Arab. J. Math. (2023) 12:667–684
https://doi.org/10.1007/s40065-023-00433-0

Arabian Journal of Mathematics

Lino F. Reséndis O · Luis M. Tovar S · Yesenia Bravo O

Conjugate complex harmonic functions

Received: 14 June 2022 / Accepted: 27 May 2023 / Published online: 15 June 2023
© The Author(s) 2023

Abstract This paper presents several properties and relations that satisfy the components of a bicomplex holomorphic function. It also exhibits several analogies and differences with the case of analytic functions.

Mathematics Subject Classification 30G35 · 31B05 · 32A30

1 Introduction

One of the best-known extensions of complex numbers to four dimensions is the algebra of quaternions, introduced by Hamilton [16,17], which, although is almost a field, loses the commutativity property. This fact makes it difficult to extend the theory of holomorphic functions. Another attempt was made in 1848 by Cockle following Hamilton's ideas. He studied an algebra called tessarines [13] where the role of the imaginary units is different from the bicomplex case. Thus, an attempt has been made to consider four-dimensional algebras which preserve commutativity and contain $\mathbb{C}$ as a subalgebra, and in which it is possible to extend the theory of holomorphic functions. Not surprisingly, this can be done by considering, for example, two imaginary units $\mathbf{i}$ and $\mathbf{j}$ such that $\mathbf{i}^2 = \mathbf{j}^2 = -1$ and introducing $\mathbf{ij} = \mathbf{ji} = \mathbf{k}$. This makes $\mathbf{k}$ an imaginary hyperbolic unit, i.e., an element such that $\mathbf{k}^2 = 1$. It was not until 1892 that the mathematician Corrado Segre, also inspired by the work of Hamilton and Clifford, introduced what he called bicomplex numbers [31]. Segre observed that $(1 - \mathbf{ij})/2$ and $(1 + \mathbf{ij})/2$ are idempotent and play a central role in the theory of bicomplex number. After Segre, other mathematicians, in particular Spampinato [33,34] and Scorza Dragoni [30], developed the first rudiments of a theory of functions over bicomplex numbers. The next great impulse in the study of bicomplex analysis was the work of Riley, in 1953 with his doctoral thesis [28], in which the theory of bicomplex functions is deepened. But the most important contribution was undoubtedly the work of Price [25], where the theory of bicomplex (as well as multicomplex) holomorphic functions is extensively developed. However, in recent years, there has been a resurgence of interest in the study of bicomplex holomorphic functions in one and several variables [2,7,10,14,20,21,24], as well as bicomplex meromorphic functions [5,11] and more recently there has been

L. F. Reséndis O (✉)
Universidad Autónoma Metropolitana Unidad Azcapotzalco Área de Análisis Matemático y sus Aplicaciones, C. P. 02128 Mexico City, Mexico
E-mail: lfro@azc.uam.mx

L. M. Tovar Sànchez · Y. Bravo O
Escuela Superior de Fisica y Matemáticas, Instituto Politécnico Nacional, Edif. 9, U.P. Adolfo López Mateos, C. P. 07338 Mexico City, Mexico
E-mail: tovar@esfm.ipn.mx

Y. Bravo O
E-mail: ybravoo.math@gmail.com

a remarkable activity in this field, dealing with the study of the bicomplex Bergman and Bloch [26] spaces, the bicomplex Stolz condition [27], the Möbius transformations in the bicomplex space [12] and the Cousin problems for the bicomplex case [8].

In this paper, we study several properties of the components of a bicomplex holomorphic function. Proposition 7.5.1 of [2] says that $F = F_1 + jF_2$ is a bicomplex holomorphic function on a domain of $\mathbb{BC}$, if and only if, their components F_1 and F_2 are related to each other by a complex Cauchy–Riemann type of conditions, very similar to the complex case. If F is expressed by its idempotent form $F = G_1\,\mathbf{e} + G_2\mathbf{e}^{\dagger}$, Sect. 3 shows how the Cauchy–Riemann type of conditions of F_1 and F_2 determine some relations between G_1 and G_2.

In classical complex analysis, it is well known that if u is a real harmonic function on a domain D in the complex plane, there exists a harmonic conjugate function v on the same domain such that $f = u + iv$ is holomorphic in D, and except for a constant this function f is unique and one has a classical integral formula to obtain v in terms of u. Sect. 4 shows that a similar result is true for the bicomplex case. In addition, two different ways of obtaining its complex harmonic conjugate F_2 are presented and illustrated with examples.

The orthogonality between the contour lines of the components of an analytic function and their corresponding gradients are well known. Section 5 shows that similar results exist for the different level sets of the components of a holomorphic bicomplex function and illustrate this with an example.

In complex analysis if $f = u + iv$, there exists a relationship between the magnitude of the square of the derivative of f and the norm of the square of the gradient of u and v. Section 6 shows what happens in the bicomplex holomorphic case. That is, if $F = F_1 + jF_2$, we exhibit various relations between the norm of the bicomplex derivative of F and the complex gradients of F_1 and F_2, even if F is expressed in its idempotent form.

In complex analysis, the conformal transplants are used to obtain various relations on the gradient, Laplacian, and integral from the conformal transplantation of an analytic bijective mapping. These results are very useful to obtain important classical applications in Physics ([18], Chapter 5). Section 7 shows that also in the bicomplex case, it is possible to generalize such relationships.

We expect that the generalizations to the bicomplex case presented in this article of classical results in complex analysis will lead to applications in mathematics and physics in the immediate future.

2 Preliminaries

This section presents several common facts about bicomplex numbers and bicomplex holomorphic functions. We will use freely results and notation of [2].

The set of bicomplex numbers $\mathbb{BC}$ is defined as

$$\mathbb{BC} := \{\, z_1 + \mathbf{j}z_2 \;\; z_1,\; z_2 \in \mathbb{C}(i),\; \mathbf{j}^2 = -1\}.$$

The sum and product of bicomplex numbers are made in the expected way. A bicomplex number $Z = z_1 + \mathbf{j}z_2$ admits several other forms of writing; however, in this work we use only two representations: their normal form $Z = z_1 + \mathbf{j}z_2$, with $z_l = x_l + iy_l \in \mathbb{C}(\mathbf{i})$, $l = 1,\ 2$, and their idempotent form, that is,

$$Z = \beta_1\,\mathbf{e} + \beta_2\mathbf{e}^{\dagger} \tag{2.1}$$

where

$$\mathbf{e} := \frac{1 + \mathbf{ij}}{2} \quad , \quad \mathbf{e}^{\dagger} := \frac{1 - \mathbf{ij}}{2}$$

and

$$\beta_1 = z_1 - \mathbf{i}z_2 \quad , \quad \beta_2 = z_1 + \mathbf{i}z_2. \tag{2.2}$$

Observe that $\mathbf{e}\,\mathbf{e}^{\dagger} = 0$; $1 = \mathbf{e} + \mathbf{e}^{\dagger}$ or more general $\lambda = \lambda(\mathbf{e} + \mathbf{e}^{\dagger})$ with $\lambda \in \mathbb{C}(\mathbf{i})$. Moreover,

$$z_1 = \frac{\beta_1 + \beta_2}{2} \quad \text{and} \quad z_2 = \frac{\beta_2 - \beta_1}{2\mathbf{i}}. \tag{2.3}$$

In the special case that $Z = \beta_1\,\mathbf{e} + \beta_2\mathbf{e}^{\dagger}$ and $W = \gamma_1\,\mathbf{e} + \gamma_2\mathbf{e}^{\dagger}$, with $\beta_l,\ \gamma_l$ real numbers, we consider the partial order

$$W \preceq Z \quad \text{if and only if} \quad \gamma_l \le \beta_l, \quad l = 1,\ 2.$$

There are several conjugations of bicomplex numbers; however, we consider here only two conjugations $Z^\dagger = z_1 - \mathbf{j}z_2$ and $Z^* = \overline{z_1} - \mathbf{j}\overline{z_2} = \overline{\beta_1}\,\mathbf{e} + \overline{\beta_2}\mathbf{e}^\dagger$, where $\overline{z_1}$ and $\overline{z_2}$ are the usual complex conjugates of $z_1, z_2 \in \mathbb{C}(\mathbf{i})$. With these conjugations,

$$|Z|_{\mathbf{i}}^2 = Z \cdot Z^\dagger = z_1^2 + z_2^2 \quad , \quad |Z|_{\mathbf{k}}^2 = Z \cdot Z^* = |\beta_1|^2\,\mathbf{e} + |\beta_2|^2\mathbf{e}^\dagger$$

and the inverse

$$Z^{-1} = \frac{Z^\dagger}{|Z|_{\mathbf{i}}^2} \quad , \quad Z^{-1} = \frac{Z^*}{|Z|_{\mathbf{k}}^2},$$

when $0 \neq Z$ is not a zero-divisor, that is, $Z \notin \{\beta\,\mathbf{e} + 0\,\mathbf{e}^\dagger\} \cup \{0\,\mathbf{e} + \beta\,\mathbf{e}^\dagger\}$ for $0 \neq \beta \in \mathbb{C}(\mathbf{i})$.

The **i**-norm and **k**-norm of Z are defined as (see Sections 1.4 and 2.7 in [20])

$$|Z|_{\mathbf{i}} = \sqrt{Z \cdot Z^\dagger} = \sqrt{z_1^2 + z_1^2} \quad , \quad |Z|_{\mathbf{k}} = \sqrt{Z \cdot Z^*} = |\beta_1|\,\mathbf{e} + |\beta_2|\mathbf{e}^\dagger.$$

Let Ω be a domain in $\mathbb{BC}$ and $F : \Omega \to \mathbb{BC}$ a bicomplex function. The complex partial derivatives of F at Z_0 are defined by the following limits (if they exist):

$$F'_{z_1}(Z_0) := \lim_{h_1 \to 0} \frac{F(Z_0 + h_1) - F(Z_0)}{h_1},$$

$$F'_{z_2}(Z_0) := \lim_{h_2 \to 0} \frac{F(Z_0 + \mathbf{j}h_2) - F(Z_0)}{h_2},$$

where h_1 and h_2 are complex increments and $Z_0 + h_1,\ Z_0 + \mathbf{j}h_2 \in \Omega$.

The bicomplex derivative $F'(Z_0)$ of the function F at a point $Z_0 \in \Omega$ is the following limit, if it exists that

$$F'(Z_0) := \lim_{Z \to Z_0} \frac{F(Z) - F(Z_0)}{Z - Z_0} = \lim_{H \to 0} \frac{(Z_0 + H) - F(Z_0)}{H}, \quad Z \in \Omega$$

such that $H = Z - Z_0$ is an invertible bicomplex number, or equivalently is not a zero divisor.

