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On some curves in three-dimensional β-Kenmotsu manifolds

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Abstract This paper is devoted to examine necessary and sufficient conditions for a Frenet curve to be f-harmonic, f-biharmonic, bi-f-harmonic and f-biminimal in three-dimensional β-Kenmotsu manifolds. In addition, such conditions are investigated for slant curves.

Mathematics Subject Classification 53C25 · 53C43 · 58E20

1 Introduction

The concept of f-Kenmotsu manifold was defined for the first time in [9] by Jannsens and Vanhecke, where f is a real constant. Subsequently, Olszak and Rosca [15] investigated normal locally conformal almost cosymplectic manifolds and gave a differential geometric interpretation of such manifolds which are called f-Kenmotsu manifolds, where f is a function on M, [15].

On the other hand, in [7], Eells and Sampson defined harmonic maps between Riemannian manifolds, and in [6], Lemaire and Eells studied various topics in harmonic maps. On the other hand, Mangione published a paper which he considered harmonic maps in f-Kenmotsu manifold, in [13]. These maps are widely studied as they have an comprehensive field of study due to their wide applications.

In [7], Eells and Sampson studied not only harmonic maps, but also biharmonic maps between the Riemannian manifolds by generalizing harmonic maps. Besides, in [20], Perktaş et al. studied biharmonic curves in three-dimensional f-Kenmotsu manifold for the first time.

f-Harmonic maps between Riemannian manifolds were introduced by Lichnerowicz in 1970 and then examined by Eells and Lemaire, in [6]. f-Harmonic maps, as the solution of inhomogeneous Heisenberg spin systems and continuous spin systems, are of interest not only for mathematicians but also for physicists [2].

In [12], Lu defined f-biharmonic maps, which are the generalization of biharmonic maps. He also studied f-biharmonic maps between Riemannian manifolds, in [5]. Besides, Ou [16] gave a complete classification of f-biharmonic curves in three-dimensional Euclidean space and characterization of f-biharmonic curves in n-dimensional space forms.

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Bi-f-harmonic maps as a generalization of biharmonic and f-harmonic maps were introduced by Ouakkas et al. [17]. In addition, Roth defined a non-f-harmonic, f-biharmonic map called as a proper f-biharmonic map [21]. In [19], Perktaş et al. obtained bi-f -harmonicity conditions for curves in Riemannian manifolds and discussed the particular cases of the Euclidean space, unit sphere and hyperbolic space.

Finally, Loubeau and Montaldo [11] studied biminimal curves in a Riemannian manifold. Moreover, Perktaş et al. handled these types of curves in f-Kenmotsu manifolds in [20]. On the other hand, Karaca and Özgür defined f-biminimal immersions and they handled f -biminimal curves in a Riemannian manifold, in [8].

This paper, which we prepared with the inspiration got from these studies, is organized as follows. In Sects. 2 and 3, we give basic definitions and properties of Frenet curves in three-dimensional β-Kenmotsu manifolds which will be needed in other sections, respectively. In Sect. 4, we prove that there is no proper f-harmonic Frenet curve in three-dimensional β -Kenmotsu manifold. In Sect. 5, we derive the f-biharmonicity conditions for a Frenet curve in a three-dimensional β-Kenmotsu manifold and give a nonexistence theorem. In Sect. 6, we get bi-f-harmonicity conditions not only for a Frenet curve but also a slant and a Legendre curve in three-dimensional β-Kenmotsu manifolds. Finally, in the last section, we investigate f-biminimality conditions.

2 Preliminaries

In this section, we remind some definitions and propositions which will be needed throughout the paper.

A differentiable manifold M^{2n+1} is called an almost contact metric manifold with the almost contact metric structure (φ, ξ, η, g) if it admits a tensor field φ of type $(1, 1)$, a vector field ξ , a 1-form η, and a Riemannian metric tensor field g satisfying the following conditions [3]:

$$
\begin{aligned}
&\varphi^2 = -I + \eta \otimes \xi,\\
&\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(X) = g(X, \xi),\\
&g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),
\end{aligned} \tag{2.1}
$$

where I denotes the identity transformation and $X, Y \in \Gamma(TM)$.

An almost contact metric manifold is said to be an f-Kenmotsu manifold if the Levi–Civita connection ∇ of g satisfies

$$
(\nabla_X \varphi) Y = f\,(g(\varphi X, Y)\,\xi - \eta(Y)\varphi X), \tag{2.2}
$$

$$
\nabla_X \xi = f(X - \eta(X)\xi), \tag{2.3}
$$

where f is a strictly positive differentiable function on M and $X, Y \in \Gamma(TM)$, [9]. Here, if f is equal to a nonzero constant β, then the manifold is called a β-Kenmotsu manifold [9,14]. In particular, if $\beta = 1$, then the manifold is known as a Kenmotsu manifold [10].

For an f-Kenmotsu manifold, the curvature tensor field equation is given as

$$
\begin{aligned}
R(X, Y)Z = &\left(\frac{r}{2} + 2(f^2 + f^{'})\right)(g(Y, Z)X - g(X, Z)Y)\\
&-\left(\frac{r}{2} + 3(f^2 + f^{'})\right)(g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi - \eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X),
\end{aligned} \tag{2.4}
$$

where $X, Y, Z \in TM$ and r is the scalar curvature of M. [13].

Definition 2.1 $\gamma : I \subset \mathbb{R} \longrightarrow M$ is called a slant curve if the contact angle $\theta : I \to [0, 2\pi)$ of given by $\cos\theta(s) = g(T(s), \xi)$ is a constant function.

In particular, if $\theta = \dfrac{\pi}{2}$ (or $\dfrac{3\pi}{2}$), then is called a Legendre curve, [4].

Remark 2.2 For a slant curve in a β-Kenmotsu manifold, we have [4]

$$
\eta(N) = -\frac{\beta}{k_1}(\sin\theta)^2, \tag{2.5}
$$

where $|\sin\theta| \le \min \dfrac{k_1}{\beta}$ and

$$
\eta(B) = \frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - \beta^2(\sin\theta)^2}. \tag{2.6}
$$

Remark 2.3 For a Legendre curve in a β-Kenmotsu manifold, we have

$$N = -\xi, \quad k_1 = \beta, \quad k_2 = 0. \tag{2.7}$$

In particular, a Legendre curve in a β-Kenmotsu manifold is a circle [4,20].

Definition 2.4 Let (M, g) and $(\bar{M}, \bar{g})$ be Riemannian manifolds. Then, a harmonic map $\psi : (M, g) \to (\bar{M}, \bar{g})$ is defined as the critical point of the energy functional

$$E(\psi) = \frac{1}{2}\int_M |\mathrm{d}\psi|^2 v_g,$$

where v_g is the volume element of (M, g). Using Euler–Lagrange equation of the energy functional $E(\psi)$, where $\tau(\psi)$ is the tension field of map ψ, a map is called as harmonic if

$$\tau(\psi) := \mathrm{trace}\nabla \mathrm{d}\psi = 0.$$

Here, ∇ is the connection induced from the Levi–Civita connection $\nabla^{\bar{M}}$ of $\bar{M}$ and the pull-back connection ∇^{ψ} [7,8].

As a natural generalization of harmonic maps, biharmonic maps are defined as below.

