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Fatima Zahra Arioui

Weighted fractional stochastic integro-differential equation with infinite delay

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Abstract In this paper, we consider a weighted fractional stochastic integro-differential equation with infinite delay and nonzero initial values involving a Riemann–Liouville fractional derivative of order $1/2 < \alpha < 1$. The existence of a mild solution is investigated using fractional calculus, stochastic analysis, and the fixed point theorem. An example is also provided to illustrate the obtained result.

Mathematics Subject Classification 35A01 · 60H10 · 60H20

1 Introduction

Fractional calculus provides an excellent set of tools for describing memory and hereditary properties of many materials and processes, allowing for the modeling of non-local and diffuse effects that frequently occur in natural phenomena. The application of fractional calculus tools and techniques is widespread in almost all areas of engineering and science in general. Viscoelasticity, robotics, control theory, and other fields have numerous applications (see [2,4,12,13,21]).

Phenomena are precisely modeled when the influences that processes may encounter along their course are considered. For example, it is well known that time delays may exist in a variety of technical systems where the derivative of a state variable depends not only on its current state but also on information from the past.

When investigating fractional differential equations, the properties of the derivative type, such as the condition at the initial moment, are of interest. In a fractional differential equation with the Caputo derivative, the initial condition is the same as in an ordinary differential equation, whereas for the Riemann–Liouville derivative, the initial condition must be well taken. In this regard, before addressing our problem, we will go over some papers that deal with Riemann–Liouville fractional differential equations with delay. Benchohra et al. [4] studied the following model with a zero initial condition:

$$\begin{cases} {}^L D_0^{\alpha}[y(t) - g(t, y_t)] = f(t, y_t), \quad t \in (0, b], \ 0 < \alpha < 1, \\ y(t) = \varphi(t), \quad t \in (-\infty, 0], \end{cases} \tag{1.1}$$

where ${}^L D_0^{\alpha}$ is the standard Riemann–Liouville derivative, $\varphi \in \mathcal{B}$ with $\varphi(0) = 0$, where $\mathcal{B}$ is the phase space defined axiomatically by Hale and Kato. Several other papers have investigated the case where $y(0) = 0$ with a finite or infinite delay (see [1,2,4,10,11,13,16,21]). It is well known that when the initial condition is nonzero, i.e., $y(0) \neq 0$, the solution to the equations discussed in the preceding papers may not be bounded at some neighborhood of the initial point $t = 0$. As a result, some researchers have considered weighted fractional

Fatima Zahra Arioui (✉)
Laboratory of Statistics and Stochastic Processes, Djillali Liabes University, P.O. Box 89, 22000 Sidi-Bel-Abbes, Algeria
E-mail: arioufatimazahra@gmail.com; fatimazahra.arioui@univ-sba.dz

differential equations; for example, Dong [10] studied the weighted functional fractional differential equation with infinite delay of the form

$$\begin{cases} {}^{L}D_0^{\alpha} y(t) = f(t, \widetilde{y}_t), \quad t \in (0, b], \; 0 < \alpha \leqslant 1, \\ \widetilde{y}_0 = \varphi \in \mathcal{B}, \quad t \in (-\infty, 0], \end{cases} \tag{1.2}$$

where the existence and continuous dependence results of solutions are obtained, Dong et al. [11] investigated the model (1.2) in an abstract Banach space and obtained the existence and uniqueness results using the measure of non-compactness and the fixed point theorems, Abdo et al. [1] studied the weighted fractional neutral functional differential equation of the form

$$\begin{cases} {}^{L}D_0^{\alpha}[y(t) - g(t, \widetilde{y}_t)] = f(t, \widetilde{y}_t), \quad t \in (0, b], \; 0 < \alpha \leqslant 1, \\ \widetilde{y}_0 = \varphi, \quad t \in (-\infty, 0]. \end{cases} \tag{1.3}$$

The Banach and Schauder fixed point theorems were used to achieve the required results.

Since the presence of the stochastic term (random term) is important due to the possibility of unpredictability in the characteristics of natural systems, stochastic differential equations are more adaptable to real-world phenomena than deterministic ones. Fractional stochastic differential equations, in particular, are used to study the memory and genetic properties of stochastic systems and have become a hot topic in recent decades. Numerous academics have shown interest in these equations due to their applications in many fields of research, such as disease transmission, option pricing, quantitative finance, and so on (see [8,9,15,16,18,20]).

Inspired by the works mentioned above, we consider the weighted fractional stochastic integro-differential equation of the form

$$\begin{cases} {}^{L}D_0^{\alpha} u(t) = f(t, \widetilde{u}_t, \int_0^t a(t, s, \widetilde{u}_s)\mathrm{d}s) + g(t, \widetilde{u}_t)\frac{\mathrm{d}w(t)}{\mathrm{d}t}, \quad t \in (0, b], \\ \widetilde{u}_0 = \varphi \in \mathcal{B}, \quad t \in (-\infty, 0], \end{cases} \tag{1.4}$$

where ${}^{L}D_0^{\alpha}$ denotes the Riemann–Liouville derivative of order $1/2 < \alpha < 1$, $t \in (0, b]$, $u(\cdot)$ is a random variable takes its values in a separable real Hilbert space $\mathcal{H}$. f, a and g are appropriate functions to be specified later. $\mathcal{B}$ is the phase space of functions mapping from $(-\infty, 0]$ into $\mathcal{H}$ described axiomatically in Sect. 2. The notation $\widetilde{u}_t$ represents the function defined by $\widetilde{u}_t(\theta) = \widetilde{u}(t + \theta)$ for $\theta \in (-\infty, 0]$. $\widetilde{u}(t) = t^{1-\alpha}u(t)$ and $\widetilde{\varphi}(t) = \varphi(t)$ for $t \in (-\infty, 0]$.

The purpose of this study is to prove the existence of a solution to the model (1.4) with nonzero initial values using the Kuratowski measure of non-compactness and the Mönch fixed point theorem.

The paper is organized as follows: Section 2 introduces some fundamental notations and preliminaries. Section 3 provides some sufficient conditions for the existence of a mild solution for the model (1.4). Section 4 includes an example to demonstrate the obtained result.

2 Preliminaries

In this section, we introduce some basic definitions, lemmas, and notations that will be used to answer the existence problem for our model (1.4).

Let $\mathcal{H}$ and $\mathcal{K}$ two real separable Hilbert spaces, and $L(\mathcal{K}, \mathcal{H})$ the space of all linear and bounded operators from $\mathcal{K}$ to $\mathcal{H}$, we use the same notation $\|\cdot\|$ to denote the norms in $\mathcal{K}$, $\mathcal{H}$ and $L(\mathcal{K}, \mathcal{H})$, and we use $(\cdot, \cdot)$ to denote the inner product of $\mathcal{K}$ and $\mathcal{H}$. $L^1_{loc}((0, b), \mathcal{H})$ is the space of Bochner integrable functions from $(0, b)$ into $\mathcal{H}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0}, P)$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ satisfying the usual conditions. An $\mathcal{H}$-valued random variable is an $\mathcal{F}$-measurable function $u(t, \cdot) : \Omega \to \mathcal{H}$; in the rest of the paper, we write $u(t)$ instead of $u(t, \varpi)$ for all $\varpi \in \Omega$.

Let $\{w(t)\}_{t \geqslant 0}$ be a Q-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0}, P)$ with the covariance operator Q such that $Tr(Q) < \infty$. It is assumed that there exists a complete orthonormal system ς_k in $\mathcal{K}$, and positive real numbers λ_k such that $Q\varsigma_k = \lambda_k \varsigma_k$, $k = 1, 2, \ldots$, and a sequence of independent Brownian motions such that $(w(t), e) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}(\varsigma_k, e)\beta_k(t), \; e \in \mathcal{K}, \; t \geqslant 0$.

