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Monge–Ampère measures associated with semi-exhaustive functions

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Abstract In this paper, we study the current $T \wedge dd^c\psi$ for positive currents T and semi-exhaustive, not necessarily plurisubharmonic, functions ψ. The study leads to new definitions of capacity and Lelong–Demailly numbers with respect to the weight ψ.

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1 Introduction

Let Ω be a domain in $\mathbb{C}^n$ and T be a current of bi-dimension (p, p) on Ω. Recall that T is said to be closed if $dT = 0$, and is said to be plurisubharmonic (resp. plurisuperharmonic) if $dd^cT \geq 0$ (resp. $dd^cT \leq 0$). Consider a non-negative function ψ of class $\mathcal{C}^2$ on Ω and set the following notations for every reals $r_1 < r_2$

$$\begin{aligned} B_\psi(r_1) &:= \{z \in \Omega;\ \psi(z) < r_1\},\\ S_\psi(r_1) &:= \{z \in \Omega;\ \psi(z) = r_1\},\\ B_\psi(r_2, r_1) &:= B_\psi(r_2) \setminus B_\psi(r_1),\\ \beta_\psi &:= dd^c\psi,\ \alpha_\psi = dd^c \log \psi. \end{aligned}$$

Throughout this paper, we assume that $d\psi(z) \neq 0$ on $\{z \in \Omega, \psi(z) \neq 0\}$ and that ψ is semi-exhaustive, which means that there exists $R_\psi > 0$ so that $B_\psi(R_\psi)$ is relatively compact in Ω. The paper consists of two parts. The first one concerns with obtaining Lelong–Jensen formula and Lelong–Demailly numbers related to ψ. More precisely, we show the following result.

Theorem. (Theorem 3.7) *If T and dd^cT are of order zero and $0 < r_1 < r_2 < R_\psi$, then*

$$\begin{aligned} \frac{1}{r_2^p}\int_{B_\psi(r_2)} T \wedge \beta_\psi^p - \frac{1}{r_1^p}\int_{B_\psi(r_1)} T \wedge \beta_\psi^p &= \int_{r_1}^{r_2}\left(\frac{1}{t^p} - \frac{1}{r_2^p}\right)\int_{B_\psi(t)} dd^cT \wedge \beta_\psi^{p-1} dt\\ &\quad + \left(\frac{1}{r_1^p} - \frac{1}{r_2^p}\right)\int_0^{r_1}\int_{B_\psi(t)} dd^cT \wedge \beta_\psi^{p-1} dt\\ &\quad + \int_{B_\psi(r_2, r_1)} T \wedge \alpha_\psi^p. \end{aligned} \tag{1.1}$$

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Moreover, Theorem 3.8 shows that the previous formula remains true when T is positive (or negative) plurisubharmonic and ψ is plurisubharmonic of class $\mathcal{C}^1$. These results generalize some classical conclusions of [2,5,8]. As a consequence of these formulas, one can obtain the Lelong–Demailly number $\nu(T,\psi)$ with respect to the weight ψ for positive plurisubharmonic current T and plurisubharmonic function ψ of class $\mathcal{C}^1$.

The second part is devoted to study the Monge–Ampère measure $T\wedge dd^c\psi$. Namely, the contribution of this section is stated as follows.

Theorem. (Theorem 4.1) *Let T be a positive current. If ψ is of class $\mathcal{C}^1$ and $d^c\psi\wedge T$ is well defined on $S_\psi(r)$ for all $0<r<R_\psi$. Then we have*

$$\int_{S_\psi(r)} T\wedge d^c\psi\wedge\beta^{p-1}\geq 0,\ \ \beta=dd^c|z|^2. \tag{1.2}$$

If, in addition, T is plurisuperharmonic, then $\int_{B_\psi(r)} T\wedge dd^c\psi\wedge\beta^{p-1}\geq 0$.

The above inequalities make possible to introduce different capacities, each originating from a different source.

