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Exact sequences for dual Toeplitz algebras on hypertori

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Abstract In this paper, we construct a symbol calculus yielding short exact sequences for the dual Toeplitz algebra generated by all bounded dual Toeplitz operators on the Hardy space associated with the polydisk $\mathbb{D}^n$ in the unitary space $\mathbb{C}^n$, that have been introduced and well studied in our earlier paper (Benaissa and Guediri in Taiwan J Math 19: 31–49, 2015), as well as for the C*-subalgebra generated by dual Toeplitz operators with symbols continuous on the associated hypertorus $\mathbb{T}^n$.

Mathematics Subject Classification 47B35 · 47L80

1 Introduction

Dual Toeplitz operators on the orthogonal complement of the Bergman space on the unit disk have been introduced and well studied by K. Stroethoff and D. Zheng [11]. Since then, various algebraic and spectral properties of dual Toeplitz operators on the orthogonal complements of various Hilbert spaces of analytic functions have been attracting more and more attention of many authors; see for instance [1,2,7,9,13] and the references therein. In particular, in our earlier paper [1], we have introduced and started a systematic study of dual Toeplitz operators on the Hardy space of the polydisk $\mathbb{D}^n$ in $\mathbb{C}^n$, where we have established some of their algebraic properties, including commutativity and products. In particular, we have established a Brown–Halmos type theorem and characterized zero divisors among dual Toeplitz operators, as well as normal dual Toeplitz operators. Moreover, we have studied Hankel products and mixed Toeplitz–Hankel products. All of these results hinge on a crucial transformation of operators, namely the operator $\mathscr{S}_w$, which will play again an important role here too.
Banach algebra techniques turn out to be very useful for the investigation of various classes of linear operators on function spaces. The monograph of R. Douglas [5] contains most fundamental results in this direction. In particular, symbol calculi and structure theorems play a fundamental role in studying spectra and Fredholm properties of Toeplitz operators and related facts. In the Hardy space setting, L.A. Coburn [3] obtained a short exact sequence for the Toeplitz algebra generated by symbols that are continuous on the sphere. More related structure theorems have been well explored in [5]. In particular, the following short exact sequence has been established:

$$(0) \longrightarrow \mathrm{Comm}(\mathscr{I}(L^\infty(\mathbb{T})) \longrightarrow \mathscr{I}(L^\infty(\mathbb{T})) \longrightarrow L^\infty(\mathbb{T}) \longrightarrow (0), \qquad (1.1)$$

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where $\mathscr{I}(L^\infty(\mathbb{T}))$ is the Toeplitz algebra generated by all bounded Toeplitz operators on the Hardy space of the unit circle, and $\text{Comm}(\mathscr{I}(L^\infty(\mathbb{T}))$ denotes its commutator ideal. On the other hand, A.M. Davie and N.P. Jewell [4] have remarkably established a short exact sequence for the Toeplitz algebra generated by all bounded Hardy space Toeplitz operators on the unit sphere $\mathbb{S}^n$, and obtained a spectral inclusion theorem, as well as many other interesting properties. N. P. Jewell and S. Krantz [8] have generalized some of the latter results to the setting of pseudoconvex domains in $\mathbb{C}^n$, while M. Engliš [6] has constructed a symbol calculus for Toeplitz operators on the Bergman space.
Returning to dual Toeplitz operators and structure theorems of their algebras, K. Stroethoff and D. Zheng [11] have constructed a symbol calculus for their Bergman space dual Toeplitz operators on the unit disk, and obtained short exact sequences related to some dual Toeplitz algebras, and deduced many corresponding spectral and Fredholm properties. In particular, the following short exact sequence has been established:

$$(0) \longrightarrow \text{Semi}(\mathscr{I}(L^\infty(\mathbb{D})) \longrightarrow \mathscr{I}(L^\infty(\mathbb{D})) \longrightarrow L^\infty(\mathbb{D}) \longrightarrow (0), \tag{1.2}$$

where $\mathscr{I}(L^\infty(\mathbb{D}))$ denotes the dual Toeplitz algebra generated by bounded dual Toeplitz operators and $\text{Semi}(\mathscr{I}(L^\infty(\mathbb{D}))$ is its semi-commutator ideal. On the other hand, Z. Yang, B. Zhang, and Y. Lu [13] obtained analogue short exact sequences in the setting of the Bergman space of the unit ball in $\mathbb{C}^n$, while L. Kong and Y. Lu [9] obtained similar structure theorems. In the framework of the polydisk, G. Z. Cheng and T. Yu [2] have obtained analogue short exact sequences for certain dual Toeplitz algebras on the Bergman space.
The purpose of the present paper is to establish more algebraic and spectral properties of our dual Toeplitz operators on the Hardy space associated with the hypertorus which have been introduced in our earlier paper [1]. In particular, in the aim of constructing a corresponding symbol calculus, we introduce a symbol map ρ and show that it is a contractive C*-homomorphism from the dual Toeplitz algebra $\mathscr{I}(L^\infty(\mathbb{T}^n))$ into the algebra $L^\infty(\mathbb{T}^n)$, whose null-space $\ker\rho$ coincides with the semi-commutator ideal of the dual Toeplitz algebra $\mathscr{I}(L^\infty(\mathbb{T}^n))$. This represents our first main result, namely Theorem 3.1. Consequently, these facts enable us to establish our second main result, namely Theorem 3.2, consisting of a short exact sequence analogue to (1.1) and (1.2) above. Furthermore, making appeal once more to our crucial transformation $\mathscr{S}_w$ constructed in [1], we show that the semi-commutator ideal $\text{Semi}(\mathscr{I}(L^\infty(\mathbb{T}^n))$ is irreducible and contains the ideal $\mathscr{K}$ of all compact operators in $\mathscr{B}\left(\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp\right)$, and then, we derive many important consequences. On the other hand, our last main result, namely Theorem 3.9, deals with the C*-algebra $\mathscr{I}(\mathscr{C}(\mathbb{T}^n))$ generated by dual Toeplitz operators with symbols in $\mathscr{C}(\mathbb{T}^n)$. More precisely, we show first that its semi-commutator ideal coincides with the ideal $\mathscr{K}$ of all compact operators on $\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$. Then, we establish a corresponding short exact sequence analogue to the above ones (1.1) and (1.2).
Our paper is organized as follows: in Sect. 2, we establish some auxiliary material needed for our structure theorems. Section 3 is devoted to the main results, where we construct the desired symbol map and derive short exact sequences and deduce interesting consequences.

