Arab. J. Math. (2023) 12:127–138
https://doi.org/10.1007/s40065-022-00404-x

Arabian Journal of Mathematics

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Isometries on almost Ricci–Yamabe solitons

Received: 14 April 2022 / Accepted: 14 October 2022 / Published online: 26 October 2022

Abstract The purpose of the present paper is to examine the isometries of almost Ricci–Yamabe solitons. Firstly, the conditions under which a compact gradient almost Ricci–Yamabe soliton is isometric to Euclidean sphere $S^n(r)$ are obtained. Moreover, we have shown that the potential f of a compact gradient almost Ricci–Yamabe soliton agrees with the Hodge–de Rham potential h. Next, we studied complete gradient almost Ricci–Yamabe soliton with $\alpha \neq 0$ and non-trivial conformal vector field with non-negative scalar curvature and proved that it is either isometric to Euclidean space E^n or Euclidean sphere S^n. Also, solenoidal and torse-forming vector fields are considered. Lastly, some non-trivial examples are constructed to verify the obtained results.

Mathematics Subject Classification 53C15 · 53C24 · 53C44

1 Introduction

One of the most significant approaches to understanding the geometric structure in Riemannian geometry is to study the theory of geometric flows. The Ricci flow is a well-known geometric flow introduced by Hamilton [15], who used it to prove a three-dimensional sphere theorem [14]. The idea of the Ricci flow is contributed to the proof of Thurston's conjecture, including as a special case, the Poincaré conjecture. The Ricci soliton on a Riemannian manifold (M, g) are self-limiting solutions to Ricci flow and is defined by

$$\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric} = \lambda g, \tag{1.1}$$

where $\mathcal{L}_V g$ denotes the Lie-derivative of g along potential vector field V, Ric is the Ricci curvature of M^{2n+1} and λ, a real constant. When the vector field V is the gradient of a smooth function f on M^{2n+1}, that is, $V = \nabla f$, then we say that Ricci soliton is gradient (for details see [9,20]). According to Petersen and Wylie [20], a gradient Ricci soliton is rigid if it is a flat $N \times_\Gamma \mathbb{R}^k$, where N is Einstein and gave certain classification. The notion of almost Ricci soliton was introduced by Pigola et al. [21] by taking λ as a smooth function in the definition of Ricci soliton (1.1). The authors in [2] studied the rigidity of gradient almost Ricci solitons and showed that it is isometric to the Euclidean space $\mathbb{R}^n$ or sphere $\mathbb{S}^n$. Barros et al. [3], Yang and Zhang [28], Cao et al. [8] obtain several rigidity results.

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To tackle the Yamabe problem on manifolds of positive conformal Yamabe invariant, Hamilton introduced the geometric flow known as Yamabe flow. The Yamabe soliton is a self-similar solution to the Yamabe flow. On a Riemannian manifold (M, g), a Yamabe soliton is given by

$$\frac{1}{2}\mathcal{L}_V g = (R - \lambda), \tag{1.2}$$

where R is the scalar curvature of the manifold and λ, a real constant. Even though both the Ricci and Yamabe solitons are similar in dimension $n = 2$, the solitons behave differently for dimension $n > 2$ as the Yamabe soliton preserves the conformal class of the metric but the Ricci soliton does not in general. If λ is a smooth function in (1.2), then it is called almost Yamabe soliton. Alkhaldi et al. [1] gave a characterization of almost Yamabe soliton with conformal vector field. Barbosa and Ribeiro [4] gave some rigidity results for Yamabe almost soliton.

Güler and Crasmareanu [13], in 2019, introduced the notion of the Ricci–Yamabe map which is a scalar combination of Ricci and Yamabe flow. In [13], the authors define the following:

Definition 1.1 [13] The map $RY^{(\alpha,\beta,g)} : I \to T_2^s(M)$ given by:

$$RY^{(\alpha,\beta,g)} = \frac{\partial g}{\partial t}(t) + 2\alpha \mathrm{Ric}(t) + \beta R(t) g(t),$$

is called the (α, β)-Ricci–Yamabe map of the Riemannian flow (M, g). If

$$RY^{(\alpha,\beta,g)} \equiv 0,$$

then $g(.)$ will be called an (α, β)-Ricci–Yamabe flow.

The Ricci–Yamabe flow can be Riemannian or semi-Riemannian or singular Riemannian flow due to the involvement of scalars α and β. This kind of different choices can be useful in some physical models such as relativity theory. The Ricci–Yamabe soliton emerges as the limit of the solution of Ricci–Yamabe flow.

Definition 1.2 A Riemannian manifold $(M^n, g), n > 2$ is said to admit almost Ricci–Yamabe soliton $(g, V, \lambda, \alpha, \beta)$ if there exist smooth function λ such that

$$\mathcal{L}_V g + 2\alpha \mathrm{Ric} = (2\lambda - \beta R) g, \tag{1.3}$$

where $\alpha, \beta \in \mathbb{R}$. Almost Ricci–Yamabe soliton is of particular interest as it generalizes a large group of well-known solitons such as:

- Ricci almost soliton ($\alpha = 1, \beta = 0$).
- almost Yamabe soliton ($\alpha = 0, \beta = 1$).
- Ricci–Bourguignon almost soliton ($\alpha = 1, \beta = -2\rho$).

Also, if λ is constant, then it includes Ricci soliton, Yamabe soliton and Ricci–Bourguignon soliton among others.

If V is a gradient of some smooth function f on M, then the above notion is called gradient almost Ricci–Yamabe soliton and then (1.3) reduces to

$$\nabla^2 f + \alpha \mathrm{Ric} = \left(\lambda - \frac{1}{2}\beta R\right) g, \tag{1.4}$$

where $\nabla^2 f$ is the Hessian of f.