If F is derivable for all $Z \in \Omega$, then F is a bicomplex holomorphic function on Ω.

If $F = F_1 + \mathbf{j}F_2$ is bicomplex derivable at Z_0, the next result follows.

Theorem 2.1 ([20], Theorem 7.3.6) *Consider a bicomplex function F derivable at Z_0. Then we have*

- *The $\mathbb{C}(\mathbf{i})-$complex partial derivatives F'_{z_l} exist, for $l = 1, 2$.*
- *The complex partial derivatives verify the identity*

$$F'(Z_0) = F'_{z_1}(Z_0) = -\mathbf{j}F'_{z_2}(Z_0),$$

which is the $\mathbb{C}(\mathbf{i})-$complex Cauchy–Riemann system for F, that is,

$$\frac{\partial F_1}{\partial z_1} = \frac{\partial F_2}{\partial z_2} \quad \textit{and} \quad \frac{\partial F_1}{\partial z_2} = -\frac{\partial F_2}{\partial z_1}. \tag{2.4}$$

We would also like to mention the next comment and definition ([20], Definition 7.4.1, p. 153) The $\mathbb{C}(\mathbf{i})-$complex partial derivatives are denoted by the symbols F'_{z_l} for $l = 1,\ 2$, instead of the symbols $\frac{\partial F}{\partial z_1}(Z)$ and $\frac{\partial F}{\partial z_2}(Z)$, because the first ones define the limits of suitable difference quotients, while the latter ones are operators acting on $\mathcal{C}^1$ functions. The relationship between these two notations is justified by the following definition: a bicomplex $\mathcal{C}^1-$ function F is called $\mathbb{C}(\mathbf{i})-$bicomplex differentiable if

$$F(Z + H) - F(Z) = F'_{z_1}(Z) \cdot h_1 + F'_{z_2}(Z) \cdot h_2 + o(H),$$

where $o(H)$ denotes a function of the form $\alpha(H)|H|$ with $\lim_{H \to 0} \alpha(H) = 0$, ([20] Remark[7.2.3]).

Theorem 2.2 ([20], Theorem 7.4.2) *A $\mathcal{C}^1-$bicomplex function F is $\mathbb{C}(\mathbf{i})-$bicomplex differentiable if and only if both the components $F_1,\ F_2$ are holomorphic functions in the sense of two complex variables.*

Consider the following bicomplex differential operators:

$$\frac{\partial}{\partial Z} := \frac{1}{2}\left(\frac{\partial}{\partial z_1} - \mathbf{j}\frac{\partial}{\partial z_2}\right) \quad , \quad \frac{\partial}{\partial Z^{\dagger}} := \frac{1}{2}\left(\frac{\partial}{\partial z_1} + \mathbf{j}\frac{\partial}{\partial z_2}\right).$$

Theorem 2.3 ([20], Theorem 7.4.3) *Let $F \in \mathcal{C}^1(\Omega, \mathbb{BC})$. If F is a bicomplex holomorphic function, then* $\frac{\partial F}{\partial Z^{\dagger}}(Z) = 0$.

In bicomplex analysis, there are different ways of characterizing the bicomplex holomorphic functions as shown in the following result.

Theorem 2.4 ([20], Theorem 7.6.3) *Let $\Omega \subset \mathbb{BC}$ be a domain. A bicomplex function $F : \Omega \to \mathbb{BC}$ of class $\mathcal{C}^1$ on Ω with idempotent representation*

$$F = G_1\,\mathbf{e} + G_2\,\mathbf{e}^{\dagger} \tag{2.5}$$

is $\mathbb{BC}$-holomorphic if and only if the following two conditions hold:

(a) *The component G_1, seen as a $\mathbb{C}(\mathbf{i})$-valued function of the complex variables (β_1, β_2), is holomorphic; moreover it does not depend on the variable β_2 and thus G_1 is a holomorphic function of the variable β_1.*
(b) *The component G_2, seen as a $\mathbb{C}(\mathbf{i})$-valued function of the complex variables (β_1, β_2) is holomorhic; moreover, it does not depend on the variable β_1 and thus G_2 is a holomorphic function of the variable β_2.*

Its derivatives of any order are given by

$$F^{(n)}(Z) = G_1^{(n)}(\beta_1)\,\mathbf{e} + G_2^{(n)}(\beta_2)\mathbf{e}^{\dagger}, \qquad n = 0,\ 1,\ 2\ldots. \tag{2.6}$$

The rules of derivability are the usual ones.

3 Relations between the idempotent components of a bicomplex holomorphic function

If we express the $\mathbb{BC}$-holomorphic function F in its idempotent form (2.5), what kind of relationships must satisfy G_1 and G_2 between them?

As we can obtain a $\mathbb{BC}$-holomorphic function $F = G_1\,\mathbf{e}+G_2\mathbf{e}^{\dagger} : \Omega_1\,\mathbf{e}+\Omega_2\,\mathbf{e}^{\dagger} \to \mathbb{BC}$ by just taking any pair of holomorphic functions $G_1 : \Omega_1 \to \mathbb{C}$ and $G_2 : \Omega_2 \to \mathbb{C}$ and setting $F = G_1\,\mathbf{e}+G_2\mathbf{e}^{\dagger} : \Omega_1\,\mathbf{e}+\Omega_2\,\mathbf{e}^{\dagger} \to \mathbb{BC}$, then we cannot expect some relation like (2.4). However, we will see that the complex Cauchy–Riemann type of condition of F_1 and F_2 determines some relationship between G_1 and G_2.

Theorem 3.1 *Let $\Omega \subset \mathbb{BC}$ be a domain and $F = F_1 + \mathbf{j}F_2 : \Omega \to \mathbb{BC}$ a bicomplex function of class $\mathcal{C}^1$ with idempotent representation*

$$F = G_1\,\mathbf{e} + G_2\,\mathbf{e}^{\dagger}.$$

Then, F is a bicomplex holomorphic function on Ω if and only if

$$\frac{\partial G_1}{\partial z_1} + \frac{\partial G_2}{\partial z_1} = \mathbf{i}\left(\frac{\partial G_1}{\partial z_2} - \frac{\partial G_2}{\partial z_2}\right) \quad \textit{and} \quad \frac{\partial G_1}{\partial z_2} + \frac{\partial G_2}{\partial z_2} = \mathbf{i}\left(\frac{\partial G_2}{\partial z_1} - \frac{\partial G_1}{\partial z_1}\right). \tag{3.7}$$

In particular,

$$\frac{\partial G_1}{\partial z_1} + \frac{\partial G_2}{\partial z_2} = \mathbf{i}\left(\frac{\partial G_1}{\partial z_2} + \frac{\partial G_2}{\partial z_1}\right). \tag{3.8}$$

Proof By (2.2),

$$G_1 = F_1 - \mathbf{i}F_2 \qquad \text{and} \qquad G_2 = F_1 + \mathbf{i}F_2.$$

Then,

$$\frac{\partial G_1}{\partial z_1} = \frac{\partial F_1}{\partial z_1} - \mathbf{i}\frac{\partial F_2}{\partial z_1}, \quad \frac{\partial G_2}{\partial z_1} = \frac{\partial F_1}{\partial z_1} + \mathbf{i}\frac{\partial F_2}{\partial z_1},$$

which implies that

$$\frac{\partial F_1}{\partial z_1} = \frac{1}{2}\left(\frac{\partial G_1}{\partial z_1} + \frac{\partial G_2}{\partial z_1}\right) \quad ; \frac{\partial F_2}{\partial z_1} = \frac{1}{2\mathbf{i}}\left(\frac{\partial G_2}{\partial z_1} - \frac{\partial G_1}{\partial z_1}\right).$$

Similarly,

$$\frac{\partial F_1}{\partial z_2} = \frac{1}{2}\left(\frac{\partial G_1}{\partial z_2} + \frac{\partial G_2}{\partial z_2}\right) \quad ; \frac{\partial F_2}{\partial z_2} = \frac{1}{2\mathbf{i}}\left(\frac{\partial G_2}{\partial z_2} - \frac{\partial G_1}{\partial z_2}\right).$$

If F is a bicomplex holomorphic function, from (2.4), we have

$$\frac{\partial F_1}{\partial z_1} = \frac{1}{2}\left(\frac{\partial G_1}{\partial z_1} + \frac{\partial G_2}{\partial z_1}\right) = \frac{1}{2\mathbf{i}}\left(\frac{\partial G_2}{\partial z_2} - \frac{\partial G_1}{\partial z_2}\right) = \frac{\partial F_2}{\partial z_2}$$

$$\frac{\partial F_1}{\partial z_2} = \frac{1}{2}\left(\frac{\partial G_1}{\partial z_2} + \frac{\partial G_2}{\partial z_2}\right) = \frac{1}{2\mathbf{i}}\left(\frac{\partial G_1}{\partial z_1} - \frac{\partial G_2}{\partial z_1}\right) = -\frac{\partial F_2}{\partial z_1}$$

and from here we get (3.7). Reciprocally, if (3.7) holds, then (2.4) is satisfied and F is a bicomplex holomorphic function. If we add the equalities in (3.7) and associate them, we get

$$(1+\mathbf{i})\left(\frac{\partial G_1}{\partial z_1} + \frac{\partial G_2}{\partial z_2}\right) + (1-\mathbf{i})\left(\frac{\partial G_1}{\partial z_2} + \frac{\partial G_2}{\partial z_1}\right) = 0;$$

therefore, (3.8) follows. □

4 The conjugate of a $\mathbb{C}(\mathbf{i})$ complex harmonic function

This section defines the concepts of $\mathbb{C}(\mathbf{i})-$complex harmonic and bicomplex harmonic function. We will show that there is a connection between them.

Let $D \subset \mathbb{C}^2$ be an open subset and $f : D \to \mathbb{C}$ a holomorphic function. A function f is said to be a $\mathbb{C}(\mathbf{i})-$complex harmonic function if it satisfies a Laplace-type equation

$$\frac{\partial^2 f}{\partial z_1^2} + \frac{\partial^2 f}{\partial z_2^2} = 0.$$

Let Ω be an open subset of $\mathbb{BC}$. A function $F = F_1 + \mathbf{j}F_2 : \Omega \to \mathbb{BC}$ is bicomplex harmonic if F has continuous second partial derivatives and

$$\frac{\partial^2 F}{\partial z_1^2} + \frac{\partial^2 F}{\partial z_2^2} = 0. \tag{4.9}$$

This equation is a bicomplex $\mathbb{C}(\mathbf{i})-$Laplace-type equation in two complex variables z_1 and z_2. The operator $\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2}$ is denoted by $\triangle_{\mathbb{C}(\mathbf{i})^2}$ and is called $\mathbb{C}(\mathbf{i})-$Laplacian. $\mathbb{C}(\mathbf{i})-$Laplace equation is abbreviated as $\triangle_{\mathbb{C}(\mathbf{i})^2} f = 0$ or $\triangle_{\mathbb{C}(\mathbf{i})^2} F = 0$.