The space of all strongly measurable, square integrable, $\mathcal{H}$-valued random variables, denoted by $\mathcal{L}_2(\Omega,\mathcal{H})$,is a Banach space equipped with the norm $\|u(\cdot)\|_{\mathcal{L}_2(\Omega,\mathcal{H})} = \left(\mathbb{E}\,\|u(\cdot)\|^2\right)^{\frac{1}{2}}$, where $\mathbb{E}u = \int_\Omega u(w)\mathrm{d}P$. The family of all $\mathcal{F}_0$-measurable, $\mathcal{H}$-valued random variables $u(0)$ is denoted by $\mathcal{L}_2^0(\Omega,\mathcal{H})$. Let $C_{1-\alpha}((0,b];\mathcal{L}_2(\Omega,\mathcal{H}))$ be the Banach space of all continuous $\mathcal{F}_t$-adapted measurable process from $(0,b]$ into $\mathcal{L}_2(\Omega,\mathcal{H})$ such that $\lim_{t\to 0} t^{1-\alpha}u(t)$ exists, with the norm $\|u\|_{C_{1-\alpha}} = \sup_{t\in(0,b]}\left(\mathbb{E}\left\|t^{1-\alpha}u(t)\right\|^2\right)^{\frac{1}{2}}$.

In this paper, we consider the phase space $\mathcal{B}$, which fulfills the following fundamental axioms, which are similar to those introduced by Hale and Kato [12] and Hino et al. [14]:

Definition 2.1 [1] $\mathcal{B}$ is a linear space of $\mathcal{F}_0$-measurable functions mapping $(-\infty,0]$ into $\mathcal{H}$ endowed with the semi-norm $\|\cdot\|_{\mathcal{B}}$, which satisfies the following axioms:

(i) if $x:(-\infty,b]\to\mathcal{H}$ $(b>0)$ is continuous on $(0,b]$ and $x_0\in\mathcal{B}$, then for each $t\in(0,b]$ the following conditions hold
  (a) $x_t\in\mathcal{B}$;
  (b) $\|x(t)\|\leqslant L\,\|x_t\|_{\mathcal{B}}$, where $L>0$ is a constant;
  (c) $\|x_t\|_{\mathcal{B}}\leqslant K(t)\sup\{\|x(s)\|:0\leqslant s\leqslant t\}+N(t)\,\|x_0\|_{\mathcal{B}}$, where $K,N:[0,\infty)\to[0,\infty)$, K is continuous, N is locally bounded and K,N are independent of $x(\cdot)$. Denote

$$\widehat{K}=\sup\{K(t):t\in(0,b]\}\quad\text{and}\quad\widehat{N}=\sup\{N(t):t\in(0,b]\};$$

(ii) for the function $x(\cdot)$ in (i), the function $t\to x_t$ is continuous from $(0,b]$ into $\mathcal{B}$;
(iii) the space $\mathcal{B}$ is complete.

Now, consider the following space $\mathcal{P}$:

$$\mathcal{P}=\{u:(-\infty,b]\to\mathcal{H}:\ u|_{(-\infty,0]}\in\mathcal{B},\ u|_{(0,b]}\in C_{1-\alpha}((0,b];\mathcal{L}_2(\Omega,\mathcal{H}))\},$$

where $u|_{(0,b]}$ is the restriction of u over $(0,b]$.

Lemma 2.2 [20] *For $\sigma\in(0,1]$ and $0<a\leqslant b$, we have $|a^\sigma-b^\sigma|\leqslant(b-a)^\sigma$.*

Lemma 2.3 [5] *Let the space $\mathcal{M}(\mathcal{K},\mathcal{H})=\{\Phi(\cdot,\cdot)\ :\ \Phi$ is an $L(\mathcal{K},\mathcal{H})$-valued process on $[0,b]\times\Omega$ such that $\Phi(t)$ is $\mathcal{F}_t$-measurable for all $t\in[0,b]\}$. If $\Phi\in\mathcal{M}(\mathcal{K},\mathcal{H})$ with $\int_0^b\mathbb{E}\,\|\Phi(t)\|^2\,\mathrm{d}t<\infty$, then*

$$\mathbb{E}\left\|\int_0^b\Phi(t,w)\mathrm{d}w(t)\right\|^2\leqslant\mathrm{Tr}(Q)\int_0^b\mathbb{E}\|\Phi(t)\|^2\mathrm{d}t.$$

Let us now present some fundamental fractional calculus definitions and lemmas (see [7,17]).

Definition 2.4 [7] Let E be a Banach space. The Riemann–Liouville fractional integral of order $\alpha>0$ of a function $u:[a,b]\to E$ is defined by

$$I^\alpha_{0+}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)\mathrm{d}s,\quad t\in[a,b],$$

provided that the right hand-side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ denotes the Euler's Gamma function defined by $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}\mathrm{d}t$.

Definition 2.5 [7] Let $\alpha>0$ be fixed and $n=[\alpha]+1$. The Riemann–Liouville fractional derivative of order α of $u:[0,\infty)\to E$ at the point t is defined by

$${}^LD^\alpha_{0+}u(t)=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t(t-s)^{n-\alpha-1}u(s)\mathrm{d}s,\quad t\in[a,b],$$

provided the right side is point-wisely defined, where $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.6 [7] *Let $0 < \alpha < 1$, then the unique solutions to the equation $D_0^{\alpha}u(t) = 0$ are given by the formula*

$$u(t) = ct^{\alpha-1}$$

for $t > 0$, where $c \in \mathbb{R}$ is a constant, provided $u \in C(0, b] \cap L^1_{loc}(0, b]$. Further, if $f \in C(0, b] \cap L^1_{\text{loc}}(0, b]$ such that $D_0^{\alpha} f \in C((0, b]) \cap L^1_{\text{loc}}(0, b]$, then

$$I_0^{\alpha} D_0^{\alpha} f(t) = f(t) + ct^{\alpha-1}$$

for $t > 0$ and some constant $c \in \mathbb{R}$.

Next, we introduce the concept of a measure of non-compactness.

Definition 2.7 [3] The Kuratowski measure of non-compactness $\mu(\cdot)$ defined on a bounded subset A of a Banach space E is:

$$\mu(A) = \inf \left\{ \epsilon > 0 : A = \bigcup_{i=1}^{k} A_i, \text{ and } \operatorname{diam}(A_i) \leqslant \epsilon \right\}.$$

Definition 2.8 [3] Let μ denote the Kuratowski measure of non-compactness on the real Banach space X and $A, B \subseteq X$ be bounded. The following properties are satisfied:

(i) $\mu(A) = 0$ if and only if A is relatively compact,
(ii) $\mu(A) = \mu(\bar{A})$, where $\bar{A}$ denotes the closure of A,
(iii) $\mu(A \cup B) = \max(\mu(A), \mu(B))$,
(iv) If $A \subseteq B$, then $\mu(A) \leqslant \mu(B)$,
(v) $\mu(\lambda A) = |\lambda|\mu(A)$, with $\lambda \in \mathbb{R}$,
(vi) $\mu(A + B) \leqslant \mu(A) + \mu(B)$, where $A + B = \{x + y : x \in A, y \in B\}$,
(vii) $\mu(A + x_0) \leqslant \mu(A)$ for all $x_0 \in E$,
(viii) $\mu(\overline{\text{conv}}(A)) = \mu(A)$, where $\overline{\text{conv}}(A)$ is the closed convex hull of A.

Lemma 2.9 [3] *If $D \subset C([0, b], E)$ is bounded, then*

$$\mu(D(t)) \leqslant \mu_C(D)$$

for all $t \in [0, b]$, where $D(t) = \{u(t) : u \in D\} \subseteq E$. Furthermore, if D is equicontinuous on $[0, b]$, then $\mu(D(t))$ is continuous on $[0, b]$, and

$$\mu_C(D) = \sup_{t \in [0,b]} \mu(D(t)).$$

Lemma 2.10 [8] *Let $D = \{u_n\} \subset C([0, b], E)$ be a bounded and countable set. Then $\mu(D(t))$ is the Lebesgue integral on $[0, b]$, and*

$$\mu\left(\left\{\int_0^b u_n(t)\mathrm{d}t | n \in \mathbb{N}\right\}\right) \leqslant 2\int_0^b \mu\left(D(t)\right)\mathrm{d}t.$$

Theorem 2.11 [19] *Let D be a bounded, closed and convex subset of E and $x_0 \in D$. Let $A : D \to D$ be a continuous mapping. If the implication*

$$C = \overline{\text{conv}}(\{x_0 \cup A(C)\}) \Longrightarrow \mu(C) = 0$$

holds for every subset $C \subset D$, then A has a fixed point in D.