2 Preliminaries and notations

Let $\mathcal{D}_{p,q}(\Omega,k)$ be the space of $\mathcal{C}^k$ compactly supported differential forms of bi-degree (p,q) on Ω. A form $\varphi\in\mathcal{D}_{p,p}(\Omega,k)$ is said to be strongly positive form if φ can be written as

$$\varphi(z)=\sum_{j=1}^{N}\gamma_j(z)\, i\alpha_{1,j}\wedge\overline{\alpha}_{1,j}\wedge\ldots\wedge i\alpha_{p,j}\wedge\overline{\alpha}_{p,j},$$

where $\gamma_j\geq 0$ and $\alpha_{s,j}\in\mathcal{D}_{1,0}(\Omega,k)$. Then, $\mathcal{D}_{p,p}(\Omega,k)$ admits a basis consisting of strongly positive forms. The dual space $\mathcal{D}'_{p,q}(\Omega,k)$ is the space of currents of bi-dimension (p,q) or bi-degree $(n-p,n-q)$ and of order k. If $T\in\mathcal{D}'_{p,p}(\Omega,k)$, then it can be written as

$$T=i^{(n-p)^2}\sum_{|I|=|J|=n-p} T_{I,J}\,dz_I\wedge d\overline{z}_J,$$

where the coefficients $T_{I,J}$ are distributions on Ω. If these coefficients are measures, then T is called of order zero. Remember that when T and dd^cT are of order zero, then T is called $\mathbb{C}$-normal. The current $T\in\mathcal{D}'_{p,p}(\Omega,k)$ is said to be positive if $\langle T,\varphi\rangle\geq 0$ for all forms $\varphi\in\mathcal{D}_{p,p}(\Omega,k)$ that are strongly positive. For such currents T, the mass is denoted by $\|T\|$ and defined by $\sum|T_{I,J}|$, where $|T_{I,J}|$ are the total variations of the measures $T_{I,J}$. Let $\beta=dd^c|z|^2$ be the Kähler form on $\mathbb{C}^n$ (where $d=\partial+\overline{\partial}$ and $d^c=i(-\partial+\overline{\partial})$, thus $dd^c=2i\partial\overline{\partial}$), then for each open subset $\Omega_1\subset\Omega$, there exists a constant $C>0$ depends only on n and p such that

$$T\wedge\frac{\beta^p}{2^p p!}(\Omega_1)\leq\|T\|_{\Omega_1}\leq C\,T\wedge\beta^p(\Omega_1).$$

3 Lelong–Jensen Formula

We start this section with some basic facts that will be used frequently in this paper.

Lemma 3.1 *Let E be a domain in $\mathbb{R}^n$ and $f:E\to\mathbb{R}$ be a function of class $\mathcal{C}^1$ so that $df(x)\neq 0$ for all $x\in E$. If φ is a locally bounded $(n-1)$-form and compactly supported, then*

$$\int_E df\wedge\varphi=\int_{-\infty}^{\infty}dt\int_{f=t}\varphi.$$

Lemma 3.2 *Let Ω be a domain in $\mathbb{C}^n$ and $\varphi : \Omega \to [0, \infty)$ be a function of class $\mathcal{C}^2$. Let $t > 0$ be a regular value of φ and set $S(r) = \{z \in \Omega,\ \varphi(z) = r\}$. Then,*

$$j_t^* dd^c(Log\varphi) = \frac{1}{t} j_t^* dd^c\varphi,$$

where $j_t^* : S(t) \to \Omega$ is the canonical injection.

Lemma 3.3 *Let φ be a function of class $\mathcal{C}^1$. If T and γ are two $\mathcal{C}^1$-form of bi-degree $(n-p, n-p)$ and $(p-1, p-1)$, respectively, then*

$$d\varphi \wedge d^c T \wedge \gamma = -d^c\varphi \wedge dT \wedge \gamma.$$

Lemma 3.4 *Let u be a $\mathcal{C}^1$-function on Ω. If T is a $\mathbb{C}$-normal current of bi-dimension (p, p), then the current $T \wedge dd^c u$ is well defined.*

Proof Take a test form φ in Ω and let $(u_j)_{j\in\mathbb{N}}$ be a sequence of smooth functions converges in $\mathcal{C}^1(\Omega)$ to u. Then,

$$\int_\Omega \mathrm{dd}^c(u_j\varphi) \wedge T \wedge \beta^{p-1} = \int_\Omega u_j\varphi \mathrm{dd}^c T \wedge \beta^{p-1}. \tag{3.1}$$

