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A note on the inhomogeneous fractional nonlinear Schrödinger equation

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Abstract This paper investigates some well-posedness issues of the fractional inhomogeneous Schrödinger equation

$$i\dot{u} - (-\Delta)^{\gamma} u = \pm |x|^{\rho} |u|^{p-1} u,$$

where $0 < \gamma < 1$ and $\rho < 0$. Here, one considers the inter-critical regime $0 < s_c := \frac{N}{2} - \frac{2\gamma+\rho}{p-1} < \gamma$, where s_c is the energy critical exponent, which is the only one real number satisfying $\|\kappa^{\frac{2\gamma+\rho}{p-1}} u_0(\kappa\cdot)\|_{\dot{H}^{s_c}} = \|u_0\|_{\dot{H}^{s_c}}$. In order to avoid a loss of regularity in Strichartz estimates, one assumes that the datum is spherically symmetric. First, using a sharp Gagliardo–Nirenberg-type estimate, one develops a local theory in the space $\dot{H}^{\gamma} \cap \dot{H}^{s_c}$. Then, one investigates the $L^{\frac{N(p-1)}{\rho+2\gamma}}$ concentration of finite-time blow-up solutions bounded in $\dot{H}^{s_c}$. Finally, one proves the existence of non-global solutions with negative energy. Since one considers the homogeneous Sobolev space $\dot{H}^{s_c}$, the main difficulty here is to avoid the mass conservation law.

Mathematics Subject Classification 35Q55

1 Introduction

This work deals with the initial value problem for an inhomogeneous nonlinear Schrödinger equation

$$\begin{cases} i\dot{u} - (-\Delta)^{\gamma} u + \epsilon |x|^{\rho} |u|^{p-1} u = 0; \\ u_{|t=0} = u_0. \end{cases} \tag{1.1}$$

The nonlinear equations of Schrödinger type have a deep influence in physical modeling. The fractional Schrödinger equation was derived in Refs. [8,9] by extending the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths. It is a fundamental equation of fractional quantum mechanics. If $\rho = 0$, the homogeneous fractional Schrödinger equation (1.1) arises in plasma physics, fluid mechanics and nonlinear optics [1]. If $\rho \neq 0$, it can model the laser beam propagation in some inhomogeneous medium [2,6,11,19].

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Here and hereafter, $N \geq 2$ and u is a complex valued function of the variable $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$. The defocusing or focusing regime is given by $\epsilon \in \{\pm 1\}$. The fractional Laplacian exponent is $\gamma \in (0, 1)$. The inhomogeneous unbounded term is $|\cdot|^\rho$, $\rho < 0$. The equation (1.1) satisfies the scaling invariance

$$u_\kappa := \kappa^{\frac{2\gamma+\rho}{p-1}} u(\kappa^{2\gamma}\cdot, \kappa\cdot), \quad \kappa > 0.$$

The following homogeneous Sobolev norm gives the critical Sobolev index:

$$\|u_\kappa\|_{\dot{H}^s} = \kappa^{s-\frac{N}{2}+\frac{2\gamma+\rho}{p-1}} \|u(\kappa^{2\gamma}\cdot)\|_{\dot{H}^s} := \kappa^{s-s_c} \|u(\kappa^{2\gamma}\cdot)\|_{\dot{H}^s}.$$

The mass-critical case $s_c = 0$ corresponds to $p = p_* =: 1 + \frac{2(2\gamma+\rho)}{N}$, which is related to the mass conservation law

$$M[u(t)] := \int_{\mathbb{R}^N} |u(t, x)|^2 \,\mathrm{d}x = M[u_0].$$

The energy-critical case $s_c = \gamma$ which corresponds to $p = p^* =: 1 + \frac{2(2\gamma+\rho)}{N-2\gamma}$ is related to the energy conservation law

$$E[u(t)] := \int_{\mathbb{R}^N} \left(\frac{1}{2}|(-\Delta)^{\frac{\gamma}{2}} u(t, x)|^2 - \frac{\epsilon}{1+p}|x|^\rho |u(t, x)|^{1+p}\right) \mathrm{d}x = E[u_0].$$

It is standard that if $\epsilon < 0$, the energy is non-negative and the problem (1.1) is said to be defocusing. In such a case, an energy sub-critical solution is claimed to be a global one. Otherwise, it is focusing and the Sobolev norm $\|\cdot\|_{\dot{H}^\gamma}$ of a local solution is no longer estimated with use of the conserved laws. In such a case, a local solution may concentrate in finite time.

To the authors knowledge, the inhomogeneous nonlinear fractional Schrödinger equation was considered in few papers. Indeed, for $\rho < 0$, the first author [15] developed a local theory in the energy space H^γ. Indeed, using a sharp Gagliardo–Nirenberg estimate, the existence of energy local solutions was established. Moreover, taking account of the Potential-well theory, the local solution extends to a global one, via the existence of ground states. In the complementary case $\rho < 0$, the local theory was considered in Ref. [13]. In fact, using an inhomogeneous Gagliardo–Nirenberg-type inequality, the ground-state threshold of global existence versus finite tine blow-up was obtained. Moreover, the existence of non-global solutions was proved, for negative energy and spherically symmetric data, following the method of Ref. [3]. Some blow-up dynamics of mass-critical focusing inhomogeneous fractional nonlinear Schrödinger equation, with a mass larger than the ground-state one, were investigated in Ref. [14].

The purpose of this manuscript is to develop a local theory of the fractional Schrödinger problem (1.1) in the space $\dot{H}^\gamma \cap \dot{H}^{s_c}$. The main difference with the previous work [13] is the lack of a mass conservation, which gives some technical problems. The limiting case $s = 1$ was considered in a recent note [4]. Finally, one needs to deal with the non-local free operator and the unbounded inhomogeneous term $|\cdot|^\rho$. Note that in the previous work [16], the first author studied similar questions for the non-fractional regime, namely $\gamma = 1$ and a non-local source term. Here, one needs to deal with the non-local fractional Laplacian operator which gives serious complications. In particular, there is no classical variance identity and one uses a localized one in the spirit of Ref. [3].

The note is organized as follows. In Sect. 2, one gives the contribution and some standard estimates. Section 3 contains a Gagliardo–Nirenberg estimate. Section 4 deals with the local well-posedness. Sections 5 and 6 deal with the finite-time blow-up of solutions.

Here and hereafter, one denotes for simplicity the Lebesgue and Sobolev spaces and their standard norms by

$$L^p := L^p(\mathbb{R}^N), \quad \|\cdot\|_p := \|\cdot\|_{L^p} \quad \text{and} \quad \|\cdot\| := \|\cdot\|_2;$$
$$\dot{H}^{\gamma,p} := (-\Delta)^{-\frac{\gamma}{2}} L^p, \quad \dot{H}^\gamma := \dot{H}^{\gamma,2} \quad \text{and} \quad \|\cdot\|_{\dot{H}^\gamma} := \|(-\Delta)^{\frac{\gamma}{2}}\cdot\|.$$

If $T > 0$ and Y is a Lebesgue or Sobolev space, one defines

$$C_T(Y) := C([0, T], Y), \quad L^p_T(Y) := L^p([0, T], Y);$$
$$\dot{H}^\gamma_{rd} := \{f \in \dot{H}^\gamma, \quad f(\cdot) = f(|\cdot|)\}.$$

Eventually, $[0, T^*)$ is the maximal existence interval of an eventual solution of (1.1).

2 Main results and background

This section contains the contribution of this work and some standard estimates needed in the sequel.

2.1 Notations

One denotes, here and hereafter, the real numbers

$$B := B(N, p, b, s) := \frac{Np - N - 2b}{2s};$$
$$A := A(N, p, b, s) := 1 + p - B(p, b);$$
$$p_c := \frac{2N}{N - 2s_c} = \frac{N(p-1)}{\rho + 2\gamma}.$$

In the spirit of [3], denote $\zeta_R := R^2\zeta(\frac{\cdot}{R})$, where $\zeta \in C_0^\infty(\mathbb{R}^N)$ is spherically symmetric and

$$\zeta : r \longmapsto \begin{cases} \frac{1}{2}r^2, & r \leq 1; \\ 0, & r \geq 10, \end{cases} \quad \text{and} \quad \zeta'' \leq 1.$$

With a direct calculus

$$\zeta_R'(r) \leq r, \quad \zeta_R'' \leq 1 \quad \text{and} \quad \Delta\zeta_R \leq N.$$

Moreover, $|\nabla^j \zeta_R| \lesssim R^{2-j}$ for $0 \leq j \leq 4$ and

$$supp(\nabla^j \zeta_R) \subset \begin{cases} |x| \leq 10R, & j = 1, 2; \\ R \leq |x| \leq 10R, & j = 3, 4. \end{cases}$$

Denote the localized Virial

$$M_\zeta[u] := 2\Im \int_{\mathbb{R}^N} \bar{u}\nabla\zeta \cdot \nabla u \, \mathrm{d}x = 2\Im \int_{\mathbb{R}^N} \bar{u}\partial_k\zeta\,\partial_k u \, \mathrm{d}x.$$

Let the differential operator acting on functions as follows:

$$\Gamma_\zeta u := -i\Big[\nabla\zeta \cdot \nabla u + \nabla \cdot (u\nabla\zeta)\Big].$$

Thus, $\langle u, \Gamma_\zeta u\rangle = M_\zeta[u]$. Eventually, one denotes the sequence of functions

$$u_n = \left(\frac{\sin(\pi s)}{\pi}\right)^{\frac{1}{2}} \mathcal{F}^{-1}\left(\frac{\mathcal{F}u}{|\cdot|^2 + n}\right).$$

2.2 Main results

Let us give the Theorems established in this note. First, one derives an inhomogeneous Gagliardo–Nirenberg estimate.