Definition 2.5 A map $\psi : (M, g) \to (\bar{M}, \bar{g})$ is defined as biharmonic if it is a critical point, for all variations, of the bienergy functional

$$E_2(\psi) = \frac{1}{2}\int_M |\tau(\psi)|^2 v_g.$$

Namely, ψ is a biharmonic map if $\tau_2(\psi)$ which is the bitension field of ψ equals to

$$\tau_2(\psi) = \mathrm{trace}(\nabla^{\psi}\nabla^{\psi} - \nabla^{\psi}_{\nabla})\tau(\psi) - \mathrm{trace}(R^{\bar{M}}(\mathrm{d}\psi, \tau(\psi))\mathrm{d}\psi) = 0.$$

Here, $R^{\bar{M}}$, the curvature tensor field of $\bar{M}$, is defined as

$$R^{\bar{M}}(X, Y)Z = \nabla^{\bar{M}}_X \nabla^{\bar{M}}_Y Z - \nabla^{\bar{M}}_Y \nabla^{\bar{M}}_X Z - \nabla^{\bar{M}}_{[X,Y]} Z,$$

for any $X, Y, Z \in \Gamma(T\bar{M})$ and ∇^{ψ} is the pull-back connection [7,8].

Note that harmonic maps are always biharmonic and biharmonic maps which are not harmonic are called proper biharmonic maps [18].

Definition 2.6 A map $\psi : (M, g) \to (\bar{M}, \bar{g})$ is said to be an f-harmonic if it is critical point of f-energy functional

$$E_f(\psi) = \frac{1}{2}\int_M f|\mathrm{d}\psi|^2 v_g,$$

where $f \in C^{\infty}(M, \mathbb{R})$ is a positive smooth function. Then, the f-harmonic map equation obtained using Euler–Lagrange equation as follows:

$$\tau_f(\psi) = f\tau(\psi) + \mathrm{d}\psi(\mathrm{grad} f) = 0,$$

where $\tau_f(\psi)$ is the f-tension field of the map ψ.
Note that f-harmonic maps are generalizations of harmonic maps [1].

Definition 2.7 A map $\psi : (M, g) \to (\bar{M}, \bar{g})$ is said to be an f-biharmonic if it is critical point of the f-bienergy functional

$$E_{2,f}(\psi) = \frac{1}{2}\int_M f|\tau(\psi)|^2 v_g.$$

The Euler–Lagrange equation for the f-biharmonic map is given by

$$\tau_{2,f}(\psi) = f\tau_2(\psi) + \Delta f\tau(\psi) + 2\nabla^{\psi}_{\mathrm{grad} f}\tau(\psi) = 0,$$

where $\tau_{2,f}(\psi)$ is the f-bitension field of the map ψ [5,8].

Remark 2.8 An f-biharmonic map turns into a biharmonic map if f is a constant.

Definition 2.9 A map $\psi : (M, g) \to (\bar{M}, \bar{g})$ is said to be a bi-f-harmonic if it is a critical point of the bi-f-energy functional

$$E_{f,2}(\psi) = \frac{1}{2}\int_M |\tau_f(\psi)|^2 v_g.$$

The Euler–Lagrange equation for the bi-f-harmonic map is given by

$$\tau_{f,2}(\psi) = \text{trace}\big((\nabla^\psi f(\nabla^\psi \tau_f(\psi)) - f\nabla^\psi_{\nabla^M}\tau_f(\psi) + fR^{\bar{M}}(\tau_f(\psi), \mathrm{d}\psi)\mathrm{d}\psi\big) = 0,$$

where $\tau_{f,2}(\psi)$ is the bi-f-tension field of the map ψ [19].

Definition 2.10 An immersion $\psi : (M, g) \to (\bar{M}, \bar{g})$ is called biminimal if it is critical point of the bienergy functional $E_2(\psi)$ for variations normal to the image $\psi(M) \subset \bar{M}$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$, such that ψ is a critical point of the λ-bienergy

$$E_{2,\lambda}(\psi) = E_2(\psi) + \lambda E(\psi)$$

for any smooth variation of the map ψ_t $:]-\epsilon, +\epsilon[$, $\psi_0 = \psi$, such that $V = \mathrm{d}\psi_t/\mathrm{d}t|_{t_0}$ is normal to $\psi(M)$ [11]. The Euler–Lagrange equation for a λ-biminimal immersion is

$$[\tau_{2,\lambda}(\psi)]^\perp = [\tau_2(\psi)]^\perp - \lambda[\tau(\psi)]^\perp = 0,$$

for some value of $\lambda \in \mathbb{R}$, where $[.]^\perp$ denotes the normal component of $[.]$.
An immersion is called free biminimal if it is biminimal for $\lambda = 0$ [8,11].

Definition 2.11 An immersion $\psi : (M, g) \to (\bar{M}, \bar{g})$ is called f-biminimal if it is a critical point of the f-bienergy functional $E_{2,f}(\psi)$ for variations normal to the image $\psi(M) \subset \bar{M}$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$, such that ψ is a critical point of the λ-f-bienergy

$$E_{2,\lambda,f}(\psi) = E_{2,f}(\psi) + \lambda E_f(\psi),$$

for any smooth variation of the map ψ_t $:]-\epsilon, +\epsilon[$, $\psi_0 = \psi$. Using the Euler–Lagrange equations for f-harmonic and f-biharmonic maps, an immersion is f-biminimal if

$$[\tau_{2,\lambda,f}(\psi)]^\perp = [\tau_{2,f}(\psi)]^\perp - \lambda[\tau_f(\psi)]^\perp = 0,$$

for some value of $\lambda \in \mathbb{R}$.
An immersion is called free f-biminimal if it is f-biminimal for $\lambda = 0$. If f is a constant, then the immersion is biminimal [8].

3 Frenet curves in three-dimensional β-Kenmotsu manifold

Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic Frenet curve parametrized by arclength s. The Serret–Frenet frame defined on γ denoted by $T = \gamma'(s)$, N, B which are the tangent, the principal normal, and the binormal vector fields, respectively. Here, Serret–Frenet formulas are given as

$$\begin{cases} \nabla_T T = k_1 N \\ \nabla_T N = -k_1 T + k_2 B \\ \nabla_T B = -k_2 N, \end{cases} \tag{3.1}$$

where k_1 and k_2 are the curvature and the torsion of the curve, respectively.
Using these Serret–Frenet formulas, we get

$$\nabla_T T = k_1 N, \tag{3.2}$$
$$\nabla^2_T T = -k_1^2 T + k_1' N + k_1 k_2 B, \tag{3.3}$$
$$\nabla^3_T T = -3k_1 k_1' T + (-k_1^3 - k_1 k_2^2 + k_1'')N + (2k_1' k_2 + k_1 k_2')B, \tag{3.4}$$

and by substituting (3.2) to the curvature tensor formula (2.4), we have

$$R(T,\nabla_T T)T = -k_1\left(\frac{r}{2}+2\beta^2\right)N \\ -k_1\left(\frac{r}{2}+3\beta^2\right)[\eta(N)\eta(T)T-\eta(T)^2N-\eta(N)\xi]. \tag{3.5}$$

With the help of these calculations, we shall present f-harmonicity, f-biharmonicity, bi-f-harmonicity, and f-biminimality conditions of a Frenet curve in a three-dimensional β-Kenmotsu manifold as in the following sections.