Hence, by a simple computation, one can deduce that

$$\begin{aligned}\int_\Omega \varphi \mathrm{dd}^c u_j \wedge T \wedge \beta^{p-1} = &\int_\Omega u_j\varphi \mathrm{dd}^c T \wedge \beta^{p-1} - 2\int_\Omega \mathrm{d}u_j \wedge \mathrm{d}^c\varphi \wedge T \wedge \beta^{p-1}\\ &- \int_\Omega u_j \mathrm{dd}^c\varphi \wedge T \wedge \beta^{p-1}.\end{aligned} \tag{3.2}$$

This shows that $\lim_{j\to\infty} \mathrm{dd}^c u_j \wedge T$ exists as the right-hand side terms of the previous equality are convergent. □

Lemma 3.5 *Let $u_1, ..., u_q,\ 1 \le q \le p$ be plurisubharmonic functions of class $\mathcal{C}^1$ on Ω. If T is positive (or negative) plurisubharmonic, then the current $T \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_q$ is well defined.*

Proof By the precedent lemma, $T \wedge \mathrm{dd}^c u_j$ is well defined for all $j \in \{1, \ldots, q\}$. Now, the result is induced by induction and the fact that each $T \wedge \mathrm{dd}^c u_j$ is positive (or negative) plurisubharmonic. □

Theorem 3.6 *(See [6]) Let T be an $(n-p, n-p)$-form of class $\mathcal{C}^2$ on Ω. Then for all $0 < r_1 < r_2 < R_\psi$, we have*

$$\begin{aligned}\int_{r_1}^{r_2} \frac{dt}{t^p} \int_{B_\psi(t)} dd^c T \wedge \beta_\psi^{p-1} = &\frac{1}{r_2^p} \int_{S_\psi(r_2)} T \wedge d^c\psi \wedge \beta_\psi^{p-1}\\ &- \frac{1}{r_1^p} \int_{S_\psi(r_1)} T \wedge d^c\psi \wedge \beta_\psi^{p-1} - \int_{B_\psi(r_2, r_1)} T \wedge \alpha_\psi^p.\end{aligned} \tag{3.3}$$

Proof By Stokes' theorem, we have

$$\begin{aligned}\int_{B_\psi(t)} \mathrm{dd}^c T \wedge \beta_\psi^{p-1} = \int_{B_\psi(t)} \mathrm{d}(\mathrm{d}^c T \wedge \beta_\psi^{p-1}) &= \int_{S_\psi(t)} \mathrm{d}^c T \wedge \beta_\psi^{p-1}\\ &= t^{p-1} \int_{S_\psi(t)} \mathrm{d}^c T \wedge \alpha_\psi^{p-1}.\end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned}\int_{r_1}^{r_2}\frac{\mathrm{d}t}{t}\int_{S_\psi(t)}\mathrm{d}^cT\wedge\alpha_\psi^{p-1} &= \int_{r_1}^{r_2}\frac{\mathrm{d}t}{t^p}\int_{B_\psi(t)}\mathrm{dd}^cT\wedge\beta_\psi^{p-1}\\ &= \int_{B_\psi(r_2,r_1)}\mathrm{d}Log\psi\wedge\mathrm{d}^cT\wedge\alpha_\psi^{p-1}\\ &= \int_{B_\psi(r_2,r_1)}\mathrm{d}T\wedge\mathrm{d}^cLog\psi\wedge\alpha_\psi^{p-1}\\ &= \int_{S_\psi(r_2)}T\wedge\mathrm{d}^cLog\psi\wedge\alpha_\psi^{p-1}-\int_{S_\psi(r_1)}T\wedge\mathrm{d}^cLog\psi\wedge\alpha_\psi^{p-1}\\ &\quad -\int_{B_\psi(r_2,r_1)}T\wedge\alpha_\psi^{p-1}. \end{aligned} \tag{3.5}$$