With this notation,

$$\triangle_{\mathbb{C}(\mathbf{i})^2}[F] = \triangle_{\mathbb{C}(\mathbf{i})^2}[F_1] + \mathbf{j}\triangle_{\mathbb{C}(\mathbf{i})^2}[F_2].$$

Moreover, if F is a bicomplex holomorphic function, then it satisfies the Cauchy–Riemann type of system given by (2.4) and so it is clear that each component is a $\mathbb{C}(\mathbf{i})-$complex harmonic function, that is,

$$\triangle_{\mathbb{C}(\mathbf{i})^2} F_1 = 0 \quad \text{and} \quad \triangle_{\mathbb{C}(\mathbf{i})^2} F_2 = 0.$$

Given a $\mathbb{C}(\mathbf{i})-$complex harmonic function on a domain of $\mathbb{C}^2$, we want to determine its bicomplex conjugate harmonic function. Then the following results appear to be natural generalizations of the complex cases. Recall that a closed form is locally exact.

Theorem 4.1 *Let $\Omega \subset \mathbb{C}^2$ be a domain and $F_i : \Omega \to \mathbb{C}$, $i = 1, 2$ be two holomorphic functions. If*

$$\frac{\partial F_1}{\partial z_2} = \frac{\partial F_2}{\partial z_1}, \tag{4.10}$$

then the differential form

$$\omega = F_1\, dz_1 + F_2\, dz_2 \tag{4.11}$$

is closed.

Proof Let $F_i = u_i + \mathbf{i}v_i$ be the real and imaginary parts of F_i for $i = 1, 2$. If $z_i = x_i + \mathbf{i}y_i$, $i = 1, 2$, the Cauchy–Riemann relations are written as

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial v_i}{\partial y_j} \quad \text{and} \quad \frac{\partial u_i}{\partial y_j} = -\frac{\partial v_i}{\partial x_j} \quad \text{for } i, j = 1, 2$$

and (4.10) rewritten as

$$\frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1} \quad \text{and} \quad \frac{\partial u_1}{\partial y_2} = \frac{\partial u_2}{\partial y_1}.$$

Now, the differential form (4.11) is

$$\begin{aligned} w &= F_1\, dz_1 + F_2\, dz_2 = (u_1 + \mathbf{i}v_1)(dx_1 + \mathbf{i}dy_1) + (u_2 + \mathbf{i}v_2)(dx_2 + \mathbf{i}dy_2) \\ &= u_1 dx_1 - v_1 dy_1 + u_2 dx_2 - v_2 dy_2 + \mathbf{i}(v_1 dx_1 + u_1 dy_1 + v_2 dx_2 + u_2 dy_2). \end{aligned}$$

To prove that the real part of ω is closed, it is necessary to verify

$$\begin{aligned} &\frac{\partial u_1}{\partial y_1} = -\frac{\partial v_1}{\partial x_1}, \quad \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1}, \quad \frac{\partial u_1}{\partial y_2} = -\frac{\partial v_2}{\partial x_1}, \\ -&\frac{\partial v_1}{\partial x_2} = \frac{\partial u_2}{\partial y_1}, \quad -\frac{\partial v_1}{\partial y_2} = -\frac{\partial v_2}{\partial y_1}, \quad \frac{\partial u_2}{\partial y_2} = -\frac{\partial v_2}{\partial x_2}. \end{aligned}$$

These equalities are a straightforward consequence from the previous Cauchy–Riemann relations and the hypothesis. In the same way, one can prove that the imaginary part of ω is closed. □

Corollary 4.2 *Let $\Omega \subset \mathbb{C}^2$ be a domain. Let $F_1 : \Omega \to \mathbb{C}$ be a $\mathbb{C}(\mathbf{i})-$complex harmonic function, that is,*

$$\frac{\partial^2 F_1}{\partial z_1^2} + \frac{\partial^2 F_1}{\partial z_2^2} = 0.$$

Then the differential form

$$\omega = -\frac{\partial F_1}{\partial z_2}\, dz_1 + \frac{\partial F_1}{\partial z_1}\, dz_2 \tag{4.12}$$

is closed.

The curves involved in the following formulas (4.13) and (4.15) are subsets of Ω.

Theorem 4.3 *Let $\Omega \subset \mathbb{C}^2$ be a domain and $F_1 : \Omega \to \mathbb{C}$ a $\mathbb{C}(\mathbf{i})-$complex harmonic function. If the first homotopy group of Ω is trivial, then all the conjugate complex harmonic functions of F_1 are given by*

$$F_2(z_1, z_2) = \int_{(z_1^0, z_2^0)}^{(z_1, z_2)} -\frac{\partial F_1}{\partial \zeta_2}(\zeta_1, \zeta_2)\, d\zeta_1 + \frac{\partial F_1}{\partial \zeta_1}(\zeta_1, \zeta_2)\, d\zeta_2 + c, \tag{4.13}$$

where $Z_0 = z_1^0 + \mathbf{j}z_2^0 = (z_1^0, z_2^0) \in \Omega$ is a fixed point and $c \in \mathbb{C}$. In particular, $F_1 + \mathbf{j}F_2$ is a bicomplex holomorphic function on Ω.

Proof By Corollary 4.2 and the hypothesis, the function F_2 is well defined, that is, it does not depend on the path joining Z_0 with Z. Then we can suppose that $Z,\ Z_0 \in \mathbb{B}_r(z_1, z_0) \subset \Omega$, and if $h \in \mathbb{C}$ is small enough, the differential quotient can be written as (omiting evaluations)

$$\frac{F_2(z_1+h, z_2) - F_2(z_1, z_2)}{h} = \frac{\int_{(z_1^0, z_2^0)}^{(z_1+h, z_2)} -\frac{\partial F_1}{\partial \zeta_2}\, d\zeta_1 + \frac{\partial F_1}{\partial \zeta_1}\, d\zeta_2 - \int_{(z_1^0, z_2^0)}^{(z_1, z_2)} -\frac{\partial F_1}{\partial \zeta_2}\, d\zeta_1 + \frac{\partial F_1}{\partial \zeta_1}\, d\zeta_2}{h}$$

$$= \frac{\int_{(z_1, z_2)}^{(z_1+h, z_2)} -\frac{\partial F_1}{\partial \zeta_2}\, d\zeta_1 + \frac{\partial F_1}{\partial \zeta_1}\, d\zeta_2}{h},$$

where the last integral is calculated using the path $\gamma : [0, 1] \to \Omega$ defined by $\gamma(t) = (z_1 + th, z_2)$. Since $\gamma'(t) = (h, 0)$, we have

$$\int_{(z_1, z_2)}^{(z_1+h, z_2)} -\frac{\partial F_1}{\partial \zeta_2}(\zeta_1, \zeta_2)\, d\zeta_1 + \frac{\partial F_1}{\partial \zeta_1}(\zeta_1, \zeta_2)\, d\zeta_2 = \int_0^1 -\frac{\partial F_1}{\partial \zeta_2}(z_1 + th, z_2) h\, dt.$$

Thus,

$$\frac{\partial F_2}{\partial z_1}(z_1, z_2) = \lim_{h\to 0} \frac{F_2(z_1+h, z_2) - F_2(z_1, z_2)}{h} = \lim_{h\to 0} \int_0^1 -\frac{\partial F_1}{\partial \zeta_2}(z_1 + th, z_2)\, dt = -\frac{\partial F_1}{\partial z_2}(z_1, z_2),$$

since F_1 is uniformly continuous on the segment. Using the path $\eta : [0, 1] \to \Omega$ defined by $\eta(t) = (z_1, z_2+th)$, we obtain

$$\frac{\partial F_2}{\partial z_2}(z_1, z_2) = \frac{\partial F_1}{\partial z_1}(z_1, z_2).$$

□

Example 4.4 Let $F_1(z_1, z_2) = z_1^3 - 3z_1 z_2^2$ and $Z_0 = z_1^0 + \mathbf{j} z_2^0 = (z_1^0, z_2^0) \in \mathbb{BC}$. Clearly, F_1 is complex harmonic. Then we integrate $\frac{\partial F_1}{\partial z_1}$ from (z_1^0, z_2^0) to (z_1, z_2), through integrals in segments from (z_1^0, z_2^0) to (z_1^0, z_2) and from (z_1^0, z_2) to (z_1, z_2). As in the first integral $d\zeta_1 = 0$, while in the second integral $d\zeta_2 = 0$, we have

$$\begin{aligned}\int_{(z_1^0, z_2^0)}^{(z_1^0, z_2)} -\frac{\partial F_1}{\partial \zeta_2}(\zeta_1, \zeta_2)\, d\zeta_1 + \frac{\partial F_1}{\partial \zeta_1}(\zeta_1, \zeta_2)\, d\zeta_2 &= \int_{(z_1^0, z_2^0)}^{(z_1^0, z_2)} \frac{\partial F_1}{\partial \zeta_1}(\zeta_1, \zeta_2)\, d\zeta_2 \\ &= \int_{(z_1^0, z_2^0)}^{(z_1^0, z_2)} (3\zeta_1^2 - 3\zeta_2^2) d\zeta_2 = 3\zeta_1^2 \zeta_2\Big]_{(z_1^0, z_2^0)}^{(z_1^0, z_2)} - \zeta_2^3\Big]_{(z_1^0, z_2^0)}^{(z_1^0, z_2)} \\ &= 3(z_1^0)^2 (z_2) - 3(z_1^0)^2 (z_2^0) - (z_2)^3 + (z_2^0)^3.\end{aligned}$$

Likewise,

$$\begin{aligned}\int_{(z_1^0, z_2)}^{(z_1, z_2)} -\frac{\partial F_1}{\partial z_2}(\zeta_1, \zeta_2) d\zeta_1 &= \int_{(z_1^0, z_2)}^{(z_1, z_2)} (6\zeta_1 \zeta_2) d\xi_1 = 3\zeta_2 \zeta_1^2\Big]_{(z_1^0, z_2)}^{(z_1, z_2)} \\ &= 3z_1^2 z_2 - 3(z_1^0)^2 z_2.\end{aligned}$$

Therefore, $F_2(z_1, z_2) = 3z_1 z_2^2 - z_2^3 + c$ with $c \in \mathbb{C}$. Moreover, $F(Z) = z_1^3 - 3z_1 z_2^2 + \mathbf{j}(3z_1 z_2^2 - z_2^3 + c)$ is a bicomplex holomorphic function, more precisely $F(Z) = (z_1 + \mathbf{j} z_2)^3 + \mathbf{j} c$.