4 f-Harmonic curves in three-dimensional β-Kenmotsu manifold

In this section, we investigate the f-harmonicity condition for a curve in a three-dimensional β-Kenmotsu manifold. Let $\gamma : I \subset \mathbb{R} \to M$ be a curve in a three-dimensional β-Kenmotsu manifold. Then, via definition (2.6), the f-harmonicity condition given as below

$$\tau_f(\gamma) = f^{'}T + f\nabla_T T = f^{'}T + f(k_1N) = 0. \tag{4.1}$$

From (4.1), we get following nonexistence theorem.

Theorem 4.1 *There is no proper f-harmonic Frenet curve in a three-dimensional β-Kenmotsu manifold.*

Proof Using the condition given in (4.1), it is easy to see that $f^{'} = 0$, so f is a constant. This situation contradicts the definition of a proper f-harmonic curve. □

5 f-Biharmonic curves in three-dimensional β-Kenmotsu manifold

Here, we derive the f-biharmonicity condition for a curve in a three-dimensional β-Kenmotsu manifold. By substituting (3.2)–(3.5) in the equation of f-bitension field $\tau_{2,f}(\gamma)$, f-biharmonicity condition is obtained as below

$$\begin{aligned}\tau_{2,f}(\gamma) &= f\tau_2(\gamma) + (\Delta f)\tau(\gamma) + 2\nabla^{\gamma}_{\operatorname{grad} f}\tau(\gamma)\\ &= f(\nabla^3_T T - R(T,\nabla_T T)T) + f^{''}\nabla_T T + 2f^{'}\nabla^2_T T\\ &= \left[-3k_1k_1^{'}f + k_1 f\left(\frac{r}{2}+3\beta^2\right)\eta(N)\eta(T) - 2k_1^2 f^{'}\right]T\\ &\quad+\left[\left(-k_1^3 - k_1k_2^2 + k_1^{''} + k_1\left(\frac{r}{2}+2\beta^2\right) - k_1\left(\frac{r}{2}+3\beta^2\right)\eta(T)^2\right)f + 2k_1^{'}f^{'} + k_1f^{''}\right]N\\ &\quad+\left[(2k_1^{'}k_2 + k_1k_2^{'})f + 2k_1k_2f^{'}\right]B\\ &\quad-k_1 f\left(\frac{r}{2}+3\beta^2\right)\eta(N)\xi\\ &= 0.\end{aligned} \tag{5.1}$$

Taking the scalar product of (5.1) with T, N and B, respectively, we can state the following theorem.

Theorem 5.1 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \to M$ be a non-geodesic Frenet curve parametrized by arclength s. Then, γ is an f-biharmonic curve if and only if*

$$\begin{cases} 3k_1^{'}f + 2k_1f^{'} = 0,\\ k_1^2 + k_2^2 - \dfrac{k_1^{''}}{k_1} - \dfrac{r}{2} - 2\beta^2 + \left(\dfrac{r}{2}+3\beta^2\right)(\eta(T)^2+\eta(N)^2) - 2\dfrac{k_1^{'}}{k_1}\dfrac{f^{'}}{f} - \dfrac{f^{''}}{f} = 0,\\ \dfrac{2k_1^{'}k_2}{k_1} + k_2^{'} - \left(\dfrac{r}{2}+3\beta^2\right)\eta(N)\eta(B) + 2k_2\dfrac{f^{'}}{f} = 0.\end{cases} \tag{5.2}$$

From Theorem 5.1, we obtain the following nonexistence theorems about f-biharmonic curves in three-dimensional β-Kenmotsu manifolds.

Theorem 5.2 *There does not exist an f-biharmonic Frenet curve with constant curvature k_1 in a three-dimensional β-Kenmotsu manifold.*

Proof Let k_1 be a constant. Then, the first equation of (5.2) reduces to $2k_1 f^{'} = 0$. Here, it is easy to see that f becomes a constant. This situation contradicts the definition of an f-biharmonic curve. □

Theorem 5.3 *There does not exist an f-biharmonic Legendre curve in a three-dimensional β-Kenmotsu manifold.*

Proof For a Legendre curve in a β-Kenmotsu manifold, it is well known that $k_1 = \beta$ where β is a constant, [4]. Therefore, the assumption $k_1 \neq$ constant contradicts the definition of a β-Kenmotsu manifold. □

Theorem 5.4 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic Frenet curve parametrized by arclength s. Then, γ is an f-biharmonic curve if and only if r, k_1 and k_2 satisfies the following conditions:*

$$\begin{cases} k_1^2 + k_2^2 = \dfrac{3}{4}\left(\dfrac{k_1^{'}}{k_1}\right)^2 - \dfrac{k_1^{''}}{2k_1} + \dfrac{r}{2} + 2\beta^2 - \left(\dfrac{r}{2} + 3\beta^2\right)(\eta(T)^2 + \eta(N)^2), \\ k_2^{'} - k_2\dfrac{k_1^{'}}{k_1} - \left(\dfrac{r}{2} + 3\beta^2\right)\eta(N)\eta(B) = 0, \end{cases} \tag{5.3}$$

where $f = ck_1^{-\frac{3}{2}}$ and c is the integration constant.

Proof From the first equation of (5.2), it is easy to see that f equals to $ck_1^{-\frac{3}{2}}$. Then, by substituting f and its derivatives into the second and third equation of (5.2), the proof is completed. □

Next, we shall examine some special cases for an f-biharmonic curve in a three-dimensional β-Kenmotsu manifold.

Case 5-I: If $k_2 = 0$, then (5.3) reduces to

$$\begin{cases} k_1^2 = \dfrac{3}{4}\left(\dfrac{k_1^{'}}{k_1}\right)^2 - \dfrac{k_1^{''}}{2k_1} + \dfrac{r}{2} + 2\beta^2 - \left(\dfrac{r}{2} + 3\beta^2\right)(\eta(T)^2 + \eta(N)^2), \\ \left(\dfrac{r}{2} + 3\beta^2\right)\eta(N)\eta(B) = 0. \end{cases} \tag{5.4}$$

Here, if we assume that $\eta(N) = 0$, then we obtain that γ is a Legendre curve. However, it is well known that for a Legendre curve in a three-dimensional β-Kenmotsu manifold $\eta(N) = -1$, which contradicts our assumption. Therefore, in the second equation of (5.4), $\eta(N)$ cannot be zero. In this case, we have following two subcases:

Subcase 5-I-1: If $(\frac{r}{2} + 3\beta^2) = 0$, then (5.4) reduces to

$$k_1^2 = \frac{3}{4}\left(\frac{k_1^{'}}{k_1}\right)^2 - \frac{k_1^{''}}{2k_1} + \frac{r}{2} + 2\beta^2.$$

Then, we conclude the following theorem.

Theorem 5.5 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold of constant scalar curvature $r = -6\beta^2$ and $\gamma : I \longrightarrow M$ be a non-geodesic Frenet curve parametrized by arclength s with $k_1 \neq$ constant and $k_2 = 0$. Then, γ is an f-biharmonic curve if and only if*

$$k_1 = \sqrt{\frac{3}{4}\left(\frac{k_1^{'}}{k_1}\right)^2 - \frac{k_1^{''}}{2k_1} - \beta^2},$$

where $f = ck_1^{-\frac{3}{2}}$ and c is the integration constant.

