Arab. J. Math. (2024) 13:35–43
https://doi.org/10.1007/s40065-023-00448-7

Arabian Journal of Mathematics

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Classification of additive mappings on certain rings and algebras

Received: 6 December 2022 / Accepted: 25 October 2023 / Published online: 14 November 2023

Abstract The objective of this research is to prove that an additive mapping $\Delta : \mathcal{A} \to \mathcal{A}$ will be a generalized derivation associated with a derivation $\partial : \mathcal{A} \to \mathcal{A}$ if it satisfies the following identity $\Delta(r^{m+n+p}) = \Delta(r^m)r^{n+p} + r^m\partial(r^n)r^p + r^{m+n}\partial(r^p)$ for all $r \in \mathcal{A}$, where $m, n \geq 1$ and $p \geq 0$ are fixed integers and $\mathcal{A}$ is a semiprime ring. Another analogous has been done where an additive mapping behaves like a generalized left derivation associated with a left derivation on $\mathcal{A}$ satisfying certain algebraic identity. The proofs of these advancements are derived employing algebraic concepts. These theorems have been validated by offering an example that shows they are not insignificant. Furthermore, we provide an application in the framework of Banach algebra.

Mathematics Subject Classification 16N60 · 16B99 · 16W25

1 Introduction

$\mathcal{A}$ will be used to signify an associative ring with identity throughout this paper. The center of $\mathcal{A}$ is $Z_{\mathcal{A}}$, the left Martindale ring of quotients is $Q_l(\mathcal{A}_C)$, and the extended centroid is $\mathcal{C}$. A ring $\mathcal{A}$ is known as t-torsion free if $tr = 0$ implies $r = 0$ for all $r \in \mathcal{A}$, where $t > 1$ is an integer. $[r, s]$ denotes the commutator $rs - sr$. A ring $\mathcal{A}$ is known as prime if $r\mathcal{A}s = \{0\}$ implies either $r = 0$ or $s = 0$, and is said to be a semiprime if $s\mathcal{A}s = \{0\}$ implies $s = 0$. A mapping ∂ from a ring $\mathcal{A}$ to itself is known as a derivation if it is additive and $\partial(rs) = \partial(r)s + r\partial(s)$ for every pair $r, s \in \mathcal{A}$ and particularly, ∂ is known as a Jordan derivation if $\partial(r^2) = \partial(r)r + r\partial(r)$ is fulfilled for all $r \in \mathcal{A}$. From the definition, it is obvious to notice that every derivation will be a Jordan derivation, but in general, the converse does not hold. An essential outcome from Herstein [6] affirms that a derivation and Jordan derivation are identically the same on a prime ring with characteristic other than 2. An extension of this conclusion is presented by Cusack [5]. Next, we define a generalization of a derivation due to [4]. An additive mapping Δ from $\mathcal{A}$ to itself is known as a generalized derivation if there occurs a derivation ∂ on $\mathcal{A}$ such that $\Delta(rs) = \Delta(r)s + r\partial(s)$ for every $r, s \in \mathcal{A}$ and particularly if $r = s$, then Δ is a generalized Jordan derivation associated with a Jordan derivation ∂. It is straightforward to authenticate that every generalized derivation will be a generalized Jordan derivation. However, this is not always the case. Encouraged by the definition of Jordan derivation, Bresar [3] has started to study Jordan triple derivations on rings which reads: an additive mapping ∂ from $\mathcal{A}$ to itself is termed as Jordan triple derivations if $\partial(rsr) = \partial(r)sr + r\partial(s)r + rs\partial(r)$ for all $r, s \in \mathcal{A}$. It is obvious that Jordan's triple derivation is an extension of Jordan's derivation. Motivated by the concept of generalized derivation and Jordan triple derivation, Jing and Lu [7] defined a new notion of $\mathcal{A}$ that is called a generalized Jordan triple derivation which is as follows: If a Jordan triple derivation ∂ on $\mathcal{A}$ exists, then an additive mapping Δ from $\mathcal{A}$ to itself is known as a generalized

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Jordan triple derivation associated with ∂ such that $\Delta(rsr) = \Delta(r)sr + r\partial(s)r + rs\partial(r)$ holds for every $r, s \in \mathcal{A}$. Every generalized Jordan triple derivation is a generalized derivation on a 2-torsion-free prime ring, according to the authors of [7]. If Δ is a generalized derivation associated with a derivation ∂ on $\mathcal{A}$, then the algebraic identity $\Delta(r^{m+n+p}) = \Delta(r^m)r^{n+p} + r^m\partial(r^n)r^p + r^{m+n}\partial(r^p)$ holds for all $r \in \mathcal{A}$ but the converse is not true in general. It is natural to ask whether the mappings Δ and ∂ from a ring $\mathcal{A}$ to itself satisfying the above generalized identity, will be a generalized derivation and a derivation on $\mathcal{A}$, respectively. The answer is a resounding yes. In this research paper, we have studied that under what condition, this mathematical statement is true. More precisely, we proved the following: an additive mapping Δ is generalized derivation associated with a derivation ∂ on $\mathcal{A}$ if the following algebraic identity $\Delta(r^{m+n+p}) = \Delta(r^m)r^{n+p} + r^m\partial(r^n)r^p + r^{m+n}\partial(r^p)$ holds for every $r \in \mathcal{A}$, where $n, m \geq 1$ and $p \geq 0$ be any fixed integers and $\mathcal{A}$ be any $(n+m+p-1)!$-torsion free semiprime ring.

