K_s}(\gamma_2, \gamma_4)\} \\ &- g_{K_s}(\mathcal{T}_{\gamma_1}\gamma_4, \mathcal{T}_{\gamma_2}\gamma_3) + g_{K_s}(\mathcal{T}_{\gamma_2}\gamma_4, \mathcal{T}_{\gamma_1}\gamma_3), \qquad (3.3) \\ K^{\ker\vartheta_*}(\gamma_1, \gamma_2) &= \frac{\varepsilon - 3}{4}\{g^2_{K_s}(\gamma_1, \gamma_2) - \|\gamma_1\|^2\|\gamma_2\|^2\} \\ &- \|\mathcal{T}_{\gamma_1}\gamma_2\|^2 + g_{K_s}(\mathcal{T}_{\gamma_2}\gamma_2, \mathcal{T}_{\gamma_1}\gamma_1), \qquad (3.4) \end{aligned}$$

here $K^{\ker\vartheta_*}$ is called bi-sectional curvature of vertical distribution $\ker\vartheta_*$.

For $(\ker\vartheta_*)^\perp$, since ϑ is an anti-invariant Riemannian submersion, and using (2.5), (2.8), (2.11) and (2.14) we obtain:

Lemma 3.3 *$(K_s(\varepsilon), g_{K_s})$ and (R_m, g_{R_m}) denote a Kenmotsu space form and a Riemannian manifold and let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})$ be an AIRS such that ξ is vertical. Then, for $\chi_1, \chi_2, \chi_3, \chi_4 \in \Gamma((\ker\vartheta_*)^\perp)$ we have*

$$\begin{aligned} R^{(\ker\vartheta_*)^\perp}(\chi_1, \chi_2, \chi_3, \chi_4) &= \frac{\varepsilon - 3}{4}\left\{g_{K_s}(\chi_1, \chi_4)g_{K_s}(\chi_2, \chi_3) - g_{K_s}(\chi_1, \chi_3)g_{K_s}(\chi_2, \chi_4)\right\} \\ &+ \frac{\varepsilon + 1}{4}\left\{\begin{array}{c}-g_{K_s}(\beta\chi_1, \chi_3)g_{K_s}(\beta\chi_2, \chi_4) + g_{K_s}(\beta\chi_2, \chi_3)g_{K_s}(\beta\chi_1, \chi_4) \\ +2g_{K_s}(\chi_1, \beta\chi_2)g_{K_s}(\beta\chi_3, \chi_4)\end{array}\right\} \\ &+ 2g_{K_s}(\mathcal{A}_{\chi_1}\chi_2, \mathcal{A}_{\chi_3}\chi_4) - g_{K_s}(\mathcal{A}_{\chi_2}\chi_3, \mathcal{A}_{\chi_1}\chi_4) \\ &+ g_{K_s}(\mathcal{A}_{\chi_1}\chi_3, \mathcal{A}_{\chi_2}\chi_4), \qquad (3.5) \\ B^{(\ker\vartheta_*)^\perp}(\chi_1, \chi_2) &= \frac{\varepsilon - 3}{4}\left\{g^2_{K_s}(\chi_1, \chi_2) - \|\chi_1\|^2\|\chi_2\|^2\right\} \\ &\frac{\varepsilon + 1}{4}\left\{-3g^2_{K_s}(\chi_1, \beta\chi_2)\right\} + 3\|\mathcal{A}_{\chi_1}\chi_2\|^2, \end{aligned}$$

here $B^{(\ker\vartheta_*)^\perp}$ is called bi-sectional curvature of horizontal distribution $(\ker\vartheta_*)^\perp$.

Lemma 3.4 *$(K_s(\varepsilon), g_{K_s})$ and (R_m, g_{R_m}) denote a Kenmotsu space form and a Riemannian manifold and let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})$ be an AIRS such that ξ is horizontal . Then, for $\chi_1, \chi_2, \chi_3, \chi_4 \in \Gamma((\ker\vartheta$*

$_*)^{\perp}$) *we have*

$$R^{(\ker\vartheta_*)^{\perp}}(\chi_1,\chi_2,\chi_3,\chi_4)=\frac{\varepsilon-3}{4}\left\{g_{K_s}(\chi_1,\chi_4)g_{K_s}(\chi_2,\chi_3)-g_{K_s}(\chi_1,\chi_3)g_{K_s}(\chi_2,\chi_4)\right\}$$
$$+\frac{\varepsilon+1}{4}\left\{\begin{array}{c}\eta(\chi_1)\eta(\chi_3)g_{K_s}(\chi_2,\chi_4)-\eta(\chi_2)\eta(\chi_3)g_{K_s}(\chi_1,\chi_4)\\ +\eta(\chi_2)\eta(\chi_4)g_{K_s}(\chi_1,\chi_3)-\eta(\chi_1)\eta(\chi_4)g_{K_s}(\chi_2,\chi_3)\\ -g_{K_s}(\beta\chi_1,\chi_3)g_{K_s}(\beta\chi_2,\chi_4)+g_{K_s}(\beta\chi_2,\chi_3)g_{K_s}(\beta\chi_1,\chi_4)\\ +2g_{K_s}(\chi_1,\beta\chi_2)g_{K_s}(\beta\chi_3,\chi_4)\end{array}\right\}$$
$$+2g_{K_s}(\mathcal{A}_{\chi_1}\chi_2,\mathcal{A}_{\chi_3}\chi_4)-g_{K_s}(\mathcal{A}_{\chi_2}\chi_3,\mathcal{A}_{\chi_1}\chi_4)$$
$$+g_{K_s}(\mathcal{A}_{\chi_1}\chi_3,\mathcal{A}_{\chi_2}\chi_4), \tag{3.6}$$
$$B^{(\ker\vartheta_*)^{\perp}}(\chi_1,\chi_2)=\frac{\varepsilon-3}{4}\left\{g^2_{K_s}(\chi_1,\chi_2)-\|\chi_1\|^2\|\chi_2\|^2\right\}$$
$$\frac{\varepsilon+1}{4}\left[\begin{array}{c}\eta(\chi_1)^2\|\chi_2\|)^2-2\eta(\chi_1)\eta(\chi_2)g_{K_s}(\chi_1,\chi_2)\\ +\eta(\chi_2)^2\|\chi_1\|)^2-3g_{K_s}(\chi_1,\beta\chi_2)^2\end{array}\right]+3\|\mathcal{A}_{\chi_1}\chi_2\|^2,$$

where $B^{(\ker\vartheta_*)^{\perp}}$ *is called bi-sectional curvature of horizontal distribution* $(\ker\vartheta_*)^{\perp}$.