2 Preparatory lemmata

Let $\mathbb{D} = \{z \in \mathbb{C},\ |z| < 1\}$ be the unit disk of the complex plane $\mathbb{C}$ whose boundary is denoted by $\partial\mathbb{D} := \mathbb{T}$. For $n > 1$, the polydisk $\mathbb{D}^n$ is the cartesian product of n copies of $\mathbb{D}$, it is therefore defined by

$$\mathbb{D}^n := \left\{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n :\ |z_j| < 1, : j = 1, \ldots, n\right\}.$$

Its distinguished boundary is the n-torus (hypertorus) defined by

$$\mathbb{T}^n := \left\{\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{C}^n :\ |\zeta_j| = 1, : j = 1, \ldots, n\right\}.$$

If $d\sigma(\theta) = \frac{d\theta}{2\pi}$ denotes the Lebesgue measure on the unit circle $\mathbb{T}$, then $d\sigma(\zeta) = \frac{d\theta_1 d\theta_2 \ldots d\theta_n}{(2\pi)^n}$, where $\zeta = (\zeta_1, \ldots, \zeta_n)$ with $\zeta_j = e^{i\theta_j}$, $j = 1, \ldots, n$, is the normalized Lebesgue measure on $\mathbb{T}^n$. The Lebesgue space $L^2(\mathbb{T}^n, d\sigma) = L^2(\mathbb{T}^n)$ is defined in the usual way, and it is known to be a Hilbert space with respect to the inner product $< f,\ g >:= \int_{\mathbb{T}^n} f(z)\overline{g(z)}d\sigma(z)$. The algebra $L^\infty(\mathbb{T}^n)$ is the space of all essentially bounded functions with sup norm: $||f||_\infty = \operatorname{ess.}\sup_{z\in\mathbb{T}^n}\{|f(z)|\}$. The Hardy space $\mathcal{H}^2(\mathbb{D}^n)$ is defined to be the set of all holomorphic functions $f : \mathbb{D}^n \to \mathbb{C}$ satisfying

$$\|f\|_2 := \left(\sup_{0<r<1}\int_{\mathbb{T}^n}|f(r\zeta)|^2\,d\sigma(\zeta)\right)^{\frac{1}{2}} < \infty.$$

On the other hand, denote by $\mathcal{H}^2(\mathbb{T}^n)$ the closure of analytic polynomials in $L^2(\mathbb{T}^n)$, which is itself a Hilbert subspace with respect to the induced inner product. In fact, it coincides with $\mathcal{H}^2(\mathbb{D}^n)$. Denote by $\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$ the orthogonal complement of the Hardy space $\mathcal{H}^2(\mathbb{T}^n)$ in $L^2(\mathbb{T}^n)$. Then, we have the following orthogonal decomposition:

$$L^2(\mathbb{T}^n) = \mathcal{H}^2(\mathbb{T}^n) \oplus \left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp. \tag{2.1}$$

If we define $\mathcal{H}_0^2(\mathbb{T}^n)$ as follows:

$$\mathcal{H}_0^2(\mathbb{T}^n) := \left\{ f \in \mathcal{H}^2(\mathbb{T}^n),\ \text{such that} \int\limits_{\mathbb{T}^n} f(\zeta)d\sigma(\zeta) = 0 \right\} = \{\mathbb{1}\}^\perp,$$

where $\mathbb{1}$ denotes the constant function with value 1, and then, we see that $\overline{\mathcal{H}_0^2(\mathbb{T}^n)}$ is a proper subset of $\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$. In other words, we have $\overline{\mathcal{H}_0^2(\mathbb{T}^n)} \subsetneqq \left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$. The Hardy space $\mathcal{H}^2(\mathbb{T}^n)$ is a reproducing kernel Hilbert space with kernel function given by

$$K_w(\zeta) = \prod_{j=1}^{n} \frac{1}{1-\overline{w}_j\zeta_j}, \quad \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n, \tag{2.2}$$

while its normalized reproducing kernel is given by

$$k_w(\zeta) := \frac{K_w}{\|K_w\|_2} = \prod_{j=1}^{n} \frac{\sqrt{1-|w_j|^2}}{1-\overline{w}_j\zeta_j}, \quad \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n. \tag{2.3}$$

Referring to the orthogonal decomposition (2.1), let $\mathcal{P}$ be the orthogonal projection of $L^2(\mathbb{T}^n)$ on its closed subspace $\mathcal{H}^2(\mathbb{T}^n)$, and let $\mathcal{Q} = I - \mathcal{P}$ be the orthogonal projection of $L^2(\mathbb{T}^n)$ into $\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$. For $f \in L^\infty(\mathbb{T}^n)$, the multiplication operator on $L^2(\mathbb{T}^n)$ is defined as follows:

$$\begin{aligned} M_f : L^2(\mathbb{T}^n) &\longrightarrow L^2(\mathbb{T}^n), \\ g &\longmapsto M_f g = fg. \end{aligned} \tag{2.4}$$

For a symbol $f \in L^\infty(\mathbb{T}^n)$, the Toeplitz operator on $\mathcal{H}^2(\mathbb{T}^n)$ is defined by

$$\begin{aligned} T_f : \mathcal{H}^2(\mathbb{T}^n) &\longrightarrow \mathcal{H}^2(\mathbb{T}^n), \\ g &\longmapsto T_f g = \mathcal{P}(fg) = \mathcal{P}M_f(g). \end{aligned} \tag{2.5}$$

Similarly, the "big" Hankel operator is defined by

$$\begin{aligned} H_f : \mathcal{H}^2(\mathbb{T}^n) &\longrightarrow \left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp, \\ g &\longmapsto H_f g = \mathcal{Q}(fg) = \mathcal{Q}M_f(g). \end{aligned} \tag{2.6}$$

Finally, a dual Toeplitz operator with symbol $f \in L^\infty(\mathbb{T}^n)$ is defined as follows:

$$\begin{aligned} S_f : \left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp &\longrightarrow \left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp, \\ g &\longmapsto S_f g = \mathcal{Q}(fg) = \mathcal{Q}M_f(g). \end{aligned} \tag{2.7}$$

This class of operators (2.7) has been introduced and well elaborated in our earlier paper [1].