Next part is based on another extension of a derivation which is termed as left derivation. An additive mapping ∂_1 from $\mathcal{A}$ to itself is said to be a left derivation and Jordan left derivation if $\partial_1(rs) = r\partial_1(s) + s\partial_1(r)$ and $\partial_1(r^2) = 2r\partial_1(z))$ hold respectively. An additive mapping ∂_1 from $\mathcal{A}$ to itself is known as a right derivation (respectively Jordan right derivation) if $\partial_1(rs) = \partial_1(r)s + \partial_1(s)r$ (respectively $\partial_1(r^2) = 2\partial_1(r)r$) for every pair $r, s \in \mathcal{A}$. If it is both left and right, then ∂_1 is a derivation. One can easily see that left derivation implies a Jordan left derivation on a ring $\mathcal{A}$, although converse is not always true, (see [13]). A generalization of left derivation is given by Ashraf et al. in [2] which is defined as follows: an additive mapping ∇ from $\mathcal{A}$ to itself is known as generalized left derivation if there is a Jordan left deviation ∂_1 on $\mathcal{A}$ such that $\nabla(rs) = r\nabla(s) + s\partial_1(r)$ for all $r, s \in \mathcal{A}$. Similarly, generalized Jordan left derivation is likewise defined there. Particularly, if we take $\partial_1 = 0$, then ∇ will become right centralizer due to Zalar [14] and if $\partial = 0$, then Δ is said to be a left centralizer. An additive mapping which is both, called as centralizer. Similarly, Jordan centralizer is defined by considering $r = s$. There is a strong relationship between right centralizer, generalized left derivation and left derivation i.e., if T right centralizer and ∂_1 is a left derivation, then ∇ can be written as $\partial_1 + T$ on $\mathcal{A}$. Now, if ∂_1 is Jordan left derivation and ∇ is a generalized left derivation on $\mathcal{A}$, then algebraic identity $\nabla(r^{m+n+p}) = r^{m+n}\nabla(r^p) + r^{m+p}\partial_1(r^n) + r^{n+p}\partial_1(r^p)$ holds for every $r \in \mathcal{A}$ but what about the converse of this statement? The converse of this statement is also true with some torsion restrictions on a semiprime ring $\mathcal{A}$. More precisely, let $m, n \geq 1$ and $p \geq 0$ be fixed integers and ∇, ∂_1 from $\mathcal{A}$ to itself be additive mappings which satisfy the following algebraic identity $\nabla(r^{m+n+p}) = r^{m+n}\nabla(r^p) + r^{m+p}\partial_1(r^n) + r^{n+p}\partial_1(r^p)$ for every r in a semiprime ring $\mathcal{A}$, then ∇ will be a generalized left derivation associated with a Jordan left derivation ∂_1 on a $(n+m+p-1)!$-torsion free ring $\mathcal{A}$.

2 Main results

We begin our investigation with the following problem:

Theorem 2.1 *Suppose that $m, n \geq 1$ and $p \geq 0$ are fixed integers and $\mathcal{A}$ is any $(n+m+p-1)!$-torsion-free semiprime ring with identity e. If Δ and ∂ from $\mathcal{A}$ to itself are additive mappings which satisfy the following algebraic identity $\Delta(r^{m+n+p}) = \Delta(r^m)r^{n+p} + r^m\partial(r^n)r^p + r^{m+n}\partial(r^p)$ for every $r \in \mathcal{A}$, then Δ will be a generalized derivation associated with a derivation ∂ on $\mathcal{A}$.*

Proof Given that

$$\Delta(r^{m+n+p}) = \Delta(r^m)r^{n+p} + r^m\partial(r^n)r^p + r^{m+n}\partial(r^p) \text{ for all } r \in \mathcal{A}. \tag{2.1}$$