Let $\vartheta: K_s(\varepsilon)\to R_m$ be an AIRS from a Kenmotsu space form to a Riemannian manifold. For any point $k\in K_s$, let $\{B_1,\ldots,B_\kappa,C_1,\ldots,C_d\}$ be an orthonormal basis of $T_kK_s(\varepsilon)$ such that $\ker\vartheta_*=\mathrm{Span}\{B_1,\ldots,B_\kappa\}$, $(\ker\vartheta_*)^{\perp}=\mathrm{Span}\{C_1,\ldots,C_d\}$.

Lemma 3.5 *Let* $\vartheta:(K_s(\varepsilon),g_{K_s})\to(R_m,g_{R_m})$ *be an AIRS from a Kenmotsu space form to a Riemannian manifold. Then, we have*

$$g_{K_s}(\chi_1,PC_\sigma)g_{K_s}(PC_\sigma,\chi_1)=g_{K_s}(\chi_1,\chi_1)+g_{K_s}(\chi_1,P\alpha\chi_1),$$
$$g_{K_s}(C_\sigma,PC_\sigma)g_{K_s}(PC_\sigma,\chi_1)=d+tr(P\alpha)).$$

Case 1: Assume that ξ is vertical

Now, for the $\ker\vartheta_*$ if we take $\gamma_4=\gamma_1$ and $\gamma_2=\gamma_3=B_\iota$, $\iota=1,2,\ldots,\kappa$ in (3.1), and using (2.13) then we get

$$Ric^{\ker\vartheta_*}(\gamma_1)=\frac{\varepsilon-3}{4}(\kappa-1)g_1(\gamma_1,\gamma_1)+\frac{\varepsilon+1}{4}\left\{(2-\kappa)\eta(\gamma_1)^2-g_{K_s}(\gamma_1,\gamma_1)\right\}$$
$$-\kappa g_{K_s}(\mathcal{T}_{\gamma_1}\gamma_1,\mathcal{H})+\sum_{\iota=1}^{\kappa}g_{K_s}(\mathcal{T}_{B_\iota}\gamma_1,\mathcal{T}_{\gamma_1}B_\iota). \tag{3.7}$$

From here, we get:

Proposition 3.6 *Let* $\vartheta:(K_s(\varepsilon),g_{K_s})\to(R_m,g_{R_m})$ *be an AIRS from a Kenmotsu space form to a Riemannian manifold such that* ξ *is vertical. Then, we have*

$$Ric^{\ker\vartheta_*}(\gamma_1)\geq\frac{\varepsilon-3}{4}(\kappa-1)+\frac{\varepsilon+1}{4}\left\{(2-\kappa)\eta(\gamma_1)^2-1\right\}-\kappa g_{K_s}(\mathcal{T}_{\gamma_1}\gamma_1,\mathcal{H}).$$

For a unit vertical vector $\gamma_1\in\Gamma(\ker\vartheta_*)$*, the equality status of the inequality is valid if and only if every fiber is totally geodesic.*

If we take $\gamma_1=B_\sigma,\sigma=1,\ldots,\kappa$ in (3.7) and using (2.8), then we acquire

$$2\rho^{\ker\vartheta_*}=\frac{\varepsilon-3}{4}\kappa(\kappa-1)-\frac{\varepsilon+1}{2}(\kappa-1)$$
$$-\kappa^2\|\mathcal{H}\|^2+\sum_{\iota,\sigma=1}^{\kappa}g_{K_s}(\mathcal{T}_{B_\iota}B_\sigma,\mathcal{T}_{B_\sigma}B_\iota),$$

where $\rho^{\ker\vartheta_*}=\sum_{1\leq\iota,\sigma\leq\kappa}Ric^{\ker\vartheta_*}(B_\sigma,B_\iota,B_\iota,B_\sigma)$. Therefore, we can state the following result.

Proposition 3.7 *Let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})$ be an AIRS from a Kenmotsu space form to a Riemannian manifold such that ξ is vertical. Then, we have*

$$2\rho^{\ker\vartheta_*} \geq \frac{\varepsilon-3}{4}\kappa(\kappa-1) - \frac{\varepsilon+1}{2}(\kappa-1) - \kappa^2\|\mathcal{H}\|^2.$$

The equality status of the inequality is valid if and only if every fiber is totally geodesic.

Now, for the horizontal distribution if we take $\chi_4 = \chi_1$ and $\chi_2 = \chi_3 = C_\sigma,\ \ \sigma = 1, 2, \ldots, d$ in (3.5), using (2.8), Lemma (3.5) and Lemma (2.4) then we get

$$\begin{aligned} Ric^{(\ker\vartheta_*)^\perp}(\chi_1) \qquad &= \tfrac{\varepsilon-3}{4}\{(d-1)g_{K_s}(\chi_1,\chi_1)\} \\ &+\tfrac{3\varepsilon+3}{4}\left\{g_{K_s}(\chi_1,\chi_1) + g_{K_s}(\chi_1, P\alpha\chi_1)\right\} \\ &-3\textstyle\sum_{\sigma=1}^{d} g_{K_s}(\mathcal{A}_{\chi_1}C_\sigma, \mathcal{A}_{\chi_1}C_\sigma). \end{aligned} \tag{3.8}$$

Taking $\chi_1 = C_\iota, \iota = 1, 2, \ldots, d$ in (3.8) and using Lemma (3.5) then we have:

$$2\rho^{(\ker\vartheta_*)^\perp} = \frac{\varepsilon-3}{4}d(d-1) + 3\frac{\varepsilon+1}{4}\{d + tr(P\alpha)\} - 3\sum_{\iota,\sigma=1}^{d} g_1(\mathcal{A}_{C_\iota}C_\sigma, \mathcal{A}_{C_\iota}C_\sigma), \tag{3.9}$$

where $\rho^{(\ker\vartheta_*)^\perp} = \sum_{\iota,\sigma=1}^{d} Ric^{(\ker\vartheta_*)^\perp}(C_\iota, C_\sigma, C_\sigma, C_\iota)$. Then, we can write

$$2\rho^{(\ker\vartheta_*)^\perp} \leq \frac{\varepsilon-3}{4}d(d-1) + 3\frac{\varepsilon+1}{4}\{d + tr(P\alpha)\}. \tag{3.10}$$

Thus, we can give:

Proposition 3.8 *Let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})$ be an AIRS from a Kenmotsu space form to a Riemannian manifold such that ξ is vertical. Then, we have*

$$2\rho^{(\ker\vartheta_*)^\perp} \leq \frac{\varepsilon-3}{4}d(d-1) + 3\frac{\varepsilon+1}{4}\{d + tr(P\alpha)\}.$$

The equality status of (3.10) satisfies if and only if $(\ker\vartheta_*)^\perp$ *is integrable.*

Case 2: Assume that ξ is horizontal.