Now, for a symbol $f \in L^{\infty}(\mathbb{T}^n)$, the multiplication operator M_f can be represented in the following matrix form:

$$M_f = \begin{pmatrix} T_f & H^*_{\overline{f}} \\ H_f & S_f \end{pmatrix}.$$

Thus, for $f, g \in L^{\infty}(\mathbb{T}^n)$, the identity $M_{fg} = M_f M_g$ leads to the following algebraic identities:

$$\begin{aligned} T_{fg} &= T_f T_g + H^*_{\overline{f}} H_g, \\ \mathcal{S}_{fg} &= H_f H^*_{\overline{g}} + \mathcal{S}_f \mathcal{S}_g, \\ H_{fg} &= H_f T_g + \mathcal{S}_f H_g. \end{aligned} \tag{2.8}$$

For each $\lambda \in \mathbb{D}$, let φ_λ be the linear fractional transformation of $\mathbb{D}$ into itself given by $\varphi_\lambda(u) = \dfrac{\lambda - u}{1 - \overline{\lambda} u}$, $u \in \mathbb{D}$. Every map φ_λ is an automorphism of the unit disk $\mathbb{D}$ satisfying the identity $\varphi_\lambda^{-1} = \varphi_\lambda$. For $\tau \in \mathbb{T}$, the map $\varphi_\lambda(\tau) = \dfrac{\lambda - \tau}{1 - \overline{\lambda}\tau}$ remains well defined on the unit circle $\mathbb{T}$. Moreover, we have the identity $|\varphi_\lambda(\tau)| = 1$. Thus, for any $w = (w_1, \ldots, w_n) \in \mathbb{D}^n$, the map

$$\varphi_w(\zeta) = (\varphi_{w_1}(\zeta_1), \ldots, \varphi_{w_n}(\zeta_n)), \ \zeta = (\zeta_1, ..., \zeta_n) \in \mathbb{T}^n, \tag{2.9}$$

is still well defined on $\mathbb{T}^n$, and we also have $\varphi_w \circ \varphi_w = Id$. For all f and g in $L^2(\mathbb{T}^n)$, we consider the operator of rank 1 defined on $L^2(\mathbb{T}^n)$ by

$$(f \otimes g)h = < h, g > f, \ \forall f \in L^2(\mathbb{T}^n). \tag{2.10}$$

Recall that for a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, we have

$$|\alpha| = \alpha_1 + \ \ldots \ + \alpha_n \ ; \ \alpha! = \alpha_1! \ldots \alpha_n! \ ; \ z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}. \tag{2.11}$$

Finally, for two operators $T, \ S \in \mathscr{B}\left(\mathcal{H}^2(\mathbb{T}^n)\right)$ and for all $f, g \in L^2(\mathbb{T}^n)$, we have

$$T(f \otimes g)S^* = Tf \otimes Sg. \tag{2.12}$$

For further details on function theory, operator theory, and Banach algebras related to our context, we refer to the monographs [5,10,14].
Now, by combining all of these elements, namely equations (2.9)–(2.12), we were able to prove the following crucial representation in our previous paper [1]:

Proposition 2.1 *On the Hardy space $\mathcal{H}^2(\mathbb{T}^n)$, the following operator identity holds:*

$$k_w \otimes k_w = \sum_{|\alpha|=0}^{n} (-1)^{|\alpha|} T_{\varphi_w^\alpha} T_{\overline{\varphi_w}^\alpha}, \ \forall w \in \mathbb{D}^n. \tag{2.13}$$

On the other hand, to obtain spectral properties of dual Toeplitz operators and establish structure theorems for their dual Toeplitz algebras, a curious function $g_{w,s}(z)$, $z \in \mathbb{D}$ has been introduced and well exploited by K. Stroethoff and D. Zheng [11]. Variants of this function have been analogously used by many authors to obtain similar results, see for instance [2,9,13]. Since we are working on hypertori, it is unclear for us how to adopt such type of arguments, and thus, an alternative approach is needed. To overcome this difficulty, we introduce a suitable peaking function, which will play the same central role as the domain function $g_{w,s}(z)$. Although the construction is quite different, but the idea of using peaking functions has its roots in the work of A. M. Davie and N. P. Jewell [4]. For $w \in \mathbb{T}^n$, consider the function

$$g_w(z) = \frac{1}{2^n} \prod_{j=1}^{n} \frac{\overline{z_j} + w_j \overline{z_j}^2}{\overline{w_j}}, \quad \text{for } z \in \mathbb{T}^n.$$

Then, we have the following peaking property of g_w:

Proposition 2.2 *The map $w \in \mathbb{T}^n \longrightarrow g_w \in \left(\mathcal{H}^2(\mathbb{T}^n)\right)^{\perp}$ is continuous, and the function g_w peaks at w.*

Proof First, observe that the function g_w peaks exactly at w, since each component function

$$g_{w_j}(z_j) = \frac{\overline{z_j} + w_j\overline{z_j}^2}{2\overline{w_j}}$$

peaks at w_j, as $g_{w_j}(w_j) = 1$, and $|g_{w_j}(z_j)| = \left|\overline{z_j} w_j \frac{\overline{w_j + z_j}}{2\overline{w_j}}\right| = \left|\frac{w_j+z_j}{2w_j}\right| < 1$ if $z_j \neq w_j$, due to the fact that $\frac{w_j+z_j}{2w_j}$ is a rotation of $g_1(z_j) = \frac{1}{2}(1+z_j)$ that sends the unit circle to the smaller circle of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$ and peaks at 1. On the other hand, it is not difficult to see that $g_w \in \overline{\mathcal{H}_0^2(\mathbb{T}^n)} \subset \left(\mathcal{H}^2(\mathbb{T}^n)\right)^{\perp}$.
For the continuity of this map g_w, we proceed by induction on n. For $n = 1$, we see that, given $w, w_0 \in \mathbb{T}$, we have