The almost Ricci–Yamabe soliton (ARYS) is said to be expanding, shrinking or steady if $\lambda < 0, \lambda > 0$ or $\lambda = 0$ respectively. In particular, if λ is constant, then ARYS reduces to Ricci–Yamabe soliton. Many geometers such as [10,11,22] analyzed Ricci–Yamabe solitons. In [23,26], authors studied Ricci–Yamabe soliton in different spacetimes. Singh and Khatri [16,25] studied ARYS in almost contact manifolds. Siddiqi et al. [24] consider ARYS on static spacetimes.

Motivated by the above studies, we investigated the ARYS under certain conditions. The present paper is organized as follows: In Sect. 2, several rigidity results are obtained by following the methods of Barros and Ribeiro [5] for compact almost Ricci soliton. Also, we obtained the conditions under which compact gradient ARYS is isometric to the Euclidean sphere $S^n(r)$. In Sect. 3, ARYS with conformal, solenoidal and torse-forming vector fields are considered. We showed that a complete ARYS with $\alpha \neq 0$ and potential vector field as conformal vector field is either isometric to Euclidean space E^n or Euclidean sphere $S^n(r)$. Also, complete gradient ARYS with conformal vector field is investigated. Lastly, ARYS with solenoidal and torse-forming vector fields are considered and obtained several rigidity results which are proved by constructing non-trivial examples.

2 Some rigidity results on ARYS

Before proceeding to the main results of this paper, we obtained several lemmas on ARYS and gradient ARYS which would be used later.

Lemma 2.1 *For a gradient ARYS* $(M^n, g, \nabla f, \lambda)$, *the following formula holds*:

(1) $2\Delta f + (2\alpha + n\beta)R = 2n\lambda$.
(2) $\{\alpha + (n-1)\beta\}\nabla_i R = 2(m-1)\nabla_i \lambda + 2R_{is}\nabla^s f$, $\alpha \neq 0$, $n \geq 3$.
(3) $\alpha(\nabla_j R_{ik} - \nabla_i R_{jk}) = \frac{\alpha}{\alpha+(n-1)\beta}\left[(\nabla_j\lambda)g_{ik} - (\nabla_i\lambda)g_{jk}\right] + \frac{\alpha+(n-3)\beta}{\alpha+(n-1)\beta} R_{ijks}\nabla^s f$, $\alpha + (n-1)\beta \neq 0$.
(4) *For* $\alpha + (n-1)\beta \neq 0$, *we have*

$$\frac{1}{2}\nabla(R + |\nabla f|^2) = \frac{n-1}{\alpha + (n-1)\beta}\nabla\lambda + \left(\lambda - \frac{\beta R}{2}\right)\nabla f + \frac{1 - \alpha^2 - (n-1)\alpha\beta}{\alpha + (n-1)\beta}\mathrm{Ric}(\nabla f).$$

Proof Equation (1) is directly obtained by taking trace of the soliton equation.

For Eq. (2), we consider Schur's Lemma ($n > 2$), we have

$$\frac{1}{2}\nabla_i R = \mathrm{divRic}_i = g^{jk}\nabla_k R_{ij},$$
$$\Longrightarrow \frac{\alpha}{2}\nabla_i R = g^{jk}\left\{(\nabla_k\lambda)g_{ij} - \frac{\beta}{2}(\nabla_k R)g_{ij}\right\} - g^{jk}\nabla_k\nabla_i\nabla_j f.$$

Then, using Ricci identity in the above expression gives

$$(\alpha + \beta)\nabla_i R = 2\nabla_i\lambda - 2\nabla_i(\Delta f) - 2R_{is}\nabla^s f.$$

Thus, in regard of equation (1) yields

$$[\alpha + (n-1)\beta]\nabla_i R = 2(n-1)\nabla_i\lambda + 2R_{is}\nabla^s f.$$

This gives Eq. (2).

In consequence of Eq. (2) and Ricci identity, we obtain

$$R_{jiks}\nabla^s f + \alpha(\nabla_j R_{ik} - \nabla_i R_{jk}) = (\nabla_j\lambda)g_{ik} - (\nabla_i\lambda)g_{jk} + \frac{\beta}{2}[(\nabla_i R)g_{jk} - (\nabla_j R)g_{ik}].$$

Further, inserting (2) in the above expression and then simplifying, we obtain Eq. (3). Now, using Eq. (3) and the fundamental equation as a (1, 1)-tensor, Eq. (4) follows, which thus completes the proof. □

Petersen and Wylie [20] obtained the following Bochner formula for Killing and gradient field as:

Lemma 2.2 *Given a vector field* X *on a Riemannian manifold* (M^n, g), *we have*

$$\mathrm{div}(\mathcal{L}_X g)(X) = \frac{1}{2}\Delta|X|^2 - |\nabla X|^2 + \mathrm{Ric}(X, X) + D_X \mathrm{div}X.$$

When $X = \nabla f$ *is a gradient field and* Z *is any vector field*, *we have*

$$\mathrm{div}(\mathcal{L}_{\nabla f} g)(Z) = 2\mathrm{Ric}(Z, \nabla f) + 2D_Z \mathrm{div}\nabla f,$$

or, *in* (1, 1)-*tensor notation*,

$$\mathrm{div}\nabla\nabla f = \mathrm{Ric}(\nabla f) + \nabla\Delta f.$$

Taking an inner product of Eq. (2) in Lemma 2.1 by arbitrary vector field Z gives

$$[\alpha + (n-1)\beta]g(\nabla R, Z) = 2(n-1)g(\nabla\lambda, Z) + 2\mathrm{Ric}(\nabla f, Z). \tag{2.1}$$