Replacing r by e, we obtain $\partial(e) = 0$. We proceed with the condition (2.1) and replace r by $r + qs$ to get

$$\begin{aligned}
&\Delta\left(r^{m+n+p} + \binom{m+n+p}{1}(r^{m+n+p-1})qs + \binom{m+n+p}{2}r^{m+n+p-2}q^2s^2 + \cdots + q^{m+n+p}s^{m+n+p}\right)\\
&= \Delta\left(r^m + \binom{m}{1}r^{m-1}qs + \binom{m}{2}r^{m-2}q^2s^2 + \cdots + q^m s^m\right) \cdot \left(r^{n+p} + \binom{n+p}{1}r^{n+p-1}qs\right.\\
&\quad \left.+ \binom{n+p}{2}r^{n+p-2}q^2s^2 + \cdots + q^{n+p}s^{n+p}\right) + \left(r^m + \binom{m}{1}r^{m-1}qs + \binom{m}{2}r^{m-2}q^2s^2 + \cdots + q^m s^m\right)\\
&\quad \times \partial\left(r^n + \binom{n}{1}r^{n-1}qs + \binom{n}{2}r^{n-2}q^2s^2 + \cdots + q^n s^n\right)\left(r^p + \binom{p}{1}r^{p-1}qs + \binom{p}{2}r^{p-2}q^2s^2\right.
\end{aligned}$$

$$+\cdots+q^p s^p\Big)+\left(r^{m+n}+\binom{m+n}{1}(r^{m+n-1})qs+\binom{m+n}{2}r^{m+n-2}q^2s^2+\cdots+q^{m+n}s^{m+n}\right)$$
$$\cdot\partial\left(r^p+\binom{p}{1}r^{p-1}qs+\binom{p}{2}r^{p-2}q^2s^2+\cdots+q^ps^p\right),$$

where q is any positive integer and $s\in\mathcal{A}$.

Rewrite the above expression by using (2.1) as

$$q\mathcal{P}_1(r,s)+q^2\mathcal{P}_2(r,s)+\cdots+q^{m+n+p-1}\mathcal{P}_{m+n+p-1}(r,s)=0,$$

where $\mathcal{P}_i(r,s)$ stand for the coefficients of q^i's for all $i=1,2,\ldots,n+m+p-1$. If we replace q by $1,2,\ldots,n+m+p-1$ in turn, then we find a system of $n+m+p-1$ homogeneous linear equations. It gives us a Vander Monde matrix

$$\begin{bmatrix} 1 & 1 & \ldots & 1\\ 2 & 2^2 & \ldots & 2^{m+n+p-1}\\ \ldots & & & \\ \ldots & & & \\ m+n+p-1 & (m+n+p-1)^2 & \ldots & (m+n+p-1)^{m+n+p-1}\end{bmatrix}.$$

Which yields that $\mathcal{P}_i(r,s)$ is zero for every r,s in $\mathcal{A}$ and for $i=1,2,\ldots,n+m+p-1$. Particularly, take $\mathcal{P}_1(r,s)=0$ to get the following:

$$\begin{aligned}\binom{m+n+p}{1}\Delta(r^{m+n+p-1})s&=\binom{n+p}{1}\Delta(r^m)r^{n+p-1}s+\binom{m}{1}\Delta(r^{m-1}s)r^{n+p}\\&+\binom{p}{1}r^m\partial(r^n)r^{p-1}s+\binom{n}{1}r^m\partial(r^{n-1}s)r^p\\&+\binom{p}{1}r^m\partial(r^n)r^{p-1}s+\binom{p}{1}r^{m+n}\partial(r^{p-1}s)\\&+\binom{m+n}{1}r^{m+n-1}s\partial(r^p)\end{aligned}\tag{2.2}$$

Reinstate the above equation by putting e in place of r to have $\binom{m+n+p}{1}\Delta(e)s=\binom{n+p}{1}\Delta(e)s+\binom{m}{1}\Delta(s)+\binom{p}{1}\partial(e)s+\binom{n}{1}\partial(s)+\binom{p}{1}\partial(e)s+\binom{p}{1}\partial(s)+\binom{m+n}{1}s\partial(e)$. On simplifying the last relation, we can obtain $(n+p)\Delta(s)=(n+p)\Delta(e)s+(n+p)\partial(s)$ for all $s\in\mathcal{A}$. A torsion restriction given in the hypothesis enable us to write

$$\Delta(s)=\Delta(e)s+\partial(s),\quad\text{for all } s\in\mathcal{A}.\tag{2.3}$$

Now consider $\mathcal{P}_2(r,s)=0$, which implies that

$$\begin{aligned}\binom{m+n+p}{2}\Delta(r^{m+n+p-2}s^2)&=\binom{n+p}{2}\Delta(r^m)r^{n+p-2}s^2+\binom{m}{1}\binom{n+p}{1}\Delta(r^{m-1}s)r^{n+p-1}s\\&+\binom{m}{2}\Delta(r^{m-2}s^2)r^{n+p}+\binom{p}{2}r^m\partial(r^n)r^{p-2}s^2\\&+\binom{n}{1}\binom{p}{1}r^m\partial(r^{n-1}s)r^{p-1}s+\binom{n}{2}r^m\partial(r^{n-2}s^2)r^p\\&+\binom{m}{1}\binom{p}{1}r^{m-1}s\partial(r^n)r^{p-1}s+\binom{m}{1}\binom{n}{1}r^{m-1}s\partial(r^{n-1}s)r^p\\&+\binom{m}{2}r^{m-2}s^2\partial(r^n)r^p+\binom{p}{2}r^{m+n}\partial(r^{p-2}s^2)\\&+\binom{m+n}{1}\binom{p}{1}r^{m+n-1}s\partial(r^{p-1}s)+\binom{m+n}{2}r^{m+m-2}s^2\partial(r^p)\end{aligned}$$