Now, for the vertical distribution if we take $\gamma_4 = \gamma_1$ and $\gamma_2 = \gamma_3 = B_\iota,\ \ \iota = 1, 2, \ldots, \kappa$ in (3.3), and using (2.13) then we arrive at

$$\begin{aligned} Ric^{\ker\vartheta_*}(\gamma_1) &= \frac{\varepsilon-3}{4}(\kappa-1)g_1(\gamma_1,\gamma_1) \\ &\quad - \kappa g_{K_s}(\mathcal{T}_{\gamma_1}\gamma_1, \mathcal{H}) + \sum_{\iota=1}^{\kappa} g_{K_s}(\mathcal{T}_{B_\iota}\gamma_1, \mathcal{T}_{\gamma_1}B_\iota). \end{aligned} \tag{3.11}$$

From here, we have:

Proposition 3.9 *Let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})$ be an AIRS from a Kenmotsu space form to a Riemannian manifold such that ξ is horizontal. Then, we have*

$$Ric^{\ker\vartheta_*}(\gamma_1) \geq \frac{\varepsilon-3}{4}(\kappa-1) - \kappa g_{K_s}(\mathcal{T}_{\gamma_1}\gamma_1, \mathcal{H}).$$

For a unit vertical vector $\gamma_1 \in \Gamma(\ker\vartheta_*)$, the equality status of the inequality is valid if and only if each fiber is totally geodesic.

If we take $\gamma_1 = B_\sigma, \sigma = 1, \ldots, \kappa$ in (3.11) and using (2.8), then we acquire

$$2\rho^{\ker\vartheta_*} = \frac{\varepsilon-3}{4}\kappa(\kappa-1) - \kappa^2\|\mathcal{H}\|^2 + \sum_{\iota,\sigma=1}^{\kappa} g_{K_s}(\mathcal{T}_{B_\iota}B_\sigma, \mathcal{T}_{B_\sigma}B_\iota),$$

where $\rho^{\ker\vartheta_*} = \sum_{1\leq\iota,\sigma\leq\kappa} Ric^{\ker\vartheta_*}(B_\sigma, B_\iota, B_\iota, B_\sigma)$. Therefore, we can state the following result.

Proposition 3.10 *Let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})$ be an AIRS from a Kenmotsu space form to a Riemannian manifold such that ξ is horizontal. Then, we have*

$$2\rho^{\ker\vartheta_*} \geq \frac{\varepsilon-3}{4}\kappa(\kappa-1) - \kappa^2\|\mathcal{H}\|^2.$$

The equality status of the inequality is valid if and only if each fiber is totally geodesic.

Now, for the horizontal distribution if we take $\chi_4 = \chi_1$ and $\chi_2 = \chi_3 = C_\sigma,\ \ \sigma = 1, 2, \ldots, d$ in (3.6), using (2.8), Lemma (3.5) and Lemma (2.4) then we get

$$\begin{aligned} Ric^{(\ker\vartheta_*)^\perp}(\chi_1) &= \frac{\varepsilon-3}{4}\{(d-1)g_{K_s}(\chi_1,\chi_1)\} \\ &\quad + \frac{\varepsilon+1}{4}\left\{\begin{array}{l} -\eta(C_\sigma)^2 g_{K_s}(\chi_1,\chi_1) + 2\eta(\chi_1)\eta(C_\sigma)g_{K_s}(\chi_1, C_\sigma) \\ -d\eta(\chi_1)^2 + 3\left\{g_{K_s}(\chi_1,\chi_1) + g_{K_s}(\chi_1, P\alpha\chi_1)\right\} \end{array}\right\} \\ &\quad - 3\sum_{\sigma=1}^{d} g_{K_s}(\mathcal{A}_{\chi_1}C_\sigma, \mathcal{A}_{\chi_1}C_\sigma). \end{aligned} \tag{3.12}$$

Taking $\chi_1 = C_\iota, \iota = 1, 2, \ldots, d$ in (3.12) using Lemma (3.5) then we have:

$$2\rho^{(\ker\vartheta_*)^\perp} = \tfrac{\varepsilon-3}{4}d(d-1) + \tfrac{\varepsilon+1}{4}(d + 3tr(P\alpha)) - 3\textstyle\sum_{\iota,\sigma=1}^{d} g_1(\mathcal{A}_{C_\iota}C_\sigma, \mathcal{A}_{C_\iota}C_\sigma), \tag{3.13}$$

where $\rho^{(\ker\vartheta_*)^\perp} = \sum_{\iota,\sigma=1}^{d} Ric^{(\ker\vartheta_*)^\perp}(C_\iota, C_\sigma, C_\sigma, C_\iota)$. Then, we can write

$$2\rho^{(\ker\vartheta_*)^\perp} \leq \frac{\varepsilon-3}{4}d(d-1) + \frac{\varepsilon+1}{4}(d + 3tr(P\alpha)). \tag{3.14}$$

Thus, we can give:

Proposition 3.11 *Let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})$ be an AIRS from a Kenmotsu space form to a Riemannian manifold such that ξ is horizontal. Then, we have*

$$2\rho^{(\ker\vartheta_*)^\perp} \leq \frac{\varepsilon-3}{4}d(d-1) + \frac{\varepsilon+1}{4}(d + 3tr(P\alpha)).$$

The equality status of (3.14) is valid if and only if $(\ker\vartheta_*)^\perp$ is integrable.