In particular,

$$[\alpha + (n-1)\beta]g(\nabla R, \nabla f) = 2(n-1)g(\nabla\lambda, \nabla f) + 2\mathrm{Ric}(\nabla f, \nabla f). \tag{2.2}$$

and

$$[\alpha + (n-1)\beta]|\nabla R|^2 = 2(n-1)g(\nabla\lambda, \nabla R) + 2\mathrm{Ric}(\nabla f, \nabla R). \tag{2.3}$$

Lemma 2.3 *For an ARYS* (M^n, g, X, λ) $(n \geq 3)$ *with* $\alpha \neq 0$, *we have*

$$\frac{2\alpha + n\beta}{2}\Delta|X|^2 - (2\alpha + n\beta)|\nabla X|^2 + \beta(2\alpha + n\beta)g(\nabla R, X) + (2\alpha + n\beta)\mathrm{Ric}(X, X) + 2[(n-2)\alpha - n\beta]g(\nabla\lambda, X) + n\beta D_X \mathrm{div}X = 0.$$

Proof Taking divergence of ARYS equation yields

$$\mathrm{div}(\mathcal{L}_X g)(X) + 2(\alpha + \beta)\mathrm{div}\mathrm{Ric}(X) = 2D_X\lambda. \tag{2.4}$$

We have from (1.3), $2\mathrm{div}X + (2\alpha + n\beta)R = 2n\lambda$, which gives

$$2D_X\mathrm{div}X + (2\alpha + n\beta)D_X R = 2nD_X\lambda. \tag{2.5}$$

Making use of Schur's Lemma, Lemma 2.2, (2.4) and (2.5), we get the required results. This completes the proof. □

Moreover, from (1.3) we have

$$\frac{1}{2}\mathcal{L}_X g(X, X) + \alpha\mathrm{Ric}(X, X) = \left(\lambda - \frac{\beta R}{2}\right)|X|^2.$$

In consequence of this in Lemma 2.3, we get

$$\frac{2\alpha + n\beta}{2}\left(\Delta - \frac{D_X}{\alpha}\right)|X|^2 = (2\alpha + n\beta)|\nabla X|^2 - \beta(2\alpha + n\beta)g(\nabla R, X) - \frac{2\alpha + n\beta}{\alpha}\left(\lambda - \frac{\beta R}{2}\right)|X|^2 + 2[n\beta - (n-2)\alpha]g(\nabla\lambda, X) - n\beta D_X\mathrm{div}X. \tag{2.6}$$

Corollary 2.4 *For a gradient ARYS* $(M^n, g, \nabla f, \lambda)$ $(n \geq 3)$ *with* $\alpha \neq 0$, *we have*

$$\frac{2\alpha + n\beta}{2}\Delta|\nabla f|^2 = (2\alpha + n\beta)|\mathrm{Hess}\, f|^2 - \beta(2\alpha + n\beta)g(\nabla R, \nabla f) - (2\alpha + n\beta)\mathrm{Ric}(\nabla f, \nabla f) + 2[n\beta - (n-2)\alpha]g(\nabla\lambda, \nabla f) - n\beta D_{\nabla f}\mathrm{div}(\nabla f).$$

Theorem 2.5 *Let* (M^n, g, X, λ) $(n \geq 3)$ *be a compact ARYS. If* $\alpha \neq \{0, -\frac{n\beta}{2}\}$ *and*

$$\int_M \left\{\mathrm{Ric}(X, X) + \beta g(\nabla R, X) + \frac{n\beta}{2\alpha + n\beta}\nabla_X\mathrm{div}X + \frac{2[(n-2)\alpha - n\beta]}{2\alpha + n\beta}g(\nabla\lambda, X)\right\}\mathrm{d}v_g \leq 0,$$

then X *is Killing and* M^n *is RYS.*

Proof Since M^n is compact, taking integration of Lemma 2.3 gives

$$\int_M |\nabla X|^2 \mathrm{d}v_g = \int_M \left\{\mathrm{Ric}(X, X) + \beta g(\nabla R, X) + \frac{n\beta}{2\alpha + n\beta}\nabla_X\mathrm{div}X + \frac{2[(n-2)\alpha - n\beta]}{2\alpha + n\beta}g(\nabla\lambda, X)\right\}\mathrm{d}v_g. \tag{2.7}$$

In view of our hypothesis

$$\int_M \left\{\mathrm{Ric}(X, X) + \beta g(\nabla R, X) + \frac{n\beta}{2\alpha + n\beta}\nabla_X\mathrm{div}X + \frac{2[(n-2)\alpha - n\beta]}{2\alpha + n\beta}g(\nabla\lambda, X)\right\}\mathrm{d}v_g \leq 0,$$

and (2.7), we get $\nabla X = 0$ which implies $\mathcal{L}_X g = 0$, i.e., X is Killing vector field. In this case, ARYS will be simply RYS since M^n will be Einstein manifold, which implies that λ is constant. This completes the proof.□

Corollary 2.6 *Let (M^n, g, X, λ) $(n \geq 3)$ be a compact RYS. If $\alpha \neq \{0, -\frac{n\beta}{2}\}$ and*

$$\int_M \left[\mathrm{Ric}(X, X) + \beta g(\nabla R, X) + \frac{n\beta}{2\alpha + n\beta} \nabla_X \mathrm{div} X\right] \mathrm{d}v_g \leq 0,$$

then X is Killing.

In particular, for $\alpha = 1$ and $\beta = -2\rho$ in Theorem 2.5, we recover Theorem 1.6 of [12]. Moreover, Theorem 3 in [5] for compact Ricci soliton is obtained for $\alpha = 1$, $\beta = 0$.