Now, (3.3) follows by applying Lemma 3.2. □

Theorem 3.7 *If T is $\mathbb{C}$-normal and $0<r_1<r_2<R_\psi$, then*

$$\begin{aligned}\frac{1}{r_2^p}\int_{B_\psi(r_2)}T\wedge\beta_\psi^p-\frac{1}{r_1^p}\int_{B_\psi(r_1)}T\wedge\beta_\psi^p &= \int_{r_1}^{r_2}\left(\frac{1}{t^p}-\frac{1}{r_2^p}\right)\int_{B_\psi(t)}dd^cT\wedge\beta_\psi^{p-1}dt\\ &\quad+\left(\frac{1}{r_1^p}-\frac{1}{r_2^p}\right)\int_0^{r_1}\int_{B_\psi(t)}dd^cT\wedge\beta_\psi^{p-1}dt\\ &\quad+\int_{B_\psi(r_2,r_1)}T\wedge\alpha_\psi^p. \end{aligned} \tag{3.6}$$

Notice that the previous formula is obtained without constraint on $\mathrm{d}T$ as required in [8] and [6].

Proof We first assume that T of class $\mathcal{C}^2$. Then by the previous lemma, one has

$$\begin{aligned}\int_{r_1}^{r_2}\frac{\mathrm{d}t}{t^p}\int_{B_\psi(t)}\mathrm{dd}^cT\wedge\beta_\psi^{p-1} &= \frac{1}{r_2^p}\int_{S_\psi(r_2)}T\wedge\mathrm{d}^c\psi\wedge\beta_\psi^{p-1}\\ &\quad-\frac{1}{r_1^p}\int_{S_\psi(r_1)}T\wedge\mathrm{d}^c\psi\wedge\beta_\psi^{p-1}-\int_{B_\psi(r_2,r_1)}T\wedge\alpha_\psi^p. \end{aligned} \tag{3.7}$$

But

$$\begin{aligned}\frac{1}{r_2^p}\int_{S_\psi(r_2)}T\wedge\mathrm{d}^c\psi\wedge\beta_\psi^{p-1} &= \frac{1}{r_2^p}\int_{B_\psi(r_2)}T\wedge\beta_\psi^p+\frac{1}{r_2^p}\int_{B_\psi(r_2)}\mathrm{d}T\wedge\mathrm{d}^c\psi\wedge\beta_\psi^{p-1}\\ &= \frac{1}{r_2^p}\int_{B_\psi(r_2)}T\wedge\beta_\psi^p+\frac{1}{r_2^p}\int_0^{r_2}\mathrm{d}t\int_{B_\psi(r_2)}\mathrm{dd}^cT\wedge\beta_\psi^{p-1}. \end{aligned} \tag{3.8}$$

Similarly, we have

$$\frac{1}{r_1^p}\int_{S_\psi(r_1)}T\wedge\mathrm{d}^c\psi\wedge\beta_\psi^{p-1}=\frac{1}{r_1^p}\int_{B_\psi(r_1)}T\wedge\beta_\psi^p+\frac{1}{r_1^p}\int_0^{r_1}\mathrm{d}t\int_{B_\psi(r_2)}\mathrm{dd}^cT\wedge\beta_\psi^{p-1}. \tag{3.9}$$

Thus, the result is verified for $\mathcal{C}^2$ currents T by combining the latter equalities. Now, for $\mathbb{C}$-normal currents T, set $E_T=\{r\in\mathbb{R},\ ||T||_{S(r)}*||\mathrm{dd}^cT||_{S(r)}\neq 0\}$. By the assumptions of T and dd^cT, it is clear that $||T||_K$ and $||\mathrm{dd}^cT||_K$ are bounded for all compact subset K of Ω. Hence, the set E_T is countable. Consider a regularization ρ_ε. Then for all $t\in\mathbb{R}\setminus E_T$, we have

$$\lim_{\varepsilon\to0}\int_{B_\psi(t)}T*\rho_\varepsilon\wedge\beta_\psi^p=\lim_{\varepsilon\to0}\int_{\mathbb{C}^n}\mathbb{1}_{B_\psi(t)}T*\rho_\varepsilon\wedge\beta_\psi^p=\int_{B_\psi(t)}T\wedge\beta_\psi^p, \tag{3.10}$$

where $\mathbb{1}_{B_\psi(t)}$ is the characteristic function of $B_\psi(t)$. If $r_1,\ r_2$ are elements of E_T one can take $(r_1^{(j)})_{j\in\mathbb{N}}$ increasing to r_1 and $(r_2^{(j)})_{j\in\mathbb{N}}$ increasing to r_2 so that $r_k^{(j)} \in \mathbb{R}\setminus E_T$. The result is achieved by taking the limits. □

