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Compact formula for skew-symmetric system of matrix equations

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Abstract In this paper, we consider skew-Hermitian solution of coupled generalized Sylvester matrix equations encompassing $*$-hermicity over complex field. The compact formula of the general solution of this system is presented in terms of generalized inverses when some necessary and sufficient conditions are fulfilled. An algorithm and a numerical example are provided to validate our findings. A numerical example is carried out using determinantal representations of the Moore–Penrose inverse.

Mathematics Subject Classification 15A03 · 15A09 · 15A24 · 15B33 · 15B57

1 Introduction

Throughout, $A \in \mathbb{C}^{m\times n}$ stays for a $m \times n$ matrix over a complex number field $\mathbb{C}$. Additionally, the rank of a matrix $A \in \mathbb{C}$ is denoted by $r(A)$. The conjugate transpose of A is written by A^*. An identity matrix with plausible shape is denoted by I. The Moore–Penrose inverse of A is represented by $A^\dagger = T$ and is defined as a solution to the following system:

$$ATA = A, \quad TAT = T, \quad (AT)^* = AT, \quad (TA)^* = TA.$$

Furthermore, $L_A = I - A^*A$ and $R_A = I - AA^*$ are projectors onto the kernel of A, such that $AL_A = 0$ and $R_A A = 0$, where I and 0 stand for the identity matrix and a zero matrix, respectively. Moreover,

$$L_A = (L_A)^* = (L_A)^2 = L_A^\dagger, \quad R_A = (R_A)^2 = (R_A)^* = R_A^\dagger.$$

The solution of matrix equations have backbone position in different fields of sciences and engineering like system design [49], singular system control [13], linear descriptor system [11], and sensitivity analysis [5]. For instance, Bai computed the iterative solution of $A_1X + XA_2 = B$ in [2] and $A_1X + YA_2 = B$ was considered by different researchers in [3,45].

Similarly, the solution of system of Sylvester matrix equations also has been observed by different researchers with different techniques. Recently, the general solution of

$$A_1X_1 + Z_1B_1 = C_1, \quad A_2X_2 + Z_1B_2 = C_2 \tag{1.1}$$

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was computed in [50] when this system is consistent. Some solvability conditions and condition number to (1.1) were also given in [25,27]. Wang et al. in [52] evaluated the constraint solution of (1.1). When $X_2 = X_1$ in (1.1), then some necessary and sufficient conditions of (1.1) were given in [54]. Wang and He also gave some necessary and sufficient conditions for

$$A_1X_1 + Z_1B_1 = C_1, \quad A_2Z_1 + X_2B_2 = C_2,$$

to have a solution with its general solution in [15]. Some latest research papers related to the general solution of different types of Sylvester matrix equations can be viewed in [7,29–31,33–44,52,53,55–59]

The numerical solution of two-sided Sylvester matrix equation was explored in [6]. A researcher in [16] discussed the triangular two-sided Sylvester matrix equation. The Hermitian solution of

$$A_1XA_1^* + B_1YB_1^* = C_1 \tag{1.2}$$

is presented in [28]. Some findings on (1.2) can be viewed in [12]. Very recently, an algorithm to find out the solution of

$$\begin{aligned} A_1XB_1 + C_1YD_1 = E_1,\\ A_2ZB_2 + C_2YD_2 = E_2 \end{aligned} \tag{1.3}$$

was constructed in [14], and recently, the Hermitian solution to (1.3) has been carried out in [36] with its general solution when this system is consistent.

Very recently, the authors in [43] researched the skew-Hermitian solution of the system

$$\begin{aligned} A_1UA_1^*+B_1VB_1^* = C_1,\ C_1 = -C_1^*,\\ A_2WA_2^*+B_2VB_2^* = C_2,\ C_2 = -C_2^*, \end{aligned} \tag{1.4}$$

when it is consistent. They also presented the closed form of formula for the general solution when this system is consistent over the complex plane $\mathbb{C}$.

Motivated by the above research and the formidable applications of generalized Sylvester matrix equations in the fields like feedback [48] and perturbation theory [26], we, in this paper, consider the skew-Hermitian system of Sylvester matrix equations

$$\begin{aligned} D_1X_1 - (D_1X_1)^* + E_1Y_1E_1^* + F_1Z_1F_1^* = G_1,\ G_1 = -G_1^*,\\ D_2X_2 - (D_2X_2)^* + E_2Y_2E_2^* + F_2Z_1F_2^* = G_2,\ G_2 = -G_2^*, \end{aligned} \tag{1.5}$$

over the complex number field $\mathbb{C}$. By solving (1.5) will definitely reinforce the application of system of skew-Hermitian Sylvester matrix equations into a variety of number of fields of sciences and engineering and their allied areas.

To start with, we give some significant results which will be used in the construction of the main result of this paper.

Lemma 1.1 [32]. *Let $K \in \mathbb{C}^{m\times n},\ P \in \mathbb{C}^{m\times t},\ Q \in \mathbb{C}^{l\times n}$. Then*

$$r\begin{bmatrix} K\\ Q \end{bmatrix} - r(QL_K) = r(K), \quad r\begin{bmatrix} K & P \end{bmatrix} - r(R_PK) = r(P),$$

$$r\begin{bmatrix} K & P\\ Q & 0 \end{bmatrix} - r(P) - r(Q) = r(R_PKL_Q).$$

Lemma 1.2 [51]. *Let A, B, and C be given matrices with right sizes over $\mathbb{C}$. Then*

(1) $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$.
(2) $L_A = L_A^2 = L_A^*,\ R_A = R_A^2 = R_A^*$.
(3) $L_A(BL_A)^\dagger = (BL_A)^\dagger,\ (R_AC)^\dagger R_A = (R_AC)^\dagger$.

In obtaining the general solution to (1.5), we need the general solution of

$$AX - (AX)^* + BYB^* + CZC^* = D,\ D = -D^*,\ Y = -Y^*,\ Z = -Z^*. \tag{1.6}$$

Lemma 1.3 [40]. *Let $A,\ B,\ C$, and $D = -D^*$ be given coefficient matrices in (1.6) over $\mathbb{C}$ with conformable sizes. Denote*

$$A_1 = R_A B,\ B_1 = R_A C,\ C_1 = R_A D R_A,\ M = R_{A_1} B_1,\ S = B_1 L_M.$$

Then

(1) *Eq. (1.6) has a solution (X, Y, Z), where $Y = -Y^*$ and $Z = -Z^*$.*
(2) *The coefficient matrices in (1.6) satisfy*

$$R_M R_{A_1} C_1 = 0,\ R_{A_1} C_1 R_{B_1}^* = 0.$$

(3) $MM^\dagger R_{A_1} C_1 = R_{A_1} C_1 = R_{A_1} C_1 (B_1^\dagger)^* B_1^*$.
(4)

$$r\begin{bmatrix} D & C & B & A \\ A^* & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} C & B & A \end{bmatrix} + r(A),$$
$$r\begin{bmatrix} D & B & A \\ A^* & 0 & 0 \\ C^* & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B & A \end{bmatrix} + r\begin{bmatrix} A & C \end{bmatrix}$$