3 Existence result

The mild solution of model (1.4) can be defined as follows:

Definition 3.1 A stochastic process $u \in \mathcal{P}$ is called a mild solution of model (1.4) if

(i) u satisfies the equation ${}^{L}D_0^{\alpha}u(t) = f(t, \widetilde{u}_t, \int_0^t a(t, s, \widetilde{u}_s)\mathrm{d}s) + g(t, \widetilde{u}_t)\frac{\mathrm{d}w(t)}{\mathrm{d}t}$, $t \in (0, b]$, with initial condition $\widetilde{u}_0 = \varphi \in \mathcal{B}$;
(ii) $u(t)$ is measurable and $\mathcal{F}_t$-adapted for each $t \in (0, b]$.

Using Lemma 2.6, the solution u to our model (1.4) satisfies the following stochastic integral equation

$$u(t) = t^{\alpha-1}\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, \widetilde{u}_s, \int_0^s a(s, \tau, \widetilde{u}_\tau)\mathrm{d}\tau)\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, \widetilde{u}_s)\mathrm{d}w(s).$$

Before demonstrating our existence result for the model (1.4), we will make a list of hypotheses that will be enforced in our main theorem.
(H1) The function $f : (0, b] \times \mathcal{B} \times \mathcal{H} \to \mathcal{H}$ satisfies the following conditions:

(i) $f(t, \cdot, \cdot) : \mathcal{B} \times \mathcal{H} \to \mathcal{H}$ is continuous for each $t \in (0, b]$, and for every $(\phi, \psi) \in \mathcal{B} \times \mathcal{H}$, the function $t \to f(t, \phi, \psi)$ is strongly measurable;
(ii) there exists $p_f \in L^1((0, b], [0, \infty))$ and a continuous non-decreasing function $\theta_f : [0, \infty) \to [0, \infty)$ such that for all $(t, x, y) \in (0, b] \times \mathcal{B} \times \mathcal{H}$, we have

$$\mathbb{E}\,\|f(t, x, y)\|^2 \leqslant p_f(t)\theta_f(\|x\|_{\mathcal{B}}^2 + \mathbb{E}\,\|y\|^2);$$

(iii) there exists a function $\zeta_f \in L^{\frac{1}{\alpha_1}}((0, b], [0, \infty))$, $\alpha_1 \in (0, \alpha)$ such that for each bounded set $D_1 \in \mathcal{B}$ and $D_2 \in \mathcal{H}$,

$$\mu\left(f(t, D_1, D_2)\right) \leqslant \zeta_f(t)\left(\sup_{-\infty<\theta\leqslant 0} \mu(D_1(\theta)) + \mu(D_2)\right).$$

(H2) The function $g : (0, b] \times \mathcal{B} \to L(\mathcal{K}, \mathcal{H})$ satisfies the conditions:

(i) $g(t, \cdot) : \mathcal{B} \to L(\mathcal{K}, \mathcal{H})$ is continuous for each $t \in (0, b]$, and for every $\phi \in \mathcal{B}$, the function $t \to g(t, \phi)$ is strongly measurable;
(ii) there exists $p_g \in L^{\frac{1}{\alpha_2}}((0, b], [0, \infty))$, $\alpha_2 \in (0, 2\alpha - 1)$ and a continuous non-decreasing function $\theta_g : [0, \infty) \to [0, \infty)$ such that for all $(t, x) \in (0, b] \times \mathcal{B}$, we have

$$\mathbb{E}\,\|g(t, x)\|^2 \leqslant p_g(t)\theta_g(\|x\|_{\mathcal{B}}^2);$$

(iii) there exists a function $\zeta_g \in L^{\frac{1}{\alpha_3}}((0, b], [0, \infty))$, $\alpha_3 \in (0, \frac{2\alpha-1}{2})$ such that for each bounded set $D_3 \in \mathcal{B}$,

$$\mu\left(g(t, D_3)\right) \leqslant \zeta_g(t) \sup_{-\infty<\theta\leqslant 0} \mu(D_3(\theta)).$$

(H3) The function $a : (0, b] \times [0, b] \times \mathcal{B} \to \mathcal{H}$ satisfies:

(i) for each $(t, s) \in (0, b] \times [0, b]$, the function $a(t, s, \cdot) : \mathcal{B} \to \mathcal{H}$ is continuous and for each $x \in \mathcal{B}$, the function $(t, s) \to a(t, s, x)$ is strongly measurable;
(ii) there exists $L_1 > 0$ such that

$$\mathbb{E}\left\|\int_0^t a(t, s, x)\mathrm{d}s\right\|^2 \leqslant L_1(1 + \|x\|_{\mathcal{B}}^2);$$

(iii) there exists $L_2 > 0$ such that for each bounded set $D_4 \in \mathcal{B}$,

$$\mu\left(a(t, s, D_4)\right) \leqslant L_2 \sup_{-\infty<\theta\leqslant 0} \mu(D_4(\theta)).$$

Theorem 3.2 *Assume that the hypotheses (H1)–(H3) hold, then the model* (1.4) *has a solution on* $(-\infty, b]$ *if the following condition holds:*

$$l_0 = \frac{2b^{1-\alpha_1}(1+2bL_2)\,\left\|\zeta_f\right\|_{L^{\frac{1}{\alpha_1}}}(1-\alpha_1)^{1-\alpha_1}}{\Gamma(\alpha)(\alpha-\alpha_1)^{1-\alpha_1}} + \frac{b^{\frac{1-2\alpha_3}{2}}\sqrt{Tr(Q)}\,\left\|\zeta_g\right\|_{L^{\frac{1}{\alpha_3}}}(1-2\alpha_3)^{\frac{1-2\alpha_3}{2}}}{\Gamma(\alpha)(2\alpha-1-2\alpha_3)^{\frac{1-2\alpha_3}{2}}} < 1. \tag{3.1}$$

Proof According to the Definition 3.1, u is a mild solution to (1.4) if

$$u(t) = \begin{cases} t^{\alpha-1}\varphi(0) + \frac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(s, \widetilde{u_s}, \int_0^s a(s,\tau,\widetilde{u_\tau})\mathrm{d}\tau)ds & \\ +\frac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} g(s,\widetilde{u_s})\mathrm{d}w(s), & t\in(0,b], \\ \varphi(t), & t\in(-\infty,0]. \end{cases}$$

Let $\widetilde{\varphi}$ be a function defined by

$$\widetilde{\varphi}(t) = \begin{cases} 0, & t\in(0,b], \\ \varphi(t), & t\in(-\infty,0]. \end{cases}$$

We extend $\widetilde{z}$ to $(-\infty, b]$, which is defined as

$$\widetilde{z}(t) = \begin{cases} t^{1-\alpha}z(t), & t\in(0,b], \\ 0, & t\in(-\infty,0]. \end{cases}$$

It is obvious that if u satisfies the integral equation

$$\begin{aligned} u(t) = t^{\alpha-1}\varphi(0) &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\widetilde{u_s},\int_0^s a(s,\tau,\widetilde{u_\tau})\mathrm{d}\tau)\mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,\widetilde{u_s})\mathrm{d}w(s), \quad t\in(0,b], \end{aligned}$$

then we can decompose $u(\cdot)$ as $u(t) = z(t)+\varphi(t)$, which implies that $\widetilde{u}(t) = \widetilde{z}(t)+\widetilde{\varphi}(t)$, and thus $\widetilde{u}_t = \widetilde{z}_t+\widetilde{\varphi}_t$, where $z(t)$ satisfies,

$$\begin{aligned} z(t) = t^{\alpha-1}\varphi(0) &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\widetilde{z_s}+\widetilde{\varphi_s},\int_0^s a(s,\tau,\widetilde{z_\tau}+\widetilde{\varphi_\tau})\mathrm{d}\tau)\mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,\widetilde{z_s}+\widetilde{\varphi_s})\mathrm{d}w(s), \ t>0. \end{aligned}$$

Set $\mathcal{P}_0 = \{z : (-\infty,b]\to\mathcal{H},\, z|_{(0,b]}\in C_{1-\alpha}((0,b];\mathcal{H}),\, z_0 = 0\}$. For $z\in\mathcal{P}_0$, and let $\|z\|_{\mathcal{P}_0}$ be the seminorm in $\mathcal{P}_0$ defined by

$$\|z\|_{\mathcal{P}_0} = \|z\|_{C_{1-\alpha}} + \|z_0\|_{\mathcal{B}} = \sup_{t\in(0,b]}\left(\mathbb{E}\left\|t^{1-\alpha}z(t)\right\|^2\right)^{\frac{1}{2}}.$$

Then $(\mathcal{P}_0, \|\cdot\|_{\mathcal{P}_0})$ is a Banach space. Let B_q defined by $B_q = \{z\in\mathcal{P}_0\,;\,\|z\|^2_{\mathcal{P}_0}\leqslant q\}$. The set B_q is clearly bounded, closed, and convex.