Subcase 5-I-2: If $\eta(B)=0$, then (5.4) reduces to

$$k_1^2=\frac{3}{4}\left(\frac{k_1^{'}}{k_1}\right)^2-\frac{k_1^{''}}{2k_1}+\frac{r}{2}+2\beta^2-\left(\frac{r}{2}+3\beta^2\right)(\eta(T)^2+\eta(N)^2).$$

Since $\xi=\eta(T)T+\eta(N)N$ and $(\eta(T))^2+(\eta(N))^2=1$, we give the following theorem.

Theorem 5.6 *Let (M,φ,ξ,η,g) be a three-dimensional β-Kenmotsu manifold and $\gamma: I\longrightarrow M$ be a non-geodesic Frenet curve parametrized by arclength s with $k_1\neq$ constant, $k_2=0$ and $\eta(B)=0$. Then, γ is an f-biharmonic curve if and only if k_1 satisfy the following differential equation:*

$$k_1^2=\frac{3}{4}\left(\frac{k_1^{'}}{k_1}\right)^2-\frac{k_1^{''}}{2k_1}-\beta^2,$$

where $f=ck_1^{-\frac{3}{2}}$ and c is the integration constant.

Case 5-II: If $k_2=$ constant >0, then (5.3) reduces to

$$\begin{cases} k_1^2+k_2^2=\frac{3}{4}\left(\frac{k_1^{'}}{k_1}\right)^2-\frac{k_1^{''}}{2k_1}+\frac{r}{2}+2\beta^2-\left(\frac{r}{2}+3\beta^2\right)(\eta(T)^2+\eta(N)^2),\\ k_2\frac{k_1^{'}}{k_1}+\left(\frac{r}{2}+3\beta^2\right)\eta(N)\eta(B)=0. \end{cases} \tag{5.5}$$

Hence, we have the following theorem.

Theorem 5.7 *Let (M,φ,ξ,η,g) be a three-dimensional β-Kenmotsu manifold and $\gamma: I\longrightarrow M$ be a non-geodesic Frenet curve parametrized by arclength s with $k_1\neq$ constant and $k_2=$ constant. Then, γ is an f-biharmonic curve if and only if*

$$f=c(\mathrm{e}^{\int\frac{1}{k_2}(\frac{r}{2}+3\beta^2)\eta(N)\eta(B)\mathrm{d}s})^{\frac{3}{2}}$$

and k_1, k_2, r satisfy the following differential equation:

$$k_1^2+k_2^2=\frac{3}{4}(\frac{k_1^{'}}{k_1})^2-\frac{k_1^{''}}{2k_1}+\frac{r}{2}+2\beta^2+\frac{k_2k_1^{'}(\eta(T)^2+\eta(N)^2)}{k_1\eta(N)\eta(B)}.$$

Proof From second equation of (5.5), we obtain that

$$k_1=\mathrm{e}^{-\int\frac{1}{k_2}(\frac{r}{2}+3\beta^2)\eta(N)\eta(B)\mathrm{d}s}.$$

Then, by substituting this result to the first equation of (5.5) and the formula $f=ck_1^{-\frac{3}{2}}$, the proof is completed. □

Now, assume that $\gamma: I\longrightarrow M$ is a slant curve such that N is non-parallel to ξ. By means of Definition 2.1, Remark 2.2 and Theorem 5.1, the following theorem and corollary are obtained.

Theorem 5.8 *Let (M,φ,ξ,η,g) be a three-dimensional β-Kenmotsu manifold and $\gamma: I\longrightarrow M$ be a non-geodesic slant curve. Then, γ is an f-biharmonic curve if and only if*

$$\begin{cases} 3k_1^{'}f+2k_1f^{'}=0,\\ k_1^2+k_2^2=\frac{k_1^{''}}{k_1}+2\frac{k_1^{'}}{k_1}\frac{f^{'}}{f}+\frac{f^{''}}{f}+\frac{r}{2}+2\beta^2-\left(\frac{r}{2}+3\beta^2\right)\left((\cos\theta)^2+\frac{\beta^2}{k_1^2}(\sin\theta)^4\right),\\ \frac{2k_1^{'}k_2}{k_1}+k_2^{'}+2k_2\frac{f^{'}}{f}-\left(\frac{r}{2}+3\beta^2\right)\left(\frac{\beta}{k_1}(\sin\theta)^2\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2-\beta^2(\sin\theta)^2}\right)=0, \end{cases} \tag{5.6}$$

where $k_1\neq$ constant.

Corollary 5.9 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic slant curve. Then, γ is an f-biharmonic curve if and only if*

$$\begin{cases} k_1^2 + k_2^2 = \frac{3}{4}\left(\frac{k_1^{'}}{k_1}\right)^2 - \frac{k_1^{''}}{2k_1} + \frac{r}{2} + 2\beta^2 - \left(\frac{r}{2} + 3\beta^2\right)\left((\cos\theta)^2 + \frac{\beta^2}{k_1^2}(\sin\theta)^4\right), \\ k_2^{'} - \frac{k_1^{'}k_2}{k_1} - \left(\frac{r}{2} + 3\beta^2\right)\left(\frac{\beta}{k_1}(\sin\theta)^2\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - \beta^2(\sin\theta)^2}\right) = 0, \end{cases} \tag{5.7}$$

where $k_1 \neq \text{constant},\ f = ck_1^{-\frac{3}{2}}$, and c is the integration constant.

Now, we discuss some special cases for a slant f-biharmonic curve in a three-dimensional β-Kenmotsu manifold.

Case 5-III: If $k_1 \neq \text{constant}$ and $k_2 = 0$, then (5.6) reduces to

$$\begin{cases} 3k_1^{'}f + 2k_1 f^{'} = 0, \\ k_1^2 = \frac{k_1^{''}}{k_1} + 2\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{f^{''}}{f} + \frac{r}{2} + 2\beta^2 - \left(\frac{r}{2} + 3\beta^2\right)\left((\cos\theta)^2 + \frac{\beta^2}{k_1^2}(\sin\theta)^4\right), \\ \left(\frac{r}{2} + 3\beta^2\right)\left(\frac{\beta}{k_1}(\sin\theta)^2\right)\left(\frac{|sin\theta|}{k_1}\sqrt{k_1^2 - \beta^2(sin\theta)^2}\right) = 0. \end{cases} \tag{5.8}$$

Then, we get the following theorem:

Theorem 5.10 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic slant curve. Then, for $k_1 \neq \text{constant}$ and $k_2 = 0$, γ is an f-biharmonic curve if and only if M is of constant scalar curvature $r = -6\beta^2$ and*

$$f = ck_1^{-\frac{3}{2}}, \quad k_1 = \sqrt{\frac{3}{4}\left(\frac{k_1^{'}}{k_1}\right)^2 - \frac{k_1^{''}}{2k_1} - \beta^2}.$$

Case 5-IV: If $k_1 \neq \text{constant}$ and $k_2 = \text{constant} > 0$, then (5.6) reduces to

$$\begin{cases} 3k_1^{'}f + 2k_1 f^{'} = 0, \\ k_1^2 + k_2^2 = \frac{k_1^{''}}{k_1} + 2\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{f^{''}}{f} + \frac{r}{2} + 2\beta^2 - \left(\frac{r}{2} + 3\beta^2\right)\left((\cos\theta)^2 + \frac{\beta^2}{k_1^2}(\sin\theta)^4\right), \\ \frac{2k_1^{'}k_2}{k_1} + 2k_2\frac{f^{'}}{f} - \left(\frac{r}{2} + 3\beta^2\right)\left(\frac{\beta}{k_1}(\sin\theta)^2\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - \beta^2(\sin\theta)^2}\right) = 0. \end{cases} \tag{5.9}$$

Using first equation of (5.9), we get $f = ck_1^{-\frac{3}{2}}$. Then, by substituting this result to the second and third equation of (5.9), we conclude the following.