are equivalent statements. Under these conditions, the general solution to the system (1.6) can be demonstrated as

$$\begin{aligned} Y = -Y^* &= A_1^\dagger C_1 (A_1^\dagger)^* - \frac{1}{2} A_1^\dagger B_1 M^\dagger C_1 [I + (B_1^\dagger)^* S^*](A_1^\dagger)^* \\ &\quad - \frac{1}{2} A_1^\dagger [I + S B_1^\dagger] C_1 (M^\dagger)^* B_1^* (A_1^\dagger)^* - A_1^\dagger S W_2 S^* (A_1^\dagger)^* - L_{A_1} U + U^* L_{A_1}^*, \\ Z = -Z^* &= \frac{1}{2} M^\dagger C_1 ({B_1}^\dagger)^* [I + S^\dagger S] + \frac{1}{2} [I + S^\dagger S] B_1^\dagger C_1 (M^\dagger)^* \\ &\quad + L_M W_2 L_M^* - V L_{B_1}^* + L_{B_1} V^* + L_M L_S W_1 - W_1^* (L_M L_S)^*, \\ X &= A^\dagger [D - BYB^* - CZC^*] - \frac{1}{2} A^\dagger [D - BYB^* - CZC^*](A^\dagger)^* A^* \\ &\quad - L_A U_1 + U_2^* (A^\dagger)^* A^* + A^\dagger U_2 A^*, \end{aligned}$$

where $U_1,\ U_2,\ W_1,\ U,\ V$, and $W_2^* = -W_2$ are arbitrary matrices over $\mathbb{C}$.

The skew-Hermitian solution to the system (1.5) will be expressed in terms of the Moore–Penrose (MP-) inverse. Thanks to the important role of generalized inverses in many application fields, considerable effort has been exerted toward the numerical algorithms for fast and accurate calculation of matrix generalized inverse. In general, most existing methods for their obtaining are iterative algorithms for approximating generalized inverses of complex matrices (some recent papers, see, e.g., [1,46]). There are only several direct methods finding MP-inverse for an arbitrary complex matrix. The most famous is method based on singular value decomposition (SVD), i.e., if $A = U\Sigma V^*$, then $A^\dagger = V\Sigma^\dagger U^*$. Another approach is constructing determinantal representations of the MP-inverse $A^\dagger$. There are various determinantal representations of generalized inverses (for the MP-inverse, see, e.g., [4,47]). Because of the complexity of the previously obtained expressions of determinantal representations of the MP-inverse, they do not found a wide applicability.

In this paper, it is used the determinantal representations of the MP-inverse recently derived by one of authors in [17].

Lemma 1.4 [17, Theorem 2.2] *If $A \in \mathbb{C}_r^{m\times n}$, then the MP-inverse $A^\dagger = \left(a_{ij}^\dagger\right) \in \mathbb{C}^{n\times m}$ possesses the following determinantal representations:*

$$a_{ij}^\dagger = \frac{\sum_{\beta\in J_{r,n}\{i\}} \left|(A^*A)_{.i}\left(a_{.j}^*\right)\right|_\beta^\beta}{\sum_{\beta\in J_{r,n}} |A^*A|_\beta^\beta} = \frac{\sum_{\alpha\in I_{r,m}\{j\}} \left|(AA^*)_{j.}(a_{i.}^*)\right|_\alpha^\alpha}{\sum_{\alpha\in I_{r,m}} |AA^*|_\alpha^\alpha}. \tag{1.7}$$

Here, $|A|_{\alpha}^{\alpha}$ denotes a principal minor of A whose rows and columns are indexed by $\alpha := \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots, m\}$

$$L_{k,m} := \{\alpha : \ 1 \le \alpha_1 < \cdots < \alpha_k \le m\}, \text{ and } I_{r,m}\{i\} := \left\{\alpha : \alpha \in L_{r,m}, i \in \alpha\right\}.$$

Also, $a^*_{.j}$ and $a^*_{i.}$ denote the jth column and the ith row of A^*, and $A_{i.}(b)$ and, respectively, and $A_{.j}(c)$ stand for the matrices obtained from A by replacing its ith row with the row vector $b \in \mathbb{C}^{1\times n}$ and its jth column with the column vector $c \in \mathbb{C}^m$.

The formulas (1.7) mean calculations of sum of all principal minors of r order of the matrices A^*A or AA^* in denominators and sum of principal minors of r order of the matrices $(A^*A)_{.i}(a^*_{.j})$ or $(AA^*)_{j.}(a^*_{i.})$ that contain the ith column or the jth row, respectively, in numerators.

Note that these new determinantal representations of the Moore–Penrose inverse have been extended over quaternion matrices [18] as well. This method was successfully applied for constructing determinantal representations of other generalized inverses in both cases for complex and quaternion matrices (see, e.g., [20,21]). It also yields Cramer's rules of various matrix equations [19,22–24,34,41,42].

Our paper is composed of four sections. The general solution to (1.5) is constituted in Sect. 2 with a special case. The algorithm and numerical example of finding the anti-Hermitian solution of (1.5) are presented in Sect. 3. A conclusion to this paper is given in Sect. 4.

2 Main result

Now, we present the main Theorem of this paper.

Theorem 2.1 *Given* $D_1,\ D_2,\ E_1,\ E_2,\ F_1,\ F_2,\ G_1 = G_1^*,\ G_2 = -G_2^*$ *be matrices of conformable shapes over* $\mathbb{C}$*. Assign*

$$\begin{aligned}
&A_1 = R_{D_1}E_1,\ B_1 = R_{D_1}F_1,\ C_1 = R_{D_1}G_1R_{D_1},\ A_2 = R_{D_2}E_2,\ B_2 = R_{D_2}F_2,\ C_2 = R_{D_2}G_2R_{D_2},\\
&M_1 = R_{A_1}B_1,\ S_1 = B_1L_{M_1},\ M_2 = R_{A_2}B_2,\ S_2 = B_2L_{M_2},\ A_4 = R_{A_3}L_{M_1},\ A_5 = R_{A_3}L_{M_2},\\
&W^* = \left[\, U_2^*\ U_3\ U_{22}^*\ U_{33} \,\right],\ A_3 = \left[\, L_{M_1}L_{S_1}\ L_{B_1}\ -L_{M_2}L_{S_2}\ -L_{B_2} \,\right],\ M_3 = R_{A_4}A_5,\\
&S_3 = A_5L_{M_3},\ Z_{02} = \frac{1}{2}M_2^{\dagger}C_2(B_2^{\dagger})^*(I + S_2^{\dagger}S_2) + \frac{1}{2}(I + S_2^{\dagger}S_2)B_2^{\dagger}C_2(M_2^{\dagger})^*,\ E_{11} = Z_{02} - Z_{01},\\
&Z_{01} = \frac{1}{2}M_1^{\dagger}C_1(B_1^{\dagger})^*(I + S_1^{\dagger}S_1) + \frac{1}{2}(I + S_1^{\dagger}S_1)B_1^{\dagger}C_1(M_1^{\dagger})^*,\ E_{22} = R_{A_3}E_{11}R_{A_3}.
\end{aligned} \tag{2.1}$$