4 Chen-Ricci inequalities

In this section, we aim to derive the Chen-Ricci inequality in vertical and horizontal distributions for AIRSs from Kenmotsu space forms to Riemannian manifold. Equality situations will also be evaluated.

Let $(K_s(\varepsilon), g_{K_s})$ be a Kenmotsu space form, (R_m, g_{R_m}) a Riemannian manifold and $\vartheta : K_s(\varepsilon) \to R_m$ be an AIRS. For every point $k \in K_s$, let $\{B_1, \ldots, B_\kappa, C_1, \ldots, C_d\}$ be an orthonormal basis of $T_k K_s(\varepsilon)$ such that $\ker\vartheta_* = \mathrm{span}\{B_1, \ldots, B_\kappa\}$ and $(\ker\vartheta_*)^\perp = \mathrm{span}\{C_1, \ldots, C_d\}$. Let's denote $\mathcal{T}^p_{\iota\sigma}$ by

$$\mathcal{T}^p_{\iota\sigma} = g_1(\mathcal{T}_{B_\iota}B_\sigma, C_p), \tag{4.1}$$

where $1 \leq \iota, \sigma \leq \kappa$ and $1 \leq p \leq d$. Similarly, let's denote $\mathcal{A}^\alpha_{\iota\sigma}$ by

$$\mathcal{A}^\alpha_{\iota\sigma} = g_1(\mathcal{A}_{C_\iota}C_\sigma, B_\alpha), \tag{4.2}$$

in which $1 \leq \iota, \sigma \leq d$ and $1 \leq \alpha \leq \kappa$ and we employee

$$\delta(\mathcal{N}) = \sum_{k=1}^{\kappa}\sum_{\iota=1}^{d}((\nabla^1_{C_\iota}\mathcal{T})_{B_k}B_k, C_\iota). \tag{4.3}$$

Case 1: Assume that ξ is vertical
Now, from (3.1), we acquire

$$2\rho^{\ker\vartheta_*} = \frac{\varepsilon-3}{4}\kappa(\kappa-1) - \frac{\varepsilon+1}{2}(\kappa-1) - \kappa^2\|\mathcal{H}\|^2 + \sum_{\iota,\sigma=1}^{\kappa} g_{K_s}(\mathcal{T}_{B_\iota}B_\sigma, \mathcal{T}_{B_\sigma}B_\iota).$$

Using (2.8) and (4.1), we obtain

$$2\rho^{\ker\vartheta_*} = \frac{\varepsilon-3}{4}\kappa(\kappa-1) - \frac{\varepsilon+1}{2}(\kappa-1) - \kappa^2\|\mathcal{H}\|^2 + \sum_{p=1}^{d}\sum_{\iota,\sigma=1}^{\kappa}(\mathcal{T}_{\iota\sigma}^{p})^2. \tag{4.4}$$

On the other hand, from [11], we know that

$$\sum_{p=1}^{d}\sum_{\iota,\sigma=1}^{\kappa}(\mathcal{T}_{\iota\sigma}^{p})^2 = \frac{1}{2}\kappa^2\|\mathcal{H}\|^2 + \frac{1}{2}\sum_{p=1}^{d}\left[\mathcal{T}_{11}^{p} - \mathcal{T}_{22}^{p} - \cdots - \mathcal{T}_{\kappa\kappa}^{p}\right]^2$$
$$+ 2\sum_{p=1}^{d}\sum_{\sigma=2}^{\kappa}(\mathcal{T}_{1\sigma}^{p})^2 - 2\sum_{p=1}^{d}\sum_{2\le\iota<\sigma\le\kappa}\left[\mathcal{T}_{\iota\iota}^{p}\mathcal{T}_{\sigma\sigma}^{p} - \left(\mathcal{T}_{\iota\sigma}^{p}\right)^2\right]. \tag{4.5}$$

If we put (4.5) in (4.4), we get

$$2\rho^{\ker\vartheta_*} = \frac{\varepsilon-3}{4}\kappa(\kappa-1) - \frac{\varepsilon+1}{2}(\kappa-1) - \frac{\kappa^2}{2}\|\mathcal{H}\|^2$$
$$+ \frac{1}{2}\sum_{p=1}^{d}\left[\mathcal{T}_{11}^{p} - \mathcal{T}_{22}^{p} - \cdots - \mathcal{T}_{\kappa\kappa}^{p}\right]^2 + 2\sum_{p=1}^{d}\sum_{\sigma=2}^{\kappa}(\mathcal{T}_{1\sigma}^{p})^2$$
$$- 2\sum_{p=1}^{d}\sum_{2\le\iota<\sigma\le\kappa}\left[\mathcal{T}_{\iota\iota}^{p}\mathcal{T}_{\sigma\sigma}^{p} - \left(\mathcal{T}_{\iota\sigma}^{p}\right)^2\right].$$

From here, we get

$$2\rho^{\ker\vartheta_*} \ge \frac{\varepsilon-3}{4}\kappa(\kappa-1) - \frac{\varepsilon+1}{2}(\kappa-1) - \frac{\kappa^2}{2}\|\mathcal{H}\|^2$$
$$-2\sum_{p=1}^{d}\sum_{2\le\iota<\sigma\le\kappa}\left[\mathcal{T}_{\iota\iota}^{p}\mathcal{T}_{\sigma\sigma}^{p} - \left(\mathcal{T}_{\iota\sigma}^{p}\right)^2\right]. \tag{4.6}$$

Also, from (2.10), taking $\gamma_1 = \gamma_4 = B_\iota$, $\gamma_2 = \gamma_3 = B_\sigma$ and using (4.1), we get

$$2\sum_{2\le\iota<\sigma\le\kappa} R^{K_s}(B_\iota, B_\sigma, B_\sigma, B_\iota) = 2\sum_{2\le\iota<\sigma\le\kappa} R^{\ker\vartheta_*}(B_\iota, B_\sigma, B_\sigma, B_\iota)$$
$$+2\sum_{p=1}^{d}\sum_{2\le\iota<\sigma\le\kappa}\left[\mathcal{T}_{\iota\iota}^{p}\mathcal{T}_{\sigma\sigma}^{p} - \left(\mathcal{T}_{\iota\sigma}^{p}\right)^2\right].$$