Theorem 5.11 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic slant curve with $k_1 \neq \text{constant}$ and $k_2 = \text{constant} > 0$. Then, γ is an f-biharmonic curve if and only if*

$$\begin{cases} k_1^2 + k_2^2 = \frac{3}{4}\left(\frac{k_1^{'}}{k_1}\right)^2 - \frac{k_1^{''}}{2k_1} + \frac{r}{2} + 2\beta^2 - \left(\frac{r}{2} + 3\beta^2\right)\left((\cos\theta)^2 + \frac{\beta^2}{k_1^2}(\sin\theta)^4\right), \\ \frac{k_1^{'}k_2}{k_1} + \left(\frac{r}{2} + 3\beta^2\right)\left(\frac{\beta}{k_1}(\sin\theta)^2\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - \beta^2(\sin\theta)^2}\right) = 0, \end{cases} \tag{5.10}$$

where $f = ck_1^{-\frac{3}{2}}$ and c is the integration constant.

6 Bi-f-harmonic curves in three-dimensional β-Kenmotsu manifold

In this section, we derive the bi-f-harmonicity condition for a Frenet curve in a three-dimensional β-Kenmotsu manifold. Using Eqs. (3.2)–(3.5) in the equation of bi-f-tension field $\tau_{f,2}(\gamma)$, see [19], we obtain bi-f-harmonicity condition as below

$$\begin{aligned}
\tau_{f,2}(\gamma) &= \mathrm{trace}\big(\nabla^{\gamma} f(\nabla^{\gamma}\tau_f(\gamma)) - f\nabla^{\gamma}_{\nabla}\tau_f(\gamma) + fR(\tau_f(\gamma), d\gamma)d\gamma\big)\\
&= (ff^{''})^{'}T + (3ff^{''} + 2(f^{'})^2)\nabla_T T + 4ff^{'}\nabla_T^2 T + f^2\nabla_T^3 T + f^2 R(\nabla_T T, T)T\\
&= \big[(ff^{''})^{'} - 4k_1^2 ff^{'} - 3k_1k_1^{'}f^2 + f^2k_1(\frac{r}{2} + 3\beta^2)\eta(N)\eta(T)\big]T\\
&\quad + \big[(3ff^{''} + 2(f^{'})^2)k_1 + 4ff^{'}k_1^{'} - f^2(k_1^3 + k_1k_2^2 - k_1^{''}) + f^2k_1(\frac{r}{2} + 2\beta^2 - \Big(\frac{r}{2} + 3\beta^2\Big)\eta(T)^2)\big]N\\
&\quad + \big[4ff^{'}k_1k_2 + f^2(2k_1^{'}k_2 + k_1k_2^{'})\big]B\\
&\quad - f^2k_1\Big(\frac{r}{2} + 3\beta^2\Big)\eta(N)\xi\\
&= 0. \qquad (6.1)
\end{aligned}$$

Therefore, we can state the following theorem:

Theorem 6.1 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic Frenet curve parametrized by arclength s. Then, γ is a bi-f-harmonic curve if and only if*

$$\begin{cases}
(ff^{''})^{'} - 4k_1^2 ff^{'} - 3k_1k_1^{'}f^2 = 0,\\
\begin{cases}(3ff^{''} + 2(f^{'})^2)k_1 + 4ff^{'}k_1^{'} - f^2(k_1^3 + k_1k_2^2 - k_1^{''})\\
+f^2k_1[\frac{r}{2} + 2\beta^2 - \Big(\frac{r}{2} + 3\beta^2\Big)(\eta(T)^2 + \eta(N)^2)] = 0,\end{cases}\\
4f^{'}k_1k_2 + f(2k_1^{'}k_2 + k_1k_2^{'}) - fk_1\Big(\frac{r}{2} + 3\beta^2\Big)\eta(N)\eta(B) = 0.
\end{cases} \qquad (6.2)$$

Now, we shall examine some special cases for the bi-f-harmonic curves in a three-dimensional β-Kenmotsu manifold.

Case 6-I: If $k_1 = \text{constant} > 0$ and $k_2 = 0$, then (6.2) reduces to

$$\begin{cases}
(ff^{''})^{'} - 4k_1^2 ff^{'} = 0,\\
(3ff^{''} + 2(f^{'})^2)k_1 + f^2k_1\Big(-k_1^2 + \frac{r}{2} + 2\beta^2 - \Big(\frac{r}{2} + 3\beta^2\Big)(\eta(T)^2 + \eta(N)^2)\Big) = 0,\\
\Big(\frac{r}{2} + 3\beta^2\Big)\eta(N)\eta(B) = 0.
\end{cases} \qquad (6.3)$$

Since any of $(\frac{r}{2} + 3\beta^2)$ or $\eta(B)$ in the third equation of (6.3) can be equal to zero, we examine Case 6-I in two subcases.

Subcase 6-I-1: If $(\frac{r}{2} + 3\beta^2) = 0$, then (6.3) reduces to

$$\begin{cases}
(ff^{''})^{'} - 4k_1^2 ff^{'} = 0,\\
(3ff^{''} + 2(f^{'})^2) - f^2(k_1^2 + \beta^2) = 0.
\end{cases}$$

Then, we have the following theorem.

Theorem 6.2 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic Frenet curve with $k_1 = \text{constant} > 0$, $k_2 = 0$ and $r = -6\beta^2$. Then, γ is a bi-f-harmonic curve if and only if k_1, f, β satisfy the following differential equation:*

$$2f^{'}f^{''} + (5k_1^2 - \beta^2)ff^{'} = 0.$$

Corollary 6.3 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic Frenet curve with $k_1 = \text{constant} > 0$, $k_2 = 0$ and $r = -6\beta^2$. Then, γ is a bi-f-harmonic curve if and only if either*

$$f(s) = c_1 \cos\left(\sqrt{\frac{\beta^2 - 5k_1^2}{2}}s\right) + c_2 \sin\left(\sqrt{\frac{\beta^2 - 5k_1^2}{2}}s\right),$$

where $\beta^2 - 5k_1^2 < 0$, or

$$f(s) = c_3 e^{-\sqrt{\frac{\beta^2 - 5k_1^2}{2}}s} + c_4 e^{\sqrt{\frac{\beta^2 - 5k_1^2}{2}}s},$$

where $\beta^2 - 5k_1^2 > 0$, c_i $(1 \le i \le 4)$ are real constants.