The next theorem generalizes Theorem 3.5 of [12] which is obtained for compact gradient Ricci–Bourguignon almost soliton, which is the case for $\alpha = 1$ and $\beta = -2\rho$.

Theorem 2.7 *Let $(M^n, g, \nabla f, \lambda)$ $(n \geq 3)$ be a compact ARYS with $\alpha \neq 0$ and $\alpha + (n-1)\beta \neq 0$. Then we have*

(1) $\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 \mathrm{d}v_g = \frac{\alpha(n-2)}{2n} \int_M g(\nabla R, \nabla f) \mathrm{d}v_g$.
(2) $\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 \mathrm{d}v_g = \frac{\alpha(n-2)}{2n[\alpha+(n-1)\beta]} \int_M [2(n-1) g(\nabla\lambda, \nabla f) + 2\mathrm{Ric}(\nabla f, \nabla f)] \mathrm{d}v_g$.

Proof From the gradient ARYS, from (1.4) we have

$$(\mathrm{Hess}\ f)(\nabla f) + \alpha \mathrm{Ric}(\nabla f) = \left(\lambda - \frac{\beta R}{2}\right) \nabla f. \tag{2.8}$$

Combining second argument of Lemma 2.1 and (2.8), then taking divergence of the obtained expression yields

$$\begin{aligned} \alpha[\alpha + (n-1)\beta]\Delta R = 2\alpha(n-1)\Delta\lambda + (2\lambda - \beta R)\Delta f - \Delta|\nabla f|^2 \\ + 2g(\nabla\lambda, \nabla f) - \beta g(\nabla R, \nabla f). \end{aligned} \tag{2.9}$$

Now, using commuting covariant derivative and Ricci identity, we have

$$\begin{aligned} \nabla_i \nabla_i (g(\nabla_j f, \nabla_j f)) &= 2\nabla_i (g(\nabla_i \nabla_j f, \nabla_j f)), \\ &= 2g(\nabla_i \nabla_i \nabla_j f, \nabla_j f) + 2|\nabla^2 f|^2, \\ &= 2g(\nabla_i \nabla_i \nabla_j f + R_{iijs} \nabla^s f, \nabla_j f) + 2|\nabla^2 f|^2, \\ &= 2g(\nabla(\Delta f), \nabla f) + 2\mathrm{Ric}(\nabla f, \nabla f) + 2|\nabla^2 f|^2. \end{aligned}$$

Making use of the above expression in (2.9), we get

$$\begin{aligned} \alpha\{\alpha + (n-1)\beta\}\Delta R + 2g(\nabla\Delta f, \nabla f) + 2\mathrm{Ric}(\nabla f, \nabla f) + 2|\nabla^2 f|^2 \\ = 2\alpha(n-1)\Delta\lambda + (2\lambda - \beta R)\Delta f + 2g(\nabla\lambda, \nabla f) - \beta g(\nabla R, \nabla f). \end{aligned} \tag{2.10}$$

Combining first argument of Lemma 2.1, (2.2) and (2.10), we obtain

$$\alpha\{\alpha + (n-1)\beta\}\Delta R - \alpha g(\nabla R, \nabla f) + 2|\nabla^2 f|^2 = 2\alpha(n-1)\Delta\lambda + (2\lambda - \beta R)\Delta f. \tag{2.11}$$

Making use of the fact that $|\nabla^2 f - \frac{\Delta f}{n} g|^2 = |\nabla^2 f|^2 - \frac{(\Delta f)^2}{n}$ in (2.11) gives

$$\alpha\{\alpha + (n-1)\beta\}\Delta R + 2\left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 - \alpha g(\nabla R, \nabla f) = 2\alpha(n-1)\Delta\lambda + \frac{2\alpha}{n} R\Delta f. \tag{2.12}$$

By hypothesis, since M^n is compact, we get

$$\int_M \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 \mathrm{d}v_g = \frac{\alpha}{2} \int_M g(\nabla R, \nabla f) \mathrm{d}v_g + \frac{\alpha}{n} \int_M R\Delta f \mathrm{d}v_g. \tag{2.13}$$

Also, we know that $\int_M R\Delta f \mathrm{d}v_g = -\int_M g(\nabla R, \nabla f) \mathrm{d}v_g$, then (2.13) becomes

$$\int_M \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 \mathrm{d}v_g = \frac{\alpha(n-2)}{2n} \int_M g(\nabla R, \nabla f) \mathrm{d}v_g. \tag{2.14}$$

Combining (2.2) in (2.14) proves the second part provided $\alpha + (n-1)\beta \neq 0$. This completes the proof. □

Now, for a gradient ARYS $(M^n, g, \nabla f, \lambda)$, from (1.4) and Lemma 2.1 we can write

$$\alpha\left(\mathrm{Ric} - \frac{R}{n}g\right) = \left(\lambda - \frac{\beta R}{2}\right)g - \nabla^2 f - \frac{\alpha R}{n}g,$$
$$= \lambda g - \frac{(2\alpha + n\beta)R}{2n}g - \nabla^2 f,$$
$$= \frac{\Delta f}{n}g - \nabla^2 f.$$

Now, using the foregoing equation in (2.14) yields

$$\int_M \left|\mathrm{Ric} - \frac{R}{n}g\right|^2 \mathrm{d}v_g = \frac{\alpha(n-2)}{2n|\alpha|^2}\int_M g(\nabla R, \nabla f)\mathrm{d}v_g. \tag{2.15}$$

Corollary 2.8 *Let $(M^n, g, \nabla f, \lambda)$ $(n \geq 3)$ be a gradient ARYS with $\alpha \neq 0$. Then we have*

(1) $\{\alpha + (n-1)\beta\}\Delta R + 2\alpha|\mathrm{Ric} - \frac{R}{n}g|^2 - g(\nabla R, \nabla f) = 2(n-1)\Delta\lambda + \frac{2}{n}R\Delta f$.
(2) *If M^n is compact, then* $\int_M |\mathrm{Ric} - \frac{R}{n}g|^2 \mathrm{d}v_g = \frac{(n-2)}{2n\alpha}\int_M g(\nabla R, \nabla f)\mathrm{d}v_g$.