Then, the following conditions are equivalent:

(1) *System (1.5) is consistent.*
(2) *The following equalities hold:*

$$\begin{aligned}
&R_{A_1}C_1R^*_{B_1} = 0,\quad R_{M_1}R_{A_1}C_1 = 0,\\
&R_{A_2}C_2R^*_{B_2} = 0,\quad R_{M_2}R_{A_2}C_2 = 0,\\
&R_{A_4}E_{22}R^*_{A_5} = 0,\quad R_{M_3}R_{A_4}E_{22} = 0.
\end{aligned} \tag{2.2}$$

(3) *The following rank equalities hold:*

$$r\begin{bmatrix} G_1 & E_1 & D_1 \\ F_1^* & 0 & 0 \\ D_1^* & 0 & 0 \end{bmatrix} = r[D_1 \quad E_1] + r[D_1 \quad F_1], \tag{2.3}$$

$$r\begin{bmatrix} G_1 & E_1 & F_1 & D_1 \\ D_1^* & 0 & 0 & 0 \end{bmatrix} = r\left[\, D_1\ E_1\ F_1 \,\right] + r(D_1), \tag{2.4}$$

$$r\begin{bmatrix} G_2 & E_2 & D_2 \\ F_2^* & 0 & 0 \\ D_2^* & 0 & 0 \end{bmatrix} = r[D_2 \quad E_2] + r[D_2 \quad F_2], \tag{2.5}$$

$$r\begin{bmatrix} G_2 & E_2 & F_2 & D_2 \\ D_2^* & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} D_2 & E_2 & F_2 \end{bmatrix} + r(D_2), \tag{2.6}$$

$$r\begin{bmatrix} 0 & B_2^* & 0 & 0 & B_2^* & 0 & 0 \\ 0 & 0 & B_1^* & 0 & B_2^* & 0 & 0 \\ 0 & 0 & 0 & B_1^* & B_2* & 0 & 0 \\ 0 & 0 & -C_1 & 0 & 0 & B_1 & A_1 \\ B_2 & -C_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2^* & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 & 0 & 0 & A_1 \\ 0 & B_1 & 0 & 0 \\ 0 & B_2 & -B_2 & 0 \\ B_2 & 0 & B_2 & 0 \end{bmatrix} + r\begin{bmatrix} B_2 & B_2 & -A_2 \\ B_1 & 0 & 0 \\ 0 & B_1 & 0 \end{bmatrix}, \tag{2.7}$$

$$r\begin{bmatrix} 0 & 0 & B_1^* & 0 & 0 & B_2^* & 0 & 0 \\ 0 & 0 & 0 & B_1^* & 0 & B_2^* & 0 & 0 \\ 0 & 0 & 0 & 0 & B_2^* & -B_2^* & 0 & 0 \\ -B_2 & -B_2 & 0 & 0 & 0 & -C_2 & A_2 & 0 \\ B_1 & 0 & C_1 & 0 & 0 & 0 & 0 & A_1 \\ 0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_2 & B_2 & -A_2 & 0 \\ B_1 & 0 & 0 & A_1 \\ 0 & B_1 & 0 & 0 \end{bmatrix} + r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_2 \end{bmatrix}. \tag{2.8}$$

Under these conditions, the general solution to (1.5) is

$$\begin{aligned} X_1 &= D_1^\dagger(G_1 - E_1Y_1E_1^* - F_1Z_1F_1^*) - \frac{1}{2}D_1^\dagger(G_1 - E_1Y_1E_1^* - F_1Z_1F_1^*)D_1D_1^\dagger \\ &\quad + D_1^\dagger T_1 D_1^* + T_2^* D_1 D_1^\dagger - L_{D_1}T_3, \\ X_2 &= D_2^\dagger(G_2 - E_2Y_2E_2^* - F_2Z_1F_2^*) - \frac{1}{2}D_2^\dagger(G_2 - E_2Y_2E_2^* - F_2Z_1F_2^*)D_2D_2^\dagger \\ &\quad + D_2^\dagger T_{11} D_2^* + T_{22}^* D_2 D_2^\dagger - L_{D_2}T_{33}, \\ Y_1 &= -Y_1^* = A_1^\dagger C_1(A_1^\dagger)^* - \frac{1}{2}A_1^\dagger B_1 M_1^\dagger C_1[I + (B_1^\dagger)^* S_1^*](A_1^\dagger)^* \\ &\quad - \frac{1}{2}A_1^\dagger[I + S_1B_1^\dagger]C_1(M_1^\dagger)^* B_1^*(A_1^\dagger)^* - A_1^\dagger S_1 U_1 S_1^*(A_1^\dagger)^* - L_{A_1}V_1 + V_1^* L_{A_1}, \\ Y_2 &= -Y_2^* = A_2^\dagger C_2(A_2^\dagger)^* - \frac{1}{2}A_2^\dagger B_2 M_2^\dagger C_2[I + (B_2^\dagger)^* S_2^*](A_2^\dagger)^* \\ &\quad - \frac{1}{2}A_2^\dagger[I + S_2B_2^\dagger]C_2(M_2^\dagger)^* B_2^*(A_2^\dagger)^* - A_2^\dagger S_2 U_{11} S_2^*(A_2^\dagger)^* - L_{A_2}V_{11} + V_{11}^* L_{A_2}, \\ Z_1 &= \frac{1}{2}M_1^\dagger C_1(B_1^\dagger)^*(I + S_1^\dagger S_1) + \frac{1}{2}(I + S_1^\dagger S_1)B_1^\dagger C_1(M_1^\dagger)^* + L_{M_1}U_1L_{M_1} \\ &\quad + L_{M_1}L_{S_1}U_2 - U_2^* L_{S_1}L_{M_1} + U_{33}L_{B_1} - L_{B_1}U_{33}^*, \end{aligned} \tag{2.9}$$

or

$$\begin{aligned} Z_1 &= \frac{1}{2}M_2^\dagger C_2(B_2^\dagger)^*(I + S_2^\dagger S_2) + \frac{1}{2}(I + S_2^\dagger S_2)B_2^\dagger C_2(M_2^\dagger)^* + L_{M_2}U_{11}L_{M_2} \\ &\quad + L_{M_2}L_{S_2}U_{22} - U_{22}^* L_{S_2}L_{M_2} + U_3L_{B_2} - L_{B_2}U_3^*, \end{aligned} \tag{2.10}$$