$$\begin{aligned} |g_w(z) - g_{w_0}(z)| &= \frac{1}{2}\left|z\frac{w_0 + \overline{w}w_0 z - w - w\overline{w_0}z}{ww_0}\right|, \\ &= \frac{1}{2}\left|(w_0 - w) + z(w_0(\overline{w} - \overline{w_0}) + w(\overline{w} - \overline{w_0}))\right| \leq \frac{3}{2}\left|w - w_0\right|, \end{aligned}$$

whence the continuity of the mapping follows for $n = 1$.
Next, suppose that the continuity is satisfied for $\mathbb{T}^{n-1}$. Put $w = (w', w_n) \in \mathbb{T}^n$, with $w' = (w_1, ..., w_{n-1}) \in \mathbb{T}^{n-1}$, and observe that $g_w(z) = g_{w'}(z')g_{w_n}(z_n)$. It follows that, for $w, v \in \mathbb{T}^n$, we have:

$$\begin{aligned} |g_w(z) - g_v(z)| &= \left|g_{w'}(z')g_{w_n}(z_n) - g_{v'}(z')g_{v_n}(z_n)\right|, \\ &= \left|g_{w'}(z')g_{w_n}(z_n) - g_{w'}(z')g_{v_n}(z_n) + g_{w'}(z')g_{v_n}(z_n) - g_{v'}(z')g_{v_n}(z_n)\right|, \\ &\leq \left|g_{w'}(z')\right|\left|g_{w_n}(z_n) - g_{v_n}(z_n)\right| + \left|g_{v_n}(z_n)\right|\left|g_{w'}(z') - g_{v'}(z')\right|, \\ &\leq \left|g_{w_n}(z_n) - g_{v_n}(z_n)\right| + \left|g_{w'}(z') - g_{v'}(z')\right|. \end{aligned}$$

Thus, the continuity follows from the above inequality and the induction hypothesis. □

Next, put

$$\lambda_k := \int_{\mathbb{T}^n} |g_w(z)|^{2k}\, d\sigma(z),$$

which is independent of $w \in \mathbb{T}^n$, owing to the fact that the measure $d\sigma$ is rotation invariant on hypertori. The following crucial assertion is the analogue of Theorem 2.1 of [4] and Lemma 2.3 of [7]. See also Theorem 2 of [8].

Proposition 2.3 *For every $f \in \mathscr{C}(\mathbb{T}^n)$ and any $w \in \mathbb{T}^n$, we have*

$$\frac{1}{\lambda_k}\int_{\mathbb{T}^n} f(z)\,|g_w(z)|^{2k}\, d\sigma(z) \longrightarrow f(w), \quad as \quad k \longrightarrow \infty.$$

Proof Without loss of generality, we may suppose that f is real-valued. By continuity of f, for any $\varepsilon > 0$, there is an open neighbourhood $\mathscr{V}_w \subset \mathbb{T}^n$ of the point $w \in \mathbb{T}^n$ satisfying

$$\sup_{z\in\mathscr{V}_w} |f(z) - f(w)| < \varepsilon. \tag{2.14}$$

Note also that

$$\sup_{z\in\mathbb{T}^n\setminus\mathscr{V}_w} |f(z) - f(w)| < \infty. \tag{2.15}$$

By the peaking property of g_w, stated in Proposition 2.2, there is a smaller neighbourhood $\mathscr{U}_w \subset \mathscr{V}_w$, such that

$$\sup_{z\in\mathbb{T}^n\setminus\mathscr{V}_w} |g_w(z)| < \inf_{z\in\mathscr{U}_w} |g_w(z)| < 1. \tag{2.16}$$

Now, let us show that

$$\frac{1}{\lambda_k}\int_{\mathbb{T}^n\setminus\mathscr{V}_w} |g_w(z)|^{2k}\ d\sigma(z) \longrightarrow 0,\ \text{ as } k \longrightarrow \infty. \tag{2.17}$$

Clearly, we have

$$\lambda_k = \int_{\mathbb{T}^n} |g_w(z)|^{2k}\ d\sigma(z) \geq \int_{\mathscr{U}_w} |g_w(z)|^{2k}\ d\sigma(z) \geq \sigma(\mathscr{U}_w) \inf_{z\in\mathscr{U}_w} \left\{|g_w(z)|^{2k}\right\}. \tag{2.18}$$

Hence, from (2.16) and (2.18), we see that

$$\left|\frac{1}{\lambda_k}\int_{\mathbb{T}^n\setminus\mathscr{V}_w} |g_w(z)|^{2k}\ d\sigma(z)\right| \leq \frac{\sigma\left(\mathbb{T}^n\setminus\mathscr{V}_w\right)}{\sigma\left(\mathscr{U}_w\right)}\left(\frac{\sup_{z\in\mathbb{T}^n\setminus\mathscr{V}_w} |g_w(z)|}{\inf_{z\in\mathscr{U}_w} |g_w(z)|}\right)^{2k} \longrightarrow 0,\ \text{ as } k \longrightarrow \infty. \tag{2.19}$$

Whence (2.17) follows. It also follows that we have:

$$\frac{1}{\lambda_k}\int_{\mathscr{V}_w} |g_w(z)|^{2k}\ d\sigma(z) \longrightarrow 1,\ \text{ as } k \longrightarrow \infty. \tag{2.20}$$

As mentioned above, since the measure $d\sigma$ is rotation invariant, the quantity λ_k is independent of $w \in \mathbb{T}^n$. Thus, for any $f \in \mathscr{C}(\mathbb{T}^n)$, we have from (2.14)- -(2.16) and (2.18)- -(2.19) that

$$\begin{aligned}\left|\frac{1}{\lambda_k}\int_{\mathbb{T}^n} f(z)\,|g_w(z)|^{2k}\, d\sigma(z) - f(w)\right| &= \left|\frac{1}{\lambda_k}\int_{\mathbb{T}^n} (f(z) - f(w)\,|g_w(z)|^{2k}\, d\sigma(z)\right|,\\ &\leq \sup_{z\in\mathbb{T}^n\setminus\mathscr{V}_w} |f(z) - f(w)|\frac{1}{\lambda_k}\int_{\mathbb{T}^n\setminus\mathscr{V}_w} |g_w(z)|^{2k}\,\sigma(z)\\ &\quad + \sup_{z\in\mathscr{V}_w} |f(z) - f(w)|\frac{1}{\lambda_k}\int_{\mathscr{V}_w} |g_w(z)|^{2k}\, d\sigma(z).\end{aligned}$$

Finally, the continuity of f along with the properties (2.17) and (2.20) yields the desired result. □

An immediate corollary can be obtained.