Theorem 2.1 *Let $N \geq 2$, $\gamma \in (0, 1)$, $-2\gamma < \rho < 0$ and $p > 1$. Then,*

1. *there exists a positive constant $C(N, p, \rho, \gamma)$, such that for any $u \in \dot{H}^\gamma \cap L^{p_c}$,*

$$\int_{\mathbb{R}^N} |u|^{1+p}|x|^\rho \, dx \leq C(N, p, \rho, \gamma)\|u\|_{p_c}^{p-1}\|(-\Delta)^{\frac{\gamma}{2}}u\|^2. \tag{2.1}$$

2. *Moreover, if* $1+\frac{2\rho}{N}<p<p^*$*, then*
   a. *The minimization problem*

$$\frac{1}{C_{opt}}=\inf\left\{\frac{\|u\|_{p_c}^{p-1}\|(-\Delta)^{\frac{\gamma}{2}}u\|^2}{\int_{\mathbb{R}^N}|u|^{1+p}|x|^\rho\,dx},\quad 0\neq u\in\dot H^\gamma\cap L^{p_c}\right\}$$

is attained in some $\psi\in H^\gamma$ *satisfying* $C_{opt}=\int_{\mathbb{R}^N}|u|^{1+p}|x|^\rho\,dx$ *and*

$$2(-\Delta)^\gamma\psi-(p-1)|\psi|^{p_c-2}\psi+\frac{p+1}{C_{opt}}|x|^\rho|\psi|^{p-1}\psi=0; \tag{2.2}$$

   b. *Furthermore*

$$C_{opt}=\frac{1+p}{2}\|\phi\|_{p_c}^{-(p-1)}, \tag{2.3}$$

where ϕ *is a ground-state solution to*

$$(-\Delta)^\gamma\phi+|\phi|^{p_c-2}\phi-|x|^\rho|\phi|^{p-1}\phi=0,\quad 0\neq\phi\in\dot H^\gamma\cap L^{p_c}. \tag{2.4}$$

Remarks 2.2 1. The proof follows the method of Ref. [20];
2. A comparable estimate using the L^2 in the place of L^{p_c} was proved in Ref. [13, Theorem 2.2];
3. Thanks to the Sobolev embedding $\dot H^{s_c}\hookrightarrow L^{p_c}$, the above estimate is adapted to the study of (1.1) in $\dot H^\gamma\cap\dot H^{s_c}$.

The Schrödinger problem (1.1) is locally well posed in $\dot H^\gamma_{rd}\cap\dot H^{s_c}$.

Theorem 2.3 *Let* $N\geq 2$, $\gamma\in\left(\frac{N}{2N-1},1\right)$, $-2\gamma<\rho<0$, $p_*<p<p^*$ *and* $u_0\in\dot H^\gamma_{rd}\cap\dot H^{s_c}$*. Then, there is a unique local solution to* (1.1),

$$u\in C_T(\dot H^\gamma_{rd}\cap\dot H^{s_c})\cap L^q_T(\dot W^{\gamma,r}\cap\dot W^{s_c,r})\cap L^{q_1}_T(L^{r_1}),$$

where $(q,r)\in\Gamma$ *and* $(q_1,r_1)\in\Gamma_{s_c}$*. Moreover, the energy is conserved and u is global*

1. *In the defocusing case;*
2. *If* $\|u\|_{L^\infty_{T^*}(\dot H^{s_c})}<\|\phi\|_{p_c}$*, where* ϕ *is a ground state of* (2.4).

Remarks 2.4 1. The sets Γ and Γ_{s_c} are defined in Remark 2.11;
2. The proof is based on a fixed point argument via Strichartz estimates and the fractional chain rules;
3. The main difficulty is to estimate the source term in some Sobolev norms;
4. The spherically symmetric assumption avoids a loss of regularity in Strichartz estimates [7].

Now, one investigates the finite-time blow-up of solutions in the repulsive regime.

Theorem 2.5 *Take* $\epsilon=1$*. Let* $N\geq 2$, $\gamma\in\left(\frac{N}{2N-1},1\right)$, $-2\gamma<\rho<0$, $p_*<p<p^*$ *and* $u_0\in\dot H^\gamma_{rd}\cap\dot H^{s_c}$*. Let u be the maximal solution to* (1.1) *given by the above result. Assume that* $T^*<\infty$ *and* $\|u\|_{L^\infty_{T^*}(\dot H^{s_c})}<\infty$*. If*

$$\lim_{t\to T^*}\lambda(t)\|(-\Delta)^{\frac{\gamma}{2}}u(t)\|^{\frac{1}{\gamma-s_c}}=\infty$$

then,

$$\liminf_{t\to T^*}\int_{|x|\leq\lambda(t)}|u(t,x)|^{p_c}\,dx\geq\|\phi\|_{p_c}^{p_c},$$

where ϕ *is a ground state of* (2.4).

Remarks 2.6 1. The above result studies the L^{p_c} concentration of the non-global solutions, which blow-up for finite time in $\dot H^\gamma$;

2. Take for $0 < t < T^*$, the scaled function $v_t(\tau, x) := (\mu(t))^{\frac{2\gamma+\rho}{p-1}} u(t + (\mu(t))^{2\gamma}\tau, \mu(t)x)$, defined for $0 < \tau < \frac{1}{(\mu(t))^{2\gamma}}(T^* - t)$. Thus, v_t satisfies (1.1) with datum $v_t(0, x) = (\mu(t))^{\frac{2\gamma+\rho}{p-1}} u(t, \mu(t)x)$. Therefore, $\|v_t(0)\|_{\dot H^\gamma} = (\mu(t))^{\gamma - s_c}\|u(t)\|_{\dot H^\gamma}$. Let us choose $\mu(t) := \|u(t)\|_{\dot H^\gamma}^{\frac{1}{s_c-\gamma}}$ so that $\|v_t(0)\|_{\dot H^\gamma} = 1$. The local existence theory gives the existence of $0 < \tau_1 < \frac{1}{(\mu(t))^{2\gamma}}(T^* - t)$ such that v_t is defined on $[0, \tau_1]$. This gives the blow-up rate

$$\|u(t)\|_{\dot H^\gamma} \geq \frac{C}{(T^* - t)^{\frac{\gamma - s_c}{2\gamma}}};$$

3. the concentration happens at the origin because of the radial assumption.

Finally, one gives a finite-time blow-up solutions result in $L^\infty_{T^*}(\dot H^{s_c})$ for negative energy.

Theorem 2.7 *Take $\epsilon = 1$. Let $N \geq 2$, $\gamma \in \left(\frac{N}{2N-1}, 1\right)$, $-2\gamma < \rho < 0$, $p_* < p < \min\{1 + 4\gamma, p^*\}$ and a solution of (1.1) denoted by $u \in C_T(\dot H^\gamma_{rd} \cap \dot H^{s_c})$ such that $u \in L^\infty_{T^*}(\dot H^{s_c})$. Then,*

1. *For any $R > 0$ and any $\beta > 0$, holds in $[0, T)$,*

$$\frac{d}{dt} M_{\zeta_R}[u] \leq 4BE(u_0) + 4(\gamma - B)\|u\|^2_{\dot H^\gamma} + \beta\|u\|^2_{\dot H^\gamma(|x|>R)} + C_\beta R^{-2(\gamma - s_c)}.$$

2. *If $E(u_0) < 0$, then $T^* < \infty$.*

Remarks 2.8 1. The above result gives some sufficient conditions to have the existence of blowing-up solutions in $\dot H^\gamma$, which are bounded in $\dot H^{s_c}$;
2. The extra assumption $p < 1 + 4\gamma$ is due to the lack of a variance identity for the Schrödinger equation with fractional Laplacian;
3. The above result gives a meaning to Theorem 2.5.

2.3 Tools

Here, one lists some standard estimates needed along this manuscript.

Definition 2.9 One call admissible pair $(q, r) \in [2, \infty]^2$ if

$$q \in \left[\frac{4N+2}{2N-1}, \infty\right], \quad \frac{2}{q} + \frac{2N-1}{r} \leq N - \frac{1}{2},$$

or

$$q \in \left[2, \frac{4N+2}{2N-1}\right], \quad \frac{2}{q} + \frac{2N-1}{r} < N - \frac{1}{2}.$$

Recall the so-called Strichartz estimate [7].

Proposition 2.10 *Let $N \geq 2$, $s \in \mathbb{R}$, $\frac{N}{2N-1} < \gamma < 1$ and $u_0 \in H^s_{rd}$. Then,*

$$\|u\|_{L^q_t(L^r) \cap L^\infty_t(\dot H^s)} \lesssim \|u_0\|_{\dot H^s} + \|i\dot u - (-\Delta)^\gamma u\|_{L^{\tilde q'}_t(L^{\tilde r'})},$$

if (q, r) and $(\tilde q, \tilde r)$ are s-admissible pairs such that $(\tilde q, \tilde r, N) \neq (2, \infty, 2)$ or $(q, r, N) \neq (2, \infty, 2)$ and satisfy the condition

$$\frac{2\gamma}{q} + s = N\Big(\frac{1}{2} - \frac{1}{r}\Big), \quad \frac{2\gamma}{\tilde q} - s = N\Big(\frac{1}{2} - \frac{1}{\tilde r}\Big).$$

Remark 2.11 For simplicity, one denotes the sets $\Gamma_s := \{(q, r),\ s\text{-admissible}\}$, $\Gamma := \Gamma_0$ and the norms

$$\|\cdot\|_{S(\dot H^s)} := \sup_{(q,r)\in\Gamma_s} \|\cdot\|_{L^q(L^r)}, \quad \|\cdot\|_{S'(\dot H^{-s})} := \inf_{(q,r)\in\Gamma_{-s}} \|\cdot\|_{L^{q'}(L^{r'})}.$$

The next fractional chain rule [5] will be useful.