From the last equality, (4.6) can be written as

$$2\rho^{\ker\vartheta_*} \ge \frac{\varepsilon-3}{4}\kappa(\kappa-1) - \frac{\varepsilon+1}{2}(\kappa-1) - \frac{\kappa^2}{2}\|\mathcal{H}\|^2$$
$$+2\sum_{2\le\iota<\sigma\le\kappa} R^{\ker\vartheta_*}(B_\iota, B_\sigma, B_\sigma, B_\iota) - 2\sum_{2\le\iota<\sigma\le\kappa} R^{K_s}(B_\iota, B_\sigma, B_\sigma, B_\iota). \tag{4.7}$$

Furthermore, we know that

$$2\rho^{\ker\vartheta_*}=2\sum_{2\le\iota<\sigma\le\kappa}R^{\ker\vartheta_*}(B_\iota,B_\sigma,B_\sigma,B_\iota)$$
$$+2\sum_{\sigma=1}^{\kappa}R^{\ker\vartheta_*}(B_1,B_\sigma,B_\sigma,B_1). \tag{4.8}$$

If we put the last equality in (4.7) and taking trace, then we get

$$2Ric^{\ker\vartheta_*}(B_1)\ge\frac{\varepsilon-3}{4}\kappa(\kappa-1)-\frac{\varepsilon+1}{2}(\kappa-1)-\frac{\kappa^2}{2}\|\mathcal{H}\|^2$$
$$-2\sum_{2\le\iota<\sigma\le\kappa}R^{K_s}(B_\iota,B_\sigma,B_\sigma,B_\iota).$$

Since K_s is a Kenmotsu space form, then curvature tensor R^{K_s} of K_s satisfies equation (2.5), so, we obtain

$$Ric^{\ker\vartheta_*}(B_1)\ge\frac{\varepsilon-3}{4}(\kappa-1)+\frac{\varepsilon+1}{4}\left[(2-\kappa)\eta(B_1)^2-1\right]-\frac{\kappa^2}{4}\|\mathcal{H}\|^2.$$

So, we can construct the following theorem:

Theorem 4.1 *$(K_s(\varepsilon),g_{K_s})$ and (R_m,g_{R_m}) denote a Kenmotsu space form and a Riemannian manifold and let $\vartheta:(K_s(\varepsilon),g_{K_s})\to(R_m,g_{R_m})$ be an AIRS such that ξ is vertical. Then we have*

$$Ric^{\ker\vartheta_*}(B_1)\ge\frac{\varepsilon-3}{4}(\kappa-1)+\frac{\varepsilon+1}{4}\left[(2-\kappa)\eta(B_1)^2-1\right]$$
$$-\frac{\kappa^2}{4}\|\mathcal{H}\|^2.$$

The equality status of the inequality is valid if and only

$$\mathcal{T}_{11}^{p}=\mathcal{T}_{22}^{p}+\cdots+\mathcal{T}_{\kappa\kappa}^{p}$$
$$\mathcal{T}_{1\sigma}^{p}=0,\sigma=2,\ldots,\kappa.$$

From (3.9), we have

$$2\rho^{(\ker\vartheta_*)^\perp}=\frac{\varepsilon-3}{4}d(d-1)+3\frac{\varepsilon+1}{4}\{d+tr(P\alpha)\}-3\sum_{\iota,\sigma=1}^{d}g_1(\mathcal{A}_{C_\iota}C_\sigma,\mathcal{A}_{C_\iota}C_\sigma).$$

Using (2.14) and (4.2), then we get

$$2\rho^{(\ker\vartheta_*)^\perp}=\frac{\varepsilon-3}{4}d(d-1)+3\frac{\varepsilon+1}{4}\{d+tr(P\alpha)\}-3\sum_{\alpha=1}^{\kappa}\sum_{\iota,\sigma=1}^{d}(\mathcal{A}_{\iota\sigma}^{\alpha})^2. \tag{4.9}$$

From (2.8), then (4.9) turns into

$$2\rho^{(\ker\vartheta_*)^\perp}=\frac{\varepsilon-3}{4}d(d-1)+3\frac{\varepsilon+1}{4}\{d+tr(P\alpha)\}$$
$$-6\sum_{\alpha=1}^{\kappa}\sum_{\sigma=2}^{d}(\mathcal{A}_{1\sigma}^{\alpha})^2-6\sum_{\alpha=1}^{\kappa}\sum_{2\le\iota<\sigma\le d}(\mathcal{A}_{\iota\sigma}^{\alpha})^2. \tag{4.10}$$

Furthermore, from (2.11), taking $\chi_1=\chi_4=C_\iota$, $\chi_2=\chi_3=C_\sigma$ and using (4.2) we obtain

$$2\sum_{2\le\iota<\sigma\le d}R^{K_s}(C_\iota,C_\sigma,C_\sigma,C_\iota)=2\sum_{2\le\iota<\sigma\le d}R^{(\ker\vartheta_*)^\perp}(C_\iota,C_\sigma,C_\sigma,C_\iota)$$
$$+6\sum_{\alpha=1}^{\kappa}\sum_{2\le\iota<\sigma\le d}(\mathcal{A}_{\iota\sigma}^{\alpha})^2. \tag{4.11}$$

If we consider (4.11) in (4.10), then we have

$$2\rho^{(\ker\vartheta_*)^\perp}=\frac{\varepsilon-3}{4}d(d-1)+3\frac{\varepsilon+1}{4}\{d+tr(P\alpha)\}-6\sum_{\alpha=1}^{\kappa}\sum_{\sigma=2}^{d}(\mathcal{A}_{1\sigma}^{\alpha})^2$$
$$+2\sum_{2\le\iota<\sigma\le d}R^{(\ker\vartheta_*)^\perp}(C_\iota,C_\sigma,C_\sigma,C_\iota)-2\sum_{2\le\iota<\sigma\le d}R^{K_s}(C_\iota,C_\sigma,C_\sigma,C_\iota).$$

Since K_s is a Kenmotsu space form, curvature tensor R^{K_s} of K_s satisfies (2.5), thus we obtain

$$2Ric^{(\ker\vartheta_*)^\perp}(C_1)=\tfrac{\varepsilon-3}{4}(d-1)+3\tfrac{\varepsilon+1}{4}\left\|\beta C_1\right\|^2-6\textstyle\sum_{\alpha=1}^{\kappa}\sum_{\sigma=2}^{d}(\mathcal{A}_{1\sigma}^{\alpha})^2.$$