Now, we are going to present a second way to calculate the conjugate of a complex harmonic function. For that, we need the following equality:

$$\int_{(z_1^0, z_2^0)}^{(z_1, z_2)} -\frac{\partial F_1}{\partial \zeta_2}\, d\zeta_1 = \int_{(z_1^0, z_2^0)}^{(z_1^0, z_2)} -\frac{\partial F_1}{\partial \zeta_2}\, d\zeta_1 + \int_{(z_1^0, z_2)}^{(z_1, z_2)} -\frac{\partial F_1}{\partial \zeta_2}\, d\zeta_1 = \int_{(z_1^0, z_2)}^{(z_1, z_2)} -\frac{\partial F_1}{\partial \zeta_2}\, d\zeta_1. \tag{4.14}$$

Proposition 4.5 *Let $\Omega \subset \mathbb{C}^2$ be a domain such that the first homotopy group of Ω is trivial. Let $F_1 : \Omega \to \mathbb{C}$ be a $\mathbb{C}(\mathbf{i})-$complex harmonic function. Then there exists an infinite number of conjugate complex harmonic functions of F_1 defined in Ω, given by*

$$F_2(z_1, z_2) = \int_{(z_1^0, z_2)}^{(z_1, z_2)} -\frac{\partial F_1}{\partial \zeta_2}(\zeta_1, \zeta_2)\, d\zeta_1 + \int_{(z_1^0, z_2^0)}^{(z_1, z_2)} \left(\frac{\partial F_1}{\partial \zeta_1}(\zeta_1, \zeta_2) + \frac{\partial}{\partial \zeta_2} \int_{(z_1^0, z_2)}^{(\zeta_1, \zeta_2)} \frac{\partial F_1}{\partial \eta_2}(\eta_1, \eta_2)\, d\eta_1 \right) d\zeta_2 + c, \tag{4.15}$$

where $(z_1^0, z_2^0),\ (z_1, z_2) \in \Omega$ and $c \in \mathbb{C}$. Moreover,

$$F_1(Z) + \mathbf{j} F_2(Z)$$

is a family of bicomplex holomorphic functions.

Proof Let $(z_1^0, z_2^0),\ (z_1, z_2) \in \Omega$, using (4.14) define

$$F_2(z_1, z_2) = \int_{(z_1^0, z_2)}^{(z_1, z_2)} -\frac{\partial F_1}{\partial \tau_2}(\tau_1, \tau_2)\, d\tau_1 + \varphi(z_2), \tag{4.16}$$

where φ is a holomorphic function that must be determined. Since the first homotopy group of Ω is trivial, we have that the integral in the definition of the function F_2 is well defined and by uniform continuity on the linear path

$$\frac{\partial F_2}{\partial z_1}(z_1, z_2) = \lim_{h \to 0} \frac{\int_{(z_1, z_2)}^{(z_1+h, z_2)} -\frac{\partial F_1}{\partial \tau_2}(\tau_1, \tau_2)\, d\tau_1}{h} = \lim_{h \to 0} \frac{\int_0^1 -\frac{\partial F_1}{\partial \tau_2}(z_1 + th, z_2) h\, dt}{h}$$
$$= -\frac{\partial F_1}{\partial z_2}(z_1, z_2).$$

In particular, F_2 is holomorphic on z_1 and since $\frac{\partial F_2}{\partial z_1}$ is holomorphic on z_2, then F_2 is also holomorphic on z_2. Since φ is a holomorphic function on z_2 only, its derivative $\frac{d\varphi}{dz_2}$ must also be independent of z_1. Since we need that $\frac{\partial F_2}{\partial z_2} = \frac{\partial F_1}{\partial z_1}$, we get

$$\frac{\partial}{\partial z_1} \left(\frac{\partial F_2}{\partial z_2}(z_1, z_2) + \frac{\partial}{\partial z_2} \int_{(z_1^0, z_2)}^{(z_1, z_2)} \frac{\partial F_1}{\partial \tau_2}(\tau_1, \tau_2)\, d\tau_1 \right)$$

$$= \frac{\partial^2 F_1}{\partial z_1 \partial z_1}(z_1, z_2) + \frac{\partial^2}{\partial z_1 \partial z_2} \int_{(z_1^0, z_2)}^{(z_1, z_2)} \frac{\partial F_1}{\partial \tau_2}(\tau_1, \tau_2)\, d\tau_1$$

$$= \frac{\partial^2 F_1}{\partial z_1^2}(z_1, z_2) + \frac{\partial^2}{\partial z_2 \partial z_1} \int_{(z_1^0, z_2)}^{(z_1, z_2)} \frac{\partial F_1}{\partial \tau_2}(\tau_1, \tau_2)\, d\tau_1$$

$$= \frac{\partial^2 F_1}{\partial z_1^2}(z_1, z_2) + \frac{\partial^2 F_1}{\partial z_2^2}(z_1, z_2) = 0,$$

since F_1 is a $\mathbb{C}(\mathbf{i})-$complex harmonic function. Thus,

$$\varphi(z_2) = \int_{(z_1^0, z_2^0)}^{(z_1, z_2)} \left(\frac{\partial F_1}{\partial \zeta_1}(\zeta_1, \zeta_2) + \frac{\partial}{\partial \zeta_2} \int_{(z_1^0, \zeta_2)}^{(\zeta_1, \zeta_2)} \frac{\partial F_1}{\partial \tau_2}(\tau_1, \tau_2)\, d\tau_1 \right) d\zeta_2 + c.$$

Substituting φ in (4.16), we have

$$\begin{aligned}\frac{\partial F_2}{\partial z_2}(\zeta_1,\zeta_2) &= \frac{\partial}{\partial z_2}\int_{(z_1^0,z_2)}^{(z_1,z_2)} -\frac{\partial F_1}{\partial \tau_2}(\tau_1,\tau_2)\,d\tau_1 + \frac{\partial}{\partial z_2}\varphi(z_2)\\ &= \frac{\partial}{\partial z_2}\int_{(z_1^0,z_2)}^{(z_1,z_2)} -\frac{\partial F_1}{\partial \tau_2}(\tau_1,\tau_2)\,d\tau_1 + \frac{\partial F_1}{\partial z_1}(z_1,z_2)\\ &\quad + \frac{\partial}{\partial z_2}\int_{(z_1^0,z_2)}^{(z_1,z_2)} \frac{\partial F_1}{\partial \eta_2}(\eta_1,\eta_2)\,d\eta_1\\ &= \frac{\partial F_1}{\partial z_1}(z_1,z_2).\end{aligned}$$

Thus, F_2 is a harmonic complex conjugate of F_1. □

The previous result gives us a classical way to calculate the harmonic conjugate of one function.

Example 4.6 Let $F_1(z_1,z_2) = z_1^3 - 3z_1z_2^2$. We know that F_1 is a $\mathbb{C}(\mathbf{i})$−complex harmonic function. Since $\frac{\partial F_1}{\partial z_2} = -\frac{\partial F_2}{\partial z_1}$, then

$$F_2(z_1,z_2) = \int 6z_1z_2\,dz_1 = 3z_1^2z_2 + \varphi(z_2).$$

Since $\frac{\partial F_1}{\partial z_1} = \frac{\partial F_2}{\partial z_2}$, we have

$$\frac{\partial F_2}{\partial z_2}(z_1,z_2) = 3z_1^2 + \varphi'(z_2) = 3z_1^2 - 3z_2^2.$$

Therefore, $\varphi'(z_2) = -3z_2^2$ and $\varphi(z_2) = -z_2^3 + c$. Finally,

$$F_2(z_1,z_2) = 3z_1^2z_2 - z_2^3 + c,$$

and the formula

$$z_1^3 - 3z_1z_2^2 + \mathbf{j}(3z_1^2z_2 - z_2^3 + c) = (z_1 + \mathbf{j}z_2)^3 + \mathbf{j}c$$

gives a family of bicomplex holomorphic functions, see Example 4.4.

5 Bicomplex level sets of bicomplex harmonic conjugate

In complex analysis, one of the consequences of holomorphicity is that the level curves of the harmonic components result in orthogonals among them, see [37]. We will see what happens in the bicomplex case.

Definition 5.1 Let $F(Z) = F_1(Z) + \mathbf{j}F_2(Z)$ be a bicomplex function in some domain $\Omega \subset \mathbb{BC}$, where $Z = x_1 + iy + \mathbf{j}(x_2 + iy_2)$. Suppose that $F_l = u_l + \mathbf{i}v_l$ where u_l, v_l are real functions for $l = 1,\ 2$. The equations:

$$u_l(x_1,y_1,x_2,y_2) = a_l, \qquad v_l(x_1,y_1,x_2,y_2) = b_l, \quad l = 1,\ 2,$$

where a_l, b_l are arbitrary real constants, which define the level sets of u_l and v_l, $l = 1,\ 2$, respectively.

Consider the following basic fact that characterizes the tangent space of hypersurfaces, see [35].

Theorem 5.2 *Let U be an open set in $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ in $C^\infty(U)$. Let $p \in U$ such that $\nabla f(p) \neq 0$, and let $c = f(p)$. Then the set of all vectors tangent to the level set $f^{-1}(c)$ is equal to $[\nabla f(p)]^\perp$, where $\nabla f(p)$ denotes the usual gradient. In particular if $\nabla f(p) \neq 0$ for all $p \in U$, $f^{-1}(c)$ is a hypersurface.*

Two hypersurfaces M and N are orthogonal if for each $p \in M \cap N$, their respective tangent spaces $T_p(M) = T_p(N)$ are orthogonal, ([6], formula (3.2.4)).

Theorems 5.3 and 5.4 may be summarized by saying that the level sets of the real parts of F_1 and F_2 are two orthogonal families as well as the level sets of the imaginary parts of them. Corollary 5.5describes the intersection of these orthogonal families in $\mathbb{BC}$.