Choose q satisfies

$$q \geqslant 3\mathbb{E}\|\varphi(0)\|^2 + \frac{3b}{(2\alpha-1)\Gamma^2(\alpha)}\theta_f(\widetilde{M_1})\left\|p_f\right\|_{L^1} + \frac{3b^{1-\alpha_2}\mathrm{Tr}(Q)}{\Gamma^2(\alpha)}\theta_g\left(\widetilde{M_2}\right)\left(\frac{1-\alpha_2}{2\alpha-1-\alpha_2}\right)^{1-\alpha_2}\left\|p_g\right\|_{L^{\frac{1}{\alpha_2}}},$$

where

$$\begin{aligned} \widetilde{M_1} &= 2\widehat{K}^2q + 2\widehat{N}\,\|\varphi\|^2_{\mathcal{B}} + L_1(1+2\widehat{K}^2q+2\widehat{N}\,\|\varphi\|^2_{\mathcal{B}}), \\ \widetilde{M_2} &= 2\widehat{K}^2q + 2\widehat{N}\,\|\varphi\|^2_{\mathcal{B}}. \end{aligned}$$

Define the operator $\Psi : \mathcal{P}_0 \to \mathcal{P}_0$ as follows:

$$(\Psi z)(t) = t^{\alpha-1}\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, \widetilde{z_s} + \widetilde{\varphi_s}, \int_0^s a(s,\tau,\widetilde{z_\tau} + \widetilde{\varphi_\tau})\mathrm{d}\tau)\mathrm{d}s$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, \widetilde{z_s} + \widetilde{\varphi_s})\mathrm{d}w(s).$$

We will prove by Mönch's fixed point theorem that Ψ has a fixed point.

Step 1: Ψ maps B_q into itself.

Using Hölder's inequality and Lemma 2.3, we obtain for $t \in (0, b]$,

$$\begin{aligned}
&\mathbb{E}\left\|t^{1-\alpha}(\Psi z)(t)\right\|^2 \\
&\leqslant 3\mathbb{E}\|\varphi(0)\|^2 + 3\mathbb{E}\Big\|\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, \widetilde{z_s} + \widetilde{\varphi_s}, \int_0^s a(s,\tau,\widetilde{z_\tau} + \widetilde{\varphi_\tau})\mathrm{d}\tau)\mathrm{d}s\Big\|^2 \\
&\quad +3\mathbb{E}\Big\|\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, \widetilde{z_s} + \widetilde{\varphi_s})\mathrm{d}w(s)\Big\|^2 \\
&\leqslant 3\mathbb{E}\|\varphi(0)\|^2 + \frac{3t^{2(1-\alpha)}}{\Gamma^2(\alpha)}\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1} f(s, \widetilde{z_s} + \widetilde{\varphi_s}, \int_0^s a(s,\tau,\widetilde{z_\tau} + \widetilde{\varphi_\tau})\mathrm{d}\tau)\mathrm{d}s\Big\|^2 \\
&\quad +\frac{3t^{2(1-\alpha)}}{\Gamma^2(\alpha)}\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1} g(s, \widetilde{z_s} + \widetilde{\varphi_s})\mathrm{d}w(s)\Big\|^2 \\
&\leqslant 3\mathbb{E}\|\varphi(0)\|^2 + \frac{3t^{2(1-\alpha)}}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2(\alpha-1)}\mathrm{d}s\int_0^t \mathbb{E}\Big\|f(s, \widetilde{z_s} + \widetilde{\varphi_s}, \int_0^s a(s,\tau,\widetilde{z_\tau} + \widetilde{\varphi_\tau})\mathrm{d}\tau)\Big\|^2\mathrm{d}s \\
&\quad +\frac{3t^{2(1-\alpha)}Tr(Q)}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2(\alpha-1)}\mathbb{E}\|g(s, \widetilde{z_s} + \widetilde{\varphi_s})\|^2\,\mathrm{d}s.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\|\widetilde{z_s} + \widetilde{\varphi_s}\|^2_{\mathcal{B}} &\leqslant 2\|\widetilde{z_s}\|^2_{\mathcal{B}} + 2\|\widetilde{\varphi_s}\|^2_{\mathcal{B}} \\
&\leqslant 2\widehat{K}^2\sup_{\tau\in[0,s]}\mathbb{E}\|\widetilde{z}(\tau)\|^2 + 2\widehat{N}^2\|\widetilde{z}_0\|^2_{\mathcal{B}} + 2\widehat{K}^2\sup_{\tau\in[0,s]}\mathbb{E}\|\widetilde{\varphi}(\tau)\|^2 + 2\widehat{N}\|\widetilde{\varphi}_0\|^2_{\mathcal{B}} \\
&\leqslant 2\widehat{K}^2\sup_{\tau\in[0,s]}\mathbb{E}\|\widetilde{z}(\tau)\|^2 + 2\widehat{N}\|\varphi\|^2_{\mathcal{B}} \\
&\leqslant 2\widehat{K}^2 q + 2\widehat{N}\|\varphi\|^2_{\mathcal{B}}.
\end{aligned} \tag{3.2}$$

As a result of hypotheses (H1, H2, H3)-(ii) and inequality (3.2), we get

$$\begin{aligned}
&\mathbb{E}\left\|t^{1-\alpha}(\Psi z)(t)\right\|^2 \\
&\leqslant 3\mathbb{E}\|\varphi(0)\|^2 + \frac{3t}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t p_f(s)\theta_f\Big(\|\widetilde{z_s} + \widetilde{\varphi_s}\|^2_{\mathcal{B}} + \mathbb{E}\Big\|\int_0^s a(s,\tau,\widetilde{z_s} + \widetilde{\varphi_s})\mathrm{d}\tau\Big\|^2\Big)\mathrm{d}s \\
&\quad +\frac{3t^{2(1-\alpha)}Tr(Q)}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2(\alpha-1)}p_g(s)\theta_g\left(\|\widetilde{z_s} + \widetilde{\varphi_s}\|^2_{\mathcal{B}}\right)\mathrm{d}s \\
&\leqslant 3\mathbb{E}\|\varphi(0)\|^2 + \frac{3t}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t p_f(s)\theta_f\big(2\widehat{K}^2 q + 2\widehat{N}\|\varphi\|^2_{\mathcal{B}} + L_1(1 + 2\widehat{K}^2 q + 2\widehat{N}\|\varphi\|^2_{\mathcal{B}})\big)\mathrm{d}s \\
&\quad +\frac{3t^{2(1-\alpha)}Tr(Q)}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2(\alpha-1)}p_g(s)\theta_g\left(2\widehat{K}^2 q + 2\widehat{N}\|\varphi\|^2_{\mathcal{B}}\right)\mathrm{d}s \\
&\leqslant 3\mathbb{E}\|\varphi(0)\|^2 + \frac{3t}{(2\alpha-1)\Gamma^2(\alpha)}\theta_f(\widetilde{M_1})\left(\int_0^t p_f(s)\mathrm{d}s\right) \\
&\quad +\frac{3t^{2(1-\alpha)}Tr(Q)}{\Gamma^2(\alpha)}\theta_g\left(\widetilde{M_2}\right)\left(\int_0^t (t-s)^{\frac{2(\alpha-1)}{1-\alpha_2}}\mathrm{d}s\right)^{1-\alpha_2}\left(\int_0^t (p_g(s))^{\frac{1}{\alpha_2}}\mathrm{d}s\right)^{\alpha_2}
\end{aligned}$$

$$\leqslant 3\mathbb{E}\,\|\varphi(0)\|^2 + \frac{3t}{(2\alpha-1)\Gamma^2(\alpha)}\theta_f(\widetilde{M}_1)\,\|p_f\|_{L^1} + \frac{3t^{2(1-\alpha)}Tr(Q)}{\Gamma^2(\alpha)}\theta_g(\widetilde{M}_2)\left(\frac{1-\alpha_2}{2\alpha-1-\alpha_2}\right)^{1-\alpha_2} t^{2\alpha-1-\alpha_2}\,\|p_g\|_{L^{\frac{1}{\alpha_2}}}.$$