Then, we can write

$$Ric^{(\ker\vartheta_*)^\perp}(C_1)\le\frac{\varepsilon-3}{4}(d-1)+3\frac{\varepsilon+1}{4}\left\|\beta C_1\right\|^2.$$

So, we can construct the following theorem:

Theorem 4.2 *$(K_s(\varepsilon),g_{K_s})$ and (R_m,g_{R_m}) denote a Kenmotsu space form and a Riemannian manifold and let $\vartheta:(K_s(\varepsilon),g_{K_s})\to(R_m,g_{R_m})$ be an AIRS such that ξ is vertical. Then we have*

$$Ric^{(\ker\vartheta_*)^\perp}(C_1)\le\frac{\varepsilon-3}{4}(d-1)+\frac{\varepsilon+1}{4}3\left\|\beta C_1\right\|^2,$$

the equality status of the inequality is valid if and only

$$\mathcal{A}_{1\sigma}=0,\sigma=2,\ldots,d.$$

Next, for the case of ξ is vertical, we can specify the inequality of Chen Ricci between the $\ker\vartheta_*$ and $(\ker\vartheta_*)^\perp$. The ρ scalar curvature of $K_s(\varepsilon)$ is given by

$$2\rho=\sum_{p=1}^{d}Ric(C_p,C_p)+\sum_{k=1}^{\kappa}Ric(B_k,C_k),$$
$$2\rho=\sum_{\sigma,k=1}^{\kappa}R^{K_s}(B_\sigma,B_k,B_k,B_\sigma)+\sum_{\iota=1}^{d}\sum_{k=1}^{\kappa}R^{K_s}(C_\iota,B_k,B_k,C_\iota)$$
$$+\sum_{\iota,p=1}^{d}R^{K_s}(C_\iota,C_p,C_p,C_\iota)+\sum_{p=1}^{d}\sum_{\sigma=1}^{\kappa}R^{K_s}(B_\sigma,C_p,C_p,B_\sigma).\tag{4.12}$$

Using (4.12), (2.5) and since $K_s(\varepsilon)$ is a Kenmotsu space form, we get

$$2\rho=\frac{\varepsilon-3}{4}\{(\kappa+d)(\kappa+d-1)\}+\frac{\varepsilon+1}{4}\{4\kappa+d-4+3tr(P\alpha)\}.\tag{4.13}$$

Furthermore, using (2.10), (2.11) and (2.12), we acquire the ρ scalar curvature of $K_s(\varepsilon)$ as:

$$\begin{aligned}
2\rho = {} & 2\rho^{\ker\vartheta_*} + 2\rho^{(\ker\vartheta_*)^\perp} + \kappa^2\|\mathcal{H}\|^2 \\
& + \sum_{\sigma,k=1}^{\kappa} g_1(\mathcal{T}_{B_k}B_\sigma, \mathcal{T}_{B_k}B_\sigma) + 3\sum_{\iota,p=1}^{d} g_1(\mathcal{A}_{C_\iota}C_p, \mathcal{A}_{C_\iota}C_p) \\
& - \sum_{\iota=1}^{d}\sum_{k=1}^{\kappa} g_1((\nabla^1_{C_\iota}\mathcal{T})_{B_k}B_k, C_\iota) \\
& + \sum_{\iota=1}^{d}\sum_{k=1}^{\kappa} \left\{ g_1(\mathcal{T}_{B_k}C_\iota, \mathcal{T}_{B_k}C_\iota) - g_1(\mathcal{A}_{C_\iota}B_k, \mathcal{A}_{C_\iota}B_k) \right\} \\
& - \sum_{p=1}^{d}\sum_{\sigma=1}^{\kappa} g_1((\nabla^1_{C_p}\mathcal{T})_{B_\sigma}B_\sigma, C_p) \\
& + \sum_{p=1}^{d}\sum_{\sigma=1}^{\kappa} \left\{ g_1(\mathcal{T}_{B_\sigma}C_p, \mathcal{T}_{B_\sigma}C_p) - g_1(\mathcal{A}_{C_p}B_\sigma, \mathcal{A}_{C_p}B_\sigma) \right\}.
\end{aligned} \tag{4.14}$$

Using (4.8), (4.3), (4.11) and (4.13), we obtain

$$\begin{aligned}
2\rho = {} & 2\rho^{\ker\vartheta_*} + 2\rho^{(\ker\vartheta_*)^\perp} + \frac{1}{2}\kappa^2\|\mathcal{H}\|^2 \\
& - \frac{1}{2}\sum_{p=1}^{d}\left[\mathcal{T}^p_{11} - \mathcal{T}^p_{22} - \cdots - \mathcal{T}^p_{\kappa\kappa}\right]^2 \\
& - 2\sum_{p=1}^{d}\sum_{\sigma=2}^{\kappa}\left(\mathcal{T}^p_{1\sigma}\right)^2 + 2\sum_{p=1}^{d}\sum_{2\le\sigma<k\le\kappa}\left[\mathcal{T}^p_{\sigma\sigma}\mathcal{T}^p_{kk} - \left(\mathcal{T}^p_{\sigma k}\right)^2\right] \\
& + 6\sum_{\alpha=1}^{\kappa}\sum_{p=2}^{d}(\mathcal{A}^\alpha_{1p})^2 + 6\sum_{\alpha=1}^{\kappa}\sum_{2\le\iota<p\le d}(\mathcal{A}^\alpha_{\iota p})^2 \\
& + \sum_{\iota=1}^{d}\sum_{k=1}^{\kappa}\left\{ g_1(\mathcal{T}_{B_k}C_\iota, \mathcal{T}_{B_k}C_\iota) - g_1(\mathcal{A}_{C_\iota}B_k, \mathcal{A}_{C_\iota}B_k) \right\} \\
& - 2\delta(\mathcal{N}) + \sum_{p=1}^{d}\sum_{\sigma=1}^{\kappa}\left\{ g_1(\mathcal{T}_{B_\sigma}C_p, \mathcal{T}_{B_\sigma}C_p) - g_1(\mathcal{A}_{C_p}B_\sigma, \mathcal{A}_{C_p}B_\sigma) \right\}.
\end{aligned} \tag{4.15}$$