with

$$
\begin{aligned}
&U_2^* = [\,I_m\ 0\ 0\ 0\,]W,\\
&U_3^* = [\,0\ I_m\ 0\ 0\,]W,\\
&U_{22} = [\,0\ 0\ I_m\ 0\,]W,\\
&U_{33} = [\,0\ 0\ 0\ I_m\,]W,
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
W &= A_3^{\dagger}(E_{11} - L_{M_1}U_1L_{M_1} - L_{M_2}U_{11}L_{M_2}) - \frac{1}{2}A_3^{\dagger}(E_{11} - L_{M_1}U_1L_{M_1}\\
&\quad - L_{M_2}U_{11}L_{M_2})A_3A_3^{\dagger} - A_3^{\dagger}W_1A_3^* + W_1^*A_3A_3^{\dagger} + L_{A_3}W_5,\\
U_1 &= -U_1^* = A_4^{\dagger}E_{22}(A_4^{\dagger})^* - \frac{1}{2}A_4^{\dagger}A_5M_3^{\dagger}E_{22}(I + (A_5^{\dagger})^*S_3^*)(A_4^{\dagger})^*\\
&\quad -\frac{1}{2}A_4^{\dagger}(I + S_3A_5^{\dagger})E_{22}(M_3^{\dagger})^*A_5^*(A_4^{\dagger})^* - A_4^{\dagger}S_3W_6(A_4^{\dagger}S_3)^* + L_{A_4}W_7 - W_7^*L_{A_4},\\
U_{11} &= -U_{11}^* = \frac{1}{2}M_3^{\dagger}E_{22}(A_5^{\dagger})^*(I + S_3^{\dagger}S_3) + \frac{1}{2}(I + S_3^{\dagger}S_3)A_5^{\dagger}E_{22}(M_3^{\dagger})^* + L_{M_3}W_6L_{M_3}\\
&\quad +L_{M_3}L_{S_3}W_8 - W_8^*L_{S_3}L_{M_3} - W_9L_{A_5} + L_{A_5}W_9^*,
\end{aligned}
\tag{2.12}
$$

where T_1, T_2, T_3, *and* $W_1,\ W_5, \cdots, W_9,\ W_6^* = -W_6$ *are any matrices of acceptable shapes over* $\mathbb{C}$.

Proof By writing the equations in (1.5) as follows:

$$
D_1X_1 - (D_1X_1)^* + E_1Y_1E_1^* + F_1Z_1F_1^* = G_1,\ G_1^* = -G_1 \tag{2.13}
$$

and

$$
D_2X_2 - (D_2X_2)^* + E_2Y_2E_2^* + F_2Z_1F_2^* = G_2,\ G_2^* = -G_2. \tag{2.14}
$$

By the support of Lemma 1.3, Eqs. (2.13–2.14) have solution if and only if

$$
\begin{aligned}
&R_{A_1}C_1R_{B_1}^*, = 0, \quad R_{M_1}R_{A_1}C_1 = 0,\\
&R_{A_2}C_2R_{B_2}^*, = 0, \quad R_{M_2}R_{A_2}C_2 = 0,\\
&R_{A_4}E_{22}R_{A_5}^* = 0, \quad R_{M_3}R_{A_4}E_{22} = 0.
\end{aligned}
$$

In this case, the general solution to (2.13) and (2.14) can be described as

$$
\begin{aligned}
X_1 &= D_1^{\dagger}(G_1 - E_1Y_1E_1^* - F_1Z_1F_1^*) - \frac{1}{2}D_1^{\dagger}(G_1 - E_1Y_1E_1^* - F_1Z_1F_1^*)D_1D_1^{\dagger}\\
&\quad + D_1^{\dagger}T_1D_1^* + T_2^*D_1D_1^{\dagger} - L_{D_1}T_3,\\
Y_1 &= A_1^{\dagger}C_1(A_1^{\dagger})^* - \frac{1}{2}A_1^{\dagger}B_1M_1^{\dagger}C_1[I + (B_1^{\dagger})^*S_1^*](A_1^{\dagger})^*\\
&\quad -\frac{1}{2}A_1^{\dagger}[I + S_1B_1^{\dagger}]C_1(M_1^{\dagger})^*B_1^*(A_1^{\dagger})^* - A_1^{\dagger}S_1U_1S_1^*(A_1^{\dagger})^* - L_{A_1}V_1 + V_1^*L_{A_1},\\
Z_1 &= \frac{1}{2}M_1^{\dagger}C_1(B_1^{\dagger})^*(I + S_1^{\dagger}S_1) + \frac{1}{2}(I + S_1^{\dagger}S_1)B_1^{\dagger}C_1(M_1^{\dagger})^*\\
&\quad +L_{M_1}U_1L_{M_1}+L_{M_1}L_{S_1}U_2-U_2^*L_{S_1}L_{M_1} + U_{33}L_{B_1} - L_{B_1}U_{33}^*,\\
X_2 &= D_2^{\dagger}(G_2 - E_2Y_2E_2^* - F_2Z_1F_2^*) - \frac{1}{2}D_2^{\dagger}(G_2 - E_2Y_2E_2^* - F_2Z_1F_2^*)D_2D_2^{\dagger}\\
&\quad +D_2^{\dagger}T_{11}D_2^* + T_{22}^*D_2D_2^{\dagger} - L_{D_2}T_{33},\\
Y_2 &= A_2^{\dagger}C_2(A_2^{\dagger})^* - \frac{1}{2}A_2^{\dagger}B_2M_2^{\dagger}C_2[I + (B_2^{\dagger})^*S_2^*](A_2^{\dagger})^*\\
&\quad -\frac{1}{2}A_2^{\dagger}[I + S_2B_2^{\dagger}]C_2(M_2^{\dagger})^*B_2^*(A_2^{\dagger})^*-A_2^{\dagger}S_2U_{11}S_2^*(A_2^{\dagger})^* - L_{A_2}V_{11} + V_{11}^*L_{A_2},\\
Z_1 &= \frac{1}{2}M_2^{\dagger}C_2(B_2^{\dagger})^*(I + S_2^{\dagger}S_2) + \frac{1}{2}(I + S_2^{\dagger}S_2)B_2^{\dagger}C_2(M_2^{\dagger})^*
\end{aligned}
\tag{2.15}
$$

$$+L_{M_2}U_{11}L_{M_2}+L_{M_2}L_{S_2}U_{22}-U_{22}^*L_{S_2}L_{M_2}+U_3L_{B_2}-L_{B_2}U_3^*, \tag{2.16}$$

where $V_1,\ U_1^*=-U_1,\ U_2,\ U_3,\ U_{11}^*=-U_{11},\ U_{22},\ U_{33}$ and T_1, T_2, T_3 are free matrices of plausible sizes over $\mathbb{C}$.

Equating (2.15) and (2.16), we get

$$A_3W-(A_3W)^*+L_{M_1}U_1L_{M_1}+L_{M_2}U_{11}L_{M_2}=E_{11}. \tag{2.17}$$

Solving Eq. (2.17) with respect to unknowns W, U_1, and U_{11} by Lemma 1.3, we have that it has a solution (2.12) if and only if (2.2) is satisfied. In this case, its general solution can be expressed by (2.9–2.10).