Lemma 2.12 *Let* $N \geq 1$, $0 < \gamma \leq 1$, $\frac{1}{p} = \frac{1}{p_i} + \frac{1}{q_i}$, $i = 1, 2$ *and* $F \in C^1(\mathbb{C})$. *Then,*

$$\|(-\Delta)^{\frac{\gamma}{2}} F(u)\|_p \lesssim \|(-\Delta)^{\frac{\gamma}{2}} u\|_{q_1} \|F'(u)\|_{p_1}, \tag{2.5}$$

and

$$\|(-\Delta)^{\frac{\gamma}{2}} (uv)\|_p \lesssim \|(-\Delta)^{\frac{\gamma}{2}} u\|_{p_1} \|v\|_{q_1} + \|(-\Delta)^{\frac{\gamma}{2}} v\|_{p_2} \|u\|_{q_2}. \tag{2.6}$$

Let us give a fractional Gagliardo–Nirenberg inequality [12].

Lemma 2.13 *Let* $1 < p, p_2 < \infty$, $0 < \gamma < N$, $0 < \theta < p < \infty$, *and* $1 < p_1 < \frac{N}{\gamma}$. *Then, the fractional inequality*

$$\|u\|_p \lesssim \|u\|_{p_2}^{1-\frac{\theta}{p}} \|(-\Delta)^{\frac{\gamma}{2}} u\|_{p_1}^{\frac{\theta}{p}},$$

holds whenever

$$1 = \frac{p-\theta}{p_2} + \theta\left(\frac{1}{p_1} - \frac{\gamma}{N}\right).$$

Let us recall a fractional Strauss type inequality [18].

Lemma 2.14 *Let* $N \geq 2$ *and* $\frac{1}{2} < \gamma < \frac{N}{2}$. *Then,*

$$\sup_{x \neq 0} |x|^{\frac{N}{2}-\gamma} |u(x)| \leq C(N, \gamma) \|(-\Delta)^{\frac{\gamma}{2}} u\|, \tag{2.7}$$

for any $u \in \dot{H}^{\gamma}_{rd}(\mathbb{R}^N)$, *where* Γ *is the Gamma function and*

$$C(N, \gamma) = \left(\frac{\Gamma(2\gamma - 1)\Gamma(\frac{N}{2} - \gamma)\Gamma(\frac{N}{2})}{2^{2\gamma}\pi^{\frac{N}{2}}\Gamma^2(\gamma)\Gamma(\frac{N}{2} - 1 + \gamma)}\right)^{\frac{1}{2}}.$$

The next Sobolev injections [10,17] will be useful.

Lemma 2.15 *Let* $N \geq 1$ *and* $1 < p \leq q < \infty$.

1. *If* $0 < s < N$ *and* $\mu \geq 0$ *such that*

$$\mu < \frac{N}{q} \quad \text{and} \quad s = \frac{N}{p} - \frac{N}{q} + \mu.$$

Then, for any $u \in W^{s,p}$, *one has*

$$\||x|^{-\mu} u\|_q \leq C(\mu, p, q, N, s)\|(-\Delta)^{\frac{s}{2}} u\|_p.$$

2. *If* $0 < 2s < N$, *then*
   a. $H^s \hookrightarrow L^q$ *for any* $q \in [2, \frac{2N}{N-2s}]$;
   b. $H^s_{rd} \hookrightarrow\hookrightarrow L^q$ *is compact for* $q \in (2, \frac{2N}{N-2s})$.

Finally, the next Sobolev injection is proved in the appendix.

Lemma 2.16 *Let* $N \geq 2$, $\gamma \in (0, 1)$, $-2\gamma < \rho < 0$ *and* $1 + \frac{2\rho}{N} < p < p^*$. *Then, the following injection is compact:*

$$\dot{H}^{\gamma}_{rd} \cap L^{p_c} \hookrightarrow\hookrightarrow L^{1+p}(|x|^{\rho}\, dx). \tag{2.8}$$

3 Proof of Theorem 2.1

One proceeds in three steps.

3.1 Proof of the interpolation inequality (2.1)

Thanks to Lemma 2.15, one has

$$\begin{aligned}\int_{\mathbb{R}^N} |u(x)|^{1+p}|x|^{\rho}\,dx &\le \||x|^{\frac{\rho}{2}}u\|^2_{\frac{2p_c}{p_c-(p-1)}}\|u\|_{p_c}^{-1+p}\\ &\lesssim \|(-\Delta)^{\frac{\gamma}{2}}u\|^2\|u\|_{p_c}^{-1+p}.\end{aligned}$$

The proof is ended.

3.2 Proof of the equation (2.2)

One denotes by

$$\inf_{\dot H^{\gamma}\cap L^{s_c}} \frac{\|(-\Delta)^{\frac{\gamma}{2}}u\|^2\|u\|_{p_c}^{-1+p}}{\int_{\mathbb{R}^N} |u(x)|^{1+p}|x|^{\rho}\,dx} := \frac{1}{C_{opt}} := \beta.$$

Taking account of (2.1), there is a sequence (v_n) in $\dot H^{\gamma}\cap L^{s_c}$ satisfying

$$\beta = \lim_n \frac{\|(-\Delta)^{\frac{\gamma}{2}}u_n\|^2\|u_n\|_{p_c}^{-1+p}}{\int_{\mathbb{R}^N} |x|^{\rho}|u_n(x)|^{1+p}\,dx} := \lim_n I(v_n).$$

Letting $u^{a,b} := au(b\cdot)$, one computes

$$\begin{aligned}a^2b^{2\gamma-N}\|(-\Delta)^{\frac{\gamma}{2}}u\|^2 &= \|(-\Delta)^{\frac{\gamma}{2}}u^{a,b}\|^2;\\ ab^{-\frac{N}{p_c}}\|u\|_{p_c} &= \|u^{a,b}\|_{p_c};\\ a^{1+p}b^{-N-\rho}\int_{\mathbb{R}^N}|u(x)|^{1+p}|x|^{\rho}\,\mathrm{d}x &= \int_{\mathbb{R}^N}|u^{a,b}(x)|^{1+p}|x|^{\rho}\,\mathrm{d}x.\end{aligned}$$

Thus $I(u) = I(u^{a,b})$. Let us pick

$$\mu_n := \left(\frac{\|v_n\|_{p_c}}{\|(-\Delta)^{\frac{\gamma}{2}}v_n\|}\right)^{\frac{1}{\gamma-s_c}} \quad \text{and} \quad \lambda_n := \frac{\|v_n\|^{\frac{N-2\gamma}{2(\gamma-s_c)}}}{\|(-\Delta)^{\frac{\gamma}{2}}v_n\|^{\frac{2\gamma+\rho}{(p-1)(\gamma-s_c)}}}.$$

Thus, $\psi_n := v_n^{\lambda_n,\mu_n}$ satisfies

$$\|\psi_n\|_{p_c} = \|(-\Delta)^{\frac{\gamma}{2}}\psi_n\| = 1 \quad \text{and} \quad \beta = \lim_n I(\psi_n).$$

Therefore, $\psi_n \rightharpoonup \psi$ in $\dot H^{\gamma}\cap L^{p_c}$ and (2.8) implies that for a sub-sequence denoted also (ψ_n), as $n\to\infty$,

$$I(\psi_n) = \frac{1}{\int_{\mathbb{R}^N}|\psi_n|^{1+p}|x|^{\rho}\,\mathrm{d}x} \to \frac{1}{\int_{\mathbb{R}^N}|\psi|^{1+p}|x|^{\rho}\,\mathrm{d}x}.$$

The lower semi-continuity of the $\dot H^{\gamma}\cap L^{p_c}$ norm gives

$$\max\{\|\psi\|_{p_c}, \|(-\Delta)^{\frac{\gamma}{2}}\psi\|\} \le 1.$$

Then, $I(\psi) < \beta$ if $\|\psi\|\|(-\Delta)^{\frac{\gamma}{2}}\psi\| < 1$. Thus,

$$\|\psi\|_{p_c} = 1 = \|(-\Delta)^{\frac{\gamma}{2}}\psi\|.$$

Therefore,

$$\lim_n \|\psi_n - \psi\|_{\dot{H}^\gamma \cap L^{p_c}} = 0, \quad \beta = I(\psi) = \frac{1}{\int_{\mathbb{R}^N} |\psi|^{1+p}|x|^\rho \, dx}.$$

Let us write the Euler–Lagrange equation satisfied by the minimizer

$$\partial_\varepsilon I(\psi + \varepsilon\eta)_{|\varepsilon=0} = 0, \quad \forall \eta \in C_0^\infty(\mathbb{R}^N).$$

Hence, ψ satisfies

$$2(-\Delta)^\gamma \psi + (p-1)|\psi|^{p_c-2}\psi - \beta(1+p)|x|^\rho |\psi|^{p-1}\psi = 0.$$

This proof is complete.