Finally, we conclude,

$$\begin{aligned}\|\Psi z\|^2_{\mathcal{P}_0} &= \sup_{t\in(0,b]}\mathbb{E}\left\|t^{1-\alpha}(\Psi z)(t)\right\|^2\\ &\leqslant 3\mathbb{E}\,\|\varphi(0)\|^2 + \frac{3b}{(2\alpha-1)\Gamma^2(\alpha)}\theta_f(\widetilde{M}_1)\,\|p_f\|_{L^1}\\ &\quad + \frac{3b^{1-\alpha_2}Tr(Q)}{\Gamma^2(\alpha)}\theta_g\left(\widetilde{M}_2\right)\left(\frac{1-\alpha_2}{2\alpha-1-\alpha_2}\right)^{1-\alpha_2}\|p_g\|_{L^{\frac{1}{\alpha_2}}}\\ &\leqslant q.\end{aligned}$$

Step 2: $\Psi(B_q)$ is equicontinuous.
Let $z \in B_q$ and $t_1, t_2 \in (0, b]$ such that $0 < t_1 < t_2 \leqslant b$, then we have

$$\begin{aligned}&\mathbb{E}\left\|t_2^{1-\alpha}(\Psi z)(t_2) - t_1^{1-\alpha}(\Psi z)(t_1)\right\|^2\\ &\leqslant 2\mathbb{E}\left\|\Big(\frac{t_2^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1} - \frac{t_1^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\Big)f(s,\widetilde{z_s}+\widetilde{\varphi_s},\int_0^s a(s,\tau,\widetilde{z_\tau}+\widetilde{\varphi_\tau})d\tau)\mathrm{d}s\right\|^2\\ &\quad +2\mathbb{E}\left\|\Big(\frac{t_2^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1} - \frac{t_1^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\Big)g(s,\widetilde{z_s}+\widetilde{\varphi_s})\mathrm{d}w(s)\right\|^2\\ &\leqslant 4\mathbb{E}\left\|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left(t_2^{1-\alpha}(t_2-s)^{\alpha-1} - t_1^{1-\alpha}(t_1-s)^{\alpha-1}\right)f(s,\widetilde{z_s}+\widetilde{\varphi_s},\int_0^s a(s,\tau,\widetilde{z_\tau}+\widetilde{\varphi_\tau})\mathrm{d}\tau)\mathrm{d}s\right\|^2\\ &\quad +4\mathbb{E}\left\|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}t_2^{1-\alpha}(t_2-s)^{\alpha-1}f(s,\widetilde{z_s}+\widetilde{\varphi_s},\int_0^s a(s,\tau,\widetilde{z_\tau}+\widetilde{\varphi_\tau})\mathrm{d}\tau)\mathrm{d}s\right\|^2\\ &\quad +4\mathbb{E}\left\|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left(t_2^{1-\alpha}(t_2-s)^{\alpha-1} - t_1^{1-\alpha}(t_1-s)^{\alpha-1}\right)g(s,\widetilde{z_s}+\widetilde{\varphi_s})\mathrm{d}w(s)\right\|^2\\ &\quad +4\mathbb{E}\left\|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}t_2^{1-\alpha}(t_2-s)^{\alpha-1}g(s,\widetilde{z_s}+\widetilde{\varphi_s})\mathrm{d}w(s)\right\|^2\\ &\leqslant \sum_{i=1}^4 I_i. \end{aligned}\tag{3.3}$$

First, we calculate I_1 and I_2 using hypotheses (H1, H3)-(ii), inequality (3.2), and Hölder's inequality.

$$\begin{aligned}I_1 &= 4\mathbb{E}\left\|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left(t_2^{1-\alpha}(t_2-s)^{\alpha-1} - t_1^{1-\alpha}(t_1-s)^{\alpha-1}\right)f(s,\widetilde{z_s}+\widetilde{\varphi_s},\int_0^s a(s,\tau,\widetilde{z_\tau}+\widetilde{\varphi_\tau})\mathrm{d}\tau)\mathrm{d}s\right\|^2\\ &\leqslant \frac{4}{\Gamma^2(\alpha)}\int_0^{t_1}\left|t_2^{1-\alpha}(t_2-s)^{\alpha-1} - t_1^{1-\alpha}(t_1-s)^{\alpha-1}\right|^2\mathrm{d}s\int_0^{t_1}\mathbb{E}\left\|f(s,\widetilde{z_s}+\widetilde{\varphi_s},\int_0^s a(s,\tau,\widetilde{z_\tau}+\widetilde{\varphi_\tau})\mathrm{d}\tau)\right\|^2\mathrm{d}s\\ &\leqslant \frac{4}{\Gamma^2(\alpha)}\int_0^{t_1}\left|t_2^{1-\alpha}\left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right) + (t_1-s)^{\alpha-1}\left(t_2^{1-\alpha}-t_1^{1-\alpha}\right)\right|^2\mathrm{d}s\\ &\quad\times\int_0^{t_1}\mathbb{E}\left\|f(s,\widetilde{z_s}+\widetilde{\varphi_s},\int_0^s a(s,\tau,\widetilde{z_\tau}+\widetilde{\varphi_\tau})\mathrm{d}\tau)\right\|^2\mathrm{d}s\\ &\leqslant \frac{8}{\Gamma^2(\alpha)}\Big(\int_0^{t_1}t_2^{2(1-\alpha)}\left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right)^2\mathrm{d}s + \left(t_2^{1-\alpha}-t_1^{1-\alpha}\right)^2\int_0^{t_1}(t_1-s)^{2(\alpha-1)}\mathrm{d}s\Big)\\ &\quad\times\int_0^{t_1}\mathbb{E}\left\|f(s,\widetilde{z_s}+\widetilde{\varphi_s},\int_0^s a(s,\tau,\widetilde{z_\tau}+\widetilde{\varphi_\tau})\mathrm{d}\tau)\right\|^2\mathrm{d}s\end{aligned}$$

$$\leqslant \frac{8}{\Gamma^2(\alpha)}\Big(t_2^{2(1-\alpha)}\int_0^{t_1}(t_1-s)^{2(\alpha-1)}-(t_2-s)^{2(\alpha-1)}\mathrm{d}s+\left(t_2^{1-\alpha}-t_1^{1-\alpha}\right)^2\int_0^{t_1}(t_1-s)^{2(\alpha-1)}\mathrm{d}s\Big)$$
$$\times\int_0^{t_1}p_f(s)\theta_f(\widetilde{M}_1)\mathrm{d}s$$
$$\leqslant \frac{8}{\Gamma^2(\alpha)}\left(\frac{t_2^{2(1-\alpha)}\left(t_1^{2\alpha-1}+(t_2-t_1)^{2\alpha-1}-t_2^{2\alpha-1}\right)}{2\alpha-1}+\frac{t_1^{2\alpha-1}(t_2^{1-\alpha}-t_1^{1-\alpha})^2}{2\alpha-1}\right)\|p_f\|_{L^1}\theta_f(\widetilde{M}_1)$$
$$\leqslant \frac{8\|p_f\|_{L^1}\theta_f(\widetilde{M}_1)}{(2\alpha-1)\Gamma^2(\alpha)}\left(b^{2(1-\alpha)}\left(t_1^{2\alpha-1}+(t_2-t_1)^{2\alpha-1}-t_2^{2\alpha-1}\right)+b^{2\alpha-1}(t_2^{1-\alpha}-t_1^{1-\alpha})^2\right),$$

and

$$I_2=4\mathbb{E}\Big\|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}t_2^{1-\alpha}(t_2-s)^{\alpha-1}f(s,\widetilde{z}_s+\widetilde{\varphi}_s,\int_0^s a(s,\tau,\widetilde{z}_\tau+\widetilde{\varphi}_\tau)\mathrm{d}\tau)\mathrm{d}s\Big\|^2$$
$$\leqslant \frac{4t_2^{2(1-\alpha)}}{\Gamma^2(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{2(\alpha-1)}\mathrm{d}s\int_{t_1}^{t_2}\mathbb{E}\Big\|f(s,\widetilde{z}_s+\widetilde{\varphi}_s,\int_0^s a(s,\tau,\widetilde{z}_\tau+\widetilde{\varphi}_\tau)\mathrm{d}\tau)\Big\|^2\mathrm{d}s$$
$$\leqslant \frac{4t_2^{2(1-\alpha)}(t_2-t_1)^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\|p_f\|_{L^1}\theta_f(\widetilde{M}_1)$$
$$\leqslant \frac{4b^{2(1-\alpha)}(t_2-t_1)^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\|p_f\|_{L^1}\theta_f(\widetilde{M}_1).$$