Subcase 6-I-2: If $\eta(B) = 0$, then (6.3) reduces to

$$\begin{cases} (ff'')' - 4k_1^2 ff' = 0, \\ (3ff'' + 2(f')^2)k_1 + f^2k_1\left(-k_1^2 + \frac{r}{2} + 2\beta^2 - \left(\frac{r}{2} + 3\beta^2\right)(\eta(T)^2 + \eta(N)^2)\right) = 0. \end{cases}$$

Since $\xi = \eta(T)T + \eta(N)N$ and $(\eta(T))^2 + (\eta(N))^2 = 1$, we give the following theorem.

Theorem 6.4 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic Frenet curve parametrized by arclength s with $k_1 = \text{constant} > 0$, $k_2 = 0$ and $\eta(B) = 0$. Then, γ is a bi-f-harmonic curve if and only if*

$$\begin{cases} (ff'')' - 4k_1^2 ff' = 0, \\ (3ff'' + 2(f')^2) - f^2\left(k_1^2 + \beta^2\right) = 0. \end{cases}$$

Case 6-II: If $k_1 = \text{constant} > 0$ and $k_2 = \text{constant} > 0$, then (6.2) reduces to

$$\begin{cases} (ff'')' - 4k_1^2 ff' = 0, \\ (3ff'' + 2(f')^2)k_1 - f^2k_1\left(k_1^2 + k_2^2 - \frac{r}{2} - 2\beta^2 + \left(\frac{r}{2} + 3\beta^2\right)(\eta(T)^2 + \eta(N)^2)\right) = 0, \\ 4f'k_1k_2 - fk_1\left(\frac{r}{2} + 3\beta^2\right)\eta(N)\eta(B) = 0. \end{cases} \tag{6.4}$$

Then, we have the following.

Theorem 6.5 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic Frenet curve with $k_1 = \text{constant} > 0$ and $k_2 = \text{constant} > 0$. Then, γ is a bi-f-harmonic curve if and only if*

$$\begin{cases} (ff'')' - 4k_1^2 ff' = 0, \\ 3ff'' + 2(f')^2 - f^2(k_1^2 + k_2^2) + f^2\left[\frac{r}{2} + 2\beta^2 - \left(\frac{r}{2} + 3\beta^2\right)(\eta(T)^2 + \eta(N)^2)\right] = 0, \\ 4f'k_2 - f\left(\frac{r}{2} + 3\beta^2\right)\eta(N)\eta(B) = 0. \end{cases}$$

Now, assume that $\gamma : I \longrightarrow M$ is a slant curve, such that N is non-parallel to ξ. By means of Definition 2.1, Remark 2.2 and Theorem 6.1, the following theorem is obtained.

Theorem 6.6 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic slant curve. Then, γ is a bi-f-harmonic curve if and only if*

$$\begin{cases} (ff'')' - 4k_1^2 ff' - 3k_1k_1' f^2 = 0, \\ \begin{cases} (3ff'' + 2(f')^2)k_1 + 4ff'k_1' - f^2(k_1^3 + k_1k_2^2 - k_1'') \\ + f^2k_1\left[\frac{r}{2} + 2\beta^2 - \left(\frac{r}{2} + 3\beta^2\right)\left((\cos\theta)^2 + \frac{\beta^2}{k_1^2}(\sin\theta)^4\right)\right] = 0, \end{cases} \\ \begin{cases} 4ff'k_1k_2 + f^2(2k_1'k_2 + k_1k_2') \\ + f^2k_1\left(\frac{r}{2} + 3\beta^2\right)\left(\frac{\beta}{k_1}(\sin\theta)^2\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - \beta^2(\sin\theta)^2}\right) = 0. \end{cases} \end{cases} \tag{6.5}$$

We shall consider some special cases for bi-f-harmonic slant curves in a three-dimensional β-Kenmotsu manifold.

Case 6-III: If k_1 = constant > 0 and $k_2 = 0$, then (6.5) reduces to

$$\begin{cases} (ff^{''})^{'} - 4k_1^2 ff^{'} = 0, \\ (3ff^{''} + 2(f^{'})^2)k_1 - f^2k_1^3 + f^2k_1\left[\frac{r}{2} + 2\beta^2 - \left(\frac{r}{2} + 3\beta^2\right)\left((\cos\theta)^2 + \frac{\beta^2}{k_1^2}(sin\theta)^4\right)\right] = 0, \\ f^2k_1\left(\frac{r}{2} + 3\beta^2\right)\left(\frac{\beta}{k_1}(\sin\theta)^2\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - \beta^2(\sin\theta)^2}\right) = 0. \end{cases}$$

Hence, we give the following.

Theorem 6.7 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a slant curve. Then, for k_1 = constant > 0 and $k_2 = 0$, γ is a bi-f-harmonic curve if and only if M is of constant scalar curvature $r = -6\beta^2$ and*

$$2f^{'}f^{''} + (5k_1^2 - \beta^2)ff^{'} = 0.$$

Case 6-IV: If k_1 = constant > 0 and k_2 = constant > 0, then (6.5) reduces to

$$\begin{cases} (ff^{''})^{'} - 4k_1^2 ff^{'} = 0, \\ \begin{cases} (3ff^{''} + 2(f^{'})^2)k_1 - f^2(k_1^3 + k_1k_2^2) \\ +f^2k_1\left[\frac{r}{2} + 2\beta^2 - \left(\frac{r}{2} + 3\beta^2\right)\left((\cos\theta)^2 + \frac{\beta^2}{k_1^2}(\sin\theta)^4\right)\right] = 0, \end{cases} \\ 4ff^{'}k_1k_2 - f^2k_1\left(\frac{r}{2} + 3\beta^2\right)\left(\frac{\beta}{k_1}(\sin\theta)^2\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - \beta^2(\sin\theta)^2}\right) = 0. \end{cases}$$

We have the following theorem.

Theorem 6.8 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic slant curve with k_1 = constant > 0 and k_2 = constant > 0. Then, γ is a bi-f-harmonic if and only if*

$$\begin{cases} (ff^{''})^{'} - 4k_1^2 ff^{'} = 0, \\ 3ff^{''} + 2(f^{'})^2 + f^2\left[-k_1^2 - k_2^2 + \frac{r}{2} + 2\beta^2 - \left(\frac{r}{2} + 3\beta^2\right)\left((\cos\theta)^2 + \frac{\beta^2}{k_1^2}(sin\theta)^4\right)\right] = 0, \\ 4ff^{'}k_2 - f\left(\frac{r}{2} + 3\beta^2\right)\left(\frac{\beta}{k_1}(\sin\theta)^2\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - \beta^2(\sin\theta)^2}\right) = 0. \end{cases}$$

Now, assume that $\gamma : I \longrightarrow M$ is a Legendre curve. By means of Definition 2.1, Remark 2.3, and Theorem 6.1, the following theorem is obtained.