Substituting e for r in the above expression and using the fact that $\partial(e) = 0$, we obtain

$$\begin{aligned}\binom{m+n+p}{2}\Delta(s^2) &= \binom{n+p}{2}\Delta(e)s^2 + \binom{m}{1}\binom{n+p}{1}\Delta(s)s + \binom{m}{2}\Delta(s^2)\\ &+\binom{n}{1}\binom{p}{1}\partial(s)s + \binom{n}{2}\partial(s^2) + \binom{m}{1}\binom{n}{1}s\partial(s)\\ &+\binom{p}{2}\partial(s^2) + \binom{m+n}{1}\binom{p}{1}s\partial(s)\end{aligned}$$

A quiet manipulation yields that

$$\begin{aligned}(m+n+p)(m+n+p-1)\Delta(s^2) &= (n+p)(n+p-1)\Delta(e)s^2 + 2m(n+p)\Delta(s)s\\ &+m(m-1)\Delta(s^2)2np\partial(s)s + n(n-1)\partial(s^2)\\ &+2mns\partial(s) + p(p-1)\partial(s^2) + 2p(m+n)s\partial(s).\end{aligned}$$

This implies that

$$\begin{aligned}[(n+m+p)(n+m+p-1) - m(m-1)]\Delta(s^2) &= (n+p)(n+p-1)[\Delta(s^2) - \partial(s^2)]\\ &+2m(n+p)\Delta(s)s + 2pn\partial(s)s\\ &+(2mn+2np+2pm)s\partial(s)\\ &+(n^2-n+p^2-p)\partial(s^2).\end{aligned}$$

This entails

$$\begin{aligned}&[(n+m+p)(n+m+p-1) - m(m-1) - (n+p)(n+p-1)]\Delta(s^2)\\ &\quad = -2np\partial(s^2) + 2m(n+p)\Delta(s)s + (2mn+2np+2pm)s\partial(s) + 2pn\partial(s)s.\end{aligned}$$

This gives

$$\begin{aligned}(2mn+2mp)\Delta(s^2) &= -2np\partial(s^2) + 2m(n+p)\Delta(s)s\\ &+(2mn+2np+2pm)s\partial(s) + 2pn\partial(s)s.\end{aligned}$$

Replacing s by s^2 in (2.3) and use it the above relation, we receive that

$$\begin{aligned}(2mn+2mp)[\Delta(e)s^2 + \partial(s^2)] &= -2np\partial(s^2) + 2m(n+p)[\Delta(e)s^2 + \partial(s)s]\\ &+(2mn+2np+2pm)s\partial(s) + 2pn\partial(s)s.\end{aligned}$$

Subsequently a few calculation, we arrive at

$$\begin{aligned}(2mn+2mp+2np)\partial(s^2) &= 2m(n+p)\partial(s)s + (2mn+2np+2pm)s\partial(s)\\ &+2pn\partial(s)s\text{for every } s \in \mathcal{A}.\end{aligned}$$

Since $\mathcal{A}$ is $2(mn+np+pm)$-torsion free ring, then we find $\partial(s^2) = \partial(s)s + s\partial(s)$ for every $s \in \mathcal{A}$. Hence, ∂ is a Jordan derivation on $\mathcal{A}$. Since, $\mathcal{A}$ is 2 torsion free semiprime ring, then from [5], ∂ will be a derivation on $\mathcal{A}$. Next, consider from (2.3), we have

$$\begin{aligned}\Delta(s^2) &= \Delta(e)s^2 + \partial(s^2)\\ &= \big[\Delta(e)s + \partial(s)\big]s + s\partial(s)\\ &= \Delta(s)s + s\partial(s)\end{aligned}$$

Hence Δ is generalized Jordan derivation on $\mathcal{A}$ associated with the derivation ∂. Using Theorem from [15], we obtain the desired outcome. □

The following are some direct ramifications of the preceding theorem:

Corollary 2.2 *Suppose that $m, n \geq 1$ and $\mathcal{A}$ are fixed integers and $(m+n-1)!$-torsion free semiprime ring with identity e, respectively. If Δ and ∂ from $\mathcal{A}$ to itself are additive mappings that fulfill the algebraic identity $\Delta(r^{m+n}) = \Delta(r^m)r^n + r^m\partial(r^n)$ for every r in $\mathcal{A}$, then Δ will be a generalized derivation associated with ∂, a derivation on $\mathcal{A}$.*