3.3 Proof of the equation (2.3)

One keeps the notations in the previous subsection ψ satisfies (2.2) and $C_{opt} = \frac{1}{\beta} = \int_{\mathbb{R}^N} |\psi(x)|^{1+p}|x|^\rho \, dx$. Let $\psi = \phi^{a,b} := a\phi(b\cdot)$. Then, the equation

$$2(-\Delta)^\gamma \psi + (p-1)|\psi|^{p_c-2}\psi - \beta(1+p)|x|^\rho |\psi|^{p-1}\psi = 0$$

gives

$$\frac{2}{p-1} a^{2-p_c} b^{2\gamma} (-\Delta)^\gamma \phi + |\phi|^{p_c-2}\phi - \frac{\beta(1+p)}{p-1} a^{p-p_c+1} b^{-\rho} |x|^\rho |\phi|^{p-1}\phi = 0.$$

Choosing

$$\begin{aligned} a &:= \left(\beta \frac{1+p}{2} \left(\frac{2}{p-1}\right)^{\frac{\rho+2\gamma}{2\gamma}}\right)^{\frac{2N\gamma}{(\rho+2\gamma)[p_c(N-2\gamma)-2N]}}; \\ b &:= \left(\frac{p-1}{2} a^{p_c-2}\right)^{\frac{1}{2\gamma}} \\ &= \left(\frac{p-1}{2}\right)^{\frac{1}{2\gamma}} \left(\beta \frac{1+p}{2} \left(\frac{2}{p-1}\right)^{\frac{\rho+2\gamma}{2\gamma}}\right)^{\frac{N(p_c-2)}{(\rho+2\gamma)[p_c(N-2\gamma)-2N]}}. \end{aligned}$$

It follows that

$$-(-\Delta)^\gamma \phi + |\phi|^{p_c-2}\phi - |x|^\rho |\phi|^{p-1}\phi = 0.$$

Finally, $\|\psi\|_{p_c} = 1 = ab^{-\frac{N}{p_c}} \|\phi\|$ gives $\beta = \frac{2}{1+p} \|\phi\|_{p_c}^{p-1}$ and finishes the proof.

4 Proof of Theorem 2.3

This section establishes the local well-posedness of the fractional inhomogeneous Schrödinger equation (1.1) in $\dot{H}^\gamma_{rd} \cap \dot{H}^{s_c}$.

4.1 Local existence

One starts with some nonlinear estimates.

Lemma 4.1 *Let $N \geq 2$, $0 < -\rho < 2\gamma$ and $p_* < p < p^*$. Then, there exist $c, \theta, \theta_1 > 0$ and $0 < \theta_2 < p - 1$ such that*

1. $\|(-\Delta)^{\frac{\gamma}{2}}(|x|^\rho |u|^{p-1}u)\|_{S'(I,L^2)} \leq c(T^\theta + T^{\theta_1})\|(-\Delta)^{\frac{\gamma}{2}}u\|^p_{S(I,L^2)}$;
2. *If $N = 2$ and $p < 1 + \frac{\gamma - \sqrt{\gamma^2 - 4(2+\rho)\gamma - 4\rho}}{2(1-\gamma)}$ or $N \geq 3$, one has*
$$\|(-\Delta)^{\frac{s_c}{2}}(|x|^\rho |u|^{p-1}u)\|_{S'(I,L^2)} \leq c(T^\theta + T^{\theta_1})\|(-\Delta)^{\frac{s_c}{2}}u\|^{p-1}_{S(I,L^2)}\|(-\Delta)^{\frac{\gamma}{2}}u\|_{S(I,L^2)};$$
3. $\||x|^\rho |u|^{p-1}u\|_{S'(I,\dot{H}^{-s_c})} \leq c(T^\theta + T^{\theta_1})\|(-\Delta)^{\frac{\gamma}{2}}u\|^\theta_{L^\infty(I,L^2)}\|u\|^{p-\theta}_{S(I,\dot{H}^{s_c})}$.

Proof 1. Let the admissible pair

$$(q, r) := \left(\frac{4\gamma(1+p)}{(N-2\gamma)(p-1)}, \frac{N(1+p)}{N+\gamma(p-1)}\right) \in \Gamma.$$

One denotes here and hereafter the centered unit ball of $\mathbb{R}^N$ by $B(1)$ and its complementary by $B^c(1)$. By Lemma 2.12 about the fractional chain rule, via Hölder estimate and Sobolev injections

$$\begin{aligned}\|(-\Delta)^{\frac{\gamma}{2}}(|x|^\rho |u|^{p-1}u)\|_{L^{r'}(B^c(1))} &\lesssim \||x|^\rho\|_{L^a(B^c(1))}\|(-\Delta)^{\frac{\gamma}{2}}u\|_r\|u\|^{p-1}_{\frac{Nr}{N-r\gamma}} \\ &\quad + \||x|^{\rho-\gamma}\|_{L^c(B^c(1))}\|u\|^p_{\frac{Nr}{N-r\gamma}} \\ &\lesssim \left(\||x|^\rho\|_{L^a(B^c(1))} + \||x|^{\rho-\gamma}\|_{L^c(B^c(1))}\right)\|(-\Delta)^{\frac{\gamma}{2}}u\|^p_r.\end{aligned}$$

Here,

$$\frac{1}{r'} = \frac{1}{a} + \frac{1}{r} + \frac{(p-1)(N-r\gamma)}{Nr} = \frac{1}{c} + \frac{p(N-r\gamma)}{Nr}.$$

Thus,

$$\begin{aligned}1 &= \frac{1}{a} + \frac{2}{r} + \frac{(p-1)(N-r\gamma)}{Nr}; \\ 1 &= \frac{1}{c} + \frac{1}{r} + \frac{p(N-r\gamma)}{Nr}.\end{aligned}$$

This gives $\frac{N}{c} = \gamma + \frac{N}{a}$. Choosing a such that $\frac{N}{a} < -\rho$, then we have

$$\|(-\Delta)^{\frac{\gamma}{2}}(|x|^\rho |u|^{p-1}u)\|_{L^{q'}_T(L^{r'}(B^c(1)))} \lesssim T^\theta\|(-\Delta)^{\frac{\gamma}{2}}u\|^p_{L^q_T(L^r)}.$$

Here, $\theta := 1 - \frac{1+p}{q} > 0$. Now, one estimates the term on the complementary of the unit ball. Let the admissible pair

$$(q_1, r_1) := \left(\frac{2(\rho+N)}{N-2\gamma}, \frac{2N(\rho+N)}{N(N-2\gamma)+4\gamma^2+\rho N}\right) \in \Gamma.$$

By Lemma 2.12 about the fractional chain rule and Hölder estimates via Sobolev injections

$$\begin{aligned}\|(-\Delta)^{\frac{\gamma}{2}}(|x|^\rho |u|^{p-1}u)\|_{L^{r_1'}(B(1))} &\lesssim \||x|^\rho\|_{L^{a_1}(B(1))}\|(-\Delta)^{\frac{\gamma}{2}}u\|_{r_1}\|u\|^{p-1}_{\frac{Nr_1}{N-r_1\gamma}} \\ &\quad + \||x|^{\rho-\gamma}\|_{L^{c_1}(B(1))}\|u\|^p_{\frac{Nr_1}{N-r_1\gamma}} \\ &\lesssim \left(\||x|^\rho\|_{L^{a_1}(B(1))} + \||x|^{\rho-\gamma}\|_{L^{c_1}(B(1))}\right)\|(-\Delta)^{\frac{\gamma}{2}}u\|^p_{r_1}.\end{aligned}$$

Here,

$$\frac{1}{r_1'} = \frac{1}{a_1} + \frac{1}{r_1} + \frac{(p-1)(N-r_1\gamma)}{Nr_1} = \frac{1}{c_1} + \frac{p(N-r_1\gamma)}{Nq_1}.$$

The integrability condition $\||x|^\rho\|_{L^{a_1}(B(1))} < \infty$ and $\||x|^{\rho-\gamma}\|_{L^{c_1}(B^c(1))} < \infty$ read

$$N\left(1 - \frac{1+p}{r_1}\right) + \gamma(p-1) > -\rho.$$

A direct computation via the fact that $p < p^*$ gives the above condition and so

$$\|(-\Delta)^{\frac{\gamma}{2}}(|x|^\rho|u|^{p-1}u)\|_{L_T^{q_1'}(L^{r_1'}(B(1)))} \lesssim T^{\theta_1}\|(-\Delta)^{\frac{\gamma}{2}}u\|^p_{L_T^{q_1}(L_1^r)},$$

where one takes $\theta_1 := 1 - \frac{1+p}{q_1} > 0$. This first point is proved.