(2) $\Leftrightarrow$ (3) : From Lemma 1.3, we have

$$R_{A_1}C_1R_{B_1}^*=0\Leftrightarrow r\begin{bmatrix}C_1 & A_1\\ B_1^* & 0\end{bmatrix}=r(A_1)+r(B_1),$$

$$\Leftrightarrow r\begin{bmatrix}R_{D_1}G_1R_{D_1}^* & R_{D_1}E_1\\ F_1^*R_{D_1}^* & 0\end{bmatrix}=r(R_{D_1}E_1)+r(R_{D_1}F_1)$$

$$\Leftrightarrow r\begin{bmatrix}G_1 & E_1 & D_1\\ F_1^* & 0 & 0\\ D_1^* & 0 & 0\end{bmatrix}=r[D_1\quad E_1]+r[D_1\quad F_1],$$

$$R_{M_1}R_{A_1}C_1=0\Leftrightarrow r\begin{bmatrix}R_{A_1}C_1 & M_1\end{bmatrix}=r(M_1)\Leftrightarrow r\begin{bmatrix}R_{A_1}C_1 & R_{A_1}B_1\end{bmatrix}=r(R_{A_1}B_1)$$

$$\Leftrightarrow r\begin{bmatrix}C_1 & B_1 & A_1\end{bmatrix}=r\begin{bmatrix}A_1 & B_1\end{bmatrix}$$

$$\Leftrightarrow r\begin{bmatrix}R_{D_1}G_1R_{D_1}^* & R_{D_1}E_1 & R_{D_1}F_1\end{bmatrix}=r[R_{D_1}E_1\quad R_{D_1}F_1]$$

$$\Leftrightarrow r\begin{bmatrix}G_1 & E_1 & F_1 & D_1\\ D_1^* & 0 & 0 & 0\end{bmatrix}=r[D_1\quad E_1\quad F_1]+r(D_1),$$

$$R_{A_2}C_2R_{B_2}^*=0\Leftrightarrow r\begin{bmatrix}C_2 & A_2\\ B_2^* & 0\end{bmatrix}=r(A_2)+r(B_2)$$

$$\Leftrightarrow r\begin{bmatrix}R_{D_2}G_2R_{D_2}^* & R_{D_2}E_2\\ F_2^*R_{D_2}^* & 0\end{bmatrix}=r(R_{D_2}E_2)+r(R_{D_2}F_2)$$

$$\Leftrightarrow r\begin{bmatrix}G_2 & E_2 & D_2\\ F_2^* & 0 & 0\\ D_2^* & 0 & 0\end{bmatrix}=r[D_2\quad E_2]+r[D_2\quad F_2],$$

$$R_{M_2}R_{A_2}C_2=0\Leftrightarrow r\begin{bmatrix}R_{A_2}C_2 & M_2\end{bmatrix}=r(M_2)\Leftrightarrow r\begin{bmatrix}R_{A_2}C_2 & R_{A_2}B_2\end{bmatrix}=r(R_{A_2}B_2)$$

$$\Leftrightarrow r\begin{bmatrix}C_2 & B_2 & A_2\end{bmatrix}=r\begin{bmatrix}A_2 & B_2\end{bmatrix}$$

$$\Leftrightarrow r\begin{bmatrix}R_{D_2}G_2R_{D_2}^* & R_{D_2}E_2 & R_{D_2}F_2\end{bmatrix}=r[R_{D_2}E_2\quad R_{D_2}F_2]$$

$$\Leftrightarrow r\begin{bmatrix}G_2 & E_2 & F_2 & D_2\\ D_2^* & 0 & 0 & 0\end{bmatrix}=r[D_2\quad E_2\quad F_2]+r(D_2),$$

$$R_{A_4}E_{22}R_{A_5}=0\Leftrightarrow r(R_{A_4}E_{22}R_{A_5})=0$$

$$\Leftrightarrow r\begin{bmatrix}E_{22} & A_4\\ A_5^* & 0\end{bmatrix}=r(A_4)+r(A_5)$$

$$\Leftrightarrow r\begin{bmatrix}R_{A_3}E_{11}R_{A_3} & R_{A_3}L_{M_1}\\ L_{M_2}R_{A_3}^* & 0\end{bmatrix}=r(R_{A_3}L_{M_1})+r[(R_{A_3}L_{M_2})^*]$$

$$\Leftrightarrow r\begin{bmatrix}E_{11} & L_{M_1} & A_3\\ L_{M_2} & 0 & 0\\ A_3^* & 0 & 0\end{bmatrix}=r\begin{bmatrix}L_{M_1} & A_3\end{bmatrix}+r\begin{bmatrix}L_{M_2} & A_3\end{bmatrix}$$

$$\Leftrightarrow r\begin{bmatrix}Z_{02}-Z_{01} & L_{M_1} & L_{M_1}L_{S_1} & L_{B_1} & -L_{M_2}L_{S_2} & -L_{B_2}\\ L_{M_2} & 0 & 0 & 0 & 0 & 0\\ L_{S_1}L_{M_1} & 0 & 0 & 0 & 0 & 0\\ L_{B_1} & 0 & 0 & 0 & 0 & 0\\ -L_{S_2}L_{M_2} & 0 & 0 & 0 & 0 & 0\\ -L_{B_2} & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$= r\begin{bmatrix} L_{M_1} & L_{B_1} & -L_{B_2} & L_{M_1}L_{S_1} & -L_{M_2}L_{S_2} \end{bmatrix} + r\begin{bmatrix} L_{M_2} & L_{M_1}L_{S_1} & L_{B_1} & -L_{M_2}L_{S_2} & -L_{B_2} \end{bmatrix}$$

$$\Leftrightarrow r\begin{bmatrix} Z_{02}-Z_{01} & L_{M_1} & L_{B_1} & -L_{M_2}L_{S_2} & -L_{B_2} \\ L_{M_2} & 0 & 0 & 0 & 0 \\ L_{S_1}L_{M_1} & 0 & 0 & 0 & 0 \\ L_{B_1} & 0 & 0 & 0 & 0 \\ -L_{B_2} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} L_{M_1} & L_{B_1} & -L_{M_2}L_{S_2} & -L_{B_2} \end{bmatrix} + r\begin{bmatrix} L_{M_2} & L_{M_1}L_{S_1} & L_{B_1} & -L_{B_2} \end{bmatrix}$$

$$\Leftrightarrow r\begin{bmatrix} Z_{02}-Z_{01} & I & I & -L_{M_2} & -I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & B_2^*L_{A_2^*} & 0 & 0 & 0 \\ L_{M_1} & 0 & 0 & 0 & 0 & 0 & S_1^* & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & B_1^* & 0 \\ -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_2^* \\ 0 & R_{A_1}B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_2L_{M_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} I & I & -L_{M_2} & -I \\ M_1 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 \\ 0 & 0 & B_2L_{M_2} & 0 \\ 0 & 0 & 0 & B_2 \end{bmatrix} + r\begin{bmatrix} I & L_{M_1} & I & -I \\ M_2 & 0 & 0 & 0 \\ 0 & B_1L_{M_1} & 0 & 0 \\ 0 & 0 & 0 & B_2 \end{bmatrix}$$