With regard to Theorem 2.7, Corollary 2.8 and Tashiro's result [27] which states that a compact Riemannian manifold (M^n, g) is conformally equivalent to $S^n(r)$ provided there exists a non-trivial function $f : M^n \to \mathbb{R}$ such that $\nabla^2 f = \frac{\Delta f}{n}g$. We obtain the following result which is a generalization of Corollary 1 of [5] and Corollary 1.10 of [12].

Corollary 2.9 *A non-trivial compact gradient ARYS $(M^n, g, \nabla f, \lambda)$ $(n \geq 3)$ with $\alpha \neq \{0, (1-n)\beta\}$ is isometric to a Euclidean sphere $S^n(r)$ if one of the following conditions hold*:

(1) *M^n has constant scalar curvature.*
(2) *M^n is a homogeneous manifold.*
(3) $\int_M [2(n-1)g(\nabla\lambda, \nabla f) + 2\mathrm{Ric}(\nabla f, \nabla f)]\mathrm{d}v_g \geq 0$ *and* $0 < \alpha < (1-n)\beta$ *or* $0 > \alpha > (1-n)\beta$.
(4) $\int_M [2(n-1)g(\nabla\lambda, \nabla f) + 2\mathrm{Ric}(\nabla f, \nabla f)]\mathrm{d}v_g \leq 0$ *with non-negative constants α and β.*

Hodge–de Rham decomposition theorem states that we may decompose the vector field X over a compact oriented Riemannian manifold as a sum of the gradient of a function h and a divergence free vector field Y, i.e.,

$$X = \nabla h + Y \tag{2.16}$$

where $\mathrm{div}\, Y = 0$.

Taking divergence of (2.16) gives $\mathrm{div}\, X = \Delta h$. From the fundamental equation, we have $2\mathrm{div}\, X + (2\alpha + n\beta)R = 2n\lambda$. Therefore, combining both equations result in the following:

$$2\Delta h + (2\alpha + n\beta)R = 2n\lambda. \tag{2.17}$$

On the other hand, if $(M^n, g, \nabla f, \lambda)$ is also a compact gradient ARYS, then from equation (1) of Lemma 2.1, we have

$$2\Delta f + (2\alpha + n\beta)R = 2n\lambda. \tag{2.18}$$

Comparing (2.17) and (2.18), we get $\Delta(h - f) = 0$. Now, by using Hopf's theorem, we see that $f = h + c$, where c is a constant. Hence, we can state the following:

Theorem 2.10 *Let (M^n, g, X, λ) be a compact ARYS. If M^n is also gradient ARYS with potential f, then up to a constant, it agrees with the Hodge–de Rham potential h.*

3 ARYS with certain conditions on the potential vector field

In this section, we consider ARYS whose potential vector field satisfies certain conditions such as conformal, solenoidal and torse-forming vector fields. First we recall the definition of conformal vector field.

A smooth vector field X on a Riemannian manifold is said to be a conformal vector field if there exists a smooth function ψ on M that satisfies

$$\mathcal{L}_X g = 2\psi g.$$

We say that X is non-trivial if X is not Killing, that is, $\psi \neq 0$. Conformal vector field under almost Ricci soliton and almost Ricci–Bourguignon solitons were considered by authors in [5,6] and obtained interesting results. Now, we state and prove the following lemma.

Lemma 3.1 *Let $(n \geq 3)$ be ARYS with $\alpha \neq 0$. If X is a conformal vector field with potential function ψ, then R and $\lambda - \psi$ are constants.*

Proof Since X is a conformal vector field, we have $\mathcal{L}_X g = 2\psi g$. Making use of this in the soliton equation (1.3) yields

$$\alpha \mathrm{Ric} = \left(\lambda - \frac{\beta R}{2} - \psi\right) g, \tag{3.1}$$

which further gives

$$(2\alpha + n\beta)R = 2n(\lambda - \psi), \tag{3.2}$$

and

$$\alpha \ \mathrm{div}\mathrm{Ric} = \nabla\left(\lambda - \frac{\beta R}{2} - \psi\right). \tag{3.3}$$

Making use of Schur's Lemma in (3.3) and inserting it in the covariant derivative of (3.2) results in $(n - 2)\alpha\nabla R = 0$. As $\alpha \neq 0$, then R is constant, which implies then from (3.2) that $\lambda - \psi$ is also constant. This completes the proof. □

Theorem 3.2 *Let (M^n, g, X, λ) $(n \geq 3)$ be a compact ARYS with $\alpha \neq 0$. If X is a non-trivial conformal vector field, then M^n is isometric to Euclidean sphere $S^n(r)$.*

Proof In regard of Lemma 3.1, we know that R and $\lambda - \psi$ are constants. Moreover, using Lemma 2.3 [29] we conclude that $R \neq 0$, otherwise $\psi = 0$, a contradiction as $\psi \neq 0$.