Theorem 5.3 *Let $F(Z) = F_1(Z) + \mathbf{j}F_2(Z)$ be a bicomplex holomorphic function in some domain $\Omega \subset \mathbb{BC}$. Suppose that $F_l = u_l + \mathbf{i}v_l$ for $l = 1,\ 2$ and $\nabla u_1(p) \neq 0,\ p \in \Omega$. Then*

$$\nabla u_1(p) \perp \nabla u_2(p)$$

and

$$\nabla v_1(p) \perp \nabla v_2(p).$$

Proof From (7.18) [2], $\nabla u_1(p) = 0$ if and only if any of the other gradients is 0. We omit the evaluation at p. Since $F_l = u_l + \mathbf{i}v_l$, then

$$\nabla u_l = \left(\frac{\partial u_l}{\partial x_1}, \frac{\partial u_l}{\partial y_1}, \frac{\partial u_l}{\partial x_2}, \frac{\partial u_l}{\partial y_2}\right) \neq 0 \tag{5.17}$$

with $l = 1,\ 2$. As $F(Z) = F_1(Z) + \mathbf{j}F_2(Z)$ is a bicomplex holomorphic function, then F_1 and F_2 satisfy the Cauchy–Riemann equations (2.4). Thus,

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} \quad , \quad \frac{\partial u_1}{\partial y_1} = \frac{\partial u_2}{\partial y_2} \tag{5.18}$$

$$\frac{\partial u_1}{\partial x_2} = -\frac{\partial u_2}{\partial x_1} \quad , \quad \frac{\partial u_1}{\partial y_2} = -\frac{\partial u_2}{\partial y_1}. \tag{5.19}$$

Applying (5.18) and (5.19) to (5.17) with $l = 2$, we obtain

$$\nabla u_2 = \left(-\frac{\partial u_1}{\partial x_2}, -\frac{\partial u_1}{\partial y_2}, \frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial y_1}\right).$$

It follows that

$$\begin{aligned}\nabla u_1 \cdot \nabla u_2 &= \left(\frac{\partial u_1}{\partial x_1}, -\frac{\partial u_1}{\partial y_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_1}{\partial y_2}\right) \cdot \left(-\frac{\partial u_1}{\partial x_2}, -\frac{\partial u_1}{\partial y_2}, \frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial y_1}\right)\\ &= -\frac{\partial^2 u_1}{\partial x_1 \partial x_2} - \frac{\partial^2 u_1}{\partial y_1 \partial y_2} + \frac{\partial^2 u_1}{\partial x_2 \partial x_1} + \frac{\partial^2 u_1}{\partial y_2 \partial y_1}\\ &= 0.\end{aligned}$$

Hence, $\nabla u_1 \perp \nabla u_2$. In the same way, we can see that $\nabla v_1 \perp \nabla v_2$ using

$$\nabla v_l = \left(\frac{\partial v_l}{\partial x_1}, \frac{\partial v_l}{\partial y_1}, \frac{\partial v_l}{\partial x_2}, \frac{\partial v_l}{\partial y_2}\right). \tag{5.20}$$

□

Theorem 5.4 *Let $F(Z) = F_1(Z) + \mathbf{j}F_2(Z)$ be a bicomplex holomorphic function in some domain $\Omega \subset \mathbb{BC}$ Suppose that $F_l = u_l + \mathbf{i}v_l$, where u_l, v_l are real functions and $l = 1, 2$. Then,*

$$\nabla u_1 \perp \nabla v_1$$

and

$$\nabla u_2 \perp \nabla v_2.$$

Proof As $F(Z) = F_1(Z) + \mathbf{j}F_2(Z)$ is a bicomplex holomorphic function, then F_1 and F_2 are holomorphic as functions of two complex variables. Then they are holomorphic on each variable separately. Thus by using Cauchy–Riemann equations, we obtain

$$\frac{\partial v_2}{\partial x_1} = -\frac{\partial v_1}{\partial x_2} = \frac{\partial u_1}{\partial y_2}; \tag{5.21}$$

$$\frac{\partial v_2}{\partial x_2} = \frac{\partial v_1}{\partial x_1} = -\frac{\partial u_1}{\partial y_1}; \tag{5.22}$$

$$\frac{\partial v_2}{\partial y_1} = \frac{\partial u_2}{\partial x_1} = -\frac{\partial u_1}{\partial x_2} = -\frac{\partial v_1}{\partial y_2}; \tag{5.23}$$

$$\frac{\partial v_2}{\partial y_2} = \frac{\partial u_2}{\partial x_2} = -\frac{\partial u_1}{\partial x_1} = -\frac{\partial v_1}{\partial y_1}. \tag{5.24}$$

Aplying (5.21), (5.22), (5.23) and (5.24) to (5.20) when $l = 1$, we have

$$\nabla v_1 = \left(-\frac{\partial u_1}{\partial y_1}, \frac{\partial u_1}{\partial x_1}, -\frac{\partial u_1}{\partial y_2}, -\frac{\partial u_1}{\partial x_2}\right).$$

Then,

$$\begin{aligned}\nabla u_1 \cdot \nabla v_1 &= \left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial y_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_1}{\partial y_2}\right) \cdot \left(-\frac{\partial u_1}{\partial y_1}, \frac{\partial u_1}{\partial x_1}, -\frac{\partial u_1}{\partial y_2}, -\frac{\partial u_1}{\partial x_2}\right)\\ &= -\frac{\partial^2 u_1}{\partial x_1 \partial y_1} + \frac{\partial^2 u_1}{\partial y_1 \partial x_1} - \frac{\partial^2 u_1}{\partial x_2 \partial y_2} + \frac{\partial^2 u_1}{\partial y_2 \partial x_2}\\ &= 0.\end{aligned}$$

Hence, $\nabla u_1 \perp \nabla v_1$. In the same way, we can see that $\nabla u_2 \perp \nabla v_2$. □

However, ∇u_1 and ∇v_2 are not orthogonal in general, because

$$\begin{aligned}\nabla u_1 \cdot \nabla v_2 &= \left(-\frac{\partial^2 u_1}{\partial x_1 \partial y_2} - \frac{\partial^2 u_1}{\partial y_1 \partial x_2} - \frac{\partial^2 u_1}{\partial x_2 \partial y_1} + \frac{\partial^2 u_1}{\partial y_2 \partial x_1}\right)\\ &= 2\left[\left(\frac{\partial^2 u_1}{\partial x_1 \partial y_2}\right) - \left(\frac{\partial^2 u_1}{\partial y_1 \partial x_2}\right)\right].\end{aligned}$$

We can see also that ∇u_2 and ∇v_1 are not orthogonal in general, because

$$\begin{aligned}\nabla u_1 \cdot \nabla v_2 &= \left(-\frac{\partial^2 u_1}{\partial x_1 \partial y_2} - \frac{\partial^2 u_1}{\partial y_1 \partial x_2} - \frac{\partial^2 u_1}{\partial x_2 \partial y_1} + \frac{\partial^2 u_1}{\partial y_2 \partial x_1}\right)\\ &= 2\left[\left(\frac{\partial^2 u_1}{\partial x_1 \partial y_2}\right) - \left(\frac{\partial^2 u_1}{\partial y_1 \partial x_2}\right)\right].\end{aligned}$$

In the same way, it is possible to see that ∇u_2 and ∇v_1 are not orthogonal in general.

The following result permits us to describe the intersection of the pairs of hypersurfaces $u_1 = a_1$ and $u_2 = a_2$; $v_1 = b_1$ and $v_2 = b_2$; $u_1 = a_1$ and $v_1 = b_1$; $u_2 = a_2$ and $v_2 = b_2$.

It is well known that we can get an n-surface as the non-empty subset of $\mathbb{R}^{n+k}$ obtained by the intersection $\cap_{i=1}^{k} f_i^{-1}(c_i)$, $c_i \in \mathbb{R}^k$, where the $f_i : D \to \mathbb{R}$ (D open in $\mathbb{R}^{n+k}$) are smooth functions such that $\{\nabla f_1(p), \ldots, \nabla f_k(p)\}$ is linearly independent for each $p \in S$. This $n-$ surface in $\mathbb{R}^{n+k}$ results in the intersection of k, $(n+k-1)-$surfaces which meet “clearly” in the sense that the normal directions are linearly independent at each point of the intersection, see [35]. Since $\mathbb{BC} \approx \mathbb{R}^4$, we have the next corollary.

Corollary 5.5 *Let $F(Z) = F_1(Z) + \mathbf{j}F_2(Z)$ be a bicomplex holomorphic function in some domain $\Omega \subset \mathbb{BC}$ with $F_l = u_l + \mathbf{i}v_l$, where u_l, v_l are real functions for $l = 1, 2$. If $\nabla u_1(p) \neq 0$ for all $p \in u^{-1}(a_1)$ and $S = u_1^{-1}(a_1) \cap u_2^{-1}(a_2)$ is not empty. Then, S is a $2-$surface in $\mathbb{BC}$ obtained by the orthogonal intersection of two hypersurfaces in $\mathbb{BC}$.*

Proof By Definition (5.1) $u_i : \mathbb{BC} \approx \mathbb{R}^{2+2} \to \mathbb{R}$ with $i = 1, 2$. By Theorem (5.3): $\nabla u_1 \perp \nabla u_2$. This implies that $\{\nabla u_1(Z_0), \nabla u_2(Z_0)\}$ is linearly independent for each $Z_0 \in S$. Therefore, S is a $2-$surface in $\mathbb{BC}$ obtained as the intersection of two orthogonal hypersurfaces. □

We get the same results if we use $\{\nabla v_1, \nabla v_2\}$ (and $(\{\nabla u_i, \nabla v_i\}$ with $i = 1, 2)$).

Example 5.6 Consider the bicomplex holomorphic function

$$\begin{aligned} F(Z) &= (z_1 + \mathbf{j}z_2)^2 = \big(x_1 + \mathbf{i}y_1 + \mathbf{j}(x_2 + \mathbf{i}y_2)\big)^2 \\ &= x_1^2 - y_1^2 - x_2^2 + y_2^2 + \mathbf{i}(2x_1y_1 - 2x_2y_2) + \mathbf{j}\big(2x_1x_2 - 2y_1y_2 + \mathbf{i}(2x_1y_2 + 2x_2y_1)\big)^2. \end{aligned}$$

To simplify our analysis, we suppose $y_1 > 0$ and so $\nabla u_1(x_1, y_1, x_2, y_2) = 2(x_1, -y_1, -x_2, y_2) \neq 0$ and $\nabla u_2(x_1, y_1, x_2, y_2) = 2(x_2, -y_2, x_1, -y_1) \neq 0$, and they are orthogonal. We consider $F(1, 0, 0, 0) = 1$ and we have the orthogonal 3-surfaces:

$$\begin{aligned} u_1^{-1}(1) &= \left\{(x_1, y_1, x_2, y_2) \;:\; x_1^2 + y_2^2 = 1 + y_1^2 + x_2^2\right\}, \\ u_2^{-1}(0) &= \{(x_1, y_1, x_2, y_2) \;:\; x_1x_2 = y_1y_2\}. \end{aligned}$$

Their intersection is the 2-surface:

$$S = u_1^{-1}(1) \cap u_2^{-1}(0) = \left\{\left(\frac{y_1\sqrt{1 + y_1^2 + x_2^2}}{\sqrt{y_1^2 + x_2^2}}, x_2, y_1, \frac{x_2\sqrt{1 + y_1^2 + x_2^2}}{\sqrt{y_1^2 + x_2^2}}\right)\right\}.$$

6 Bicomplex gradients

In complex analysis, we have that the following relationship exists between the modulus of derivatives and the gradients of the real and imaginary parts of an analytic function: $|f'(z)|^2 = \|\nabla u\|_2^2 = \|\nabla v\|_2^2$. Now, we show what happens in the bicomplex case.