Corollary 2.4 *Suppose that $f \in L^\infty(\mathbb{T}^n)$ is a nonnegative real-valued function. Then, for every $w \in \mathbb{T}^n$, there exists a sequence $(g_k)_{k\in\mathbb{N}} \subset \left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$ given by $g_k(z) = \frac{g_w^k(z)}{\|g_w^k\|_2}$, with $g_w(z)$ as above, such that $\|fg_k\|_2 \longrightarrow f(w)$, as $k \longrightarrow \infty$.*

The idea of the next key proposition goes back to A.M. Davie and N.P. Jewell [4], namely Theorem 2.2. See also Theorem 2 of [8]. Note that it has been used in a more abstract context by K. Sundberg [12] to establish his beautiful general structure theorem.

Proposition 2.5 *Let P be a polynomial in k non-commuting variables. Then, we have*

$$\|P(S_{f_1}, S_{f_2},\ \ldots, S_{f_k})\| \geq \|P(f_1, f_2,\ \ldots, f_k)\|_\infty,$$

where $f_j \in L^\infty(\mathbb{T}^n)$ for $1 \leq j \leq k$.

Proof Let $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Lambda(f_1, f_2, \ldots, f_k)$, where Λ denotes the joint spectrum of $(f_1, f_2, \ldots, f_k)$. Then, set

$$f := \sum_{j=1}^{k} |f_j - \lambda_j|^2 \in L^\infty(\mathbb{T}^n).$$

Clearly f has a zero in $\mathbb{T}^n$. By Corollary 2.4, given $\varepsilon > 0$, there exists $h \in \left(H^2(\mathbb{T}^n)\right)^\perp$ with $\|h\|_\infty = 1$, such that $\|fh\|_2 < \varepsilon$. Then

$$\sum_{j=1}^{k} \|S_{f_j}(h) - \lambda_j h\|_2^2 = \sum_{j=1}^{k} \|Q(f_j h - \lambda_j h)\|_2^2 \leq \sum_{j=1}^{k} \|f_j h - \lambda_j h\|_2^2 = \|fh\|_2^2 < \varepsilon^2,$$

whence $\|S_{f_j}(h) - \lambda_j h\|_2 < \varepsilon$, for $1 \leq j \leq k$.
Next, observe that the quantity $\|P(S_{f_1}...S_{f_k})(h) - P(\lambda_1, \ldots, \lambda_k)h\|_2$ can be made arbitrarily small by choosing ε small enough. For, consider operators of the form $S_{f_{j_1}} S_{f_{j_2}} \ldots S_{f_{j_m}} - \lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_m}$, where $1 \leq j_l \leq k$, $1 \leq l \leq m$, and observe that

$$S_{f_{j_1}} S_{f_{j_2}} \ldots S_{f_{j_m}} - \lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_m} = \sum_{l=1}^{m} S_{f_{j_1}} \ldots S_{f_{j_{l-1}}} (S_{f_{j_l}} - \lambda_{j_l})\lambda_{j_{l+1}}\ldots\lambda_{j_m}.$$

Without loss of generality, we can assume that $\|f_j\|_\infty \leq 1$ and $|\lambda_j| \leq 1$ for $1 \leq j \leq k$. It follows that:

$$\begin{aligned}\|S_{f_{j_1}}\ldots S_{f_{j_m}}(h) - \lambda_{j_1}\ldots\lambda_{j_m} h\|_2 &\leq \sum_{l=1}^{m} \|S_{f_{j_1}}\ldots S_{f_{j_{l-1}}}(S_{f_{j_l}}(h) - \lambda_{j_l} h)\lambda_{j_{l+1}}\ldots\lambda_{j_m}\|_2, \\ &\leq \sum_{l=1}^{m} \|S_{f_{j_l}}(h) - \lambda_{j_l} h\| < m\varepsilon.\end{aligned}$$

For any $\delta > 0$, we can choose $\varepsilon > 0$, such that

$$\|P(S_{f_1}...S_{f_k})(h) - P(\lambda_1, \ldots, \lambda_k)h\|_2 \leq \delta.$$

The triangle inequality implies that

$$|P(\lambda_1, \ldots, \lambda_k)| \leq \|P(S_{f_1}...S_{f_k})(h)\|_2 + \delta \leq \|P(S_{f_1}...S_{f_k})\| + \delta.$$

As δ is arbitrary, we infer that

$$|P(\lambda_1, \ldots, \lambda_k)| \leq \|P(S_{f_1}...S_{f_k})\|.$$

Since $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Lambda(f_1, f_2, \ldots, f_k)$ are arbitrarily chosen, we conclude that

$$\|P(f_1, f_2, \ldots, f_k)\|_\infty \leq \|P(S_{f_1}, S_{f_2}, \ldots, S_{f_k})\|,$$

which completes the proof. □

Denote the closed ideal generated by the semi-commutators of all bounded dual Toeplitz operators by $\mathfrak{S}$. The next assertion will be needed in the sequel.