Theorem 6.9 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic Legendre curve. Then, γ is a bi-f-harmonic curve if and only if the function f satisfies the following differential equation:*

$$(ff^{''})^{'} + 3f^{''}f + 2(f^{'})^2 - 4\beta^2 f^{'}f - 2\beta^2 f^2 = 0.$$

7 f-Biminimal curves in three-dimensional β-Kenmotsu manifold

Finally, in this section, we derive the f-biminimality condition for a Frenet curve in a three-dimensional β-Kenmotsu manifold. The f-biminimality condition, see [8], obtained as below using normal components of

f-tension and f-bitension field with the help of λ-f-bienergy functional

$$\begin{aligned}[\tau_{2,\lambda,f}(\gamma)]^{\perp} &= [\tau_{2,f}(\gamma)]^{\perp} - \lambda[\tau_f(\gamma)]^{\perp} \\ &= \Big[\Big(-k_1^3 - k_1k_2^2 + k_1^{''} + k_1\Big(\frac{r}{2} + 2\beta^2 - \Big(\frac{r}{2} + 3\beta^2\Big)\eta(T)^2 - \lambda\Big)\Big)f + 2k_1^{'}f^{'} + k_1f^{''}\Big]N \\ &\quad + [(2k_1^{'}k_2 + k_1k_2^{'})f + 2k_1k_2f^{'}]B \\ &\quad - k_1 f\Big(\frac{r}{2} + 3\beta^2\Big)\eta(N)\xi \\ &= 0. \end{aligned} \tag{7.1}$$

Using (7.1) we obtain the following.

Theorem 7.1 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic Frenet curve parametrized by arclength s. Then, γ is an f-biminimal curve if and only if*

$$\begin{cases} k_1^2 + k_2^2 = \dfrac{k_1^{''}}{k_1} + 2\dfrac{k_1^{'}f^{'}}{k_1 f} + \dfrac{f^{''}}{f} + 2\beta^2 - \lambda + \dfrac{r}{2} - \Big(\dfrac{r}{2} + 3\beta^2\Big)(\eta(T)^2 + \eta(N)^2), \\ (2k_1^{'}k_2 + k_1k_2^{'})f + 2k_1k_2f^{'} - k_1 f\Big(\dfrac{r}{2} + 3\beta^2\Big)\eta(N)\eta(B) = 0. \end{cases} \tag{7.2}$$

Now, we discuss some special cases for a f-biminimal curve in a three-dimensional β-Kenmotsu manifold.

Case 7-I: If $k_1 = \text{constant} > 0$ and $k_2 = 0$, then (7.2) reduces to

$$\begin{cases} k_1^2 = \dfrac{f^{''}}{f} - \lambda + \dfrac{r}{2} + 2\beta^2 - \Big(\dfrac{r}{2} + 3\beta^2\Big)(\eta(T)^2 + \eta(N)^2), \\ k_1 f\Big(\dfrac{r}{2} + 3\beta^2\Big)\eta(N)\eta(B) = 0. \end{cases} \tag{7.3}$$

In the third equation of (7.3), $(\frac{r}{2} + 3\beta^2)$ or $\eta(B)$ can be equal to zero, so we consider Case 7-I in two subcases.

Subcase 7-I-1: If $(\frac{r}{2} + 3\beta^2) = 0$, then (7.3) reduces to

$$k_1^2 = \frac{f^{''}}{f} + \frac{r}{2} + 2\beta^2 - \lambda. \tag{7.4}$$

Subcase 7-I-2: If $\eta(B) = 0$, we know that $\eta(T)^2 + \eta(N)^2 = 1$, which reduces (7.3) to the following:

$$k_1^2 = \frac{f^{''}}{f} - \lambda - \beta^2. \tag{7.5}$$

Since in Subcase 7-I-1, $r = -6\beta^2$, then (7.4) and (7.5) overlap.
Thus, we get the following theorem.

Theorem 7.2 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic Frenet curve with $k_1 = \text{constant} > 0$, $k_2 = 0$. Then, γ is an f-biminimal curve if and only if either $r = -6\beta^2$ or $\eta(B) = 0$ and, in both cases, f satisfies*

$$f(s) = c_1\cos((\sqrt{k_1^2 + \beta^2 + \lambda})s) + c_2\sin((\sqrt{k_1^2 + \beta^2 + \lambda})s),$$

where $k_1^2 + \beta^2 + \lambda < 0$, and

$$f(s) = c_3 e^{-(\sqrt{k_1^2+\beta^2+\lambda})s} + c_4 e^{(\sqrt{k_1^2+\beta^2+\lambda})s},$$

where $k_1^2 + \beta^2 + \lambda > 0$, c_i $(1 \le i \le 4)$ are real constants.

Case 7-II: If $k_1 = \text{constant} > 0$ and $k_2 = \text{constant} > 0$, then (7.2) reduces to

$$\begin{cases} k_1^2 + k_2^2 = \dfrac{f''}{f} - \lambda + \dfrac{r}{2} + 2\beta^2 - \left(\dfrac{r}{2} + 3\beta^2\right)(\eta(T)^2 + \eta(N)^2), \\ 2k_2 f' - f\left(\dfrac{r}{2} + 3\beta^2\right)\eta(N)\eta(B) = 0. \end{cases} \tag{7.6}$$

Using second equation of (7.6) into the first equation, we get the following theorem.

Theorem 7.3 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic Frenet curve with $k_1 = \text{constant} > 0$ and $k_2 = \text{constant} > 0$. Then, γ is an f-biminimal curve if and only if*

$$k_1^2 + k_2^2 = \frac{f''}{f} - \frac{2k_2 f'(\eta(T)^2 + \eta(N)^2)}{f\eta(N)\eta(B)} + 2\beta^2 + \frac{r}{2} - \lambda.$$

Now, assume that $\gamma : I \longrightarrow M$ is a slant curve, such that N is non-parallel to ξ. By means of Definition 2.1, Remark 2.2, and Theorem 7.1, the following theorem is obtained.

Theorem 7.4 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic slant curve. Then, γ is an f-biminimal curve if and only if*

$$\begin{cases} k_1^2 + k_2^2 = \dfrac{k_1''}{k_1} + 2\dfrac{k_1' f'}{k_1 f} + \dfrac{f''}{f} - \lambda + \dfrac{r}{2} + 2\beta^2 - \left(\dfrac{r}{2} + 3\beta^2\right)\left((\cos\theta)^2 + \dfrac{\beta^2}{k_1^2}(\sin\theta)^4\right), \\ (2k_1' k_2 + k_1 k_2')f + 2k_1 k_2 f' + k_1 f\left(\dfrac{r}{2} + 3\beta^2\right)\left(\dfrac{\beta}{k_1}(\sin\theta)^2\right)\left(\dfrac{|\sin\theta|}{k_1}\sqrt{k_1^2 - \beta^2(\sin\theta)^2}\right) = 0. \end{cases} \tag{7.7}$$

Here, we examine some cases for the f-biminimal slant curves in a three-dimensional β-Kenmotsu manifold.

Case 7-III: If $k_1 = \text{constant} > 0$ and $k_2 = 0$, then (7.7) reduces to

$$\begin{cases} k_1^2 = \dfrac{f''}{f} - \lambda + \dfrac{r}{2} + 2\beta^2 - \left(\dfrac{r}{2} + 3\beta^2\right)\left((\cos\theta)^2 + \dfrac{\beta^2}{k_1^2}(\sin\theta)^4\right), \\ k_1 f\left(\dfrac{r}{2} + 3\beta^2\right)\left(\dfrac{\beta}{k_1}(\sin\theta)^2\right)\left(\dfrac{|\sin\theta|}{k_1}\sqrt{k_1^2 - \beta^2(\sin\theta)^2}\right) = 0. \end{cases}$$

Then, we have the following.