Let F be a bicomplex function in some open subset Ω of $\mathbb{BC}$. We define the bicomplex gradient of F denoted by $\operatorname{grad} F : \Omega \to \mathbb{BC}$ as

$$\operatorname{grad} F = \frac{\partial F}{\partial z_1} + \mathbf{j}\frac{\partial F}{\partial z_2}.$$

Let $F = F_1 + \mathbf{j}F_2$ be a bicomplex holomorphic function in some domain $\Omega \subset \mathbb{BC}$. By Theorem 2.1, F'_{z_l} exists for $l = 1, 2$ and verify the identity

$$F'(Z) = F'_{z_1}(Z) = -\mathbf{j}F'_{z_2}(Z)\,, \quad Z \in \Omega.$$

Since F_1 and F_2 satisfy the Cauchy–Riemann type of equations (2.4), F' can be represented as

$$F' = \frac{\partial F_1}{\partial z_1} - \mathbf{j}\frac{\partial F_1}{\partial z_2} = \frac{\partial F_2}{\partial z_2} + \mathbf{j}\frac{\partial F_2}{\partial z_1},$$

where the partial derivatives are all evaluated at $Z = z_1 + \mathbf{j}z_2$. Thus, we obtain

$$|F'|_{\mathbb{BC}}^2 = \left|\frac{\partial F_1}{\partial z_1}\right|^2 + \left|\frac{\partial F_1}{\partial z_2}\right|^2 = \left|\frac{\partial F_2}{\partial z_1}\right|^2 + \left|\frac{\partial F_2}{\partial z_2}\right|^2, \tag{6.25}$$

where $|\cdot|_{\mathbb{BC}}$ is the Euclidean norm in $\mathbb{BC}$ and $|\cdot|$ is the Euclidean norm in $\mathbb{C}$. On the other hand, by definition of bicomplex gradient, we have

$$\operatorname{grad} F_1 = \frac{\partial F_1}{\partial z_1} + \mathbf{j}\frac{\partial F_1}{\partial z_2},$$

and

$$\operatorname{grad} F_2 = \frac{\partial F_2}{\partial z_1} + \mathbf{j}\frac{\partial F_2}{\partial z_2},$$

then

$$|\text{grad} F_1|^2_{\mathbb{BC}} = \left|\frac{\partial F_1}{\partial z_1}\right|^2 + \left|\frac{\partial F_1}{\partial z_2}\right|^2, \tag{6.26}$$

and

$$|\text{grad} F_2|^2_{\mathbb{BC}} = \left|\frac{\partial F_2}{\partial z_1}\right|^2 + \left|\frac{\partial F_2}{\partial z_2}\right|^2, \tag{6.27}$$

where $|\cdot|$ is the norm in $\mathbb{C}$. Thus,

$$|F'|^2_{\mathbb{BC}} = |\text{grad} F_1|^2_{\mathbb{BC}} = |\text{grad} F_2|^2_{\mathbb{BC}}.$$

If we write $F_1 = u_1 + \mathbf{i}v_1$ where u_1 and v_1 are real functions, by Cauchy–Riemann equations $\frac{\partial F_1}{\partial z_1}$ can be written as

$$\frac{\partial F_1}{\partial z_1} = \frac{\partial u_1}{\partial x_1} - \mathbf{i}\frac{\partial u_1}{\partial y_1} = \frac{\partial v_1}{\partial y_1} - \mathbf{i}\frac{\partial v_1}{\partial x_1},$$

which implies that

$$\left|\frac{\partial F_1}{\partial z_1}\right|^2 = \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_1}{\partial y_1}\right)^2 = \left(\frac{\partial v_1}{\partial y_1}\right)^2 + \left(\frac{\partial v_1}{\partial x_1}\right)^2 \tag{6.28}$$

and

$$\left|\frac{\partial F_1}{\partial z_2}\right|^2 = \left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(\frac{\partial u_1}{\partial y_2}\right)^2 = \left(\frac{\partial v_1}{\partial y_2}\right)^2 + \left(\frac{\partial v_1}{\partial x_2}\right)^2. \tag{6.29}$$

On the other hand, we know that

$$\|\nabla u_1\|^2_{\mathbb{R}^4} = \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_1}{\partial y_1}\right)^2 + \left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(\frac{\partial u_1}{\partial y_2}\right)^2,$$

where $\|\cdot\|^2_{\mathbb{R}^4}$ is the norm in $\mathbb{R}^4$. By (6.26), (6.28) and (6.29), we have

$$|\text{grad} F_1|^2_{\mathbb{BC}} = \|\nabla u_1\|^2_{\mathbb{R}^4} = \|\nabla v_1\|^2_{\mathbb{R}^4}.$$

In the same way, if $F_2 = u_2 + \mathbf{j}v_2$ and using (6.27),

$$|\text{grad} F_2|^2_{\mathbb{BC}} = \|\nabla u_2\|^2_{\mathbb{R}^4} = \|\nabla v_2\|^2_{\mathbb{R}^4};$$

therefore,

$$\begin{aligned} |F'|^2_{\mathbb{BC}} &= |\text{grad} F_1|^2_{\mathbb{BC}} = |\text{grad} F_2|^2_{\mathbb{BC}} \\ &= \|\nabla u_1\|^2_{\mathbb{R}^4} = \|\nabla v_1\|^2_{\mathbb{R}^4} \\ &= \|\nabla u_2\|^2_{\mathbb{R}^4} = \|\nabla v_2\|^2_{\mathbb{R}^4}. \end{aligned}$$

Now, if we consider the $\mathbf{i}-$norm, we have

$$|F'|^2_{\mathbf{i}} = \left(\frac{\partial F_1}{\partial z_1}\right)^2 + \left(\frac{\partial F_1}{\partial z_2}\right)^2 = \left(\frac{\partial F_2}{\partial z_1}\right)^2 + \left(\frac{\partial F_2}{\partial z_2}\right)^2.$$

On the other hand,

$$(\text{grad} F_1)^\dagger = \frac{\partial F_1}{\partial z_1} - \mathbf{j}\frac{\partial F_1}{\partial z_2} \quad , \quad (\text{grad} F_2)^\dagger = \frac{\partial F_2}{\partial z_1} - \mathbf{j}\frac{\partial F_2}{\partial z_2}.$$

Then,

$$\left|(\operatorname{grad} F_1)^{\dagger}\right|_{\mathbf{i}}^{2} = \left(\frac{\partial F_1}{\partial z_1}\right)^2 + \left(\frac{\partial F_1}{\partial z_2}\right)^2$$

and

$$\left|(\operatorname{grad} F_2)^{\dagger}\right|_{\mathbf{i}}^{2} = \left(\frac{\partial F_2}{\partial z_1}\right)^2 + \left(\frac{\partial F_2}{\partial z_2}\right)^2,$$

however,

$$|F'|_{\mathbf{i}}^{2} = \left|(\operatorname{grad} F_1)^{\dagger}\right|_{\mathbf{i}}^{2} = \left|(\operatorname{grad} F_2)^{\dagger}\right|_{\mathbf{i}}^{2} = |(\operatorname{grad} F_1)|_{\mathbf{i}}^{2} = |(\operatorname{grad} F_2)|_{\mathbf{i}}^{2}.$$

Suppose that $F = G_1\,\mathbf{e} + G_2\,\mathbf{e}^{\dagger}$ is a bicomplex holomorphic function in $\Omega \subset \mathbb{BC}$. By (2.6),

$$F'(Z) = G_1'(\beta_1)\,\mathbf{e} + G_2'(\beta_2)\,\mathbf{e}^{\dagger}.$$

If we consider the $\mathbf{k}-$norm, we get

$$|F'(Z)|_{\mathbf{k}}^{2} = |G_1'(\beta_1)|^2\,\mathbf{e} + |G_2'(\beta_2)|^2\,\mathbf{e}^{\dagger},$$

where $|\cdot|$ is the norm in $\mathbb{C}$.

Since G_1 and G_2 are complex holomorphic functions, G_l has the form $G_l = \mu_l + \mathbf{i}\nu_l$ and satisfies the following relations for $l = 1, 2$:

$$|G_1'|^2 = \|\nabla\mu_1\|_2^2 = \|\nabla\nu_1\|_2^2$$

and

$$|G_2'|^2 = \|\nabla\mu_2\|_2^2 = \|\nabla\nu_2\|_2^2.$$

Consequently,

$$\begin{aligned} |F'|_{\mathbf{k}}^{2} &= \|\nabla\mu_1\|_2^2\ \mathbf{e} + \|\nabla\mu_2\|_2^2\ \mathbf{e}^{\dagger} \\ &= \|\nabla\mu_1\|_2^2\ \mathbf{e} + \|\nabla\nu_2\|_2^2\ \mathbf{e}^{\dagger} \\ &= \|\nabla\nu_1\|_2^2\ \mathbf{e} + \|\nabla\mu_2\|_2^2\ \mathbf{e}^{\dagger} \\ &= \|\nabla\nu_1\|_2^2\ \mathbf{e} + \|\nabla\nu_2\|_2^2\ \mathbf{e}^{\dagger}. \end{aligned}$$

7 Bicomplex conformal transplants

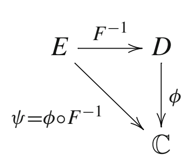
In this section, we will give the generalization to the bicomplex case of the complex conformal transplants, which are very useful in Physics, to facilitate the calculation of potentials (temperature, electricity, stress, etc.). For more information, see [18].

Let D, E be domains in $\mathbb{BC}$ and F, a bijective bicomplex holomorphic function from D to E, with inverse F^{-1} also bijective and $F'(Z) \neq 0$ for all $Z \in D$.

Let ϕ be a $\mathbb{C}(\mathbf{i})-$valued function of class $\mathcal{C}^2(D)$. We define in E a function ψ as follows: for any $W \in E$, let

$$\psi(W) = \phi(F^{-1}(W)) = \phi(z_1(w_1, w_2), z_2(w_1, w_2)).$$

The function ψ will be called the bicomplex conformal transplant of ϕ under the mapping F and its process of construction is given by the bicomplex conformal transplantation. See the diagram.



.