$$\Leftrightarrow r\begin{bmatrix} Z_{02}-Z_{01} & I & I & -L_{M_2} & -I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & B_2^*L_{A_2^*} & 0 & 0 & 0 & 0 \\ L_{M_1} & 0 & 0 & 0 & 0 & 0 & L_{M_1}B_1^* & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & B_1^* & 0 & 0 \\ -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_2^* & 0 \\ 0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_1 \\ 0 & 0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_2L_{M_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} I & I & -L_{M_2} & -I \\ R_{A_1}B_1 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 \\ 0 & 0 & B_2L_{M_2} & 0 \\ 0 & 0 & 0 & B_2 \end{bmatrix} + r\begin{bmatrix} I & L_{M_1} & I & -I \\ R_{A_2}B_2 & 0 & 0 & 0 \\ 0 & B_1L_{M_1} & 0 & 0 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & B_2 \end{bmatrix}$$

$$\Leftrightarrow r\begin{bmatrix} Z_{02}-Z_{01} & I & I & -I & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & B_2^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & B_1^* & 0 & 0 & 0 & M_1^* & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & B_1^* & 0 & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_2^* & 0 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 & 0 & A_1 & 0 \\ 0 & 0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_2{}^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} I & I & -I & -I & 0 \\ B_1 & 0 & 0 & 0 & A_1 \\ 0 & B_1 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 \\ 0 & 0 & 0 & B_2 & 0 \end{bmatrix} + r\begin{bmatrix} I & I & I & -I & 0 \\ B_2 & 0 & 0 & 0 & A_2 \\ 0 & B_1 & 0 & 0 & 0 \\ 0 & 0 & B_1 & 0 & 0 \\ 0 & 0 & 0 & B_2 & 0 \end{bmatrix}$$

$$\Leftrightarrow r\begin{bmatrix} 0 & -B_2^* & 0 & 0 & B_2^* & 0 & 0 \\ 0 & 0 & B_1^* & 0 & B_2^* & 0 & 0 \\ 0 & 0 & 0 & B_1^* & B_2^* & 0 & 0 \\ 0 & 0 & -C_1 & 0 & 0 & B_1 & A_1 \\ B_2 & -C_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2^* & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= r\begin{bmatrix} B_1 & 0 & 0 & A_1 \\ 0 & B_1 & 0 & 0 \\ 0 & B_2 & -B_2 & 0 \\ B_2 & 0 & B_2 & 0 \end{bmatrix} + r\begin{bmatrix} B_2 & B_2 & -A_2 \\ B_1 & 0 & 0 \\ 0 & B_1 & 0 \end{bmatrix} \Leftrightarrow (2.7).$$

On the same lines, $R_{M_3}R_{A_4}E_{22} = 0$ can be proved to be same as (2.8).
Hence, the theorem is finished. □

Comment 2.2 *The application of extremal rank in the area of control theory can be viewed in [8–10]. We may carry out the extremal rank of the general solution of the system (1.5).*

Now, we discuss some particular cases of our system.
Using Theorem 2.1, the general solution and the solvability conditions to (1.4) can be obtained as follows.

Corollary 2.3 *Let $A_1 \in \mathbb{C}^{m\times n}$, $A_2 \in \mathbb{C}^{m\times q}$, $B_i \in \mathbb{C}^{m\times k}$, and $C_i = -C_i^* \in \mathbb{C}^{m\times m}$ for $i = 1, 2$. Assign*

$$M_1 = R_{A_1}B_1,\ S_1 = B_1L_{M_1},\ M_2 = R_{A_2}B_2,\ S_2 = B_2L_{M_2},\ A_4 = R_{A_3}L_{M_1},\ B_4 = R_{A_3}L_{M_2},$$
$$A_3 = \begin{bmatrix} L_{B_2}^* & -L_{B_1} & L_{M_1}L_{S_1} & -L_{M_2}L_{S_2} \end{bmatrix},\ M_3 = R_{A_4}B_4,\ S_3 = B_4L_{M_3},$$
$$C_3 = V_{02} - V_{01},\ V_{02} = \frac{1}{2}M_2^\dagger C_2(B_2^\dagger)^*(I + S_2^\dagger S_2) + \frac{1}{2}(I + S_2^\dagger S_2)B_2^\dagger C_2(M_2^\dagger)^*,$$
$$V_{01} = \frac{1}{2}M_1^\dagger C_1(B_1^\dagger)^*(I + S_1^\dagger S_1) + \frac{1}{2}(I + S_1^\dagger S_1)B_1^\dagger C_1(M_1^\dagger)^*,\ C_4 = R_{A_3}C_3R_{A_3}.$$

Then, the following conditions are equivalent:

(1) *System (1.4) is consistent.*
(2) *The following equalities hold:*

$$R_{A_1}C_1R_{B_1} = 0,\quad R_{M_1}R_{A_1}C_1 = 0,$$
$$R_{A_2}C_2R_{B_2} = 0,\quad R_{M_2}R_{A_2}C_2 = 0,$$
$$R_{A_4}C_4R_{B_4} = 0,\quad R_{M_3}R_{A_4}C_4 = 0.$$

(3) *The following rank equalities hold:*

$$r\begin{bmatrix} C_1 & A_1 \\ B_1^* & 0 \end{bmatrix} = r(A_1) + r(B_1),\quad r\begin{bmatrix} C_1 & B_1 & A_1 \end{bmatrix} = r\begin{bmatrix} A_1 & B_1 \end{bmatrix},$$
$$r\begin{bmatrix} C_2 & A_2 \\ B_2^* & 0 \end{bmatrix} = r(A_2) + r(B_2),\quad r\begin{bmatrix} C_2 & B_2 & A_2 \end{bmatrix} = r\begin{bmatrix} A_2 & B_2 \end{bmatrix},$$
$$\Leftrightarrow r\begin{bmatrix} 0 & 0 & 0 & B_2^* & B_1 & 0 & 0 \\ 0 & 0 & 0 & -B_2^* & 0 & B_1^* & 0 \\ B_1 & 0 & 0 & 0 & 0 & C_1 & A_1 \\ 0 & B_2 & 0 & -C_2 & 0 & 0 & 0 \\ -B_2 & -B_2 & B_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_2^* & 0 & 0 & 0 \end{bmatrix}$$
$$= r\begin{bmatrix} -B_1 & 0 & -B_1 & A_1 \\ B_2 & B_2 & 0 & 0 \\ 0 & B_1 & 0 & 0 \\ 0 & 0 & B_2 & 0 \end{bmatrix} + r\begin{bmatrix} B_2 & 0 & 0 & A_2 \\ -B_2 & B_2 & -B_2 & 0 \\ 0 & B_1 & 0 & 0 \\ 0 & 0 & B_1 & 0 \end{bmatrix},$$

$$r\begin{bmatrix} 0 & 0 & -B_1^* & B_2^* & 0 & 0 & 0 & 0 \\ 0 & 0 & -B_1^* & 0 & B_1^* & 0 & 0 & 0 \\ 0 & 0 & -B_1^* & 0 & 0 & B_2^* & 0 & 0 \\ -B_1^* & -B_1^* & 0 & 0 & -C_1 & 0 & A_1 & 0 \\ B_2 & 0 & 0 & 0 & 0 & C_2 & A_2 & 0 \\ 0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= r\begin{bmatrix} -B_1 & 0 & -B_1 & A_1 \\ B_2 & B_2 & 0 & 0 \\ 0 & B_1 & 0 & 0 \\ 0 & 0 & B_2 & 0 \end{bmatrix} + r\begin{bmatrix} B_2 & 0 \\ B_1 & 0 \\ 0 & B_2 \end{bmatrix} + r(B_1).$$