Proposition 2.6 *For $f_1, \ldots, f_k \in L^\infty(\mathbb{T}^n)$, the operator $S_{f_1} \cdots S_{f_k} - S_{f_1 \ldots f_k}$ is in $\mathfrak{S}$.*

Proof We proceed by induction. Clearly, $S_{f_1}S_{f_2} - S_{f_1 f_2}$ is a semi-commutator, so

$$S_{f_1}S_{f_2} - S_{f_1 f_2} \in \mathfrak{S}. \tag{2.21}$$

Now, suppose that for any $f_1, \ldots, f_k \in L^\infty(\mathbb{T}^n)$, we have that

$$S_{f_1}\ldots S_{f_{k-1}} - S_{f_1\ldots f_{k-1}} \in \mathfrak{S}. \tag{2.22}$$

Thus, we conclude that

$$\begin{aligned}S_{f_1}\ldots S_{f_k} - S_{f_1\ldots f_k} &= S_{f_1}\ldots S_{f_k} - S_{f_1 \ldots f_k} + (S_{f_1}S_{f_2\ldots f_k} - S_{f_1}S_{f_2\ldots f_k}), \\ &= S_{f_1}(S_{f_2}\ldots S_{f_k} - S_{f_2\ldots f_k}) + (S_{f_1}S_{f_2\ldots f_k} - S_{f_1\ldots f_k}).\end{aligned}$$

Since $\mathfrak{S}$ is an ideal, by (2.21) and (2.22), we see that $S_{f_1}\ldots S_{f_k} - S_{f_1\ldots f_k} \in \mathfrak{S}$. □

3 Structure theorems for some dual Toeplitz algebras

First, recall that $\mathfrak{S}$ denotes the semi-commutator ideal of the dual Toeplitz algebra $\mathscr{I}_\infty := \mathscr{I}(L^\infty(\mathbb{T}^n))$. In the aim of constructing a symbol calculus theory for the C*-algebra $\mathscr{I}_\infty$, we follow the general procedure developed in Douglas' monograph [5] for the Hardy space case, and by M. Engliš [6] for the Bergman space case, as well as by K. Stroethoff and D. Zheng [11] for dual Toeplitz operators in the framework of the Bergman space of the unit disk. It consists in defining a symbol map acting from the dual Toeplitz algebra $\mathscr{I}_\infty$ into the algebra $L^\infty(\mathbb{T}^n)$ as follows. For a single dual Toeplitz operator, we just set $\rho(S_f) = f$, for all $f \in L^\infty(\mathbb{T}^n)$, while for an operator in the form of sums of finite products of dual Toeplitz operators $S = \sum_{i=1}^k S_{f_{i1}} \cdots S_{f_{ik_i}}$, we define ρ by $\rho(S) = \sum_{i=1}^k f_{i1} \, . \, . \, . \, f_{ik_i}$. Now, since $\mathscr{I}_\infty$ is the closed subalgebra of $\mathscr{B}\left(\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp\right)$ generated by $\{S_f : f \in L^\infty(\mathbb{T}^n)\}$, then each $S \in \mathscr{I}_\infty$ is the limit of a sequence of finite sums of finite products of dual Toeplitz operators, and thus, $\rho(S)$ can be defined by passing to the limit.
Furthermore, we have the following assertion whose proof is similar to that of Theorem 8.4 of [11]. Indeed, by similar arguments, we can show that ρ is linear, multiplicative, contractive, and preserves involution, and obtain the following theorem:

Theorem 3.1 *The symbol map ρ is a contractive C*-homomorphism from the dual Toeplitz algebra $\mathscr{I}_\infty$ to $L^\infty(\mathbb{T}^n)$, with null-space* $\ker \rho = \mathfrak{S}$, *such that $\rho(S_f) = f$, for all $f \in L^\infty(\mathbb{T}^n)$.*

Next, define the map $\xi : L^\infty(\mathbb{T}^n) \longrightarrow \mathscr{B}\left(\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp\right)$ by $\xi(f) = S_f$, for $f \in L^\infty(\mathbb{T}^n)$. The following theorem asserts that the mapping ξ_c from $L^\infty := L^\infty(\mathbb{T}^n)$ to $\mathscr{I}_\infty/_{\mathfrak{S}}$ induced by ξ is a $*$-isometrical isomorphism.

Theorem 3.2 *There is a short exact sequence*

$$(0) \longrightarrow \mathfrak{S} \longrightarrow \mathscr{I}_\infty \xrightarrow{\ \rho\ } L^\infty \longrightarrow (0)$$

for which ξ is an isometric cross section.

Proof The mapping ξ_c is readily seen to be linear, multiplicative, and contractive.
To complete the proof, we must show that $\|\xi_c(f)\| \geq \|f\|_\infty$, for $f \in L^\infty(\mathbb{T}^n)$, which implies that ξ_c is an isometry. First, observe that

$$\|\xi_c(f)\| = \|\xi(f) + \mathfrak{S}\| = \left\|S_f + \mathfrak{S}\right\| = \inf\{\left\|S_f + T\right\| : T \in \mathfrak{S}\}.$$

Consequently, it suffices to show that $\left\|S_f + T\right\| \geq \|f\|_\infty$, for any $T \in \mathfrak{S}$. For, let $T \in \mathfrak{S}$. Then, $T = \lim_{k\to\infty} T_k$, where T_k consists of finite sums of operators of the form

$$S_{\phi_1} . \, . \, . \, . \, S_{\phi_l}(S_\phi S_{\psi_0} - S_{\phi\psi})S_{\psi_1} . \, . \, . \, . \, S_{\psi_m}. \tag{3.1}$$

Next, consider a polynomial with non-commutative variables whose terms are of the same form as T_k. Since $\phi_1 . \, . \, . \, . \, \phi_l(\phi\psi - \phi\psi)\psi_1 . \, . \, . \, . \, \psi_m = 0$, Proposition 2.5 implies that

$$\left\|S_f + T_k\right\| \geq \|f\|_\infty . \tag{3.2}$$

Passing to the limit, we obtain the desired inequality, which completes the proof. □

Now, our main task is to establish a similar short exact sequence as in Theorem 3.2 for another dual Toeplitz algebra, namely the one generated by continuous symbols. However, first, we may establish some necessary facts. We start with the following lemma which will be needed in the sequel:

Lemma 3.3 *For all* $1 \leq i, \ j \leq n,$ *we have* $\bar{z}_i \otimes \bar{z}_j \in \mathfrak{S}$.