Using (4.7), (4.11) and (4.13) in the (4.15) then we have

$$\begin{aligned}
& \frac{\varepsilon-3}{2}\kappa d + \frac{\varepsilon+1}{2}\{3\kappa - 3 - d\} \\
& \quad + 2\sum_{\iota=1}^{\kappa} R^{K_s}(B_1, B_\iota, B_\iota, B_1) + 2\sum_{\sigma=1}^{d} R^{K_s}(C_1, C_\sigma, C_\sigma, C_1) \\
& \quad = 2Ric^{\ker\vartheta_*}(B_1) + 2Ric^{(\ker\vartheta_*)^\perp}(C_1) + \frac{1}{2}\kappa^2\|\mathcal{H}\|^2 \\
& \quad - \frac{1}{2}\sum_{p=1}^{d}\left[\mathcal{T}^p_{11} - \mathcal{T}^p_{22} - \cdots - \mathcal{T}^p_{\kappa\kappa}\right]^2 - 2\sum_{p=1}^{d}\sum_{\sigma=2}^{\kappa}\left(\mathcal{T}^p_{1\sigma}\right)^2 \\
& \quad + 6\sum_{\alpha=1}^{\kappa}\sum_{p=2}^{d}(\mathcal{A}^\alpha_{1p})^2 + \|\mathcal{T}^V\|^2 - \|\mathcal{A}^H\|^2 - 2\delta(\mathcal{N}),
\end{aligned}$$

here

$$\|\mathcal{A}^H\|^2 = \sum_{\iota=1}^{d}\sum_{k=1}^{\kappa} g_1(\mathcal{A}_{C_\iota}B_k, \mathcal{A}_{C_\iota}B_k)$$

and

$$\|\mathcal{T}^V\|^2 = \sum_{\iota=1}^{d}\sum_{k=1}^{\kappa} g_1(\mathcal{T}_{B_k}C_\iota, \mathcal{T}_{B_k}C_\iota).$$

From (2.5), since $K_s(\varepsilon)$ is a Kenmotsu space form, then we obtain the result:

Theorem 4.3 *$(K_s(\varepsilon), g_{K_s})$ and (R_m, g_{R_m}) denote a Kenmotsu space form and a Riemannian manifold and let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})$ be an AIRS such that ξ is vertical. Then we have*

$$\begin{aligned} &\frac{\varepsilon-3}{4}\{(\kappa+d+\kappa d-2)\} + \frac{\varepsilon+1}{4}\left\{\begin{array}{c} 3\kappa-d-4 \\ +(2-\kappa)\eta(B_1)^2 + 3\|\beta C_1\|^2 \end{array}\right\} \\ &\leq Ric^{\ker\vartheta_*}(B_1) + Ric^{(\ker\vartheta_*)^\perp}(C_1) + 3\sum_{\alpha=1}^{\kappa}\sum_{p=2}^{d}(\mathcal{A}_{1p}^{\alpha})^2 \\ &+\frac{1}{4}\kappa^2\|\mathcal{H}\|^2 + \|\mathcal{T}^V\|^2 - \|\mathcal{A}^H\|^2 - \delta(\mathcal{N}) \end{aligned}$$

the equality status of the inequality is valid if and only if

$$\mathcal{T}_{11}^p = \mathcal{T}_{22}^p + \cdots + \mathcal{T}_{\kappa\kappa}^p$$
$$\mathcal{T}_{1\sigma}^p = 0, \sigma = 2, \ldots, \kappa.$$

Corollary 4.4 *Let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})(K_s(\varepsilon), g_{K_s})$ be an AIRS from a Kenmotsu space form to a Riemannian manifold such that each fiber is totally geodesic and ξ is vertical. Then we have*

$$\begin{aligned} &\frac{\varepsilon-3}{4}\{(\kappa+d+\kappa d-2)\} + \frac{\varepsilon+1}{4}\left\{3\kappa-d-4+(2-\kappa)\eta(B_1)^2 + 3\|\beta C_1\|^2\right\} \\ &\leq Ric^{\ker\vartheta_*}(B_1) + Ric^{(\ker\vartheta_*)^\perp}(C_1) + 3\sum_{\alpha=1}^{\kappa}\sum_{p=2}^{d}(\mathcal{A}_{1p}^{\alpha})^2 - \|\mathcal{A}^H\|^2 \end{aligned} \tag{4.16}$$

Equality case of (4.16) holds if and only if $\mathcal{A}_{11} = \mathcal{A}_{11} = \ldots = \mathcal{A}_{dd}$ and $\mathcal{A}_{\iota\sigma} = 0$, for $\iota \neq \sigma \in \{1, 2, \ldots d\}$.

Corollary 4.5 *Let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})(K_s(\varepsilon), g_{K_s})$ be an AIRS from a Kenmotsu space form to a Riemannian manifold such that horizontal distribution is integrable and ξ is vertical. Then we have*

$$\begin{aligned} &\frac{\varepsilon-3}{4}\{(\kappa+d+\kappa d-2)\} + \frac{\varepsilon+1}{4}\left\{3\kappa-d-4+(2-\kappa)\eta(B_1)^2 + 3\|\beta C_1\|^2\right\} \\ &\leq Ric^{\ker\vartheta_*}(B_1) + Ric^{(\ker\vartheta_*)^\perp}(C_1) + \|\mathcal{T}^V\|^2 - \delta(\mathcal{N}) + \frac{1}{4}\kappa^2\|\mathcal{H}\|^2. \end{aligned} \tag{4.17}$$

Equality case of (4.17) holds if and only if the fibre of ϑ is a totally geodesic submanifold of $K_s(\varepsilon)$.