Proof Taking $p = 0$ in Theorem 2.1, we get the required result. □

Corollary 2.3 *Suppose that $m, n \geq 1$ and $\mathcal{A}$ are fixed integers and $(m+n-1)!$-torsion free semiprime ring with identity e, respectively. If Δ from $\mathcal{A}$ to itself is an additive mapping that fulfills the algebraic identity $\Delta(r^{m+n}) = \Delta(r^m)r^n$ for every r in $\mathcal{A}$, then Δ will be a centralizer on $\mathcal{A}$.*

Proof Taking $d = p = 0$ in Theorem 2.1, we find the required conclusion. □

Corollary 2.4 *Let $m, n \geq 1$ and $p \geq 0$ be fixed integers and $\mathcal{A}$ be any $(n+m+p-1)!$-torsion free semiprime ring with identity e. Suppose that $\partial : \mathcal{A} \to \mathcal{A}$ is additive mapping which satisfy the following algebraic identity $\partial(r^{m+n+p}) = \partial(r^m)r^{n+p} + r^m\partial(r^n)r^p + r^{m+n}\partial(r^p)$ for every r in $\mathcal{A}$, then ∂ will be a derivation.*

Proof Considering ∂ as Δ and using the same steps as we did in the first main theorem, we obtain the required conclusion. □

Corollary 2.5 *Let $m \geq 1$ and $\mathcal{A}$ be fixed integers and any $(3m-1)!$-torsion free semiprime ring with identity e, respectively. If Δ and ∂ from $\mathcal{A}$ to itself are additive mappings that fulfills the algebraic identity $\Delta(r^{3m}) = \Delta(r^m)r^{2m} + r^m\partial(r^m)r^m + r^{2m}\partial(r^m)$ for every r in $\mathcal{A}$, then Δ will be a generalized derivation associated with a derivation ∂.*

Proof Considering $m = n = p$ and using the same steps as we did in the first theorem, we reach at the required conclusion. □

Corollary 2.6 *Let $m \geq 1$ and $\mathcal{A}$ be fixed integers and any $(2m-1)!$-torsion-free semiprime ring with identity e, respectively. If Δ and ∂ from $\mathcal{A}$ to itself are additive mappings that fulfill the algebraic identity $\Delta(r^{2m}) = \Delta(r^m)r^m + r^m\partial(r^m)$ for every r in $\mathcal{A}$, then Δ is a generalized derivation associated with ∂, a derivation on $\mathcal{A}$.*

Proof Taking $m = n$ in Corollary 2.2, we obtain the desired outcome. □

Corollary 2.7 *Let $m \geq 1$ and $\mathcal{A}$ be fixed integers and any $(2m-1)!$-torsion free semiprime ring with identity e, respectively. If Δ from $\mathcal{A}$ to itself is an additive mapping that fulfills the algebraic identity $\Delta(r^{2m}) = \Delta(r^m)r^m$ for every r in $\mathcal{A}$, then Δ will be a centralizer on $\mathcal{A}$.*

Proof Considering $m = n$ in Corollary 2.3, we will arrive at the conclusion. □

Corollary 2.8 *Let $m \geq 1$ and $\mathcal{A}$ be fixed integers and any $(3m-1)!$-torsion free semiprime ring with identity e, respectively. If ∂ from $\mathcal{A}$ to itself is an additive mapping that fulfill the algebraic identity $\partial(r^{3m}) = \partial(r^m)r^{2m} + r^m\partial(r^m)r^m + r^{2m}\partial(r^m)$ for every r in $\mathcal{A}$, then ∂ will be a derivation.*

Proof Considering $m = n = p$ in Corollary 2.4, we will arrive at the conclusion. □

Now let us move on to the second key theorem of the paper:

Theorem 2.9 *Assume $m, n \geq 1$ and $p \geq 0$ are fixed integers and $\mathcal{A}$ is any $(n+m+p-1)!$-torsion free semiprime ring with identity e. If ∇ and ∂_1 from $\mathcal{A}$ to itself are additive mappings that fulfill the algebraic identity $\nabla(r^{m+n+p}) = r^{m+n}\nabla(r^p) + r^{m+p}\partial_1(r^n) + r^{n+p}\partial_1(r^p)$ for every r in $\mathcal{A}$, then ∇ will be generalized left derivation associated with a left derivation ∂_1.*

Proof Given that

$$\nabla(r^{m+n+p}) = r^{m+n}\nabla(r^p) + r^{m+p}\partial_1(r^n) + r^{n+p}\partial_1(r^m) \text{ for every } r \in \mathcal{A}. \tag{2.4}$$