Second, we calculate I_3 and I_4 using hypothesis (H2)-(ii), Lemma 2.3, and inequality (3.2).

$$I_3=4\mathbb{E}\Big\|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left(t_2^{1-\alpha}(t_2-s)^{\alpha-1}-t_1^{1-\alpha}(t_1-s)^{\alpha-1}\right)g(s,\widetilde{z}_s+\widetilde{\varphi}_s)\mathrm{d}w(s)\Big\|^2$$
$$\leqslant \frac{4Tr(Q)}{\Gamma^2(\alpha)}\int_0^{t_1}\left|t_2^{1-\alpha}(t_2-s)^{\alpha-1}-t_1^{1-\alpha}(t_1-s)^{\alpha-1}\right|^2\mathbb{E}\|g(s,\widetilde{z}_s+\widetilde{\varphi}_s)\|^2\mathrm{d}s$$
$$\leqslant \frac{4Tr(Q)}{\Gamma^2(\alpha)}\int_0^{t_1}\left|t_2^{1-\alpha}\left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right)+(t_1-s)^{\alpha-1}\left(t_2^{1-\alpha}-t_1^{1-\alpha}\right)\right|^2\mathbb{E}\|g(s,\widetilde{z}_s+\widetilde{\varphi}_s)\|^2\mathrm{d}s$$
$$\leqslant \frac{8\mathrm{Tr}(Q)t_2^{2(1-\alpha)}}{\Gamma^2(\alpha)}\int_0^{t_1}\left((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right)^2\mathbb{E}\|g(s,\widetilde{z}_s+\widetilde{\varphi}_s)\|^2\mathrm{d}s$$
$$+\frac{8\mathrm{Tr}(Q)\left(t_2^{1-\alpha}-t_1^{1-\alpha}\right)^2}{\Gamma^2(\alpha)}\int_0^{t_1}(t_1-s)^{2(\alpha-1)}\mathbb{E}\|g(s,\widetilde{z}_s+\widetilde{\varphi}_s)\|^2\mathrm{d}s$$
$$\leqslant \frac{8\mathrm{Tr}(Q)t_2^{2(1-\alpha)}}{\Gamma^2(\alpha)}\left(\frac{1-\alpha_2}{2\alpha-1-\alpha_2}\right)^{1-\alpha_2}\left(t_1^{2\alpha-1-\alpha_2}+(t_2-t_1)^{2\alpha-1-\alpha_2}-t_2^{2\alpha-1-\alpha_2}\right)\times\|p_g\|_{L^{\frac{1}{\alpha_2}}}\theta_g(\widetilde{M}_2)$$
$$+\frac{8\mathrm{Tr}(Q)\left(t_2^{1-\alpha}-t_1^{1-\alpha}\right)^2}{\Gamma^2(\alpha)}\left(\frac{1-\alpha_2}{2\alpha-1-\alpha_2}\right)^{1-\alpha_2}t_1^{2\alpha-1-\alpha_2}\|p_g\|_{L^{\frac{1}{\alpha_2}}}\theta_g(\widetilde{M}_2)$$
$$\leqslant \frac{8\mathrm{Tr}(Q)(1-\alpha_2)^{1-\alpha_2}\|p_g\|_{L^{\frac{1}{\alpha_3}}}\theta_g(\widetilde{M}_2)}{\Gamma^2(\alpha)(2\alpha-1-\alpha_2)^{1-\alpha_2}}$$
$$\times\Big(b^{2(1-\alpha)}\Big(t_1^{2\alpha-1-\alpha_2}+(t_2-t_1)^{2\alpha-1-\alpha_2}-t_2^{2\alpha-1-\alpha_2}\Big)+b^{2\alpha-1-\alpha_2}\left(t_2^{1-\alpha}-t_1^{1-\alpha}\right)^2\Big),$$

and

$$I_4=4\mathbb{E}\Big\|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}t_2^{1-\alpha}(t_2-s)^{\alpha-1}g(s,\widetilde{z}_s+\widetilde{\varphi}_s)\mathrm{d}w(s)\Big\|^2$$
$$\leqslant \frac{4Tr(Q)t_2^{2(1-\alpha)}}{\Gamma^2(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{2(\alpha-1)}\mathbb{E}\|g(s,\widetilde{z}_s+\widetilde{\varphi}_s)\|^2\mathrm{d}s$$

$$\leqslant \frac{4\mathrm{Tr}(Q)t_2^{2(1-\alpha)}}{\Gamma^2(\alpha)}\left(\frac{1-\alpha_2}{2\alpha-1-\alpha_2}\right)^{1-\alpha_2}(t_2-t_1)^{2\alpha-1-\alpha_2}\left\|p_g\right\|_{L^{\frac{1}{\alpha_2}}}\theta_g(\widetilde{M}_2)$$
$$\leqslant \frac{4\mathrm{Tr}(Q)b^{2(1-\alpha)}(1-\alpha_2)^{1-\alpha_2}(t_2-t_1)^{2\alpha-1-\alpha_2}}{\Gamma^2(\alpha)(2\alpha-1-\alpha_2)^{1-\alpha_2}}\left\|p_g\right\|_{L^{\frac{1}{\alpha_2}}}\theta_g(\widetilde{M}_2).$$

Substituting $(I_1)-(I_4)$ into (3.3), we get

$$\begin{aligned}
&\mathbb{E}\left\|t_2^{1-\alpha}(\Psi z)(t_2)-t_1^{1-\alpha}(\Psi z)(t_1)\right\|^2\\
&\leqslant \frac{8\left\|p_f\right\|_{L^1}\theta_f(\widetilde{M}_1)}{(2\alpha-1)\Gamma^2(\alpha)}\left(b^{2(1-\alpha)}\left(t_1^{2\alpha-1}+(t_2-t_1)^{2\alpha-1}-t_2^{2\alpha-1}\right)+b^{2\alpha-1}(t_2^{1-\alpha}-t_1^{1-\alpha})^2\right)\\
&\quad+\frac{4b^{2(1-\alpha)}(t_2-t_1)^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\left\|p_f\right\|_{L^1}\theta_f(\widetilde{M}_1)\\
&\quad+\frac{8\mathrm{Tr}(Q)(1-\alpha_2)^{1-\alpha_2}\left\|p_g\right\|_{L^{\frac{1}{\alpha_3}}}\theta_g(\widetilde{M}_2)}{\Gamma^2(\alpha)(2\alpha-1-\alpha_2)^{1-\alpha_2}}\\
&\quad\times\left(b^{2(1-\alpha)}\left(t_1^{2\alpha-1-\alpha_2}+(t_2-t_1)^{2\alpha-1-\alpha_2}-t_2^{2\alpha-1-\alpha_2}\right)+b^{2\alpha-1-\alpha_2}\left(t_2^{1-\alpha}-t_1^{1-\alpha}\right)^2\right)\\
&\quad+\frac{4\mathrm{Tr}(Q)b^{2(1-\alpha)}(1-\alpha_2)^{1-\alpha_2}(t_2-t_1)^{2\alpha-1-\alpha_2}}{\Gamma^2(\alpha)(2\alpha-1-\alpha_2)^{1-\alpha_2}}\left\|p_g\right\|_{L^{\frac{1}{\alpha_2}}}\theta_g(\widetilde{M}_2).
\end{aligned}$$

As a result, according to Lemma 2.2, the right hand side of the previous inequality tends to zero as $t_1\to t_2$. Thus, $\Psi(B_q)$ is equicontinuous.

Step 3: Ψ is continuous.

Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence such that $\{z_n\}$ converges to z in B_q as $n\to\infty$. It is clear from axiom (i) in the Definition 2.1 that $\{z_{ns}\}_{n\in\mathbb{N}}\to z_s$ uniformly for $s\in(0,b]$ as $n\to\infty$, and then $\{\widetilde{z}_{ns}\}_{n\in\mathbb{N}}\to\widetilde{z}_s$ as $n\to\infty$, we can deduce from hypotheses (H1, H2, H3)-(i),

$$\begin{aligned}
&a(s,\tau,\widetilde{z}_{n\tau}+\widetilde{\varphi}_\tau)\to a(s,\tau,\widetilde{z}_\tau+\widetilde{\varphi}_\tau),\ \text{as } n\to\infty,\\
&f\left(s,\widetilde{z}_{ns}+\widetilde{\varphi}_s,\int_0^s a(s,\tau,\widetilde{z}_{n\tau}+\widetilde{\varphi}_\tau)\mathrm{d}\tau\right)\to f\left(s,\widetilde{z}_s+\widetilde{\varphi}_s,\int_0^s a(s,\tau,\widetilde{z}_\tau+\widetilde{\varphi}_\tau)\mathrm{d}\tau\right),\ \text{as } n\to\infty,\\
&g(s,\widetilde{z}_{ns}+\widetilde{\varphi}_s)\to g(s,\widetilde{z}_s+\widetilde{\varphi}_s),\ \text{as } n\to\infty.
\end{aligned}$$

From hypotheses (H1, H2, H3)-(ii), Hölder's inequality, Lemma 2.3, and the dominated convergence theorem, we have

$$\begin{aligned}
&\mathbb{E}\left\|t^{1-\alpha}(\Psi z_n)(t)-t^{1-\alpha}(\Psi z)(t)\right\|^2\\
&\leqslant 2\mathbb{E}\left\|\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left(f\left(s,\widetilde{z}_{ns}+\widetilde{\varphi}_s,\int_0^s a(s,\tau,\widetilde{z}_\tau+\widetilde{\varphi}_\tau)\mathrm{d}\tau\right)-f\left(s,\widetilde{z}_s+\widetilde{\varphi}_s,\int_0^s a(s,\tau,\widetilde{z}_\tau+\widetilde{\varphi}_\tau)\mathrm{d}\tau\right)\right)\mathrm{d}s\right\|^2\\
&\quad+2\mathbb{E}\left\|\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left(g(s,\widetilde{z}_{ns}+\widetilde{\varphi}_s)-g(s,\widetilde{z}_s+\widetilde{\varphi}_s)\right)\mathrm{d}w(s)\right\|^2\\
&\leqslant\frac{b}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t\mathbb{E}\left\|f\left(s,\widetilde{z}_{ns}+\widetilde{\varphi}_s,\int_0^s a(s,\tau,\widetilde{z}_{n\tau}+\widetilde{\varphi}_\tau)\mathrm{d}\tau\right)-f\left(s,\widetilde{z}_s+\widetilde{\varphi}_s,\int_0^s a(s,\tau,\widetilde{z}_\tau+\widetilde{\varphi}_\tau)\mathrm{d}\tau\right)\right\|^2\mathrm{d}s\\
&\quad+\frac{2Tr(Q)t^{2(1-\alpha)}}{\Gamma^2(\alpha)}\int_0^t(t-s)^{2(\alpha-1)}\mathbb{E}\left\|g(s,\widetilde{z}_{ns}+\widetilde{\varphi}_s)-g(s,\widetilde{z}_s+\widetilde{\varphi}_s\right\|^2\mathrm{d}s\longrightarrow 0\quad\text{as}\quad n\longrightarrow\infty.
\end{aligned}$$

As a result, Ψ is continuous.

Step 4: Ψ satisfies Mönch's condition.

Assume that $D\subseteq B_q$ be countable and $D\subseteq\overline{conv}(\{0\}\cup\Psi(D))$. We prove that D is relatively compact, that is, $\mu(D)=0$.

From the above steps, it is easy to see that $\widetilde{D}=(\cdot)^{1-\alpha}D$ is bounded and equicontinuous. According to Lemma 2.9 and Definition 2.8, we can derive for $t\in(0,b]$ the following,

$$\mu(\widetilde{D}(t))\leqslant\sup_{t\in(0,b]}\mu(\widetilde{D}(t))=\sup_{t\in(0,b]}\mu(t^{1-\alpha}D(t))$$

$$
\begin{aligned}
&\leqslant \sup_{t\in(0,b]} \mu(t^{1-\alpha}(\Psi D)(t))\\
&\leqslant \mu\left(\varphi(0)\right) + \sup_{t\in(0,b]} \mu\left(\int_0^t \frac{t^{1-\alpha}}{\Gamma(\alpha)}(t-s)^{\alpha-1} f(s,\widetilde{D}_s+\widetilde{\varphi}_s,\int_0^s a(s,\tau,\widetilde{D}_\tau+\widetilde{\varphi}_\tau)\mathrm{d}\tau)\mathrm{d}s\right)\\
&\quad + \sup_{t\in(0,b]} \mu\left(\int_0^t \frac{t^{1-\alpha}}{\Gamma(\alpha)}(t-s)^{\alpha-1} g(s,\widetilde{D}_s+\widetilde{\varphi}_s)\mathrm{d}w(s)\right)\\
&\leqslant I_1 + I_2.
\end{aligned}\tag{3.4}
$$

From Lemma 2.10, hypotheses (H1, H3)-(iii), we get

$$
\begin{aligned}
I_1 &= \sup_{t\in(0,b]} \mu\Big(\int_0^t \frac{t^{1-\alpha}}{\Gamma(\alpha)}(t-s)^{\alpha-1} f(s,\widetilde{D}_s+\widetilde{\varphi}_s,\int_0^s a(s,\tau,\widetilde{D}_\tau+\widetilde{\varphi}_\tau)\mathrm{d}\tau)\mathrm{d}s\Big)\\
&\leqslant 2\sup_{t\in(0,b]} \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mu\Big(f(s,\widetilde{D}_s+\widetilde{\varphi}_s,\int_0^s a(s,\tau,\widetilde{D}_\tau+\widetilde{\varphi}_\tau)\mathrm{d}\tau)\Big)\mathrm{d}s\\
&\leqslant 2\sup_{t\in(0,b]} \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\zeta_f(s)\\
&\quad\times\Big(\sup_{-\infty<\theta\leqslant 0}\mu(\widetilde{D}_s(\theta)+\widetilde{\varphi}_s(\theta)) + \mu\Big(\int_0^s a(s,\tau,\widetilde{D}_\tau(\theta)+\widetilde{\varphi}_\tau(\theta)\Big)\mathrm{d}\tau)\Big)\mathrm{d}s\\
&\leqslant 2\sup_{t\in(0,b]} \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\zeta_f(s)\\
&\quad\times\Big(\sup_{-\infty<\theta\leqslant 0}\mu(\widetilde{D}_s(\theta)+\widetilde{\varphi}_s(\theta)) + 2\int_0^s L_2\sup_{-\infty<\theta\leqslant 0}\mu(\widetilde{D}_\tau(\theta)+\widetilde{\varphi}_\tau(\theta))\mathrm{d}\tau\Big)\mathrm{d}s\\
&\leqslant 2\sup_{t\in(0,b]} \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\zeta_f(s)\Big(\sup_{-\infty<\eta\leqslant s}\mu(\widetilde{D}(\eta)) + 2\int_0^s L_2\sup_{-\infty<\vartheta\leqslant\tau}\mu(\widetilde{D}(\vartheta))\mathrm{d}\tau\Big)\mathrm{d}s\\
&\leqslant \frac{2b^{1-\alpha}}{\Gamma(\alpha)}\Big(\sup_{0<t\leqslant b}\sup_{0<\eta\leqslant t}\mu(\widetilde{D}(\eta)) + \sup_{0<t\leqslant b}\sup_{0<\vartheta\leqslant t}\mu(\widetilde{D}(\vartheta))\times 2bL_2\Big)\sup_{t\in(0,b]}\int_0^t (t-s)^{\alpha-1}\zeta_f(s)\mathrm{d}s\\
&\leqslant \frac{2b^{1-\alpha}}{\Gamma(\alpha)}\sup_{0<\nu\leqslant b}\mu(\widetilde{D}(\nu))(1+2bL_2)\left(\frac{1-\alpha_1}{\alpha-\alpha_1}\right)^{1-\alpha_1} b^{\alpha-\alpha_1}\left\|\zeta_f\right\|_{L^{\frac{1}{\alpha_1}}}\\
&\leqslant \frac{2b^{1-\alpha_1}(1+2bL_2)\ \left\|\zeta_f\right\|_{L^{\frac{1}{\alpha_1}}}}{\Gamma(\alpha)}\left(\frac{1-\alpha_1}{\alpha-\alpha_1}\right)^{1-\alpha_1}\mu(\widetilde{D}).
\end{aligned}\tag{3.5}
$$