Taking Lie derivative of (3.1) and using the fact that R and $\lambda - \psi$ are constants give

$$\alpha \mathcal{L}_X \mathrm{Ric} = \left(\lambda - \frac{\beta R}{2} - \psi\right)\mathcal{L}_X g = \left(\lambda - \frac{\beta R}{2} - \psi\right)\psi g.$$

Now, applying Theorem 4.2 of [29] to conclude that M^n is isometric to Euclidean sphere $S^n(r)$. This completes the proof. □

Now, we look at gradient ARYS admitting conformal vector field on which we state and prove the following:

Theorem 3.3 *Let $(M^n, g, \nabla f, \lambda)$ $(n \geq 3)$ be a complete gradient ARYS with $\alpha \neq 0$. If ∇f is a non-trivial conformal vector field with non-negative scalar curvature, then either*

(1) *M^n is isometric to a Euclidean space E^n. or*
(2) *M^n is isometric to a Euclidean sphere S^n. Moreover, ψ is a first eigenfunction of Laplacian and $\lambda = \frac{2\alpha+n\beta}{2n}R - \frac{\lambda_1}{n}f + k$, where k is a constant.*

Proof Since ∇f is a non-trivial conformal vector field, we have $\mathcal{L}_{\nabla f} g = 2\psi g$, $\psi \neq 0$. Now, in consequence of argument (1) of Lemma 2.1, we get $\psi = \frac{\Delta f}{n} \neq 0$. Moreover, from Lemma 3.1, we know that R and $\lambda - \psi$ are constants. Suppose $R = 0$, then this implies that M^n is Ricci flat and by using Tashiro's theorem [27] in the fundamental equation, we conclude that M^n is isometric to a Euclidean space E^n. On the other hand, suppose $R \neq 0$. Then, making use of Lemma 2.1 in $\psi = \frac{\Delta f}{n}$ gives $\lambda = \psi + (\frac{2\alpha+n\beta}{2n})R$. As a consequence, (3.1) becomes $\mathrm{Ric} = \frac{R}{n}g$ for $\alpha \neq 0$. Therefore, by involving a theorem by Nagano and Yano [18], we can conclude that M^n is isometric to a Euclidean sphere S^n. Furthermore, taking into account of the fact that $\mathrm{Ric} = \frac{R}{n}g$, we can use Lichnerowicz's theorem [17], the first eigenvalue of the Laplacian of M^n is $\lambda_1 = \frac{R}{n-1}$. Now, we make use of well known formula by Obata and Yano [19], which gives

$$\Delta\psi + \frac{R}{n-1}\psi = 0. \tag{3.4}$$

In view of (3.4), one can easily obtain $\Delta\psi = -\lambda_1\psi$, that is, ψ is a first eigenfunction of the Laplacian. Also, we get $\Delta(\Delta f + \lambda_1 f) = 0$. Then, by Hopf theorem, we obtain $\Delta f + \lambda_1 f = c$, where c is a constant. Combining the last expression with Lemma 2.1 give us the required expression for λ. This completes the proof. □

In [6], the authors consider almost Ricci–Bourguignon soliton and almost η-Ricci–Bourguignon soliton with solenoidal and torse-forming vector field and obtained several rigidity results. Following similar methods, we examine ARYS (M^n, g, ξ, λ) with solenoidal and torse-forming vector fields.

Let ξ be a solenoidal vector field. Then, by taking trace of the ARYS equation (1.3), we get

$$R = \frac{2}{2\alpha + n\beta}(\lambda n - \mathrm{div}(\xi)), \tag{3.5}$$

provided $\alpha \neq -\frac{n\beta}{2}$. If $\alpha = -\frac{n\beta}{2}$, then $\lambda = \frac{\mathrm{div}(\xi)}{n}$. For $\alpha \neq \{0, -\frac{n\beta}{2}\}$, the soliton equation can be written as

$$\frac{1}{2}\mathcal{L}_\xi g + \alpha\mathrm{Ric} = \frac{\beta\mathrm{div}(\xi) + 2\alpha\lambda}{2\alpha + n\beta} g. \tag{3.6}$$

Taking an inner product with Ric in (3.6) gives

$$\begin{aligned}\langle\mathcal{L}_\xi g, \mathrm{Ric}\rangle &= -2\alpha|\mathrm{Ric}|^2\\ &\quad + \frac{4}{(2\alpha + n\beta)^2}[(n\beta - 2\alpha)\mathrm{div}(\xi)\lambda - \beta(\mathrm{div}(\xi))^2 + 2\alpha n\lambda^2].\end{aligned} \tag{3.7}$$

Again, taking an inner product with $\mathcal{L}_\xi g$ in (3.6) and considering $|\mathcal{L}_\xi g|^2 = 4|\nabla\xi|^2$, we have

$$\langle\mathcal{L}_\xi g, \mathrm{Ric}\rangle = -\frac{2}{\alpha}|\nabla\xi|^2 + \frac{2}{\alpha(2\alpha + n\beta)}[\beta(\mathrm{div}(\xi))^2 + 2\alpha\lambda\mathrm{div}(\xi)]. \tag{3.8}$$

Comparing (3.7) and (3.8), we get

$$|\mathrm{Ric}|^2 = \frac{1}{\alpha^2}|\nabla\xi|^2 + \frac{1}{\alpha^2(2\alpha + n\beta)^2}[4\alpha^2 n\lambda^2 - 8\alpha^2\lambda\mathrm{div}(\xi) - (4\alpha + n\beta)\beta(\mathrm{div}(\xi))^2].$$

which leads to the following:

Proposition 3.4 *For an ARYS (M^n, g, ξ, λ) with $\alpha \neq \{0, -\frac{n\beta}{2}\}$ and a solenoidal vector field ξ, we have*

$$|\mathrm{Ric}|^2 \geq \frac{1}{\alpha^2}|\nabla\xi|^2.$$

Now, let ξ be a gradient vector field. Making use of Bochner formula [7], we have

$$\mathrm{Ric}(\xi, \xi) = \frac{1}{2}\Delta(|\xi|^2) - |\nabla\xi|^2 - \xi(\mathrm{div}(\xi)). \tag{3.9}$$