2. Using Sobolev injections, Strichartz and Hölder estimates, one has

$$\begin{aligned}
&\|(-\Delta)^{\frac{s_c}{2}}(|x|^\rho|u|^{p-1}u)\|_{L^{\frac{2N}{2\gamma+N}}(B(1))}\\
&\lesssim \|(-\Delta)^{\frac{\gamma}{2}}(|x|^\rho|u|^{p-1}u)\|_{L^{\frac{N(p-1)}{2\gamma(p-1)+2\gamma+\rho}}(B(1))}\\
&\lesssim \||x|^\rho\|_{L^a(B(1))}\|(-\Delta)^{\frac{\gamma}{2}}u\|_r\|u\|^{p-1}_{\frac{Nr}{N-rs_c}}\\
&\quad + \||x|^{\rho-\gamma}\|_{L^c(B(1))}\|u\|^{p-1}_{\frac{Nr}{N-rs_c}}\|u\|_{\frac{Nr}{N-r\gamma}}\\
&\lesssim \left(\||x|^\rho\|_{L^a(B(1))} + \||x|^{\rho-\gamma}\|_{L^c(B(1))}\right)\|(-\Delta)^{\frac{\gamma}{2}}u\|_r\|u\|^{p-1}_{\frac{Nr}{N-rs_c}}.
\end{aligned}$$

Here,

$$-\rho > \frac{N}{a} = 2\gamma + \frac{2\gamma+\rho}{p-1} - \frac{Np}{r} + (p-1)s_c, \quad N > r\gamma.$$

Denote by x^+ a real number near to x such that $x^+ > x$ and x^- a real number near to x such that $x^- < x$. Let us pick $(q, r) \in \Gamma$ such that

$$\left(\frac{Np(p-1)}{2\gamma+\rho+\frac{N}{2}(p-1)^2}\right)^- := r, \quad \left(\frac{4\gamma p(p-1)}{N(p-1)-2(2\gamma+\rho)}\right)^+ := q.$$

A direct calculus gives $2 < r < \frac{2N}{N-2\gamma}$. Therefore, for $N \geq 4$, one has $\gamma r < \frac{2N}{N-2\gamma} \leq N$. For $N \in \{2, 3\}$, the condition $N > \gamma r$ is equivalent to

$$(N-2\gamma)x^2 - 2\gamma x + 2(2\gamma+\rho) > 0, \quad x := p-1. \tag{4.1}$$

• First case $N = 2$. Then, the previous inequality reads

$$P(x) := (1-\gamma)x^2 - \gamma x + 2\gamma + \rho > 0.$$

The discriminant is

$$\begin{aligned}
\Delta(P) &:= \gamma^2 - 4(2\gamma+\rho)(1-\gamma)\\
&= 9\gamma^2 - 4\gamma(2+\rho) - 4\rho\\
&:= Q(\gamma).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Delta(Q) &:= 4[(2+\rho)^2 + 9\rho]\\
&:= R(\rho).
\end{aligned}$$

Now, $\Delta(Q) < 0$ for $\rho \in (-2\gamma, \frac{-13+\sqrt{153}}{2})$ and $\Delta(Q) > 0$ for $\rho \in (\frac{-13+\sqrt{153}}{2}, 0)$. Thus, $\Delta(P) > 0$ for $\rho \in (-2\gamma, \frac{-13+\sqrt{153}}{2})$ and, because $P(1) > 0$, $\Delta(P) > 0$ for $\rho \in (\frac{-13+\sqrt{153}}{2}, 0)$. Thus, $P(x) > 0$ iif $p < 1 + \frac{\gamma-\sqrt{\gamma^2-4(2+\rho)\gamma-4\rho}}{2(1-\gamma)}$.

• Second case $N = 3$. Then, the inequality (4.1) reads

$$P(x) := (3-2\gamma)x^2 - 2\gamma x + 2(2\gamma+\rho) > 0.$$

The discriminant is

$$\begin{aligned}\Delta(P) &:= \gamma^2 - 2(2\gamma+\rho)(3-2\gamma)\\ &= 9\gamma^2 - 4(3-\rho)\gamma - 6\rho\\ &:= Q(\gamma).\end{aligned}$$

Moreover,

$$\begin{aligned}\Delta(Q) &:= 2[2(3-\rho)^2 + 27\rho]\\ &:= R(\rho).\end{aligned}$$

Now, $\Delta(Q) < 0$ for $\rho \in (-2\gamma, -\frac{3}{2})$ and $\Delta(Q) > 0$ for $\rho \in (-\frac{3}{2}, 0)$. Thus, $\Delta(P) > 0$ for $\rho \in (-2\gamma, -\frac{3}{2})$ and, since $Q(1) < 0$ and $Q(-\frac{\rho}{2}) > 0$, $\Delta(P) > 0$ for $[\gamma \in (-\frac{\rho}{2}, \frac{2(3-\rho)-\sqrt{4(3-\rho)^2+54\rho}}{9})$ and $\rho \in (-\frac{3}{2}, 0)]$ and $\Delta(P) < 0$ for $[\gamma \in (\frac{2(3-\rho)-\sqrt{4(3-\rho)^2+54\rho}}{9}, 1)$ and $\rho \in (-\frac{3}{2}, 0)]$. If $\Delta(P) < 0$, we are done. Otherwise, the roots of P are positive and the smallest one $\frac{\gamma-\sqrt{Q(\gamma)}}{3-2\gamma} < 1$. Thus, because $P(1) > 0$, the two roots are less than one. We are done. Moreover, the admissibility condition reads $\frac{2}{q} + \frac{2N-1}{r} < N - \frac{1}{2}$ and is equivalent to $p > p_*$. In conclusion,

$$\begin{aligned}\|(-\Delta)^{\frac{s_c}{2}}(|x|^\rho|u|^{p-1}u)\|_{L^2(I,L^{\frac{2N}{2\gamma+N}}(B(1)))} &\lesssim \|\|(-\Delta)^{\frac{\gamma}{2}}u\|_r\|(-\Delta)^{\frac{\gamma}{2}}u\|_r^{p-1}\|_{L^2(I)}\\ &\lesssim T^{\frac{1}{2}-\frac{p}{q}}\|(-\Delta)^{\frac{\gamma}{2}}u\|_{L^q(I,L^r)}\|(-\Delta)^{\frac{\gamma}{2}}u\|^{p-1}_{L^q(I,L^r)}.\end{aligned}$$

The condition $p < p^*$ gives $\frac{1}{2} - \frac{p}{q} > 0$. The estimation of the term on the complementary of the unit ball follows similarly by taking

$$\left(\frac{Np(p-1)}{2\gamma+\rho+\frac{N}{2}(p-1)^2}\right)^+ := r, \quad \left(\frac{4\gamma p(p-1)}{N(p-1)-2(2\gamma+\rho)}\right)^- := q.$$

3. Letting $(\tilde{q}, r) \in \Gamma_{-s_c}$ and $(q, r) \in \Gamma_{s_c}$, Hölder and Sobolev estimates give

$$\begin{aligned}\||u|^p|x|^\rho\|_{L_T^{\tilde{q}'}(L^{r'}(B(1)))} &\le c\||x|^\rho\|_{L^a(B(1))}\|u\|^\theta_{L_T^\infty\left(L^{\frac{2N}{N-2\gamma}}\right)}\|u\|^{p-\theta}_{L^q(L^r)}\\ &\le cT^{\frac{1}{\tilde{q}'}-\frac{p-\theta}{q}}\||x|^\rho\|_{L^a(B(1))}\|(-\Delta)^{\frac{\gamma}{2}}u\|^\theta_{L_T^\infty(L^2)}\|u\|^{p-\theta}_{L_T^q(L^r)}\\ &\le cT^{\frac{1}{\tilde{q}'}-\frac{p-\theta}{q}}\|(-\Delta)^{\frac{\gamma}{2}}u\|^\theta_{L_T^\infty(L^2)}\|u\|^{p-\theta}_{L_T^q(L^r)}.\end{aligned}$$

Here, $\frac{1}{\tilde{q}'} - \frac{p-\theta}{q} > 0$ and

$$\frac{N}{a} = N - \frac{\theta(N-2\gamma)}{2} - \frac{N(1+p-\theta)}{r} > -\rho. \tag{4.2}$$

The first inequality is equivalent to $q > \frac{\gamma(1+p-\theta)}{\gamma-s_c}$. Let us take $0 < \theta \ll 1$ and

$$q := \left(\frac{\gamma(1+p-\theta)}{\gamma-s_c}\right)^+, \quad r := \left(\frac{2N(1+p-\theta)}{(N-2s_c)(1+p-\theta)-4(\gamma-s_c)}\right)^-.$$

A direct computation gives (4.2). The estimation of the term on the complementary of the unit ball follows similarly by taking

$$(q,r)=\left(\infty,\frac{2N}{N-2s_c}\right).$$

This closes the proof. □

Now, using Strichartz estimates, Duhamel formula and a fixed point method, one proves Theorem 2.3. One defines the function

$$f(u):=e^{i\cdot(-\Delta)^{\frac{\gamma}{2}}}u_0+\int_0^{\cdot}e^{i(\cdot-s)(-\Delta)^{\frac{\gamma}{2}}}|x|^\rho|u|^{p-1}u\,ds.$$

One denotes by $B_T(R)$ the centered ball with radius $R>0$ of the space

$$X_T:=\Big(\cap_{(q,r)\in\Gamma}L^q_T(\dot W^{\gamma,r}\cap\dot W^{s_c,r})\Big)\cap\Big(\cap_{(q_1,r_1)\in\Gamma_{s_c}}L^{q_1}_T(L^{r_1})\Big),$$

endowed with the complete distance

$$\begin{aligned}d(u,v):=&\sup_{(q,r)\in\Gamma}\|(-\Delta)^{\frac{\gamma}{2}}(u-v)\|_{L^q_T(L^r)}+\sup_{(q,r)\in\Gamma}\|(-\Delta)^{\frac{s_c}{2}}(u-v)\|_{L^q_T(L^r)}\\&+\sup_{(q,r)\in\Gamma_{s_c}}\|u-v\|_{L^q_T(L^r)}.\end{aligned}$$