Under these conditions, the general solution to (1.4) is

$$\begin{aligned} U &= A_1^\dagger C_1 (A_1^\dagger)^* - \frac{1}{2} A_1^\dagger B_1 M_1^\dagger C_1 [I + (B_1^\dagger)^* S_1^*](A_1^\dagger)^* \\ &\quad - \frac{1}{2} A_1^\dagger [I + S_1 B_1^\dagger] C_1 (M_1^\dagger)^* B_1^* (A_1^\dagger)^* - A_1^\dagger S_1 U_1 S_1^* (A_1^\dagger)^* + L_{A_1} V_1 - V_1^* L_{A_1}, \\ W &= -W^* = A_2^\dagger C_2 (A_2^\dagger)^* - \frac{1}{2} A_2^\dagger B_2 M_2^\dagger C_2 [I + (B_2^\dagger)^* S_2^*](A_2^\dagger)^* \\ &\quad - \frac{1}{2} A_2^\dagger [I + S_2 B_2^\dagger] C_2 (M_2^\dagger)^* B_2^* (A_2^\dagger)^* - A_2^\dagger S_2 U_4 S_2^* (A_2^\dagger)^* + L_{A_2} V_2 - V_2^* L_{A_2}, \\ V &= -V^* = \frac{1}{2} M_1^\dagger C_1 (B_1^\dagger)^* (I + S_1^\dagger S_1) + \frac{1}{2} (I + S_1^\dagger S_1) B_1^\dagger C_1 (M_1^\dagger)^* \\ &\quad + L_{M_1} U_1 L_{M_1} + L_{M_1} L_{S_1} U_2 - U_2^* L_{S_1} L_{M_1} + U_3 L_{B_1} - L_{B_1} U_3^*, \end{aligned}$$

or

$$\begin{aligned} V &= -V^* = \frac{1}{2} M_2^\dagger C_2 (B_2^\dagger)^* (I + S_2^\dagger S_2) + \frac{1}{2} (I + S_2^\dagger S_2) B_2^\dagger C_2 (M_2^\dagger)^* \\ &\quad + L_{M_2} U_4 L_{M_2} + L_{M_2} L_{S_2} U_5 - U_5^* L_{S_2} L_{M_2} + U_6 L_{B_2} - L_{B_2} U_6^*, \end{aligned}$$

with

$$\begin{aligned} U_6^* &= [\, I_k \; 0 \; 0 \; 0 \,] Z, \\ U_3^* &= [\, 0 \; I_k \; 0 \; 0 \,] Z, \\ U_2 &= [\, 0 \; 0 \; I_k \; 0 \,] Z, \\ U_5 &= [\, 0 \; 0 \; 0 \; I_k \,] Z, \end{aligned}$$

where

$$\begin{aligned} Z &= A_3^\dagger (C_3 - L_{M_1} U_1 L_{M_1} - L_{M_2} U_4 L_{M_2}) - \frac{1}{2} A_3^\dagger (C_3 - L_{M_1} U_1 L_{M_1} - L_{M_2} U_4 L_{M_2}) A_3 A_3^\dagger \\ &\quad - A_3^\dagger U_7 A_3^* - U_7^* A_3 A_3^\dagger + L_{A_3} U_8, \\ U_1 &= -U_1^* = A_4^\dagger C_4 (A_4^\dagger)^* - \frac{1}{2} A_4^\dagger B_4 M_3^\dagger C_4 (I + (B_4^\dagger)^* S_3^*)(A_4^\dagger)^* \\ &\quad - \frac{1}{2} A_4^\dagger (I + S_3 B_4^\dagger) C_4 (M_3^\dagger)^* B_4^* (A_4^\dagger)^* - A_4^\dagger S_3 U_9 (A_4^\dagger S_3)^* + L_{A_4} U_{10} - U_{10}^* L_{A_4}, \\ U_4 &= -U_4^* = \frac{1}{2} M_3^\dagger C_4 (B_4^\dagger)^* (I + S_3^\dagger S_3) + \frac{1}{2} (I + S_3^\dagger S_3) B_4^\dagger C_4 (M_3^\dagger)^* + L_{M_3} U_{11} L_{M_3} \\ &\quad + L_{M_3} L_{S_3} U_{12} - U_{12}^* L_{S_3} L_{M_3} + U_{13} L_{B_4} - L_{B_4} U_{13}^*, \end{aligned}$$

where $V_1, V_2, U_7, \ldots, U_{13}, U_9 = -U_9^*, U_{11} = -U_{11}^*$ *are any matrices of acceptable shapes over* $\mathbb{C}$.

3 Algorithm with example

In this section, we construct the algorithm for finding solutions to (1.5) that is inducted by Theorem 2.1.

Algorithm 3.1 (1) *Feed the values of* $D_i,\ E_i,\ F_i,\ G_i,\ (i=1,2)$ *with conformable shapes over* $\mathbb{C}$.

(2) *Compute the matrices determined by (2.1).*

(3) *Verify the consistence equalities expressed by matrix equations (2.2) or rank equalities* (2.3)-*(2.8). If no, then return "inconsistent".*

(4) *If the consistence equalities are true, then we compute auxiliary matrices* U_1, U_{11}, *and* W *by (2.12), and* U_2, U_3, U_{22}, *and* U_3 *by (2.11).*

(5) *Finally, we find the solution* $X_i,\ Y_i,\ (i=1,2)$ *and* Z_1 *by (2.9), or another formula for* Z_1 *is (2.10).*

Using Algorithm 3.1, we consider the following example. Note that our goal is both to confirm correctness of main results from Theorem 2.1 and to demonstrate the technique of applying the determinantal representations of the MP-inverse from Lemma 1.4.