Using (3.5) in the soliton equation (1.3), we get

$$\frac{1}{2}\mathcal{L}_\xi g + \alpha\mathrm{Ric} = \left[\lambda - \frac{\beta}{2\alpha + n\beta}(\lambda n - \mathrm{div}(\xi))\right] g. \tag{3.10}$$

From (3.10), we have

$$\alpha\mathrm{Ric}(\xi, \xi) = -\frac{1}{2}\xi(|\xi|^2) + \left[\lambda - \frac{\beta}{2\alpha + n\beta}(\lambda n - \mathrm{div}(\xi))\right]|\xi|^2. \tag{3.11}$$

Comparing (3.6) and (3.11), we can state the following:

Theorem 3.5 *A gradient ARYS (M^n, g, ξ, λ) with $\alpha \neq \{0, -\frac{n\beta}{2}\}$ has the function λ expressed in terms of ξ as*

$$\lambda = \frac{2\alpha + n\beta}{4\alpha|\xi|^2}[\alpha\Delta(|\xi|^2) - 2\alpha|\nabla\xi|^2 + \xi(|\xi|^2) - 2\alpha\xi\mathrm{div}(\xi)] - \frac{\beta}{2\alpha}\mathrm{div}(\xi).$$

In particular, for $\alpha = 1$ and $\beta = -2\rho$, where $\rho \in \mathbb{R}$ and $\rho \neq \frac{1}{n}$, we recover Theorem 2.2 of [6].

If $\xi = \nabla f$ with f a smooth function on M^n and $\alpha \neq \{0, -\frac{n\beta}{2}\}$, the soliton equation becomes

$$\text{Hess } f + \alpha\text{Ric} = \left(\lambda - \frac{\beta R}{2}\right)g. \tag{3.12}$$

and (3.5) becomes

$$R = \frac{2}{2\alpha + n\beta}(\lambda n - \Delta f). \tag{3.13}$$

Differentiating the above expression gives

$$\mathrm{d}(\Delta f) = n\mathrm{d}\lambda - \frac{2\alpha + n\beta}{2}\mathrm{d}R, \tag{3.14}$$
$$\implies \nabla(\Delta f) = n\nabla\lambda - \frac{2\alpha + n\beta}{2}\nabla R.$$

Taking divergence of (3.12) and using Schur's Lemma, we get

$$\mathrm{div}(\text{Hess } f) = \mathrm{d}\lambda - \frac{\alpha + \beta}{2}\mathrm{d}R. \tag{3.15}$$

Also, from [7], we have

$$\mathrm{div}(\text{Hess } f) = \mathrm{d}(\Delta f) + i_{Q(\nabla f)}g, \tag{3.16}$$

where i denotes the interior product and Q is the Ricci operator.

Comparing (3.15) and (3.16) yields

$$\mathrm{d}(\Delta f) = \mathrm{d}\lambda - \frac{\alpha + \beta}{2}\mathrm{d}R - i_{Q(\nabla f)}g. \tag{3.17}$$

From (3.14) and (3.17), we have

$$(n-1)\mathrm{d}\lambda = \frac{\alpha + (n-1)\beta}{2}\mathrm{d}R - Q(\nabla f). \tag{3.18}$$

Therefore we can state the following:

Proposition 3.6 *For a gradient ARYS on M^n with $\alpha \neq \{0, -\frac{n\beta}{2}\}$, we have*

$$\mathrm{grad}(\lambda) = \frac{\alpha + (n-1)\beta}{2(n-1)}\mathrm{grad}(R) - \frac{1}{n-1}Q(\mathrm{grad}\ f).$$

Moreover, if grad $f \in \mathrm{Ker}(Q)$, then

$$\mathrm{grad}(\lambda) = \frac{\alpha + (n-1)\beta}{2(n-1)}\mathrm{grad}(R). \tag{3.19}$$

In the gradient case, we have $\xi = \nabla f$, if $\alpha \neq \{0, -\frac{n\beta}{2}\}$, then from (3.13), we get

$$\lambda = \frac{2\alpha + n\beta}{2n}R + \frac{\Delta f}{n}. \tag{3.20}$$

Then, (3.12) becomes

$$\text{Hess } f + \alpha\text{Ric} = \frac{\alpha R + \Delta f}{n}g. \tag{3.21}$$

Taking inner product with Ric and Hess f respectively in (3.21) yields

$$\alpha|\mathrm{Ric}|^2 = \frac{\alpha R + \Delta f}{n} R - \langle \mathrm{Hess}\ f, \mathrm{Ric}\rangle, \tag{3.22}$$

and

$$\frac{1}{\alpha}|\mathrm{Hess}\ f|^2 = \frac{\alpha R + \Delta f}{\alpha n}\Delta f - \langle \mathrm{Ric}, \mathrm{Hess}\ f\rangle. \tag{3.23}$$

On comparing (3.22) and (3.23), we get

$$\alpha|\mathrm{Ric}|^2 - \frac{1}{\alpha}|\mathrm{Hess}\ f|^2 = \frac{\alpha^2 R^2 - (\Delta f)^2}{\alpha n},$$

which leads to the following:

Theorem 3.7 *For a gradient ARYS $(M^n, g, \nabla f, \lambda)$ on M^n with $\alpha \neq \{0, -\frac{n\beta}{2}\}$, we have*

$$\frac{1}{\alpha^2}|\mathrm{Hess}\ f|^2 - \frac{(\Delta f)^2}{\alpha^2 n} \leq |\mathrm{Ric}|^2 \leq \frac{1}{\alpha^2}|\mathrm{Hess}\ f|^2 + \frac{R^2}{n}.$$