Thanks to the previous Lemma via Strichartz estimate, one has for $w:=u-v$,

$$\begin{aligned}d(f(u),f(v))\lesssim&\|(-\Delta)^{\frac{\gamma}{2}}[|x|^\rho(|u|^{p-1}+|v|^{p-1})w]\|_{S'((0,T),L^2)}\\&+\|(-\Delta)^{\frac{s_c}{2}}[|x|^\rho(|u|^{p-1}+|v|^{p-1})w]\|_{S'((0,T),L^2)}\\&+\||x|^\rho(|u|^{p-1}+|v|^{p-1})w\|_{S'((0,T),\dot H^{s_c})}\\\le&\,c(T^\theta+T^{\theta_1})\Big[\|(-\Delta)^{\frac{\gamma}{2}}u\|^{p-1}_{S(I,L^2)}+\|(-\Delta)^{\frac{s_c}{2}}u\|^{p-1}_{S(I,L^2)}\\&+\|(-\Delta)^{\frac{\gamma}{2}}u\|^\theta_{L^\infty(I,L^2)}\|u\|^{p-1-\theta}_{S(I,\dot H^{s_c})}\Big]d(u,v)\\\le&\,c(T^\theta+T^{\theta_1})R^{p-1}d(u,v).\end{aligned}$$

Moreover, taking $v=0$ in the above lines and taking account of Strichartz estimates, one writes

$$\begin{aligned}&\sup_{(q,r)\in\Gamma}\|(-\Delta)^{\frac{\gamma}{2}}f(u)\|_{L^q_T(L^r)}+\sup_{(q,r)\in\Gamma}\|(-\Delta)^{\frac{s_c}{2}}f(u)\|_{L^q_T(L^r)}+\sup_{(q,r)\in\Gamma_{s_c}}\|f(u)\|_{L^q_T(L^r)}\\&\le c\|u_0\|_{\dot H^\gamma\cap\dot H^{s_c}}+c(T^\theta+T^{\theta_1})R^p.\end{aligned}$$

Choose $R:=2c\|u_0\|_{\dot H^\gamma\cap\dot H^{s_c}}$ and $T>0$ such that $c(T^\theta+T^{\theta_1})<\frac{1}{2R^{p-1}}$.Thus, f is a contraction of $B_T(R)$. One concludes the proof by a fixed point Theorem.

4.2 Global existence

Here, one assumes that $\|u\|_{L^\infty_{T^*}(\dot H^{s_c})}<\|\phi\|_{p_c}$ and $T^*<\infty$. Then, by Theorem 2.1, one has

$$\begin{aligned}2E(t)&=\|(-\Delta)^{\frac{\gamma}{2}}u\|^2-\frac{2\epsilon}{p+1}\int_{\mathbb{R}^N}|u|^{1+p}|x|^\rho\,\mathrm{d}x\\&\ge\|(-\Delta)^{\frac{\gamma}{2}}u\|^2-\frac{2C_{opt}}{p+1}\|u\|^{p-1}_{p_c}\|(-\Delta)^{\frac{\gamma}{2}}u\|^2\\&\ge\left(1-\left[\frac{\|u\|_{p_c}}{\|\phi\|_{p_c}}\right]^{p-1}\right)\|(-\Delta)^{\frac{\gamma}{2}}u\|^2.\end{aligned}$$

Thus, $\sup_{0\le t<T^*}\|(-\Delta)^{\frac{\gamma}{2}}u(t)\|<\infty$. This contradiction closes the proof.

5 Proof of Theorem 2.5

Let the sequences

$$t_n \to T^*, \quad \beta_n := \|(-\Delta)^{\frac{\gamma}{2}} u(t_n)\|^{-\frac{1}{\gamma - s_c}}, \quad v_n := \beta_n^{\frac{2\gamma+\rho}{p-1}} u(t_n, \beta_n \cdot),$$

and compute

$$\begin{aligned} \|(-\Delta)^{\frac{s_c}{2}} v_n\| &= \|(-\Delta)^{\frac{s_c}{2}} u_n\|; \\ \|(-\Delta)^{\frac{\gamma}{2}} v_n\| &= 1; \\ E(v_n) &= \beta_n^{2(\gamma - s_c)} E(u_0). \end{aligned}$$

Thus,

$$\sup_n \|v_n\|_{\dot{H}^{s_c} \cap \dot{H}^\gamma} < \infty, \quad E(v_n) \to 0.$$

Denote by $B(R)$ the centered ball of $\mathbb{R}^N$ with radius $R > 0$ and $B(R)^c$ its complementary. Take $v_n \rightharpoonup v$ in $\dot{H}^{s_c} \cap \dot{H}^\gamma$. Since $\lambda(t_n) \gg \beta_n$, the weak limit lower semi-continuity gives for any $R > 0$,

$$\begin{aligned} \int_{B(R)} |v|^{p_c} \, \mathrm{d}x &\le \liminf_n \int_{B(R)} |v_n|^{p_c} \, \mathrm{d}x \\ &= \liminf_n \int_{B(R\beta_n)} |u(t_n)|^{p_c} \, \mathrm{d}x \\ &\le \liminf_n \int_{B(\lambda(t_n))} |u(t_n)|^{p_c} \, \mathrm{d}x. \end{aligned}$$

Finally, (2.8) gives

$$0 = \liminf_n E(v_n) \ge \frac{1}{2} \left(1 - \left[\frac{\|v\|_{p_c}}{\|\phi\|_{p_c}} \right]^{p-1} \right) \|(-\Delta)^{\frac{\gamma}{2}} v\|^2.$$

Therefore,

$$\liminf_n \int_{|x| < \lambda(t_n)} |u(t_n)|^{p_c} \, \mathrm{d}x \ge \|\phi\|_{p_c}^{p_c}.$$

The proof is achieved.

6 Proof of Theorem 2.7

This section is devoted to prove Theorem 2.7. Take for simplicity $\epsilon = 1$ and denote the inhomogeneous nonlinear term

$$\mathcal{I} := \mathcal{I}_p := -|x|^\rho |u|^{p-1} u.$$

1. Localized variance identity.

Lemma 6.1 *One has*

$$\begin{aligned} \frac{d}{dt} M_\zeta[u(t)] = &\int_0^\infty m^\gamma \int_{\mathbb{R}^N} \left(4 \overline{\partial_k u_m} \partial_{kl}^2 \zeta \, \partial_l u_m - \Delta^2 \zeta |u_m|^2 \right) dx \, dm \\ &+ \frac{4\rho}{1+p} \int_{\mathbb{R}^N} x \cdot \nabla \zeta |u|^{1+p} |x|^{\rho - 2} \, dx - \frac{2(p-1)}{1+p} \int_{\mathbb{R}^N} \Delta \zeta |u|^{1+p} |x|^\rho \, dx. \end{aligned}$$

Proof Compute using (1.1),

$$\frac{\mathrm{d}}{\mathrm{d}t}M_\zeta[u(t)] = \langle u(t), [(-\Delta)^s, i\Gamma_\zeta]u(t)\rangle + \left\langle u(t), \left[-\frac{\mathcal{I}}{u}, i\Gamma_\zeta\right]u(t)\right\rangle.$$

Here, the commutator reads $AB - BA := [A, B]$. According to computation done in [3], one has

$$\langle u(t), [(-\Delta)^s, i\Gamma_\zeta]u(t)\rangle = \int_0^\infty m^\gamma \int_{\mathbb{R}^N} \left(4\overline{\partial_k u_m}\partial^2_{kl}\zeta\,\partial_l u_m - \Delta^2\zeta|u_m|^2\right)\mathrm{d}x\,\mathrm{d}m.$$

Let us write

$$\begin{aligned}(N_p) := \left\langle u, \left[-\frac{\mathcal{I}_p}{u}, i\Gamma_\zeta\right]u\right\rangle &= \langle u, [-|u|^{p-1}|x|^\rho, i\Gamma_\zeta]u\rangle \\ &= \langle u, [-|u|^{p-1}|x|^\rho, div(\nabla\zeta\cdot) + \nabla\zeta\nabla\cdot]u\rangle \\ &= -\langle u, |x|^\rho|u|^{p-1}(div(\nabla\zeta u) + \nabla\zeta\nabla u)\rangle \\ &\quad + \langle u, div(\nabla\zeta|x|^\rho|u|^{p-1}u) + \nabla\zeta\nabla(|x|^\rho|u|^{p-1}u)\rangle.\end{aligned}$$

Then,

$$\begin{aligned}(N_p) &= -\langle u, |x|^\rho|u|^{p-1}(\Delta\zeta u + 2\nabla\zeta\nabla u)\rangle + \langle u, \Delta\zeta|x|^\rho|u|^{p-1}u + 2\nabla\zeta\nabla(|x|^\rho|u|^{p-1}u)\rangle \\ &= \langle u, \Delta\zeta|x|^\rho|u|^{p-1}u + 2\nabla\zeta\nabla(|x|^\rho|u|^{p-1}u) - |x|^\rho|u|^{p-1}(\Delta\zeta u + 2\nabla\zeta\nabla u)\rangle \\ &= 2\langle u, \nabla\zeta\nabla(|x|^\rho|u|^{p-1}u) - |x|^\rho|u|^{p-1}\nabla\zeta\nabla u\rangle \\ &= 2\left\langle u, \nabla\zeta\left(\nabla(|x|^\rho)|u|^{p-1}u + |x|^\rho\nabla(|u|^{p-1})u\right)\right\rangle.\end{aligned}$$