To get $\partial_1(e) = 0$, replace r by e. Further, by substituting r in (2.4) with $r + qs$, we obtain

$$\nabla\left(r^{m+n+p} + \binom{m+n+p}{1}(r^{m+n+p-1})qs + \binom{m+n+p}{2}r^{m+n+p-2}q^2s^2 + \cdots + q^{m+n+p}s^{m+n+p}\right)$$
$$= \left(r^{m+n} + \binom{m+n}{1}r^{m+n-1}qs + \binom{m+n}{2}r^{m+n-2}q^2s^2 + \cdots + q^{m+n}s^{m+n}\right) \cdot \nabla\left(r^p + \binom{p}{1}r^{p-1}qs\right.$$

$$+\binom{p}{2}r^{p-2}q^2s^2+\cdots+q^ps^p\Big)+\left(r^{m+p}+\binom{m+p}{1}r^{m+p-1}qs+\binom{m+p}{2}r^{m+p-2}q^2s^2\right.$$
$$\left.+\cdots+q^{m+p}s^{m+p}\right)\cdot\partial_1\left(r^n+\binom{n}{1}r^{n-1}qs+\binom{n}{2}r^{n-2}q^2s^2+\cdots+q^ns^n\right)$$
$$+\left(r^{n+p}+\binom{n+p}{1}(r^{n+p-1})qs+\binom{n+p}{2}r^{n+p-2}q^2s^2+\cdots+q^{n+p}s^{n+p}\right)$$
$$\cdot\partial_1\left(r^m+\binom{m}{1}r^{m-1}qs+\binom{m}{2}r^{m-2}q^2s^2+\cdots+q^ms^m\right).$$

Write the above equation as the sum of the terms of power of q with (2.4), we obtain

$$q\mathcal{P}_1(r,s)+q^2\mathcal{P}_2(r,s)+\cdots+q^{m+n+p-1}\mathcal{P}_{m+n+p-1}(r,s)=0,$$

where $\mathcal{P}_i(r,s)$ stance for the coefficients of q^i's for all $i=1,2,\ldots,n+m+p-1$. If we replace q by $1,2,\ldots,n+m+p-1$ in turn, then we find a system of $n+m+p-1$ homogeneous linear equations which gives a Vander Monde matrix, which turnout to $\mathcal{P}_i(r,s)=0$ for every pair r,s in $\mathcal{A}$ and for $i=1,2,\ldots,n+m+p-1$. Particularly $\mathcal{P}_1(r,s)=0$

$$\begin{aligned}\binom{m+n+p}{1}\nabla(r^{m+n+p-1})s=&\binom{p}{1}r^{m+n}\nabla(r^{p-1}s)+\binom{m+n}{1}r^{m+n-1}s\nabla(r^p)\\&+\binom{n}{1}r^{m+p}\partial_1(r^{n-1}s)+\binom{m+p}{1}r^{m+p-1}s\partial_1(r^n)\\&+\binom{m}{1}r^{n+p}\partial_1(r^{m-1}s)+\binom{n+p}{1}r^{n+p-1}s\partial_1(r^m)\end{aligned}\tag{2.5}$$

In the above equation, replace r by e and use the fact that $\partial_1(e)=0$ to have $\binom{m+n+p}{1}\nabla(e)s=\binom{p}{1}\nabla(s)+\binom{m+n}{1}s\nabla(e)+\binom{n}{1}\partial_1(s)+\binom{m}{1}\partial_1(s)$. On simplifying the previous relation we can obtain $(n+m)\nabla(s)=(n+m)s\nabla(e)+(n+m)\partial_1(s)$ for every $s\in\mathcal{A}$. Torsion restriction given in the hypothesis enable us to write

$$\nabla(s)=s\nabla(e)+\partial_1(s),\quad\text{for every } s\in\mathcal{A}.\tag{2.6}$$

Next, take $\mathcal{P}_2(r,s)=0$, which yields the following expression

$$\begin{aligned}\binom{m+n+p}{2}\nabla(r^{m+n+p-2}s^2)=&\binom{p}{2}r^{m+n}\nabla(r^{p-2}s^2)+\binom{m+n}{1}\binom{p}{1}r^{m+n-1}s\nabla(r^{p-1}s)\\&+\binom{m+n}{2}r^{m-n-2}s^2\nabla(r^p)+\binom{n}{2}r^{m+p}\partial_1(r^{n-2}s^2)\\&+\binom{m+p}{1}\binom{n}{1}r^{m+p-1}s\partial_1(r^{n-1}s)+\binom{m+p}{2}r^{m+p-2}s^2\partial_1(r^n)\\&+\binom{m}{2}r^{n+p}\partial_1(r^{m-2}s^2)+\binom{m}{1}\binom{n+p}{1}r^{n+p-1}s\partial_1(r^{m-1}s)\\&+\binom{n+p}{2}r^{n+p-2}s^2\partial_1(r^m).\end{aligned}$$

Replacing e for r in the above equation and using the fact that $\partial_1(e)=0$, we find

$$\begin{aligned}\binom{m+n+p}{2}\nabla(s^2)=&\binom{p}{2}\nabla(s^2)+\binom{m+n}{1}\binom{p}{1}s\nabla(s)+\binom{m+n}{2}s^2\nabla(e)+\binom{n}{2}\partial_1(s^2)\\&+\binom{m+p}{1}\binom{n}{1}s\partial_1(s)+\binom{m}{2}\partial_1(s^2)+\binom{m}{1}\binom{n+p}{1}s\partial_1(s).\end{aligned}$$