Theorem 3.8 *If T is positive (or negative) plurisubharmonic current and ψ is plurisubharmonic and of class $\mathcal{C}^1$, then Lelong–Jensen formula (3.6) remains valid.*

This result generalizes the formulas in [2] to the case of $\mathcal{C}^1$ functions.

Proof By regularizing ψ, one can assume that ψ is smooth. Now the result follows by applying, first, Theorem 3.7 and, second, Lemma 3.5. □

Remark 3.9 According to Theorem 3.7 and Theorem 3.8, if $T\wedge\alpha_\psi^p$ and $\mathrm{dd}^cT\wedge\beta_\psi^{p-1}$ are positive measures, then the function $r\mapsto \frac{1}{r^p}\int_{B_\psi(r)} T\wedge\beta_\psi^p$ is positive and increasing on $(0, R_\psi)$. Therefore, $\lim_{r\to 0^+}\frac{1}{r^p}\int_{B_\psi(r)} T\wedge\beta_\psi^p$ exists, and is denoted by $\nu(T,\psi)$ the Demailly–Lelong number of T with respect to the weight ψ. This show that $\nu(T,\psi)$ exists in the particular case when T is positive plurisubharmonic and ψ is plurisubharmonic and of class $\mathcal{C}^1$.

4 Capacity related to semi-exhaustive functions

In this section, we study the current $\mathrm{dd}^c\psi\wedge T$. From now on, we relax the classification of ψ to $\mathcal{C}^1$.

Theorem 4.1 *If T is positive and $d^c\psi\wedge T$ is well defined on $S_\psi(r)$ for all $0<r<R_\psi$, then we have*

$$\int_{S_\psi(r)} d^c\psi\wedge T\wedge\beta^{p-1}\geq 0. \tag{4.1}$$

If, in addition, T is plurisuperharmonic, then $\int_{B_\psi(r)} T\wedge dd^c\psi\wedge\beta^{p-1}\geq 0$.

Proof Notice first that $\mathrm{d}\psi\wedge\mathrm{d}^c\psi\wedge T$ is a positive current. Hence, the function $f(r)=\int_{B_\psi(r)}\mathrm{d}\psi\wedge\mathrm{d}^c\psi\wedge T\wedge\beta^{p-1}$ is non decreasing. So, $f'(r)\geq 0$. But

$$\begin{aligned} f'(r) &= \left[\int_{B_\psi(r)}\mathrm{d}\psi\wedge\mathrm{d}^c\psi\wedge T\wedge\beta^{p-1}\right]' \\ &= \left[\int_0^r \mathrm{d}t\int_{S_\psi(t)}\mathrm{d}^c\psi\wedge T\wedge\beta^{p-1}\right]' \\ &= \int_{S_\psi(r)}\mathrm{d}^c\psi\wedge T\wedge\beta^{p-1}. \end{aligned} \tag{4.2}$$