An integration by parts gives

$$\begin{aligned}(N_p) &= 2\int_{\mathbb{R}^N} \nabla\zeta\nabla(|x|^\rho)|u|^{1+p}\,\mathrm{d}x + 2\int_{\mathbb{R}^N} |x|^\rho\nabla\zeta\nabla(|u|^{p-1})|u|^2\,\mathrm{d}x \\ &= 2\int_{\mathbb{R}^N} \nabla\zeta\nabla(|x|^\rho)|u|^{1+p}\,\mathrm{d}x + \frac{2(p-1)}{1+p}\int_{\mathbb{R}^N} \nabla\zeta\nabla(|u|^{1+p})|x|^\rho\,\mathrm{d}x \\ &= 2\int_{\mathbb{R}^N} \nabla\zeta\nabla(|x|^\rho)|u|^{1+p}\,\mathrm{d}x - \frac{2(p-1)}{1+p}\int_{\mathbb{R}^N} |u|^{1+p}\left(\nabla(|x|^\rho)\nabla\zeta + |x|^\rho\Delta\zeta\right)\mathrm{d}x \\ &= \frac{4}{1+p}\int_{\mathbb{R}^N} \nabla\zeta\nabla(|x|^\rho)|u|^{1+p}\,\mathrm{d}x - \frac{2(p-1)}{1+p}\int_{\mathbb{R}^N} \Delta\zeta|x|^\rho|u|^{1+p}\,\mathrm{d}x \\ &= \frac{4\rho}{1+p}\int_{\mathbb{R}^N} x.\nabla\zeta|u|^{1+p}|x|^{\rho-2}\,\mathrm{d}x - \frac{2(p-1)}{1+p}\int_{\mathbb{R}^N} \Delta\zeta|u|^{1+p}|x|^\rho\,\mathrm{d}x.\end{aligned}$$

This finishes the proof. □

Now, one establishes Theorem 2.7. Using the identities

$$\int_0^\infty m^\gamma \int_{\mathbb{R}^N} |\nabla u_m|^2\,\mathrm{d}x\,\mathrm{d}m = \gamma\|(-\Delta)^{\frac{\gamma}{2}}u\|^2;$$

$$\zeta_R = \frac{|\cdot|^2}{2}, \quad \text{for} \quad |x| < R,$$

one has

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} M_{\zeta_R}[u(t)]\\
&= \int_0^\infty m^\gamma \int_{\mathbb{R}^N} \left(4\overline{\partial_k u_m}\partial_{kl}^2\zeta_R\partial_l u_m - \Delta^2\zeta_R|u_m|^2\right)\mathrm{d}x\,\mathrm{d}m\\
&\quad + \frac{4\rho}{1+p}\int_{\mathbb{R}^N} x\cdot\nabla\zeta_R|u|^{1+p}|x|^{\rho-2}\,dx - \frac{2(p-1)}{1+p}\int_{\mathbb{R}^N}\Delta\zeta_R|u|^{1+p}|x|^\rho\,\mathrm{d}x\\
&= 4\gamma\|u\|_{\dot H^\gamma}^2 - \frac{4\gamma B}{1+p}\int_{\mathbb{R}^N}|u|^{1+p}|x|^\rho\,dx - 4\gamma\|u\|_{\dot H^\gamma(|x|>R)}^2\\
&\quad - \int_0^\infty m^\gamma\int_{|x|>R}\Delta^2\zeta_R|u_m|^2\,dx\,dm + 4\int_0^\infty m^\gamma\int_{|x|>R}\overline{\partial_k u_m}\partial_{kl}^2\zeta_R\partial_l u_m\,\mathrm{d}x\,\mathrm{d}m\\
&\quad - \frac{2(p-1)}{1+p}\int_{|x|>R}(\Delta\zeta_R - N)|u|^{1+p}|x|^\rho\,dx + \frac{4\rho}{1+p}\int_{|x|>R}(|x|^2 - x\cdot\nabla\zeta_R)|u|^{1+p}|x|^{\rho-2}\,\mathrm{d}x\\
&= 4\gamma BE(u_0) - 2\gamma(B-2)\|u\|_{\dot H^\gamma}^2 - 4\gamma\|u\|_{\dot H^\gamma(|x|>R)}^2\\
&\quad - \int_0^\infty m^\gamma\int_{|x|>R}\Delta^2\zeta_R|u_m|^2\,dx\,dm + 4\int_0^\infty m^\gamma\int_{|x|>R}\overline{\partial_k u_m}\partial_{kl}^2\zeta_R\partial_l u_m\,\mathrm{d}x\\
&\quad - \frac{2(p-1)}{1+p}\int_{|x|>R}(\Delta\zeta_R - N)|u|^{1+p}|x|^\rho\,dx + \frac{4\rho}{1+p}\int_{|x|>R}(|x|^2 - x.\nabla\zeta_R)|u|^{1+p}|x|^{\rho-2}\,\mathrm{d}x.
\end{aligned}
$$

Thanks to the radial derivative formula

$$
\partial_{jk}^2 = \left(\delta_{jk} - \frac{x_j x_k}{r^2}\right)\frac{\partial_r}{r} + \frac{x_j x_k}{r^2}\partial_r^2,
$$

one has

$$
\int_0^\infty m^\gamma\int_{|x|>R}\overline{\partial_k u_m}\partial_{kl}^2\zeta_R\partial_l u_m\,\mathrm{d}x = \int_0^\infty m^\gamma\int_{|x|>R}\zeta_R''|\nabla u_m|^2\,\mathrm{d}x \le \gamma\|u\|_{\dot H^\gamma}^2.
$$

Moreover, Lemma A.2 in Ref. [3] gives via Hölder estimate and Sobolev injection via the properties of ζ,

$$
\begin{aligned}
\int_0^\infty m^\gamma\int_{|x|>R}\Delta^2\zeta_R|u_m|^2\,\mathrm{d}x\,\mathrm{d}m &\lesssim \|\Delta^2\zeta_R\|_\infty^\gamma\|\Delta\zeta_R\|_\infty^{1-\gamma}\|u\|_{L^2(|x|\le 10R)}^2\\
&\lesssim R^{-2(\gamma-s_c)}\|u\|_{\dot H^{s_c}}^2.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} M_{\zeta_R}[u(t)]\\
&= 4\gamma BE(u_0) - 2\gamma(B-2)\|u\|_{\dot H^\gamma}^2 - 4\gamma\|u\|_{\dot H^\gamma(|x|>R)}^2\\
&\quad - \int_0^\infty m^\gamma\int_{|x|>R}\Delta^2\zeta_R|u_m|^2\,\mathrm{d}x\,\mathrm{d}m + 4\int_0^\infty m^\gamma\int_{|x|>R}\overline{\partial_k u_m}\partial_{kl}^2\zeta_R\partial_l u_m\,\mathrm{d}x
\end{aligned}
$$

$$-\frac{2(p-1)}{1+p}\int\limits_{|x|>R}(\Delta\zeta_R-N)|u|^{1+p}|x|^{\rho}\,dx+\frac{4\rho}{1+p}\int\limits_{|x|>R}(|x|^2-x\cdot\nabla\zeta_R)|u|^{1+p}|x|^{\rho-2}\,\mathrm{d}x$$
$$\leq 4\gamma BE(u_0)-2\gamma(B-2)\|u\|^2_{\dot H^{\gamma}}+R^{-2(\gamma-s_c)}\|u\|^2_{\dot H^{s_c}}+c\int\limits_{|x|>R}|u|^{1+p}|x|^{\rho}\,\mathrm{d}x.$$

In order to estimate the last term, one denotes the annulus $C_A := C(A, 2A)$ with respective small radius $A > 0$ and large one $2A$. Thus, thanks to Strauss inequality (2.7) and the properties of ζ_R, one gets for $0 < s < \frac{N}{2}$,

$$\int\limits_{C_A}|u|^{1+p}|x|^{\rho}\,\mathrm{d}x\lesssim \||u|^{-1+p}|x|^{\rho}\|_{L^\infty(C_A)}\int\limits_{C_A}|u|^2\,\mathrm{d}x$$
$$\lesssim \||x|^{-(p-1)(\frac{N}{2}-s)+\rho}\|_{L^\infty(C_A)}\|u\|^{p-1}_{\dot H^s}\int\limits_{C_A}|u|^2\,\mathrm{d}x.$$

Using the interpolation inequality for $\frac{1}{2} < s < \gamma < \frac{N}{2}$ and the Sobolev estimate

$$\|(-\Delta)^{\frac{eq1}{2}}\cdot\|\lesssim\|\cdot\|^{1-\frac{s}{\gamma}}\|(-\Delta)^{\frac{\gamma}{2}}\cdot\|^{\frac{s}{\gamma}},\tag{6.1}$$
$$\|\cdot\|_{L^2(|x|\lesssim R)}\lesssim R^{s_c}\|(-\Delta)^{\frac{s_c}{2}}\cdot\|_{L^2(|x|\lesssim R)},\tag{6.2}$$

one gets

$$\int\limits_{C_A}|u|^{1+p}|x|^{\rho}\,\mathrm{d}x\lesssim A^{-(p-1)(\frac{N}{2}-s)+\rho}\|u\|^{\frac{s(p-1)}{\gamma}}_{\dot H^{\gamma}}\left(\int\limits_{C_A}|u|^2\,\mathrm{d}x\right)^{1+\frac{p-1}{2}(1-\frac{s}{\gamma})}$$
$$\lesssim A^{-(p-1)(\frac{N}{2}-s)+\rho}\|u\|^{\frac{s(p-1)}{\gamma}}_{\dot H^{\gamma}}\left(A^{s_c}\|(-\Delta)^{\frac{s_c}{2}}u\|_{L^2(|x|\lesssim R)}\right)^{2+(p-1)(1-\frac{s}{\gamma})}$$
$$\lesssim A^{-(p-1)(\frac{N}{2}-s)+\rho+s_c(2+(p-1)(1-\frac{s}{\gamma}))}\|u\|^{\frac{s(p-1)}{\gamma}}_{\dot H^{\gamma}}$$
$$\lesssim A^{-2(\gamma-s_c)(1-\frac{s(p-1)}{2\gamma})}\|u\|^{\frac{s(p-1)}{\gamma}}_{\dot H^{\gamma}}.$$