A straightforward estimation turnout the following $(m+n+p)(m+n+p-1)(p^2-p)\nabla(s^2)=2(mp+np)s\nabla(s)+2(mn+pn)s\partial_1(s)+2(mn+mp)s\partial_1(s)+(m+n)(m+n-1)[\nabla(s^2)-\partial_1(s^2)]+$

$n(n-1)\partial_1(s^2) + m(m-1)\partial_1(s^2)$ for every $s \in \mathcal{A}$, which yields that $2(mp+np)\nabla(s^2) = 2(2mn+np+pm)s\partial_1(s) - 2mn\partial_1(s^2) + 2(mn+np)s\nabla(s)$. Using Eq. (2.3) after some calculations, for all $s \in \mathcal{A}$ this entails $2(2mn+2mp+2np)\partial_1(s^2) = (2mn+2mp+2np)s\partial_1(s)$ for all $s \in \mathcal{A}$. Using torsion restrictions on $\mathcal{A}$, we find $\partial_1(s^2) = 2s\partial_1(s)$. Hence, ∂_1 is a Jordan left derivation on $\mathcal{A}$ as the ring is 2 torsion free semiprime, Now, from (2.3), we get

$$\begin{aligned}\nabla(s^2) &= s^2\nabla(e) + \partial_1(s^2)\\ &= s\big[s\nabla(e) + \partial_1(s)\big] + s\partial_1(s)\\ &= s\nabla(s) + s\partial_1(s)\end{aligned}$$

Hence, ∇ will be a generalized Jordan left derivation having associated left derivation ∂_1. Using theorem from [1], we find the required conclusion. □

The next result is a consequence of Theorem 2.9:

Theorem 2.10 *Assume $m, n \geq 1$ and $p \geq 0$ are fixed integers. If ∇ and ∂_1 from a $(n+m+p-1)!$-torsion free semiprime ring $\mathcal{A}$ with identity e to itself are additive mappings that fulfill the algebraic identity $\nabla(r^{m+n+p}) = r^{m+n}\nabla(r^p) + r^{m+p}\partial_1(r^n) + r^{n+p}\partial_1(r^p)$, then*

(1) *∂_1 is a Commuting derivation on $\mathcal{A}$, and*
(2) *$\partial_1(\mathcal{A}) = Z_{\mathcal{A}}$, and*
(3) *either ∂_1 is zero or $\mathcal{A}$ is commutative, and*
(4) *For some $q \in Q_l(\mathcal{A}_C)$, $f(r) = rq$ for all $r \in \mathcal{A}$, and*
(5) *∇ will be a generalized derivation.*

Proof (1) Using Theorem 2.9 together with Theorem 3.1 of [1], we get the required result.
(2) We find the required conclusion using Theorem 2.9 and Theorem 2 of [12].
(3) Assume that $\partial_1 \neq 0$. We have $\nabla(r^{m+n+p}) = r^{m+n}\nabla(r^p) + r^{m+p}\partial_1(r^n) + r^{n+p}\partial_1(r^p)$ for all $r \in \mathcal{A}$. Then, ∇ will be a generalized left derivation associated with ∂_1 from Theorem 2.9. Therefore, ∂_1 is a commuting derivation from (1). As a conclusion, $\mathcal{A}$ is commutative from [9, Theorem 2].
(4) In view of Theorem 2.9 and (3), we find $\partial_1 = 0$. As $\mathcal{A}$ is a semiprime ring, then ∇ will be a right centralizer on $\mathcal{A}$. Therefore, from [1, Proposition 2.10], we achieve the desired outcome.
(5) Considering $\nabla(r^{m+n+p}) = r^{m+n}\nabla(r^p) + r^{m+p}\partial_1(r^n) + r^{n+p}\partial_1(r^p)$ for all $r \in \mathcal{A}$. In perspective of (3) and Theorem 2.9, ∂_1 and $\mathcal{A}$ will be a derivation and commutative ring respectively. Hence, We obtain the desired conclusion.

□

The next example demonstrates that the theorems provided in this paper are not unjustifiable.