Proof For any i, j with $1 \leq i, \ j \leq n,$ we have

$$\bar{z}_i \otimes \bar{z}_j = \left(H_{\bar{z}_i}1\right) \otimes \left(H_{\bar{z}_j}1\right) = H_{\bar{z}_i}(1 \otimes 1)H^*_{\bar{z}_j} = H_{\bar{z}_i}(k_0 \otimes k_0)H^*_{\bar{z}_j}.$$

Owing to Proposition 2.1 and identities (2.8), we have

$$\begin{aligned}\bar{z}_i \otimes \bar{z}_j &= H_{\bar{z}_i}\left(\sum_{|\alpha|=0}^{n}(-1)^{|\alpha|}T_{z^\alpha}T_{\bar{z}^\alpha}\right)H^*_{\bar{z}_j},\\ &= \sum_{|\alpha|=0}^{n}(-1)^{|\alpha|}H_{\bar{z}_i}T_{z^\alpha}T_{\bar{z}^\alpha}H^*_{\bar{z}_j},\\ &= \sum_{|\alpha|=0}^{n}(-1)^{|\alpha|}S_{z^\alpha}H_{\bar{z}_i}H^*_{\bar{z}_j}S_{\bar{z}^\alpha},\\ &= \sum_{|\alpha|=0}^{n}(-1)^{|\alpha|}S_{z^\alpha}\left(S_{\bar{z}_i z_j} - S_{\bar{z}_i}S_{z_j}\right)S_{\bar{z}^\alpha}.\end{aligned}$$

Since $S_{\bar{z}_i z_j} - S_{\bar{z}_i}S_{z_j} \in \mathfrak{S}$, we see that $\bar{z}_i \otimes \bar{z}_j \in \mathfrak{S}$, as well. □

Next, denote the ideal of compact operators on $\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$ by $\mathscr{K}$, and recall that if T is an operator on a Hilbert space $\mathcal{H}$ and $\mathcal{N}$ is a closed subspace of $\mathcal{H}$, then $\mathcal{N}$ is called an invariant subspace for T if $T(\mathcal{N}) \subset \mathcal{N}$, and it is called a reducing subspace if, in addition, $T(\mathcal{N}^\perp) \subset \mathcal{N}^\perp$. On the other hand, a subset $\mathscr{E} \subset \mathscr{B}(\mathcal{H})$ is said to be irreducible if no proper closed subspace is reducing for all S in $\mathscr{E}$. A useful fact, (see Theorem 5.39 of [5]), asserts that, if $\mathscr{E}$ is an irreducible C*-algebra contained in $\mathscr{B}(\mathcal{H})$, such that $\mathscr{E}$ contains a compact operator, then $\mathscr{K}$ must be contained in $\mathscr{E}$. Accordingly, we have the following assertion:

Lemma 3.4 *The semi-commutator ideal $\mathfrak{S}$ is irreducible.*

Proof Let $\mathcal{N}$ be a closed linear subspace of $\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$ which is reducing for $\mathfrak{S}$. We have to show that $\mathcal{N} = \left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$. For, let us show that the function $\overline{z_i}$ is in $\mathcal{N}$, for $1 \le i \le n$. Since $\mathcal{N}$ is non-zero, it must contain a non-zero function ϕ. We know that the linear combinations of the monomials from $\left\{z^\lambda\bar{z}^\gamma : \text{ with } \lambda, \gamma \text{ are multi-indices}\right\}$ are dense in $L^2(\mathbb{T}^n)$; so there exist multi-indices α, β, such that $\left\langle\phi, z^\alpha\bar{z}^\beta\right\rangle \neq 0$. Since $\phi \in \mathcal{N} \subset \left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$, ϕ is then orthogonal to z^α. Therefore, we must have $\beta \neq 0$, i.e., there is some $1 \le i_0 \le n$, such that $\beta_{i_0} > 0$. Next, observe that

$$\begin{aligned}\left\langle\phi, z^\alpha\bar{z}^\beta\right\rangle\bar{z}_i &= \left\langle\phi\bar{z}^\alpha z_{i_0}^{\beta_{i_0}-1}\prod_{j\neq i_0}z_j^{\beta_j}, \bar{z}_{i_0}\right\rangle\bar{z}_i = \left\langle S_{\left(\bar{z}^\alpha z_{i_0}^{\beta_{i_0}-1}\Pi_{j\neq i_0}z_j{}^{\beta_j}\right)}(\phi), \bar{z}_{i_0}\right\rangle\bar{z}_i\\ &= \left(\bar{z}_i \otimes \bar{z}_{i_0}\right)S_{\left(\bar{z}^\alpha z_{i_0}^{\beta_{i_0}-1}\Pi_{j\neq i_0}z_j^{\beta_j}\right)}(\phi).\end{aligned}$$

Lemma 3.3 implies that $\left(\bar{z}_i \otimes \bar{z}_{i_0}\right)S_{\left(\bar{z}^\alpha z_{i_0}^{\beta_{i_0}-1}\Pi_{j\neq i_0}z_j^{\beta_j}\right)} \in \mathfrak{S}$. Since $\mathcal{N}$ reduces $\mathfrak{S}$, we see that $\left\langle\phi, z^\alpha\bar{z}^\beta\right\rangle\bar{z}_i \in \mathcal{N}$.

Now, since $\left\langle\phi, z^\alpha\bar{z}^\beta\right\rangle \neq 0$, we infer that $\bar{z}_i \in \mathcal{N}$, for any $1 \le i \le n$.

On the other hand, let ψ be a function in $\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$ which is orthogonal to $\mathcal{N}$, and let α, β be two multi-indices. If $\beta = (0, \ldots, 0)$, then $\left\langle\psi, z^\alpha\bar{z}^\beta\right\rangle = 0$. Therefore, in case $\beta \neq (0, \ldots, 0)$, we may suppose that there exists i_0, such that $\beta_{i_0} > 0$. Similarly as above, we see that

$$\left\langle\psi, z^\alpha\bar{z}^\beta\right\rangle\bar{z}_i = \left(\bar{z}_i \otimes \bar{z}_{i_0}\right)S_{\left(\bar{z}^\alpha z_{i_0}^{\beta_{i_0}-1}\Pi_{j\neq i_0}z_j^{\beta_j}\right)}(\psi).$$

Since $\mathcal{N}$ reduces $\mathfrak{S}$ and $\bar{z}_i \in \mathcal{N}$, we see that $\left\langle\psi, z^\alpha\bar{z}^\beta\right\rangle = 0$ as well. Thus, $\psi = 0$ *a.e.* on $\mathbb{T}^n$. It follows that $\mathcal{N} = \left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$. Therefore, we conclude that $\mathfrak{S}$ is irreducible in $\mathscr{B}\left(\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp\right)$.