Since $p < 1+4\gamma$, one takes $s = \left(\frac{1}{2}\right)^+$, so that $\frac{s(p-1)}{\gamma} < 2$. Therefore, by Young Lemma, for any $\beta > 0$,

$$\int\limits_{C_A}|u|^{1+p}|x|^{\rho}\,\mathrm{d}x\lesssim\beta\|u\|^2_{\dot H^{\gamma}}+C_\beta A^{-2(\gamma-s_c)}.$$

Now, using a series expansion

$$\int\limits_{\mathbb{R}^N}|u|^{1+p}|x|^{\rho}\,\mathrm{d}x=\sum_{k=0}^{\infty}\int\limits_{C_{2^kR}}|u|^{1+p}|x|^{\rho}\,\mathrm{d}x$$
$$\lesssim\beta\sum_{k=0}^{\infty}\|u\|^2_{\dot H^{\gamma}(C_{2^kR})}+C_\beta\sum_{k=0}^{\infty}(2^kR)^{-2(\gamma-s_c)}$$
$$\lesssim\beta\|u\|^2_{\dot H^{\gamma}(|x|>R)}+C_\beta R^{-2(\gamma-s_c)}.$$

Finally, since $u\in L^\infty_{T^*}(\dot H^{s_c})$, one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}M_{\zeta_R}[u(t)]\leq 4\gamma BE(u_0)-2\gamma(B-2)\|u\|^2_{\dot H^{\gamma}}+\beta\|u\|^2_{\dot H^{\gamma}}+C_\beta R^{-2(\gamma-s_c)}.$$

2. Finite time blow-up. Since $p > p_*$ and $E(u_0) < 0$, taking $0 < \beta \ll 1 \ll R$, there is $c > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} M_{\zeta_R}[u(t)] < -c\|u\|^2_{\dot{H}^\gamma}.$$

Assume, with contradiction that $T^* = \infty$. Since $E(u_0) < 0$, by Theorem 2.1, one gets $\inf_{[0,T^*)} \|u(t)\|_{\dot{H}^\gamma} > 0$. Thus, by integrating in time, there is $t_0 > 0$ such that

$$M_{\zeta_R}[u(t)] < 0, \quad \forall t \geq t_0;$$

$$M_{\zeta_R}[u(t)] < -c \int_{t_0}^{t} \|u(\tau)\|_{\dot{H}^\gamma}\, \mathrm{d}\tau, \quad \forall t \geq t_0.$$

Moreover, by Lemma 4.1 in Ref. [3], via the fact that $supp(\zeta_R) \subset \{|x| \leq 10R\}$ and (6.1)–(6.2), there is $c := c_{N,R}$ such that

$$\begin{aligned}
M_{\zeta_R}[u] &\leq c\left(\|(-\Delta)^{\frac{1}{4}}u\|^2_{L^2(|x|\lesssim R)} + \|u\|_{L^2(|x|\lesssim R)}\|(-\Delta)^{\frac{1}{4}}u\|_{L^2(|x|\lesssim R)}\right)\\
&\leq c\left(\|u\|^{2-\frac{1}{\gamma}}_{L^2(|x|\lesssim R)}\|(-\Delta)^{\frac{\gamma}{2}}u\|^{\frac{1}{\gamma}}_{L^2(|x|\lesssim R)} + \|u\|^{2-\frac{1}{2\gamma}}_{L^2(|x|\lesssim R)}\|(-\Delta)^{\frac{\gamma}{2}}u\|^{\frac{1}{2\gamma}}_{L^2(|x|\lesssim R)}\right)\\
&\leq c\left(\|(-\Delta)^{\frac{s_c}{2}}u\|^{2-\frac{1}{\gamma}}_{L^2(|x|\lesssim R)}\|(-\Delta)^{\frac{\gamma}{2}}u\|^{\frac{1}{\gamma}}_{L^2(|x|\lesssim R)}\right.\\
&\quad\left.+\|(-\Delta)^{\frac{s_c}{2}}u\|^{2-\frac{1}{2\gamma}}_{L^2(|x|\lesssim R)}\|(-\Delta)^{\frac{\gamma}{2}}u\|^{\frac{1}{2\gamma}}_{L^2(|x|\lesssim R)}\right)\\
&\leq c\left(\|(-\Delta)^{\frac{\gamma}{2}}u\|^{\frac{1}{\gamma}}_{L^2(|x|\lesssim R)} + \|(-\Delta)^{\frac{\gamma}{2}}u\|^{\frac{1}{2\gamma}}_{L^2(|x|\lesssim R)}\right)\\
&\leq c\|(-\Delta)^{\frac{\gamma}{2}}u\|^{\frac{1}{\gamma}}_{L^2(|x|\lesssim R)}.
\end{aligned}$$

In the last line, one uses

$$\inf_{0\leq t<T^*} \|u(t)\|_{\dot{H}^\gamma} > 0 \quad \text{and} \quad \sup_{0\leq t<T^*} \|u(t)\|_{\dot{H}^{s_c}} < \infty.$$

Then, for $\gamma > \frac{1}{2}$ and a finite $t_1 > 0$,

$$M_{\zeta_R}[u(t)] \leq -C_R|t-t_1|^{1-2\gamma} \to -\infty, \quad \text{when} \quad t \to t_1.$$

Finally, $T^* < \infty$.

7 Proof of Lemma 2.16

Take a functional sequence satisfying

$$\sup_n \left(\|(-\Delta)^{\frac{\gamma}{2}}u_n\| + \|u_n\|_{p_c}\right) < \infty \quad \text{and} \quad u_n \rightharpoonup 0 \quad \text{in} \quad \dot{H}^\gamma \cap L^{p_c}.$$

One will prove that

$$\int_{\mathbb{R}^N} |x|^\rho |u_n|^{1+p}\, \mathrm{d}x \to 0.$$

Since $p_c < \frac{2N}{N-2\gamma}$, with an interpolation argument, one has

$$\sup_n \|u_n\|_q < \infty, \quad \forall q \in \left(p_c, \frac{2N}{N-2\gamma}\right).$$

Let $0 < \varepsilon << 1$. Using Hölder estimate and Sobolev injection via $p < p^*$, one has

$$\begin{aligned}\int\limits_{|x|>R} |x|^\rho |u_n|^{1+p}\,\mathrm{d}x &\leq \||x|^\rho\|_{L^{\frac{N+\varepsilon}{|\rho|}}(|x|>R)} \|u_n\|^{1+p}_{(\frac{N+\varepsilon}{|\rho|})'(1+p)}\\ &\leq CR^{-\varepsilon}\|u_n\|^{1+p}_{(\frac{N+\varepsilon}{|\rho|})'(1+p)}\\ &\leq CR^{-\varepsilon}.\end{aligned}$$

Here, one needs

$$\frac{N(p-1)}{\rho+2\gamma} = p_c < \left(\frac{N+\varepsilon}{|\rho|}\right)'(1+p) = \frac{\varepsilon+N}{\varepsilon+N+\rho}(1+p) < \frac{2N}{N-2\gamma}.$$

Indeed, the above condition read

$$\begin{aligned}\varepsilon((N-\rho-2\gamma)p-(N+\rho+2\gamma)) &< N(N-2\gamma)(p^*-p);\\ \varepsilon\left(p-\frac{N+2\gamma}{N-2\gamma}\right) &< 0 < N(p-p^*).\end{aligned}$$

Take $R > (\frac{1}{\varepsilon})^{\frac{1}{\varepsilon}}$ and gets

$$\int\limits_{B(R)^c} |u_n|^{1+p}|x|^\rho\,dx \leq c\varepsilon. \tag{7.1}$$

Now, Poincare inequality and the compact Sobolev injections give for all $2 < q < \frac{2N}{N-2\gamma}$,

$$\lim n \to \infty \|u_n\|_{L^q(B(R))} = 0.$$

Moreover, by Hölder estimate

$$\int\limits_{B(R)} |u_n|^{1+p}|x|^\rho\,\mathrm{d}x \leq \||x|^\rho\|_{L^a(B(R))}\|u_n\|^{1+p}_{a'(1+p)}.$$

Here, one picks $a := \frac{N}{|\rho|} - \varepsilon$. This gives $2 < a'(1+p) < \frac{2N}{N-2\gamma}$ if $2(1+\frac{\rho}{N}) < 1+p < \frac{2(N+\rho)}{N-2\gamma}$. Taking account of (7.1), the proof id achieved because $1+\frac{2\rho}{N} < p < p^*$.

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