By definition

$$\phi(z_1, z_2) = \psi(w_1(z_1, z_2), w_2(z_1, z_2)),$$

we have $F(Z) = F(z_1 + \mathbf{j}z_2) = F_1(z_1, z_2) + \mathbf{j}F_2(z_1, z_2) = W = w_1 + \mathbf{j}w_2$.
Then we have the following results:

Theorem 7.1 *If ψ results from ϕ by bicomplex conformal transplantation by means of the mapping F, then*

$$\operatorname{grad}\phi(Z) = \operatorname{grad}\psi(W) \cdot F'(Z).$$

Proof Applying the chain rule and the Cauchy–Riemann system given by (2.4), we obtain

$$\begin{aligned}
\operatorname{grad}\phi &= \frac{\partial\psi}{\partial w_1}\frac{\partial w_1}{\partial z_1} + \frac{\partial\psi}{\partial w_2}\frac{\partial w_2}{\partial z_1} + \mathbf{j}\left(-\frac{\partial\psi}{\partial w_1}\frac{\partial w_2}{\partial z_1} + \frac{\partial\psi}{\partial w_2}\frac{\partial w_1}{\partial z_1}\right)\\
&= \frac{\partial\psi}{\partial w_1}\frac{\partial w_1}{\partial z_1} + \frac{\partial\psi}{\partial w_2}\frac{\partial w_2}{\partial z_1} - \mathbf{j}\frac{\partial\psi}{\partial w_1}\frac{\partial w_2}{\partial z_1} + \mathbf{j}\frac{\partial\psi}{\partial w_2}\frac{\partial w_1}{\partial z_1}\\
&= \frac{\partial\psi}{\partial w_1}\left(\frac{\partial w_1}{\partial z_1} - \mathbf{j}\frac{\partial w_2}{\partial z_1}\right) + \frac{\partial\psi}{\partial w_2}\left(\frac{\partial w_2}{\partial z_1} + \mathbf{j}\frac{\partial w_1}{\partial z_1}\right)\\
&= \left(\frac{\partial\psi}{\partial w_1} + \mathbf{j}\frac{\partial\psi}{\partial w_2}\right)\left(\frac{\partial w_1}{\partial z_1} - \mathbf{j}\frac{\partial w_2}{\partial z_1}\right)\\
&= \operatorname{grad}\psi(W)\left(\frac{\partial w_1}{\partial z_1} - \mathbf{j}\frac{\partial w_2}{\partial z_1}\right),
\end{aligned}$$

since by Theorem 2.1 $F'(Z) = F'_{z_1}(Z)$, which completes the proof. □

We will see what is the relationship between the $\mathbb{C}(\mathbf{i})-$Laplacian of ϕ and the $\mathbb{C}(\mathbf{i})-$ Laplacian of ψ. By definition of $\mathbb{C}(\mathbf{i})-$Laplacian, we have

$$\triangle_{\mathbb{C}(\mathbf{i})^2}\phi = \frac{\partial^2\phi}{\partial z_1^2} + \frac{\partial^2\phi}{\partial z_2^2}, \tag{7.30}$$

$$\triangle_{\mathbb{C}(\mathbf{i})^2}\psi = \frac{\partial^2\psi}{\partial w_1^2} + \frac{\partial^2\psi}{\partial w_2^2}. \tag{7.31}$$

Theorem 7.2 *Under the hypotheses of the preceeding theorem,*

$$\triangle_{\mathbb{C}(\mathbf{i})^2}\phi(Z) = \triangle_{\mathbb{C}(\mathbf{i})^2}\psi(W)|F'(Z)|^2_{\mathbf{i}}.$$

Proof Applying the chain rule again, we obtain

$$\begin{aligned}
\frac{\partial^2\phi}{\partial z_1^2} &= \frac{\partial}{\partial z_1}\left(\frac{\partial\phi}{\partial z_1}\right)\\
&= \frac{\partial}{\partial z_1}\left(\frac{\partial\psi}{\partial w_1}\frac{\partial w_1}{\partial z_1} + \frac{\partial\psi}{\partial w_2}\frac{\partial w_2}{\partial z_1}\right)\\
&= \frac{\partial}{\partial z_1}\left(\frac{\partial\psi}{\partial w_1}\right)\cdot\frac{\partial w_1}{\partial z_1} + \frac{\partial\psi}{\partial w_1}\cdot\frac{\partial}{\partial z_1}\left(\frac{\partial w_1}{\partial z_1}\right)\\
&\quad + \frac{\partial}{\partial z_1}\left(\frac{\partial\psi}{\partial w_2}\right)\cdot\frac{\partial w_2}{\partial z_1} + \frac{\partial\psi}{\partial w_2}\cdot\frac{\partial}{\partial z_1}\left(\frac{\partial w_2}{\partial z_1}\right)\\
&= \left(\frac{\partial^2\psi}{\partial w_1^2}\cdot\frac{\partial w_1}{\partial z_1} + \frac{\partial^2\psi}{\partial w_2\partial w_1}\cdot\frac{\partial w_2}{\partial z_1}\right)\cdot\frac{\partial w_1}{\partial z_1} + \frac{\partial\psi}{\partial w_1}\cdot\frac{\partial^2 w_1}{\partial z_1^2}\\
&\quad + \left(\frac{\partial^2\psi}{\partial w_1\partial w_2}\cdot\frac{\partial w_1}{\partial z_1} + \frac{\partial^2\psi}{\partial w_2^2}\cdot\frac{\partial w_2}{\partial z_1}\right)\cdot\frac{\partial w_2}{\partial z_1} + \frac{\partial\psi}{\partial w_2}\cdot\frac{\partial^2 w_2}{\partial z_1^2}
\end{aligned}$$

$$= \frac{\partial^2\psi}{\partial w_1^2}\cdot\left(\frac{\partial w_1}{\partial z_1}\right)^2 + \frac{\partial^2\psi}{\partial w_2\partial w_1}\frac{\partial w_2}{\partial z_1}\frac{\partial w_1}{\partial z_2} + \frac{\partial\psi}{\partial w_1}\frac{\partial^2 w_1}{\partial z_1^2}$$
$$+ \frac{\partial^2\psi}{\partial w_1\partial w_2}\frac{\partial w_1}{\partial z_1}\frac{\partial w_2}{\partial z_1} + \frac{\partial^2\psi}{\partial w_2^2}\cdot\left(\frac{\partial w_2}{\partial z_1}\right)^2 + \frac{\partial\psi}{\partial w_2}\frac{\partial^2 w_2}{\partial z_1^2}.$$

Since $\psi \in C^2(E)$, we have

$$\frac{\partial^2\phi}{\partial z_1^2} = \frac{\partial^2\psi}{\partial w_1^2}\cdot\left(\frac{\partial w_1}{\partial z_1}\right)^2 + 2\left(\frac{\partial^2\psi}{\partial w_1\partial w_2}\frac{\partial w_1}{\partial z_1}\frac{\partial w_2}{\partial z_1}\right) + \frac{\partial^2\psi}{\partial w_2^2}\cdot\left(\frac{\partial w_2}{\partial z_1}\right)^2 + \frac{\partial\psi}{\partial w_1}\frac{\partial^2 w_1}{\partial z_1^2} + \frac{\partial\psi}{\partial w_2}\frac{\partial^2 w_2}{\partial z_1^2}. \tag{7.32}$$

Analogously,

$$\frac{\partial^2\phi}{\partial z_2^2} = \frac{\partial^2\psi}{\partial w_1^2}\cdot\left(\frac{\partial w_1}{\partial z_2}\right)^2 + 2\left(\frac{\partial^2\psi}{\partial w_1\partial w_2}\frac{\partial w_1}{\partial z_2}\frac{\partial w_2}{\partial z_2}\right) + \frac{\partial^2\psi}{\partial w_2^2}\cdot\left(\frac{\partial w_2}{\partial z_2}\right)^2 + \frac{\partial\psi}{\partial w_1}\frac{\partial^2 w_1}{\partial z_2^2} + \frac{\partial\psi}{\partial w_2}\frac{\partial^2 w_2}{\partial z_2^2}. \tag{7.33}$$

Substituting (7.32) and (7.33) in (7.30) we get

$$\triangle_{\mathbb{C}(\mathbf{i})^2}\phi = \frac{\partial^2\psi}{\partial w_1^2}\left[\left(\frac{\partial w_1}{\partial z_1}\right)^2 + \left(\frac{\partial w_1}{\partial z_2}\right)^2\right] + \frac{\partial^2\psi}{\partial w_2^2}\left[\left(\frac{\partial w_2}{\partial z_1}\right)^2 + \left(\frac{\partial w_2}{\partial z_2}\right)^2\right]$$
$$+ 2\frac{\partial^2\psi}{\partial w_1\partial w_2}\left[\left(\frac{\partial w_1}{\partial z_1}\frac{\partial w_2}{\partial z_1} + \frac{\partial w_1}{\partial z_2}\frac{\partial w_2}{\partial z_2}\right)\right] + \frac{\partial\psi}{\partial w_1}\left(\frac{\partial^2 w_1}{\partial z_1^2} + \frac{\partial^2 w_1}{\partial z_2^2}\right)$$
$$+ \frac{\partial\psi}{\partial w_2}\left(\frac{\partial^2 w_2}{\partial z_1^2} + \frac{\partial^2 w_2}{\partial z_2^2}\right),$$

and by Cauchy–Riemann type of equations we have

$$\frac{\partial w_1}{\partial z_1}\frac{\partial w_2}{\partial z_1} + \frac{\partial w_1}{\partial z_2}\frac{\partial w_2}{\partial z_2} = 0.$$

Furthermore,

$$\triangle_{\mathbb{C}(\mathbf{i})^2}(w_1) = \frac{\partial}{\partial z_1}\left(\frac{\partial w_2}{\partial z_2}\right) + \frac{\partial}{\partial z_2}\left(-\frac{\partial w_2}{\partial z_1}\right) = \frac{\partial^2 w_2}{\partial z_1\partial z_2} - \frac{\partial^2 w_2}{\partial z_2\partial z_1}.$$

Since w_1 and w_2 represent the components F_1 and F_2 of F, it follows that $\triangle_{\mathbb{C}(\mathbf{i})^2}(w_1) = 0$ and $\triangle_{\mathbb{C}(\mathbf{i})^2}(w_2) = 0$. Hence,

$$\triangle_{\mathbb{C}(\mathbf{i})^2}\phi = \frac{\partial^2\psi}{\partial w_1^2}\left[\left(\frac{\partial w_1}{\partial z_1}\right)^2 + \left(\frac{\partial w_1}{\partial z_2}\right)^2\right] + \frac{\partial^2\psi}{\partial w_2^2}\left[\left(\frac{\partial w_2}{\partial z_1}\right)^2 + \left(\frac{\partial w_2}{\partial z_2}\right)^2\right]$$
$$= \frac{\partial^2\psi}{\partial w_1^2}\left[\left(\frac{\partial w_1}{\partial z_1}\right)^2 + \left(\frac{\partial w_2}{\partial z_1}\right)^2\right] + \frac{\partial^2\psi}{\partial w_2^2}\left[\left(\frac{\partial w_1}{\partial z_2}\right)^2 + \left(\frac{\partial w_2}{\partial z_2}\right)^2\right].$$

As $F'(z) = F'_{z_1} = -\mathbf{j}F'_{z_2}(Z)$, we have $|F'(z)|_{\mathbf{i}}^2 = |F'_{z_1}|_{\mathbf{i}}^2 = |F'_{z_2}(Z)|_{\mathbf{i}}^2$, then

$$\triangle_{\mathbb{C}(\mathbf{i})^2}\phi = \left(\frac{\partial^2\psi}{\partial w_1^2} + \frac{\partial^2\psi}{\partial w_2^2}\right)\cdot|F'(z)|_{\mathbf{i}}^2,$$

which proves the theorem. □

The next result gives the relationship between the integral of ϕ over D and the integral of ψ over E.