Now, assume that $\mathrm{dd}^cT\leq 0$. By Stokes' formula, we have

$$\begin{aligned} \int_{S_\psi(r)}\mathrm{d}^c\psi\wedge T\wedge\beta^{p-1} &= \int_{B_\psi(r)}\mathrm{dd}^c\psi\wedge T\wedge\beta^{p-1} - \int_{B_\psi(r)}\mathrm{d}^c\psi\wedge\mathrm{d}T\wedge\beta^{p-1} \\ &= \int_{B_\psi(r)}\mathrm{dd}^c\psi\wedge T\wedge\beta^{p-1} + \int_{B_\psi(r)}\mathrm{d}\psi\wedge\mathrm{d}^cT\wedge\beta^{p-1} \\ &= \int_{B_\psi(r)}\mathrm{dd}^c\psi\wedge T\wedge\beta^{p-1} + \int_0^r\mathrm{d}t\int_{S_\psi(t)}\mathrm{d}^cT\wedge\beta^{p-1} \\ &= \int_{B_\psi(r)}\mathrm{dd}^c\psi\wedge T\wedge\beta^{p-1} + \int_0^r\mathrm{d}t\int_{B_\psi(t)}\mathrm{dd}^cT\wedge\beta^{p-1}. \end{aligned} \tag{4.3}$$

This shows that

$$\int_{B_\psi(r)} dd^c\psi \wedge T \wedge \beta^{p-1} = \int_{S_\psi(r)} d^c\psi \wedge T \wedge \beta^{p-1} - \int_0^r dt \int_{B_\psi(t)} dd^c T \wedge \beta^{p-1} \geq 0. \tag{4.4}$$

□

Remark 4.2 If T is $\mathbb{C}$-normal on Ω, then the current $d^c\psi \wedge T$ is well defined on $S_\psi(r)$. Indeed, the wedge product $dd^c\psi \wedge T$ is achieved by Lemma 3.4. Hence, we set

$$\int_{S_\psi(r)} d^c\psi \wedge T \wedge \beta^{p-1} = \int_{B_\psi(r)} dd^c\psi \wedge T \wedge \beta^{p-1} + \int_0^r dt \int_{B_\psi(t)} dd^c T \wedge \beta^{p-1}. \tag{4.5}$$

As shown above, semi-exhaustive functions have things in common with plurisubharmonic functions. Despite this, we must be cautious once we deal with these semi-exhaustive functions as some important properties of Psh are not applicable to this type of functions. For example, if ψ is plurisubharmonic, then it is so obvious that $r \mapsto \int_{B_\psi(r)} dd^c\psi \wedge T \wedge \beta^{p-1}$ is increasing in r. This fact is not valid when the plurisubharmonicity is omitted. The following example shows this.

Example 4.3 In $\mathbb{C}$, set $\Omega = B(0, 1)$ and put $T = 1$. Now, take $\psi(z) = \sin(\frac{\pi}{2}|z|^2)$. Clearly, ψ is an semi-exhaustive function on Ω where $R_\psi = 1$. By a simple computation, we have

$$dd^c\psi = [-(\frac{\pi}{2})^2|z|^2 \sin(\frac{\pi}{2}|z|^2) + \frac{\pi}{2}\cos(\frac{\pi}{2}|z|^2)]\,\beta. \tag{4.6}$$

Notice that $dd^c\psi$ tends to $\frac{\pi}{2}\beta$ when $|z| \to 0$, while $dd^c\psi$ tends to $-(\frac{\pi}{2})^2\beta$ as $|z| \to 1^-$.

Let us recall a very fundamental fact about currents. When g is a locally bounded plurisubharmonic function on Ω and T is positive and closed, the current gT is well defined. The exterior derivatives lead to the current $dd^c g \wedge T$ as it is defined by $dd^c(gT)$.

Proposition 4.4 *Let T be a positive closed current of bi-dimension (p, p) on Ω and g be a locally bounded plurisubharmonic function. Then,*

$$\int_{B_\psi(r)} d\psi \wedge d^c g \wedge T \wedge \beta^{p-1} \geq 0. \tag{4.7}$$

If g is positive, then

$$\int_{B_\psi(r)} dd^c(\psi g) \wedge T \wedge \beta^{p-1} \geq 0. \tag{4.8}$$

Proof First, we show that the quantity $\int_{B_\psi(r)} d\psi \wedge d^c g \wedge T \wedge \beta^{p-1}$ is non-negative and increasing in r. By Lemma 3.1, we have

$$\begin{aligned}\int_{B_\psi(r)} d\psi \wedge d^c g \wedge T \wedge \beta^{p-1} &= \int_0^r dt \int_{S_\psi(t)} d^c g \wedge T \wedge \beta^{p-1} \\ &= \int_0^r dt \int_{B_\psi(t)} dd^c g \wedge T \wedge \beta^{p-1} \geq 0.\end{aligned} \tag{4.9}$$