Case 2: Assume that ξ is horizontal

From (3.3), similar to Theorem (4.1), we can give the following result:

Theorem 4.6 *$(K_s(\varepsilon), g_{K_s})$ and (R_m, g_{R_m}) denote a Kenmotsu space form and a Riemannian manifold and let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})$ be an AIRS such that ξ is horizontal. Then we have*

$$Ric^{\ker\vartheta_*}(B_1) \geq \frac{\varepsilon-3}{4}(\kappa-1) - \frac{\kappa^2\|\mathcal{H}\|^2}{4}.$$

The equality status of the inequality is valid if and only

$$\mathcal{T}_{11}^p = \mathcal{T}_{22}^p + \cdots + \mathcal{T}_{\kappa\kappa}^p$$
$$\mathcal{T}_{1\sigma}^p = 0, \sigma = 2, \ldots, \kappa.$$

Similar to Theorem (4.2), from (3.6), we can give the result:

Theorem 4.7 *$(K_s(\varepsilon), g_{K_s})$ and (R_m, g_{R_m}) denote a Kenmotsu space form and a Riemannian manifold and let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})$ be an AIRS such that ξ is horizontal . Then we have*

$$Ric^{(\ker\vartheta_*)^{\perp}}(C_1) \leq \frac{\varepsilon-3}{4}(d-1) + \frac{\varepsilon+1}{4}\left\{(2-\kappa)\eta(C_1)^2 + 3\|\beta C_1\|^2 - 1\right\},$$

the equality status of the inequality is valid if and only

$$\mathcal{A}_{1\sigma} = 0, \sigma = 2, \ldots, d.$$

For the case of ξ is horizontal, we can express the inequality of Chen Ricci between the $\ker\vartheta_*$ and $(\ker\vartheta_*)^{\perp}$. From (4.12) we get

$$2\rho = \frac{\varepsilon-3}{4}\{(\kappa+d)(\kappa+d-1)\} + \frac{\varepsilon+1}{4}\{4\kappa+d-7+3tr(P\alpha)\}. \tag{4.18}$$

Using (4.18), (4.3), (4.5), (4.11), (4.8) and (4.14), then we have

$$\begin{aligned}
&\frac{\varepsilon-3}{2}\kappa d + \frac{\varepsilon+1}{2}(2\kappa-3) + 2\sum_{\iota=1}^{\kappa} R^{K_s}(B_1, B_\iota, B_\iota, B_1) + 2\sum_{\sigma=1}^{d} R^{K_s}(C_1, C_\sigma, C_\sigma, C_1)\\
&= 2Ric^{\ker\vartheta_*}(B_1) + 2Ric^{(\ker\vartheta_*)^{\perp}}(C_1) + \frac{1}{2}\kappa^2\|\mathcal{H}\|^2\\
&\quad - \frac{1}{2}\sum_{p=1}^{d}\left[\mathcal{T}_{11}^p - \mathcal{T}_{22}^p - \cdots - \mathcal{T}_{\kappa\kappa}^p\right]^2 - 2\sum_{p=1}^{d}\sum_{\sigma=2}^{\kappa}(\mathcal{T}_{1\sigma}^p)^2\\
&\quad + 6\sum_{\alpha=1}^{\kappa}\sum_{p=2}^{d}(\mathcal{A}_{1p}^{\alpha})^2 - \|\mathcal{A}^H\|^2 + \|\mathcal{T}^V\|^2 - 2\delta(\mathcal{N}).
\end{aligned}$$

Then, from (2.5), we can give the following result:

Theorem 4.8 *$(K_s(\varepsilon), g_{K_s})$ and (R_m, g_{R_m}) denote a Kenmotsu space form and a Riemannian manifold and let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})$ be an AIRS such that ξ is horizontal. Then we have*

$$\begin{aligned}
&\frac{\varepsilon-3}{4}\{(\kappa+d+\kappa d-2)\} + \frac{\varepsilon+1}{4}\left\{2\kappa-4+(2-\kappa)\eta(B_1)^2 + 3\|\beta C_1\|^2\right\}\\
&\leq Ric^{\ker\vartheta_*}(B_1) + Ric^{(\ker\vartheta_*)^{\perp}}(C_1) + 3\sum_{\alpha=1}^{\kappa}\sum_{p=2}^{d}(\mathcal{A}_{1p}^{\alpha})^2\\
&\quad + \|\mathcal{T}^V\|^2 + \frac{1}{4}\kappa^2\|\mathcal{H}\|^2 - \|\mathcal{A}^H\|^2 - \delta(\mathcal{N})
\end{aligned}$$

the equality status of the inequality is valid if and only if

$$\mathcal{T}_{11}^p = \mathcal{T}_{22}^p + \cdots + \mathcal{T}_{\kappa\kappa}^p$$
$$\mathcal{T}_{1\sigma}^p = 0, \sigma = 2, \ldots, \kappa.$$

Corollary 4.9 *Let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})(K_s(\varepsilon), g_{K_s})$ be an AIRS from a Kenmotsu space form to a Riemannian manifold such that each fiber is totally geodesic and ξ is horizontal. Then we have*

$$\begin{aligned}
&\frac{\varepsilon-3}{4}\{(\kappa+d+\kappa d-2)\} + \frac{\varepsilon+1}{4}\left\{2\kappa-4+(2-\kappa)\eta(B_1)^2 + 3\|\beta C_1\|^2\right\}\\
&\leq Ric^{\ker\vartheta_*}(B_1) + Ric^{(\ker\vartheta_*)^{\perp}}(C_1) + 3\sum_{\alpha=1}^{\kappa}\sum_{p=2}^{d}(\mathcal{A}_{1p}^{\alpha})^2 - \|\mathcal{A}^H\|^2
\end{aligned} \tag{4.19}$$

Equality case of (4.19) holds if and only if $\mathcal{A}_{11} = \mathcal{A}_{11} = \ldots = \mathcal{A}_{dd}$ and $\mathcal{A}_{\iota\sigma} = 0$, for $\iota \neq \sigma \in \{1, 2, \ldots d\}$.

Corollary 4.10 *Let $\vartheta : (K_s(\varepsilon), g_{K_s}) \to (R_m, g_{R_m})(K_s(\varepsilon), g_{K_s})$ be an AIRS from a Kenmotsu space form to a Riemannian manifold such that horizontal distribution is integrable and ξ is horizontal. Then we have*

$$\begin{aligned} &\frac{\varepsilon-3}{4}\{(\kappa+d+\kappa d-2)\}+\frac{\varepsilon+1}{4}\left\{2\kappa-4+(2-\kappa)\eta(B_1)^2+3\,\|\beta C_1\|^2\right\} \\ &\leq Ric^{\ker\vartheta_*}(B_1)+Ric^{(\ker\vartheta_*)^{\perp}}(C_1)+\|\mathcal{T}^V\|^2-\delta(\mathcal{N})+\frac{1}{4}\kappa^2\|\mathcal{H}\|^2 \end{aligned} \tag{4.20}$$

Equality case of (4.20) holds if and only if the fibre of ϑ is a totally geodesic submanifold of $K_s(\varepsilon)$.

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