Theorem 7.5 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic slant curve. Then, for $k_1 = \text{constant} > 0$ and $k_2 = 0$, γ is an f-biminimal curve if and only if M is of constant curvature $r = -6\beta^2$ and either*

$$f(s) = c_1\cos((\sqrt{k_1^2 + \beta^2 + \lambda})s) + c_2\sin((\sqrt{k_1^2 + \beta^2 + \lambda}))s),$$

where $k_1^2 + \beta^2 + \lambda < 0$, or

$$f(s) = c_3 e^{\left(\sqrt{k_1^2+\beta^2+\lambda}\right)s} + c_4 e^{-\left(\sqrt{k_1^2+\beta^2+\lambda}\right)s},$$

where $k_1^2 + \beta^2 + \lambda > 0$, c_i $(1 \le i \le 4)$ are real constants.

Case 7-IV: If $k_1 = \text{constant} > 0$ and $k_2 = \text{constant} > 0$, then (7.7) reduces to

$$\begin{cases} k_1^2 + k_2^2 = \dfrac{f''}{f} - \lambda + \dfrac{r}{2} + 2\beta^2 - \left(\dfrac{r}{2} + 3\beta^2\right)\left((\cos\theta)^2 + \dfrac{f^2}{k_1^2}(\sin\theta)^4\right), \\ 2k_1 k_2 f' + k_1 f\left(\dfrac{r}{2} + 3\beta^2\right)\left(\dfrac{f}{k_1}(\sin\theta)^2\right)\left(\dfrac{|\sin\theta|}{k_1}\sqrt{k_1^2 - f^2(\sin\theta)^2}\right) = 0. \end{cases} \tag{7.8}$$

Hence, we get

Theorem 7.6 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic slant curve with $k_1 = \text{constant} > 0$ and $k_2 = \text{constant} > 0$. Then, γ is an f-biminimal curve if and only if*

$$k_1^2 + k_2^2 = -\frac{2k_2 f' \left((\cos\theta)^2 + \frac{\beta^2}{k_1^2}(\sin\theta)^4\right) - 3\beta^2 \left(f \frac{\beta}{k_1}(\sin\theta)^2\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - \beta^2(\sin\theta)^2}\right)}{f\left(\frac{\beta}{k_1}(\sin\theta)^2\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - \beta^2(\sin\theta)^2}\right)} + \frac{f''}{f} + 2\beta^2 - \lambda.$$

Now, assume that $\gamma : I \longrightarrow M$ is a Legendre curve. Via Definition 2.1, Remark 2.3, and Theorem 7.1, the following theorem is obtained.

Theorem 7.7 *Let $(M, \varphi, \xi, \eta, g)$ be a three-dimensional β-Kenmotsu manifold and $\gamma : I \longrightarrow M$ be a non-geodesic Legendre curve. Then, γ is an f-biminimal curve if and only if either*

$$f(s) = c_1 \cos((\sqrt{2\beta^2 + \lambda})s) + c_2 \sin((\sqrt{2\beta^2 + \lambda})s),$$

where $2\beta^2 + \lambda < 0$, or

$$f(s) = c_3 \mathrm{e}^{-\left(\sqrt{2\beta^2+\lambda}\right)s} + c_4 \mathrm{e}^{\left(\sqrt{2\beta^2+\lambda}\right)s},$$

where $2\beta^2 + \lambda > 0$, c_i $(1 \leq i \leq 4)$ are real constants.

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References

1. Ara, M.: Geometry of f-harmonic maps. Kodai Math. J. **22**, 243–263 (1999)
2. Baird, P.; Wood, J.C.: Harmonic Morphisms Between Riemannian Manifolds. London Mathematical Society Monographs. 29. Oxford University Press, Oxford (2003)
3. Blair, D.E.: Contact Manifolds. Contact Manifolds in Riemannian Geometry, pp. 1–16. Springer, Berlin (1976)
4. Calin, C.; Crasmareanu, M.; Munteanu, M.I.: Slant curves in three-dimensional f-Kenmotsu manifolds. J. Math. Anal. Appl. **394**(1), 400–407 (2012)
5. Chiang, Y.J.: f-Biharmonic maps between Riemannian manifolds. In: Proceedings of the Fourteenth International Conference on Geometry, Integrability and Quantization, pp. 74–86 (2013)
6. Eells, J.; Lemaire, L.: A report on harmonic maps. Bull. Lond. Math. Soc. **10**, 1–68 (1978)
7. Eells, J.; Sampson, J.H.: Harmonic mappings of Riemannian manifolds. Am. J. Math. **86**, 109–160 (1964)

8. Gürler, F.; Özgür, C.: f-Biminimal immersions. Turk. J. Math. **41**, 564–575 (2017)
9. Janssens, D.; Vanhecke, L.: Almost contact structures and curvature tensors. Kodai Math. J. **4**(1), 1–27 (1981)
10. Kenmotsu, K.: A class of almost contact Riemannian manifolds. Tohoku Math. J. Second Ser. **24**(1), 93–103 (1972)
11. Loubeau, E.; Montaldo, S.: Biminimal immersions. Proc. Edinb. Math. Soc. **51**, 421–437 (2008)
12. Lu, W.: On f-bi-harmonic maps and bi-f-harmonic maps between Riemannian manifolds. Sci. China Math. **58**(7), 1483–1498 (2015)
13. Mangione, V.: Harmonic maps and stability on f-Kenmotsu manifolds. Int. J. Math. Math. Sci. **7**, Article ID 798317 (2008)
14. Olszak, Z.: Locally conformal almost cosymplectic manifolds. Colloq. Math. Inst. Math. Pol. Acad. Sci. **57**, 73–87 (1989)
15. Olszak, Z.; Rosca, R.: Normal locally conformal almost cosymplectic manifolds. Publ. Math. Debr. **39**(3–4), 315–323 (1991)
16. Ou, Y.L.: On f-biharmonic maps and f-biharmonic submanifolds. Pac. J. Math. **271**(2), 461–477 (2014)
17. Ouakkas, S.; Nasri, R.; Djaa, M.: On the f-harmonic and f-biharmonic maps. Jpn. J. Geom. Topol **10**(1), 11–27 (2010)
18. Perktaş, S.Y.; Blaga, A.M.; Acet, B.E.; Erdoğan, F.E.: Magnetic biharmonic curves on three dimensional normal almost paracontact metric manifolds. AIP Conf. Proc. **1991**(1), 020004 (2018)
19. Perktaş, S.Y.; Blaga, A.M.; Erdoğan, F.E.; Acet, B.E.: Bi-f-harmonic curves and hypersurfaces. Filomat **33**(16), 5167–5180 (2019)
20. Perktaş, S.Y.; Acet, B.; Ouakkas, S.: On biharmonic and biminimal curves in 3-dimensional f-Kenmotsu manifold. Fundam. Contemp. Math. Sci. **1**(1), 14–22 (2020)
21. Roth, J., Upadhyay, A.: f-Biharmonic and bi-f-harmonic submanifolds of generalized space forms. arXiv preprint arXiv:1609.08599 (2016)