Example 3.2 Given the matrices:

$$E_1=\begin{bmatrix}4i & 4-4i & 4 & 4+4i\\ -4 & 4+4i & 4i & 4i-4\\ 4i-4 & 8 & 4i+4 & 8i\\ 8 & -8i-8 & -8i & 8-8i\end{bmatrix},\ F_1=\begin{bmatrix}4+8i & 8+4i & -8+4i\\ -8+4i & 8i-4 & -4-8i\\ -4+12i & 12i+4 & -12-4i\\ -8-16i & -16-8i & 16-8i\end{bmatrix}, \tag{3.1}$$

$$D_1=\begin{bmatrix}1+i & -1+i\\ -1+i & -1-i\\ 1-i & 1+i\\ -1-i & 1-i\end{bmatrix},\ G_1=92160\begin{bmatrix}i & 1 & 1+i & -2\\ -1 & i & -1+i & -2i\\ -1+i & 1+i & 2i & -2-2i\\ 2 & -2i & 2-2i & 4i\end{bmatrix}, \tag{3.2}$$

$$E_2=\begin{bmatrix}-i+1 & -1+i\\ 1+i & -i-1\\ 3+i & -3-i\\ 2+2i & -2i-2\end{bmatrix},\ F_2=\begin{bmatrix}3i & -3 & 3+3i\\ 3 & 3i & 3-3i\\ -6i & 6 & -6i-6\\ -6 & -6i & -6+6i\end{bmatrix}, \tag{3.3}$$

$$D_2=\begin{bmatrix}2+i & 2-i & -1+2i\\ -1+2i & 2i+1 & -2-i\\ 1+3i & 3+i & -3+i\\ 3-i & 1-3i & 1+3i\end{bmatrix},\ G_2=96\begin{bmatrix}i & 1 & 2+i & 2\\ -1 & i & -1+2i & 2i\\ -2+i & 1+2i & 5i & 2+4i\\ -2 & 2i & -2+4i & 4i\end{bmatrix}. \tag{3.4}$$

1. Thanks to Lemma 1.4, we calculate the Moore–Penrose inverses of given matrices and using them for compute all needed matrices from (2.1). For example

$$D_1^{\dagger}=\frac{1}{16}\begin{bmatrix}1-i & -1-i & 1+i & -1+i\\ -i-1 & -1+i & -i+1 & 1+i\end{bmatrix},\ A_1=\begin{bmatrix}3+3i & -6i & 3-3i & 6]\\ -3+3i & 6 & 3+3i & 6i\\ -5+i & 6+4i & 1+5i & -4+6i\\ 5+i & -6i-4 & 1-5i & 6-4i\end{bmatrix},$$

$$B_1=\begin{bmatrix}3+i & 3-i & -1+3i\\ -1+3i & 1+3i & -3-i\\ -11+13i & -1+17i & -13-11i\\ -7-9i & -11-3i & 9-7i\end{bmatrix},\ D_2^{\dagger}=\frac{1}{90}\begin{bmatrix}2-i & -1-2i & 1-3i & 3+i\\ 2+i & 1-2i & 3-i & 1+3i\\ -1-2i & -2+i & -3-i & 1-3i\end{bmatrix}$$

$$A_2=\begin{bmatrix}-i & i\\ 1 & -1\\ 2 & -2\\ 1+3i & -3i-1\end{bmatrix},\ B_2^{\dagger}=\begin{bmatrix}5i+2 & -5+2i & 7+3i\\ 2i+1 & -2+i & 3+i\\ -2i & 2 & -2i-2\\ -2 & -2i & -2+2i\end{bmatrix},$$

$$M_1=\frac{1}{11}\begin{bmatrix}-24+32i & 40i & -32-24i\\ -32-24i & -40 & 24-32i\\ -104+72i & -40+120i & -72-104i\\ -120-40i & -120+40i & 40-120i\end{bmatrix},\ L_{M_1}=\frac{1}{15}\begin{bmatrix}10 & 4+3i & -5i\\ -4-3i & 10 & 3-4i\\ 5i & 3+4i & 10\end{bmatrix},$$

$$M_2=\frac{1}{8}\begin{bmatrix}13+37i & -37+13i & 50+24i\\ 11+13i & -13+11i & 24+2i\\ 8-22i & 22+6i & -16-28i\\ -4+6i & -6-4i & 2+10i\end{bmatrix},\ L_{M_2}=\frac{1}{4}\begin{bmatrix}3 & -i & -1+i\\ i & 3 & 1+i\\ -1-i & 1-i & 2\end{bmatrix},$$

etc. In particular, we obtain S_1, S_2, A_4, A_5, and E_{11}, are zero matrices.

2. Confirm that (2.2) are true for given matrices.
3. To avoid a trivial singular case, we put

$$W_6 = \begin{bmatrix} 4i & 4 & -4i \\ -4 & 8i & -4 \\ -4i & 4 & 4i \end{bmatrix}, \quad W_7 = \begin{bmatrix} i & i & 2 \\ i & 1 & i \\ 1 & i & i \end{bmatrix}, V_{11} = \begin{bmatrix} 1+i & -1+2i \\ -i-1 & 2-i \end{bmatrix},$$

$$V_1 = \begin{bmatrix} 1+i & -1+i & -2+i & 2+qi \\ -1+i & -i-1 & -1-2i & -1+2i \\ 2i & -2 & -3-i & 1+3i \\ 2 & 2i & -1+3i & 3-qi \end{bmatrix}, T_3 = \begin{bmatrix} 1+i & -1+i & -3+i & 3+qi \\ -1+i & -i-1 & -1-3i & -1+3i \end{bmatrix}.$$

4. Finally, we have

$$Z_1 = \frac{1}{1200}\begin{bmatrix} 2700i & 1916-312i & 3345-1315i \\ -1916-312i & 2700i & 1257+3299i \\ -3345-1315i & -1257+3299i & 5460i \end{bmatrix}, Y_2 = \frac{1}{2}\begin{bmatrix} 24i & 1-23i \\ -1-23i & 26i \end{bmatrix},$$

$$Y_1 = \frac{1}{6}\begin{bmatrix} 974i & 953-963i & 951 & 958+965i \\ -953-963i & 1920i & -952+960i & -1927+i \\ -951 & 952+960i & 952i & -966+977i \\ -958+965i & 1927+i & 966+977i & 1898i \end{bmatrix},$$

$$X_1 = \begin{bmatrix} 1+i & -1+i & -3+i & 3+qi \\ -1+i & -i-1 & -1-3i & -1+3i \end{bmatrix},$$

$$X_2 = \frac{1}{2250}\begin{bmatrix} 33497+20254i & 20254+13243i & -39729-13243i & 13243-39729i \\ 3895+38950i & 1558+24149i & -13243-39729i & 39729i-13243i \end{bmatrix}.$$

Note that Maple 2021 was used to perform the numerical experiment.

4 Conclusion

The compact form of formula for the general solution of system of skew-Hermitian generalized Sylvester matrix equations (1.5) is established in this paper when this system obeys some solvable conditions over a complex number field $\mathbb{C}$. The Moore–Penrose inverse and the rank equalities of the coefficient matrices are used to obtain our main result. A particular case of this system is also discussed. We provide an algorithm and a numerical example to compute the general solution to (1.5) based on determinantal representations of generalized inverses.

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Declarations

Author Contribution All authors have contributed to the study conception and design, material preparation, data collection, and analysis. The first draft of the manuscript was written by AR and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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