Theorem 7.3 *Under the hypotheses of Theorem (7.1),*

$$\int_D \phi(Z)dz_1dz_2 = \int_E \psi(W)|(F^{-1})'(W)|_i^2 dw_1dw_2.$$

Proof Notice that

$$
\begin{aligned}
|\det(F^{-1})'| = \frac{\partial(z_1, z_2)}{\partial(w_1, w_2)} &= \begin{vmatrix} \frac{\partial z_1}{\partial w_1} & \frac{\partial z_1}{\partial w_2} \\ \frac{\partial z_2}{\partial w_1} & \frac{\partial z_2}{\partial w_2} \end{vmatrix} \\
&= \frac{\partial z_1}{\partial w_1}\frac{\partial z_2}{\partial w_2} - \frac{\partial z_2}{\partial w_1}\frac{\partial z_1}{\partial w_2} \\
&= \frac{\partial z_1}{\partial w_1}\frac{\partial z_1}{\partial w_1} + \frac{\partial z_1}{\partial w_2}\frac{\partial z_1}{\partial w_2} \\
&= \left(\frac{\partial z_1}{\partial w_1}\right)^2 + \left(\frac{\partial z_1}{\partial w_2}\right)^2 \\
&= |(F^{-1})'(W)|_i^2,
\end{aligned}
$$

since D and E are domains in $\mathbb{BC} \approx \mathbb{R}^4$, and F^{-1} is a bijective bicomplex function so, in particular, it is a diffeomorphism of class C^1. Moreover if $\phi \in \mathcal{L}^1(D, \mathbb{C})$, then $\left(\phi \circ F^{-1}\right)|\det(F^{-1})'| \in \mathcal{L}^1(E, \mathbb{C})$ and

$$
\int_D \phi(Z)dz_1dz_2 = \int_E \left(\phi \circ F^{-1}\right)|\det(F^{-1})'|dw_1dw_2.
$$

□

Important applications of bicomplex analysis in physics have appeared in several works during the present century, for example, to Maxwell's equations in bicomplex analysis [1], in applications of bicomplex algebra to electromagnetism [3], as well as works on bicomplex quantum mechanics [29] and on the bicomplex quantum Coulomb potential problem [22], among others. We expect that the results presented in this work will be applied in the near future to physics problems in higher dimensions.

In a forthcoming paper, we will present some consequences of the results of this paper related to Cauchy's bicomplex integral theorem and Morera's bicomplex theorem.

Acknowledgements We are very grateful to the rewievers.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Funding The authors have not disclosed any funding.

Declarations

Conflict of interest The authors have not disclosed any competing interests.

References

1. Agarwal, R.; Goswami, M.P.; Agarwal, R.P.: Sumudu transform in bicomplex space and its applications. Ann. Appl. Math. **33**(2), 239–253 (2022)
2. Alpay, D., Luna-Elizarraraz, M. E., Shapiro, M., Shapiro M., Struppa, C.: Basics on Functional Analysis with Bicomplex Scalars, and Bicomplex Schur Analysis. Springer Briefs in Mathematics (2014)
3. Anastassiu, H.T.; Atlamazoglou, P.E.; Kaklamani, D.I.: Application of bicomplex (quaternion) algebra to fundamental electromagnetics: a lower order alternative to the Helmholtz equation. IEEE Trans. Anten. Propag. **51**(8), 2130–2136 (2003). https://doi.org/10.1109/TAP.2003.810231
4. Ball, J.A.; Bolotnikov, V.: Weighted Bergman spaces: shift-invariant subspaces and input/state/output linear systems. Integr. Equ. Oper. Theory **76**, 301 (2013). https://doi.org/10.1007/s00020-013-2053-5
5. Bamdad, H.; Vahidi, A.: Extension functors of Cousin cohomology modules. Bull. Iran. Math. Soc. **44**, 253–267 (2018). https://doi.org/10.1007/s41980-018-0009-x
6. Beardon, A. F.: The Geometry of Discrete Groups, Graduate Texts in Mathematics, vol 91, Springer
7. Bidkham, M., Ahmadi, S.: Generalization of the Gauss–Lucas theorem for bicomplex polynomials. Turk. J. Math. **41** (2017). https://doi.org/10.3906/mat-1512-19

8. Bravo, Y., Reséndis, L. F., Tovar, L. M.: Cousin's problems I and II. The bicomplex case, special issue of Boletín de la Sociedad Matemática Mexicana. (To appear)
9. Charak, K. S., Rochon, D.: On Factorization of bicomplex meromorphic functions, Trends in Mathematics, Birkhäuser Verlag, Basel, 55-68 (2008)
10. Charak, K. S., Rochon, D., Sharma, N.: Normal families of bicomplex holomorphic functions. Fractals **17** (2008). https://doi.org/10.1142/S0218348X09004314
11. Charak, K. S., Rochon, D., Sharma, N.: Normal families of bicomplex meromorphic functios. Annales Polinici Mathematici **103** (2011). https://doi.org/10.4064/ap103-3-6
12. Chen, L., Dai, B.: The Fixed points and cross-ratios of hyperbolic möbius transformations in bicomplex space. Adv. Appl. Clifford Algebras **32** (2022). https://doi.org/10.1007/s00006-022-01231-1
13. Cockle, J.: On certain functions resembling quaternionsand on a new imaginary in algebra. Lond. Dublin Edinbu. Philos. Mag. Sci. J. **32**(3), 435–439 (1848)
14. Colombo, F., Sabadini, I., Struppa, D.C.: Bicomplex holomorphic functional calculus. Math. Nachrich. **287**(10), 1093–1105. https://doi.org/10.1002/mana.201200354
15. Duren, P., Schuster, A.: Bergman Spaces, Mathematical surveys and Monographs, Vol. 100. AMS (2004)
16. Hamilton, W.R.: On quaternions, or on a new system of imaginaries in algebra. Philos. Mag. **25**(3), 489–495 (1844)
17. Hamilton, W.R.: Lectures on Quaternions: Containing a Systematic Statement of a New Mathematical Method. Hodges and Smith, Dublin (1853)
18. Henrici, P.: Applied and Computational Complex Analysis Vol. 1. Wiley (1974)
19. Kumar, R., Singh, K., Saini, H., Kumar, S.: Bicomplex weighted Hardy spaces and bicomplex C^*-algebras, Adv. Appl. Cliff. Alg. **26**(1), 217–235
20. Luna-Elizarraraz, M. E., Shapiro, M., Struppa, Daniele C., Vajiac, Adrián.: Bicomplex Holomorphic Functions: The Algebra, Geometry and Analysis, Frontiers in Mathematics. Birkhäuser (2015)
21. Luna-Elizarraraz, M.E.; Pérez-Regalado, C.O.; Shapiro, M.: On the Laurent series for bicomplex holomorphic functions. Complex Variab. Ellip. Equ. **62**(9), 1266–1286 (2017)
22. Mathieu, J.; Marchildon, L.; Rochon, D.: The bicomplex quantum Coulomb potential problem. Canad. J. Phys. Worldcat **91**(12), 1093–1100 (2013). https://doi.org/10.1139/cjp-2013-0261. **ISSN 0008-4204**
23. Pérez-Regalado, C.O., Quiroga-Barranco, R.: Bicomplex Bergman Spaces on Bounded domains (2018). arXiv:1811.00150vl [math.FA]
24. Pogoruy, A., Rodríguez-Dagnino, R.: On the set of zeros of bicomplex polynomials. Complex Variab Ellip Equ. **51** (2006). https://doi.org/10.1080/17476930600757139
25. Price, G. B.: An Introduction to Multicomplex Spaces and Functions, Monographs and Textbooks in Pure and Applied Mathematics 140, Dekker, New York (1991). https://doi.org/10.1080/17476930600757139
26. Reséndis, L.F.; Tovar, L.M.: Bicomplex Bergman and Bloch spaces. Arab. J. Math. **9**, 665–679 (2020). https://doi.org/10.1007/s40065-020-00285-y
27. Reséndis, L. F, Telléz, G, Tovar, L. M.: Directional derivatives and Stolz conditions for bicomplex holomorphic functions. In: Cerejeiras, A.G.P., et al. (eds) Current Trends in Analysis, its Applications and Computation, Research Perspectives. Springer, Switzerland (2022). https://doi.org/10.1007/978-3-030-87502-2_20
28. Riley, J. D.: Contributions to the theory of functions of a bicomplex variable. J. Free Access **5** (1953). https://doi.org/10.2748/tmj/1178245302
29. Rochon, D., Tremblay, S.: Bicomplex quantum mechanics: II. The hilbert space. Adv. Appl. Cliff. Algebras **16** (2) (2006). https://doi.org/10.1007/s00006-006-0008-5
30. Scorza Dragoni, G.: Sulle funzioni olomorfe du una variabile bicomplessa. Reale Accad. I'Italia, Mem. Classe Sci. Nat Fis. Mat. **5**, 597–665 (1934)
31. Segre, C.: Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici. Math. Ann. **40**, 413–467 (1892)
32. Sharma, A.K.: Generalized weighted composition operators on the Bergman space. Demostratio Math. **44**(2), 359–372 (2017). https://doi.org/10.1515/dema-2013-0305
33. Spampinato, N.: Estensione nel campo bicomplesso di due teoremi, del Levi-Civita e del Severi, per le funzioni olomorfe di due variabili bicomplesse I, II, Reale Accad. Naz Lincei, **22**(38-43), 578–583 (1935)
34. Spampinato, N.: Sulla rappresentazione di funzioni di variabile bicomplessa totalmente derivabili. Ann. Math. Pura Appl. **14**, 305–325 (1936)
35. Thorpe, J. A.: Elementary topics in differential geometry. Springer, New York (1979)
36. Zhu, K.: Spaces of holomorphic functions in the unit ball, Graduate Texts in Mathematics, vol 226. Springer, New York (2005)
37. Zill, D. G, Shanahan, P. D.: A first course in complex analysis with applications. Jones and Bartlett Publishers, Inc (2003)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.