Again, let us consider a torse forming vector field ξ, then, $\nabla\xi = \gamma I + \psi \otimes \xi$, where γ is a smooth function, ψ is a 1-form and I is the identity endomorphism on the space of vector fields. Then, we have

$$\mathrm{div}(\xi) = n\gamma + \psi(\xi),$$
$$\mathcal{L}_\xi g = 2\gamma g + \psi \otimes \theta + \theta \otimes \psi,$$

where θ is the dual 1-form of ξ. From (1.3), we get for $\alpha \neq \{0, -\frac{n\beta}{2}\}$ that

$$\mathrm{Ric} = \frac{\beta\psi(\xi) - 2\alpha(\gamma - \lambda)}{\alpha(2\alpha + n\beta)} g - \frac{1}{2\alpha}(\psi \otimes \theta + \theta \otimes \psi). \tag{3.24}$$

Thus,

$$Q = \frac{\beta\psi(\xi) - 2\alpha(\gamma - \lambda)}{\alpha(2\alpha + n\beta)} I - \frac{1}{2\alpha}(\psi \otimes \xi + \theta \otimes \zeta),$$

which implies

$$R = \frac{n\beta\psi(\xi) - 2\alpha n(\gamma - \lambda)}{\alpha(2\alpha + n\beta)},$$

where ζ is the dual vector field of ψ.

Computing the Riemann curvature for $\nabla\xi = \gamma I + \psi \otimes \xi$, we get

$$R(X,Y)\xi = (\mathrm{d}\gamma - \gamma\psi)(X)Y - (\mathrm{d}\gamma - \gamma\psi)(Y)X + [(\nabla_X\psi)Y - (\nabla_Y\psi)X]\xi,$$

for any $X, Y \in \chi(M^n)$. If ψ is a Codazzi tensor field, i.e., $(\nabla_X\psi)Y = (\nabla_Y\psi)X$, then

$$\mathrm{Ric}(\xi,\xi) = (1-n)[\xi(\gamma) - \gamma\psi(\xi)]. \tag{3.25}$$

Also, from (3.24), we have

$$\mathrm{Ric}(\xi,\xi) = \frac{|\xi|^2}{\alpha(2\alpha + n\beta)}[2\alpha(\lambda - \gamma) - \{2\alpha + (n-1)\beta\}\psi(\xi)]. \tag{3.26}$$

Then, comparing (3.25) and (3.26) yields

Proposition 3.8 *Let (M^n, g, ξ, λ) defines an ARYS with $\alpha \neq \{0, -\frac{n\beta}{2}\}$ such that ξ is a torse forming vector field and ψ is a Codazzi tensor field, then*

$$\lambda = \gamma + \frac{2\alpha + n\beta}{2|\xi|^2}(1-n)\xi(\gamma) + \frac{1}{\alpha|\xi|^2}[\{2\alpha + (n-1)\beta\}|\xi|^2 + \alpha(n-1)(2\alpha + n\beta)\gamma]\psi(\xi).$$

Let us verify the obtained results by assuming non-trivial examples constructed by Blaga and Tastan [6].

Example On the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3,\ z > 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$ with the Riemannian metric

$$g := \frac{1}{z^2}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2).$$

Then $\left(g, \xi = \frac{\partial}{\partial z}, \lambda = \frac{3\beta}{2\alpha z} - \frac{2\alpha+3\beta}{2\alpha}(2 + \frac{1}{z})\right)$ defines a gradient ARYS.

Precisely, $\xi = \nabla f$ for $f(x, y, z) = -\frac{1}{z}$ where $|\xi|^2 = \frac{1}{z^2}$, $\xi(|\xi|^2) = -\frac{2}{z^3}$, $\Delta(|\xi|^2) = \frac{8}{z^2}$, $|\nabla\xi|^2 = \frac{3}{z^2}$, $\mathrm{div}(\xi) = -\frac{3}{z}$, $\xi(\mathrm{div}(\xi)) = \frac{3}{z^2}$. Therefore, $\lambda = \frac{3\beta}{2\alpha z} - \frac{2\alpha+3\beta}{2\alpha}(2 + \frac{1}{z})$ is obtained from Theorem 3.5.

Example Let $M = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$. Consider the Riemannian metric

$$g := \exp(2z)(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}z^2.$$

Then, $(g, \xi = \exp(z)\frac{\partial}{\partial z}, \lambda = \frac{2\alpha+3\beta}{2\alpha}(\exp(z) - 2\alpha) - \frac{3\beta}{2\alpha}\exp(z))$ defines a gradient ARYS with $\xi = \nabla f$, where $f(x, y, z) = \exp(z)$. On the other hand, one can check that $|\xi|^2 = \exp(2z)$, $\xi(|\xi|^2) = 2\exp(3z)$, $\Delta(|\xi|^2) = 8\exp(2z)$, $|\nabla\xi|^2 = 3\exp(2z)$, $\mathrm{div}(\xi) = 3\exp(z)$, $\xi(\mathrm{div}(\xi)) = 3\exp(2z)$, therefore, $\lambda = \frac{2\alpha+3\beta}{2\alpha}(\exp(z) - 2\alpha) - \frac{3\beta}{2\alpha}\exp(z)$ is immediately obtained from Theorem 3.5.

Acknowledgements The first author is thankful to the Department of Science and Technology, New Delhi, India for financial support in the form of INSPIRE Fellowship (DST/INSPIRE Fellowship/2018/IF180830).

Funding Not applicable.

Declarations

Conflict of interest The authors declared that there is no conflict of interest.

Data Availability Not applicable.

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