For any $u, v \in B_q$, applying Lemma 2.3, we obtain

$$
\begin{aligned}
&\mathbb{E}\left\|\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(g(s,\widetilde{u}_s+\widetilde{\varphi}_s)-g(s,\widetilde{v}_s+\widetilde{\varphi}_s)\right)\mathrm{d}w(s)\right\|^2\\
&\quad\leqslant \frac{t^{2(1-\alpha)}Tr(Q)}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2(\alpha-1)}\mathbb{E}\left\|g(s,\widetilde{u}_s+\widetilde{\varphi}_s)-g(s,\widetilde{v}_s+\widetilde{\varphi}_s)\right\|^2\mathrm{d}s.
\end{aligned}
$$

By the properties of the measure of the stochastic integral (see [6]) and (H2)-(iii), we have

$$
\begin{aligned}
I_2 &= \sup_{0<t\leqslant b}\mu\Big(\int_0^t \frac{t^{1-\alpha}}{\Gamma(\alpha)}(t-s)^{\alpha-1} g(s,\widetilde{D}_s+\widetilde{\varphi}_s)\mathrm{d}w(s)\Big)\\
&\leqslant \sup_{0<t\leqslant b}\frac{t^{1-\alpha}}{\Gamma(\alpha)}\Big(\mathrm{Tr}(Q)\int_0^t (t-s)^{2(\alpha-1)}\left(\mu(g(s,\widetilde{D}_s+\widetilde{\varphi}_s))\right)^2\mathrm{d}s\Big)^{\frac{1}{2}}\\
&\leqslant \sup_{0<t\leqslant b}\frac{t^{1-\alpha}\sqrt{\mathrm{Tr}(Q)}}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{2(\alpha-1)}\left(\zeta_g(s)\sup_{-\infty<\theta\leqslant 0}\mu(\widetilde{D}_s(\theta)+\widetilde{\varphi}_s(\theta))\right)^2\mathrm{d}s\Big)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \sup_{0<t\leqslant b} \frac{t^{1-\alpha}\sqrt{\operatorname{Tr}(Q)}}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{2(\alpha-1)}\left(\zeta_g(s)\sup_{0<\mu\leqslant s}\mu(\widetilde{D}(\mu))\right)^2 \mathrm{d}s\right)^{\frac{1}{2}}\\
&\leqslant \frac{b^{1-\alpha}\sqrt{\operatorname{Tr}(Q)}}{\Gamma(\alpha)} \sup_{0<t\leqslant b}\sup_{0<\mu\leqslant t}\mu(\widetilde{D}(\mu))\sup_{0<t\leqslant b}\left(\int_0^t (t-s)^{2(\alpha-1)}\zeta_g^2(s)\mathrm{d}s\right)^{\frac{1}{2}}\\
&\leqslant \frac{b^{1-\alpha}\sqrt{\operatorname{Tr}(Q)}}{\Gamma(\alpha)} \sup_{0<\mu\leqslant b}\mu(\widetilde{D}(\mu))\left(\frac{1-2\alpha_3}{2\alpha-1-2\alpha_3}\right)^{\frac{1-2\alpha_3}{2}} b^{\frac{2\alpha-1-2\alpha_3}{2}}\left\|\zeta_g\right\|_{L^{\frac{1}{\alpha_3}}}\\
&\leqslant \frac{b^{\frac{1-2\alpha_3}{2}}\sqrt{\operatorname{Tr}(Q)}\left\|\zeta_g\right\|_{L^{\frac{1}{\alpha_3}}}}{\Gamma(\alpha)}\left(\frac{1-2\alpha_3}{2\alpha-1-2\alpha_3}\right)^{\frac{1-2\alpha_3}{2}}\mu(\widetilde{D}). \qquad (3.6)
\end{aligned}
$$

Substituting (3.5) and (3.6) into (3.4), we get for $t \in (0, b]$,

$$
\begin{aligned}
&\sup_{t\in(0,b]}\mu(\widetilde{D}(t))\\
&\leqslant \left(\frac{2b^{1-\alpha_1}(1+2bL_2)\left\|\zeta_f\right\|_{L^{\frac{1}{\alpha_1}}}(1-\alpha_1)^{1-\alpha_1}}{\Gamma(\alpha)(\alpha-\alpha_1)^{1-\alpha_1}} + \frac{b^{\frac{1-2\alpha_3}{2}}\sqrt{\operatorname{Tr}(Q)}\left\|\zeta_g\right\|_{L^{\frac{1}{\alpha_3}}}(1-2\alpha_3)^{\frac{1-2\alpha_3}{2}}}{\Gamma(\alpha)(2\alpha-1-2\alpha_3)^{\frac{1-2\alpha_3}{2}}}\right)\mu(\widetilde{D}).
\end{aligned}
$$

Hence

$$\mu(\widetilde{D}) \leqslant l_0\mu(\widetilde{D}).$$

Since $l_0 < 1$ in (3.1), we get $\mu(\widetilde{D}) = 0$, which implies that $\mu(D) = 0$, proving that the set D is relatively compact. We conclude from Theorem 2.11 that Ψ has a fixed point that is a mild solution to the model (1.4). □

4 Example

Consider the following weighted fractional stochastic integro-differential equation with infinite delay:

$$
\begin{cases}
{}^L D_0^{\alpha} u(t) = \int_{-\infty}^t \rho(s-t)\widetilde{u}(s)\mathrm{d}s + \int_0^t (t-s)\int_{-\infty}^s \vartheta(\tau-s)\widetilde{u}(\tau)\mathrm{d}\tau\mathrm{d}s\\
\qquad\qquad + \int_{-\infty}^t \xi(s-t)\widetilde{u}(s)\mathrm{d}s\frac{\mathrm{d}w(s)}{\mathrm{d}s}, \quad t\in(0,b],\\
\widetilde{u}_0(t) \quad = \varphi(t), \quad t\in(-\infty,0],
\end{cases} \qquad (4.1)
$$

where $\frac{1}{2} < \alpha < 1$. We denote by $\mathcal{PC}_r \times L^p(\tilde{h}, \mathcal{H})$ the space of all functions $\varphi : (-\infty, 0] \to \mathcal{H}$ such that $\varphi|_{[-r,0]} \in \mathcal{PC}([-r, 0], \mathcal{H})$, $\varphi(\cdot)$ is Lebesgue measurable on $(-\infty, -r)$, and $\tilde{h}\,\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$ (see [14] for details). The seminorm is given by

$$\|\varphi\|_{\mathcal{B}} = \sup_{-r\leqslant s\leqslant 0}\|\varphi(s)\| + \left(\int_{-\infty}^0 \tilde{h}(s)\,\|\varphi\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}.$$

In addition, we assume that the following conditions hold:

The functions $\rho, \xi, \vartheta : \mathbb{R} \to \mathbb{R}$ are continuous, with

$$c_\rho = \int_{-\infty}^0 \frac{\rho^2(s)}{\tilde{h}(s)}\mathrm{d}s < \infty, \quad c_\xi = \int_{-\infty}^0 \frac{\xi^2(s)}{\tilde{h}(s)}\mathrm{d}s < \infty, \quad c_\vartheta = \int_{-\infty}^0 \frac{\vartheta^2(s)}{\tilde{h}(s)}\mathrm{d}s < \infty,$$

We choose $\mathcal{B} = \mathcal{PC}_0 \times L^2(\tilde{h}, \mathcal{H})$. For $t \in (0, b]$ and $\varphi \in \mathcal{B}$, we give

$$f\left(t,\varphi,\int_0^t a(t,s,\varphi)\mathrm{d}s\right) = \int_{-\infty}^0 \rho(s)\varphi(s)\mathrm{d}s + \int_0^t (t-s)\int_{-\infty}^0 \rho(\tau)\varphi(\tau)\mathrm{d}\tau\mathrm{d}s,$$

$$g(t,\varphi) = \int_{-\infty}^0 \rho(s)\varphi(s)\mathrm{d}s,$$

Obviously, $\theta_f(t) = \theta_g(t) = t$. Under the above conditions, we can represent the Example (4.1) by the model (1.4).

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