Now, assume that g is positive. Then the current gT is $\mathbb{C}$-normal. Hence by Lemma 3.1 and Theorem 4.1, we have

$$\begin{aligned}\int_{B_\psi(r)} dd^c(\psi g) \wedge T \wedge \beta^{p-1} &= \int_{S_\psi(r)} d^c(\psi g) \wedge T \wedge \beta^{p-1} \\ &= \int_{S_\psi(r)} g d^c\psi \wedge T \wedge \beta^{p-1} + r \int_{B_\psi(r)} dd^c g \wedge T \wedge \beta^{p-1} \geq 0.\end{aligned} \tag{4.10}$$

□

In virtue of [7], if $g \in Psh(\Omega) \cap L^{\infty}_{loc}(\Omega \setminus K)$ for some compact subset K of Ω, then $dd^c g \wedge T$ is well defined. Therefore, by following a similar technique as in [1], the current gT in this case can be deduced. Indeed, take neighborhoods V and W so that $K \Subset V \Subset W \subset \Omega$, and $\chi \in \mathcal{C}^{\infty}_0(W)$ such that $\chi = 1$ on V. Now, construct a decreasing sequence of smooth plurisubharmonic functions (g_j) converges point-wise to g on Ω. Then,

$$\int_W dd^c(\chi|z|^2)g_j T \wedge \beta^{p-1} = \int_W \chi|z|^2 dd^c g_j \wedge T \wedge \beta^{p-1}.$$

This implies that,

$$\begin{aligned}\int_W -\chi g_j T \wedge \beta^p = &-\int_W \chi|z|^2 dd^c g_j \wedge T \wedge \beta^{p-1}\\ &+2\int_W g_j d\chi \wedge d^c|z|^2 \wedge T \wedge \beta^{p-1}\\ &+\int_W g_j|z|^2 dd^c\chi \wedge T \wedge \beta^{p-1}.\end{aligned}$$

Thus,

$$\sup_j \int_V |g_j| T \wedge \beta^p < \infty.$$

The previous discussion yields to the fulfillment of Proposition 4.4 in the case of unbounded functions g.

Definition 4.5 A real-valued function f on Ω is called a T-Monge-Ampère of degree q, $0 \le q \le p$ on Ω (for short $f \in \mathcal{MA}^q(T, \Omega)$) if the current $(dd^c f)^q \wedge T$ is well defined on Ω. If in addition $\int_\Omega (dd^c f)^q \wedge T \wedge \beta^{p-q} \ge 0$, then f is said to be of class $\mathcal{P}^q(T, \Omega)$.

Clearly, the set $\mathcal{C}^2 \cap Psh(\Omega) \subseteq \mathcal{P}^p(T, \Omega)$. Moreover, the early studies of currents lead to many cases where the previous inclusion is proper. For instant, if T is positive and closed, then we already know that $Psh(\Omega) \cap L^{\infty}_{loc}(\Omega) \subset \mathcal{P}^p(T, \Omega)$. Also, the above study shows that the $\mathcal{C}^1$ semi-exhaustive function $\psi \in \mathcal{P}^1(T, B_\psi(r))$.

Definition 4.6 Let S be a positive current of bi-dimension (p, p) on Ω. We define the capacity $\mathcal{C}^q_S(O, \Omega)$ for all Borel set $O \Subset \Omega$ by

$$\mathcal{C}^q_S(O, \Omega) = \sup\left\{\int_O (dd^c f)^q \wedge S \wedge \beta^{p-q},\ f \in \mathcal{P}^q(S, \Omega),\ 0 \le f \le 1\right\}.$$

Observe that for positive and closed currents S, the capacity $\mathcal{C}_S$, which is introduced in [4], is dominated by $\mathcal{C}^p_S$. This is an obvious inclusion from the fact that $Psh(\Omega) \cap L^{\infty}_{loc}(\Omega) \subset \mathcal{P}^p(T, \Omega)$. We give an example where $\mathcal{C}_S < \mathcal{C}^q_S$.