Having Lemmas 3.3 and 3.4 at hand, we are now ready to state another important result. It asserts that every compact operator in the dual Toeplitz algebra must be in its semi-commutator ideal. □

Theorem 3.5 *The semi-commutator ideal $\mathfrak{S}$ in the dual Toeplitz algebra $\mathscr{I}(L^\infty(\mathbb{T}^n))$ contains the ideal $\mathscr{K}$ of all compact operators on $\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$.*

Proof Taking $\mathfrak{S} = \mathscr{E}$ in Theorem 5.39 of [5], mentioned above, and combining this fact with Lemmas 3.3 and 3.4, we conclude that $\mathscr{K} \subset \mathfrak{S}$. □

The next consequence is closely related to the commutativity properties stated in Theorem 3.1 of [1].

Corollary 3.6 *If f_1, f_2 and f_3 are functions in $L^\infty(\mathbb{T}^n)$, such that $S_{f_1}S_{f_2} - S_{f_3}$ is compact on $\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$, then $f_1 f_2 = f_3$.*

Proof Since the map ξ_c is linear and multiplicative, by Theorem 3.2, we see that

$$\xi_c(f_1 f_2 - f_3) = (S_{f_1}S_{f_2} - S_{f_3}) + \mathfrak{S}.$$

From Theorem 3.5, we know that $\mathfrak{S}$ contains the ideal $\mathscr{K}$ of compact operators on $\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$, and by assumption we see that $(S_{f_1}S_{f_2} - S_{f_3}) \in \mathfrak{S}$, i.e., we have

$$\xi_c(f_1 f_2 - f_3) = (S_{f_1}S_{f_2} - S_{f_3}) + \mathfrak{S} = \mathfrak{S}.$$

Now, since ξ_c is $*$-isometric, we infer that $f_1 f_2 - f_3$ is the zero map; whence $f_1 f_2 = f_3$. □

Remark 3.7 We can also show the latter by considering the symbol map ρ. By Theorem 3.2, $\mathfrak{S}$ contains all compact operators. Thus $S_{f_1}S_{f_2} - S_{f_3} \in \mathfrak{S}$. However, $\mathfrak{S} = \ker \rho$. Hence, we obtain $\rho\left(S_{f_1}S_{f_2} - S_{f_3}\right) = f_1 f_2 - f_3 = 0$.

Corollary 3.8 *Let $f_1, \ldots, f_k \in L^\infty(\mathbb{T}^n)$. If the product $S_{f_1} \cdots S_{f_k}$ is compact, then we have that $f_1(w) \ldots f_k(w) = 0$, for almost all w in $\mathbb{T}^n$.*

Proof If $S_{f_1} \cdots S_{f_k}$ is compact, then by Theorem 3.5, $S_{f_1} \cdots S_{f_k}$ is in $\mathfrak{S}$. By Proposition 2.6, we have that $S_{f_1} \cdots S_{f_k} - S_{f_1 \ldots f_k} \in \mathfrak{S}$. Thus, we obtain $S_{f_1} \cdots S_{f_k} - (S_{f_1} \cdots S_{f_k} - S_{f_1 \ldots f_k}) = S_{f_1 \ldots f_k} \in \mathfrak{S}$. Therefore, by Theorem 3.1, we have $\rho(S_{f_1 \ldots f_k}) = 0$. Hence, by definition of the symbol map ρ, we infer that $f_1 \ldots f_k = 0$, a.e. on $\mathbb{T}^n$. □

Now, we can extend the results of theorems 3.2 and 3.5 to another related dual Toeplitz algebra mentioned above, namely, the C*-algebra $\mathscr{I}_0 := \mathscr{I}(\mathscr{C}(\mathbb{T}^n))$ generated by dual Toeplitz operators with symbols in $\mathscr{C}(\mathbb{T}^n)$. For, denote by $\mathscr{C} := \mathscr{C}(\mathbb{T}^n)$ the algebra of all continuous functions on $\mathbb{T}^n$, and by $\mathfrak{C}$ the semi-commutator ideal of the dual Toeplitz algebra $\mathscr{I}_0$, as well as by $\mathscr{K}$ the ideal of all compact operators on $\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$. The following theorem says that the quotient algebra $\mathscr{I}(\mathscr{C}(\mathbb{T}^n))/\mathscr{K}$ is $*$-isometrically isomorphic to $\mathscr{C}(\mathbb{T}^n)$.

Theorem 3.9 *The semi-commutator ideal $\mathfrak{C}$ of the dual Toeplitz algebra $\mathscr{I}_0$ coincides with the ideal $\mathscr{K}$ of compact operators on $\left(\mathcal{H}^2(\mathbb{T}^n)\right)^\perp$. Moreover, the sequence*

$$(0) \longrightarrow \mathscr{K} \longrightarrow \mathscr{I}_0 \longrightarrow \mathscr{C} \longrightarrow (0)$$

is short exact.

Proof Arguing in the same manner as in Theorem 3.5, we see that $\mathscr{K} \subset \mathfrak{C}$. On the other hand, by the identities (2.8), for two continuous functions f and g on $\mathbb{T}^n$, the semi-commutator $S_{fg} - S_f S_g = H_f H_{\bar{g}}^*$ is compact. Since $\mathfrak{C}$ is generated by the semi-commutators of dual Toeplitz operators with symbols in $\mathscr{C}(\mathbb{T}^n)$, it follows that $\mathfrak{C}$ is contained in $\mathscr{K}$. Hence, we infer that $\mathfrak{C} \equiv \mathscr{K}$. Now, if we replace $L^\infty(\mathbb{T}^n)$ by $\mathscr{C}(\mathbb{T}^n)$ and $\mathfrak{S}$ by $\mathfrak{C}$ in the proof of Theorem 3.2, we obtain the desired short exact sequence. □

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