Example 2.11 Determine a ring $\mathcal{A} = \left\{\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix} \mid \alpha_1, \alpha_2 \in 2\mathbb{Z}_8\right\}$, $\mathbb{Z}_8$ has its usual meaning. Define mappings $\Delta, \partial, \nabla, \partial_1 : \mathcal{A} \to \mathcal{A}$ by $\Delta\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & \alpha_2\end{pmatrix}$, $\partial\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix} = \begin{pmatrix}\alpha_1 & 0\\ 0 & 0\end{pmatrix}$, $\nabla\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & \alpha_2\end{pmatrix}$ and $\partial_1\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix} = \begin{pmatrix}\alpha_1 & 0\\ 0 & 0\end{pmatrix}$. It is clear that Δ and ∇ are not a generalized derivation and generalized left derivation respectively but $\Delta, \partial, \nabla, \partial_1$ satisfy the algebraic conditions in Theorem 2.1 and Theorem 2.9. Particularly, if $m = n = p = 1$, then neither $\mathcal{A}$ is semiprime nor $\mathcal{A}$ is 2! torsion free, which shows that semiprimeness and torsion restriction on $\mathcal{A}$ are essential conditions for these theorems.

3 Application

According to the Singer–Wermer theorem, a classic Banach algebra theorem, on a commutative Banach algebra every continuous derivation maps into its Jacobson radical. Thomas demonstrated that the Singer–Wermer theorem is not invalid despite the fact that the derivation is not continuous. The Singer–Wermer conjecture refers to this generalization. The following systematic observations are motivated by the parallel line of investigation. Since every semisimple Banach algebra is semiprime, then it will be a quit interesting if one can think about Theorem 2.9 in the setting of semisimple Banach algebra. So, let $\mathfrak{A}$ be a semisimple Banach algebra and

$\mathfrak{F}, \mathfrak{D} : \mathfrak{A} \to \mathfrak{A}$ be two additive mappings fulfilling $\mathfrak{F}(A^{m+n+p}) = A^{m+n}\mathfrak{F}(A^p) + A^{m+p}\mathfrak{D}(A^n) + A^{n+p}\mathfrak{D}(A^p)$ for all $A \in \mathfrak{A}$. Then $\mathfrak{D} = 0$ on $\mathfrak{A}$, where $m, n \geq 1$ and $p \geq 0$ are fixed integers. To prove Theorem 3.4, we need the following results:

Fact 3.1 ([8]). On a semisimple Banach Algebra, every linear derivation is continuous.

Fact 3.2 ([10]). On a commutative Banach Algebra, every continuous linear derivation transforms algebra to its radical.

Thomos conclude the following from the previous two facts:

Lemma 3.3 ([11]). *On commutative semisimple Banach algebras, there does not exist any nonzero linear derivation.*

In the perspective of the above lemma, we conclude the following theorem:

Theorem 3.4 *Let $\mathfrak{A}$ be a semisimple Banach algebra. If $\mathfrak{F}$ and $\mathfrak{D}$ are two additive mappings from $\mathfrak{A}$ to itself satisfying $\mathfrak{F}(A^{m+n+p}) = A^{m+n}\mathfrak{F}(A^p) + A^{m+p}\mathfrak{D}(A^n) + A^{n+p}\mathfrak{D}(A^p)$ for all $A \in \mathfrak{A}$, then $\mathfrak{D} = 0$ on $\mathfrak{A}$, where $m, n \geq 1$ and $p \geq 0$ are fixed integers.*

Proof All the consideration of Theorem 2.10 (1) are satisfied because every semisimple Banach algebra is semiprime, on semisimple Banach algebra $\mathfrak{A}$, we have a linear derivation. As a result, according to Theorem 4 of [12], $\mathfrak{D} = 0$. □

4 Conjecture

From the above observation, one can think about the following two conjectures:

Conjecture 4.1 Suppose that $m, n \geq 1$ and $p \geq 0$ are fixed integers and $\mathcal{A}$ is any $(n+m+p-1)!$-torsion-free semiprime ring not necessarily with identity. If Δ and ∂ from $\mathcal{A}$ to itself are additive mappings which satisfy the following algebraic identity $\Delta(r^{m+n+p}) = \Delta(r^m)r^{n+p} + r^m\partial(r^n)r^p + r^{m+n}\partial(r^p)$ for every $r \in \mathcal{A}$, then Δ will be a generalized derivation associated with a derivation ∂ on $\mathcal{A}$.

Conjecture 4.2 Assume $m, n \geq 1$ and $p \geq 0$ are fixed integers and $\mathcal{A}$ is any $(n+m+p-1)!$-torsion-free semiprime ring not necessarily with identity. If ∇ and ∂_1 from $\mathcal{A}$ to itself are additive mappings that fulfill the algebraic identity $\nabla(r^{m+n+p}) = r^{m+n}\nabla(r^p) + r^{m+p}\partial_1(r^n) + r^{n+p}\partial_1(r^p)$ for every r in $\mathcal{A}$, then ∇ will be generalized left derivation associated with a left derivation D.

Acknowledgements The author is grateful for the valuable suggestions provided by the referees to enhance the manuscript's presentation. Sincere appreciation is extended to the Deanship of Scientific Research at the Islamic University of Madinah.

Funding Not applicable.

Data availability Not applicable.

Declarations

Conflict of interest The author declares that there are no conflicts of interest regarding the publication of this paper.

Informed consent Not applicable.

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