Example 4.7 In $\mathbb{C}^1$, set $\Omega = B(0, 1)$ and $S = 1$. From [3] it is very well known that $\mathcal{C}_S(B(0, \frac{1}{2})) = \frac{1}{\log 2}$. Now construct a positive smooth semi-exhaustive function ψ on $B(0, 1)$ so that $\psi(z) = \frac{2}{3}|z|^2$ on $B(0, \frac{1}{2})$, and $\psi(z) = \sin(\frac{\pi}{2}|z|^2)$ on an appropriate neighborhood of $\{|z| = 1\}$. Clearly,

$$\int_{B(0,\frac{1}{2})} dd^c\psi = \frac{2}{3}\pi. \tag{4.11}$$

This show that $\mathcal{C}_S(B(0, \frac{1}{2})) < \mathcal{C}^1_S(B(0, \frac{1}{2}))$.

Another definition of capacity is given as follows.

Definition 4.8 Let S be a positive closed current of bi-dimension (p,p) on Ω. We define the capacity $\mathcal{C}_S(\mathrm{d}\psi, r, r')$ for all $0 < r < r' < R_\psi$ by

$$\sup\left\{\int_{B_\psi(r)} \mathrm{d}\psi \wedge \mathrm{d}^c g \wedge (\mathrm{dd}^c g)^{p-1} \wedge S,\ g \in Psh(B_\psi(r')),\ 0 \le g \le 1\right\}.$$

The above definitions together with Proposition 4.4 yield to the next properties.

Proposition 4.9 *Let S be a positive closed current of bi-dimension (p,p) on Ω. Then for all $0 < r < r' < r'' < R_\psi$, we have*

(1) $\mathcal{C}_S(d\psi, r, r'') \le \mathcal{C}_S(d\psi, r', r'')$.
(2) $\mathcal{C}_S(d\psi, r, r') \ge \mathcal{C}_S(d\psi, r, r'')$.
(3) $\frac{1}{r}\mathcal{C}_S(d\psi, r, R_\psi) \le \mathcal{C}_S(B_\psi(r), B_\psi(R_\psi)) \le \mathcal{C}_S^p(B_\psi(r), B_\psi(R_\psi))$.

We end this paper with a version of Chern–Levine–Nirenberg inequality in the case of semi-exhaustive functions.

Proposition 4.10 *Let K be a compact subset of Ω so that $B_\psi(r) \Subset K \Subset \Omega$. If T is positive and plurisuperharmonic, then there exists a constant $C_K(r) > 0$ such that*

$$\int_{B_\psi(r)} dd^c\psi \wedge T \wedge \beta^{p-1} \le C_K(r)\|\psi\|_{\mathcal{L}^\infty(K)}\|T\|_K. \tag{4.12}$$

Proof By similar arguments as above, one can assume that ψ is of class $\mathcal{C}^2$. Now, set $O = \{z \in B_\psi(r), \mathrm{dd}^c\psi(z) > 0\}$. Clearly, O is an open subset of $B_\psi(r)$, and

$$\int_{B_\psi(r)} \mathrm{dd}^c\psi \wedge T \wedge \beta^{p-1} \le \int_O \mathrm{dd}^c\psi \wedge T \wedge \beta^{p-1}. \tag{4.13}$$

Thus, for $\varepsilon > 0$, there exists an open subset O_ε of O so that

$$\int_{B_\psi(r)} \mathrm{dd}^c\psi \wedge T \wedge \beta^{p-1} \le \int_{O_\varepsilon} \mathrm{dd}^c\psi \wedge T \wedge \beta^{p-1} + \varepsilon. \tag{4.14}$$

But Chern–Lieven–Nirenberg shows that

$$\begin{aligned}\int_{B_\psi(r)} \mathrm{dd}^c\psi \wedge T \wedge \beta^{p-1} &\le \int_{O_\varepsilon} \mathrm{dd}^c\psi \wedge T \wedge \beta^{p-1} + \varepsilon\\ &\le C_K(r)\|\psi\|_{\mathcal{L}^\infty(K)}\|T\|_K.\end{aligned} \tag{4.15}$$

□

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Declarations

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